Decay of semigroup for an infinite interacting particle system on continuum configuration spaces

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Outline for section 1

Brownian motion

What is the definition of Brownian motion ?

- Physics (before 20th): Brownian motion is the random motion of particles suspended in a fluid (a liquid or a gas) resulting from their collision with the fast-moving molecules in the fluid.
- Mathematics: Brownian motion is a continuous stochastic processes with stationary independent increments.

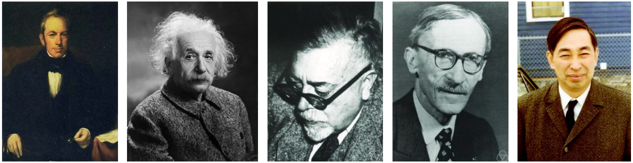
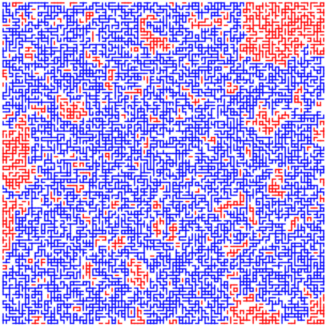


Figure: From left to right is Robert Brown, Albert Einstein, Nobert Wiener, Paul Lévy and Kiyoshi Itô.

Question: What is the gap between the two definitions ?

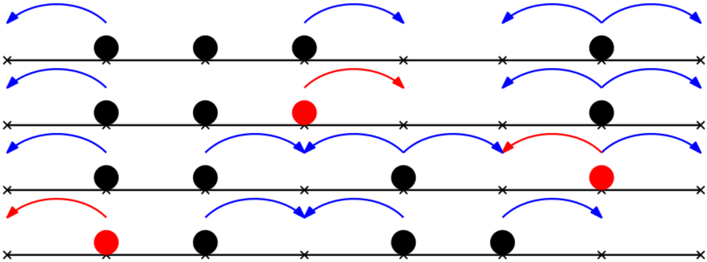
Diffusion in random environment/with interactions

- Random walk on random conductance: Invariant principle for random walk on random conductance/supercritical percolation model. See the survey *Recent progress on the random conductance model (2011)* by M. Biskup.



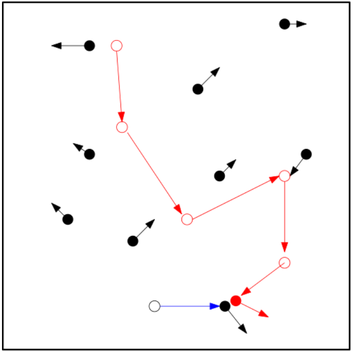
Diffusion in random environment/with interactions

- Simple symmetric exclusion process (SSEP): $\eta \in \{0,1\}^{\mathbb{T}_N^d}$, the hydrodynamic limit of empirical measure $\mu_t^N = \frac{1}{N^d}\sum_{x \in \mathbb{T}_N^d} \eta_{N^2 t}(x)$ is the solution of heat equation. See the book *Scaling limit of interacting particle systems* by C. Kipnis and C. Landim.



Diffusion in random environment/with interactions

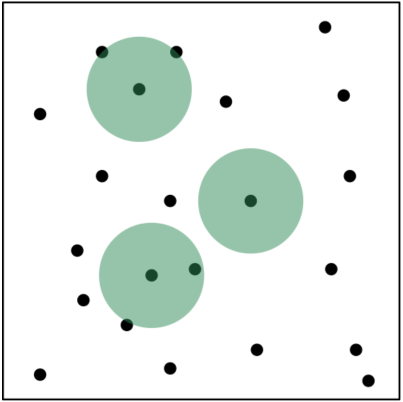
- Hard sphere model: In the system of N particles in $\mathbb{T}^d$ following the collision of Newton law, the trajectory of a tagged particle converges to Brownian motion in $[0, T]$, under the dilute region of Boltzmann-Grad scaling $\varepsilon \to 0, N \to \infty, \varepsilon^{d-1} N \to \alpha$. See the work *The Brownian motion as the limit of a deterministic system of hard-spheres (2015)* of T. Bodineau, I. Gallagher, L. Saint-Raymond.



Outline for section 2

Diffusion on continuum configuration spaces

We want to define a continuum diffusion process, that every particle evolves as a diffusion associated to the generator $-\nabla \cdot \mathbf{a}\nabla$, where the diffusion matrix depends on the local information.



Configuration spaces

- The continuum configuration space: introduced by S. Albeverio, Y.G. Kondratiev and M. Röckner. We use the point measure to define the configuration

$$\mathcal{M}_\delta(\mathbb{R}^d) := \left\{ \mu = \sum_{i \in I} \delta_{x_i} \text{ for some } I \text{ finite or countable,} \right.$$

$$\left. \text{and } x_i \in \mathbb{R}^d \text{ for any } i \in I \right\}. \quad (2.1)$$

- Filtration: for every Borel set $U \subseteq \mathbb{R}^d$, we denote by $\mathcal{F}_U$ the smallest σ-algebra such that for every Borel subset $V \subseteq U$, the mapping $\mu \in \mathcal{M}_\delta(\mathbb{R}^d) \mapsto \mu(V)$ is measurable.
- Probability: fix $\rho > 0$, and define $\mathbb{P}_\rho$ a probability measure on $(\mathcal{M}_\delta(\mathbb{R}^d), \mathcal{F}_{\mathbb{R}^d})$, to be the Poisson measure on $\mathbb{R}^d$ with density ρ. We denote by $\mathbb{E}_\rho$ the expectation, $\mathbb{V}\mathrm{ar}_\rho$ the variance associated with the law $\mathbb{P}_\rho$.

Derivative on configuration spaces

- Derivative: $\mathcal{F}_{\mathbb{R}^d}$-measurable function $f : \mathcal{M}_\delta(\mathbb{R}^d) \to \mathbb{R}$. Let $\{\mathbf{e}_k\}_{1 \leqslant k \leqslant n}$ be d canonical directions, for $x \in \mathrm{supp}(\mu)$, we define

$$\partial_k f(\mu, x) := \lim_{h \to 0} \frac{1}{h}(f(\mu - \delta_x + \delta_{x+h\mathbf{e}_k}) - f(\mu)),$$

if the limit exists, and the gradient as a vector

$$\nabla f(\mu, x) := (\partial_1 f(\mu, x), \partial_2 f(\mu, x), \cdots \partial_d f(\mu, x)).$$

- Function space:
  - $C_c^\infty(\mathcal{M}_\delta(\mathbb{R}^d))$: a function which is $\mathcal{F}_U$ supported with $U \subseteq \mathbb{R}^d$ compact Borel set. Conditioned $\mu(U) = N$, the function is C^∞ with all the coordinates.
  - $H_0^1(\mathcal{M}_\delta(\mathbb{R}^d))$: closure of $C_c^\infty(\mathcal{M}_\delta(\mathbb{R}^d))$ for the norm

$$\|f\|_{H^1(\mathcal{M}_\delta(\mathbb{R}^d))} := \left(\mathbb{E}_\rho\left[f^2\right] + \mathbb{E}_\rho\left[\int_{\mathbb{R}^d} |\nabla f|^2 \, \mathrm{d}\mu\right]\right)^{\frac{1}{2}}.$$

Derivative on configuration spaces

Example

$F \in C_c^\infty(\mathbb{R}^N), \forall 1 \leqslant i \leqslant N, g_i \in C_c^\infty(\mathbb{R}^d)$,

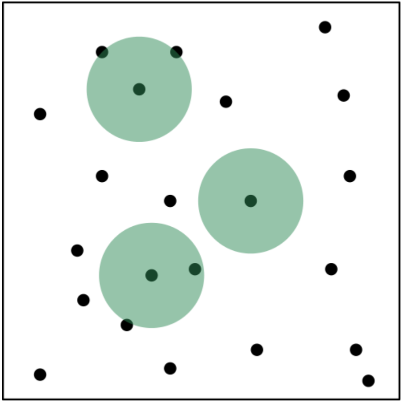
$$f(\mu) := F(\mu(g_1), \cdots \mu(g_N)).$$

Then for its derivative at $x \in \mathrm{supp}(\mu)$

$$\nabla f(\mu, x) = \sum_{i=1}^{N} \nabla_{x_i} F(\mu(g_1), \cdots \mu(g_N)) \nabla g_i(x).$$

Diffusion matrix

- Diffusion matrix: $\mathbf{a}_\circ : \mathcal{M}_\delta(\mathbb{R}^d) \to \mathbb{R}^{d\times d}_{sym}$
  - locality: $\mathcal{F}_{B_1}$-measurable;
  - uniform ellipticity: $\exists \Lambda \in [1, +\infty)$ s.t $\forall \mu \in \mathcal{M}_\delta(\mathbb{R}^d)$, $\forall \xi \in \mathbb{R}^d$, $|\xi|^2 \leqslant \xi \cdot \mathbf{a}_\circ(\mu)\xi \leqslant \Lambda|\xi|^2$.
  - stationarity:$\mathbf{a}(\mu, x) := \tau_x \mathbf{a}_\circ(\mu) = \mathbf{a}_\circ(\tau_{-x}\mu)$.



Diffusion on configuration spaces

- Diffusion defined by Dirichlet form: we define $\mathcal{M}_\delta(\mathbb{R}^d)$-valued Markov process $((\mu_t)_{t\geqslant 0}, (\mathscr{F}_t)_{t\geqslant 0}, (P_t)_{t\geqslant 0})$ by its Dirichlet form
  - Dirichlet form:

$$\mathcal{E}^{\mathbf{a}}(f,g) := \mathbb{E}_\rho\left[\int_{\mathbb{R}^d} \nabla f(\mu,x)\cdot \mathbf{a}(\mu,x)\nabla g(\mu,x)\,\mathrm{d}\mu(x)\right].$$

  - Domain: $\mathcal{D}(\mathcal{E}^{\mathbf{a}}) := H_0^1(\mathcal{M}_\delta(\mathbb{R}^d))$.
  - Characterization: let $u_t = P_t u$, for any $v \in \mathcal{D}(\mathcal{E}^{\mathbf{a}})$

$$\mathbb{E}_\rho[u_t v] - \mathbb{E}_\rho[uv] = -\int_0^t \mathcal{E}^{\mathbf{a}}(u_s, v)\,\mathrm{d}s. \tag{2.2}$$

Main theorem

Theorem (Decay of variance)

There exists two finite positive constants $\gamma := \gamma(\rho, d, \Lambda)$, $C := C(\rho, d, \Lambda)$ *such that for any* $u \in C_c^\infty(\mathcal{M}_\delta(\mathbb{R}^d))$ *supported in* Q_{l_u}, *then we have*

$$\mathbb{V}\mathrm{ar}_\rho[u_t] \leqslant C(1 + |\log t|)^\gamma \left(\frac{1 + l_u}{\sqrt{t}}\right)^d \|u\|^2_{L^\infty}. \tag{2.3}$$

A solvable example

- $\mathbf{a} = \frac{1}{2}$: independent Brownian motion issued from Poisson point process.
- $u(\mu) := \int_{\mathbb{R}^d} f \, \mathrm{d}\mu$ with $f \in C_c^\infty(\mathbb{R}^d)$.
- $\Phi_t(x) = \frac{1}{(2\pi t)^{\frac{d}{2}}} \exp\left(-\frac{|x|^2}{2t}\right)$, then $f_t(x) = \Phi_t \star f(x)$

$$u_t(\mu) = \mathbb{E}_\rho\left[u(\mu_t) | \mathscr{F}_0\right] = \mathbb{E}_\rho\left[\sum_{i \in \mathbb{N}} f\left(B_t^{(i)}\right) \middle| \mathscr{F}_0\right] = \int_{\mathbb{R}^d} f_t(x) \, \mathrm{d}\mu(x),$$

- Under this case, the variance can be calculated

$$\mathbb{V}\mathrm{ar}_\rho\left[u\right] = \rho \int_{\mathbb{R}^d} f^2(x) \, \mathrm{d}x = \rho \|f\|^2_{L^2\left(\mathbb{R}^d\right)},$$

$$\mathbb{V}\mathrm{ar}_\rho\left[u_t\right] = \rho \int_{\mathbb{R}^d} f_t^2(x) \, \mathrm{d}x = \rho \|f_t\|^2_{L^2\left(\mathbb{R}^d\right)}.$$

A solvable example

- By the heat kernel estimate for the standard heat equation, we known that $\|f_t\|^2_{L^2(\mathbb{R}^d)} \simeq C(d)t^{-\frac{d}{2}}\|f\|^2_{L^2(\mathbb{R}^d)}$, thus the scale $t^{-\frac{d}{2}}$ is the best one that we can obtain.
- Moreover, if we take $f = \mathbf{1}_{\{Q_r\}}$, and $t = r^{2(1-\varepsilon)}$ for a small $\varepsilon > 0$, then we see that the typical scale of diffusion is a ball of size $r^{1-\varepsilon}$. So for every $x \in Q_{r\left(1-r^{-\frac{\varepsilon}{2}}\right)}$, the value $f_t(x) \simeq 1 - e^{-r^{\frac{\varepsilon}{2}}}$ and we have

$$\mathbb{V}\mathrm{ar}_\rho\left[u_t\right] = \rho \int_{\mathbb{R}^d} f_t^2(x)\,\mathrm{d}x \geqslant \rho r^d(1 - r^{-\frac{\varepsilon}{2}}) = (1 - r^{-\frac{\varepsilon}{2}})\mathbb{V}\mathrm{ar}_\rho\left[u\right].$$

  It illustrates that before the scale $t = r^2$, the decay is very slow so in the Theorem 2 the factor $\left(\frac{l_u}{\sqrt{t}}\right)^d$ is reasonable.
- Remark: besides this case (linear functional + no interaction), I do not know how to calculate exactly the variance.

Outline for section 3

Zero range model

- An analogue is proved in discrete case: *Relaxation to Equilibrium of Conservative Dynamics. I: Zero-Range Processes, (1999)* by E. Janvresse, C. Landim, J. Quastel, and H. T. Yau.
- Our contributions:
    - Generalization to continuum configuration space without gradient condition.
    - Correct scaling exponent $\left(\frac{l_u}{\sqrt{t}}\right)^d$ and uniform for time.
    - Fix an error in the proof.

Decomposition of approximation and variance

- $\mathcal{Z}_K := \mathbb{Z}^d \cap \left[-\frac{K}{2}, \frac{K}{2}\right]^d$.
- A decomposition of type "approximation - variance".
- $u_t = v_t + w_t$

$$v_t := u_t - \frac{1}{|\mathcal{Z}_K|} \sum_{y \in \mathcal{Z}_K} \tau_y u_t,$$

$$w_t := \frac{1}{|\mathcal{Z}_K|} \sum_{y \in \mathcal{Z}_K} \tau_y u_t.$$

Estimate of variance

Lemma

There exists a finite positive number $C := C(d)$ *such that for any* $u \in C_c^\infty(\mathcal{M}_\delta(\mathbb{R}^d))$ *supported in* Q_{l_u} *and* $K \geqslant l_u$, *we have*

$$\mathbb{V}\mathrm{ar}_\rho\left[\left(\frac{1}{|\mathcal{Z}_K|}\sum_{y\in\mathcal{Z}_K}\tau_y u_t\right)^2\right] \leqslant C(d)\left(\frac{l_u}{K}\right)^d \mathbb{E}_\rho[u^2]. \tag{3.1}$$

Estimate of variance

Proof.

Then we can estimate the variance simply by L^2 decay that

$$\mathbb{E}_\rho[(w_t)^2] = \mathbb{E}_\rho\left[\left(P_t\left(\frac{1}{|\mathcal{Z}_K|}\sum_{y\in\mathcal{Z}_K}\tau_y u\right)\right)^2\right]$$
$$\leqslant \mathbb{E}_\rho\left[\left(\frac{1}{|\mathcal{Z}_K|}\sum_{y\in\mathcal{Z}_K}\tau_y u\right)^2\right] = \frac{1}{|\mathcal{Z}_K|^2}\sum_{x,y\in\mathcal{Z}_K}\mathbb{E}_\rho\left[(\tau_{x-y}u)u\right].$$

We know that for $|x-y| \geqslant l_u$, then the term $\tau_{x-y}u$ and u is independent so $\mathbb{E}_\rho\left[(\tau_{x-y}u)u\right] = 0$. This concludes eq. (3.1). □

Estimate of approximation

Recall that $v_t = u_t - \frac{1}{|\mathcal{Z}_K|} \sum_{y \in \mathcal{Z}_K} \tau_y u_t$.

Lemma

There exists two finite positive numbers $C := C(d, \rho), \gamma := \gamma(d, \rho)$ such that for any $u \in C_c^\infty(\mathcal{M}_\delta(\mathbb{R}^d))$ supported in Q_{l_u}, $K \geqslant l_u$ and v_t defined above, for $t_n \geqslant \max\left\{l_u^2, 16\Lambda^2\right\}, t_{n+1} = Rt_n$ with $R > 1$ we have

$$(t_{n+1})^{\frac{d+2}{2}} \mathbb{E}_\rho[(v_{t_{n+1}})^2] - (t_n)^{\frac{d+2}{2}} \mathbb{E}_\rho[(v_{t_n})^2] \leqslant C(\log(t_{n+1}))^\gamma K^2 (l_u)^d \|u\|_{L^\infty}^2 + \mathbb{E}_\rho[u^2]. \quad (3.2)$$

Proof of the main theorem from two lemmas

Iteration using $K := \sqrt{t_{n+1}}$

$$
\begin{aligned}
&\mathbb{E}_\rho[(u_{t_{n+1}})^2] \\
\leqslant & 2\mathbb{E}_\rho[(v_{t_{n+1}})^2] + 2\mathbb{E}_\rho[(w_{t_{n+1}})^2] \\
\leqslant & 2\left(\frac{t_n}{t_{n+1}}\right)^{\frac{d+2}{2}} \mathbb{E}_\rho[(v_{t_n})^2] + 2\mathbb{E}_\rho[(w_{t_{n+1}})^2] \\
& \qquad + 2(t_{n+1})^{-\frac{d+2}{2}} \left(C(\log(t_{n+1}))^\gamma t_{n+1}(l_u)^d \|u\|_{L^\infty}^2 + \mathbb{E}_\rho[u^2]\right) \\
\leqslant & 4\left(\frac{t_n}{t_{n+1}}\right)^{\frac{d+2}{2}} \mathbb{E}_\rho[(u_{t_n})^2] + 4\left(\frac{t_n}{t_{n+1}}\right)^{\frac{d+2}{2}} \mathbb{E}_\rho[(w_{t_n})^2] + 2\mathbb{E}_\rho[(w_{t_{n+1}})^2] \\
& \qquad + 2(t_{n+1})^{-\frac{d+2}{2}} \left(C(\log(t_{n+1}))^\gamma t_{n+1}(l_u)^d \|u\|_{L^\infty}^2 + \mathbb{E}_\rho[u^2]\right).
\end{aligned} \tag{3.3}
$$

Proof of the main theorem from two lemmas

- $U_n = (t_n)^{\frac{d}{2}}\mathbb{E}_\rho[(u_{t_n})^2]$ and $\theta = 4R^{-1}$

$$U_{n+1} \leqslant \theta U_n + C_2\left((\log(t_{n+1}))^\gamma (l_u)^d \|u\|_{L^\infty}^2 + (t_{n+1})^{-1}\mathbb{E}_\rho[u^2]\right) + C_3(l_u)^d\mathbb{E}_\rho[u^2]$$

- By choose R large such that $\theta \in (0,1)$ and $t_0 = (l_u)^2$

$$\begin{aligned} & U_{n+1} \\ \leqslant & \sum_{k=1}^{n}\left(C_2\left((\log(t_{n+1}))^\gamma (l_u)^d \|u\|_{L^\infty}^2 + \mathbb{E}_\rho[u^2]\right) + C_3(l_u)^d\mathbb{E}_\rho[u^2]\right)\theta^{n-k} \\ & \quad + U_0\theta^{n+1} \\ \leqslant & \frac{1}{1-\theta}\left(C_2\left((\log(t_{n+1}))^\gamma (l_u)^d \|u\|_{L^\infty}^2 + \mathbb{E}_\rho[u^2]\right) + C_3(l_u)^d\mathbb{E}_\rho[u^2]\right) \\ & \quad + (l_u)^d\mathbb{E}_\rho[u^2] \\ \Longrightarrow & \mathbb{E}_\rho[(u_{t_{n+1}})^2] \leqslant C_4(\log(t_{n+1}))^\gamma \left(\frac{l_u}{\sqrt{t_{n+1}}}\right)^d \|u\|_{L^\infty}^2. \end{aligned}$$

Outline for section 4

Localization inequality

- Recall $\mathcal{A}_s f = \mathbb{E}\left[f|\mathcal{F}_{Q_s}\right]$.
- The local information is scale $L > \sqrt{t}$ approximates u_t.

Theorem (Localization inequality)

For $u \in L^2\left(\mathcal{M}_\delta(\mathbb{R}^d)\right)$ of compact support that $\mathrm{supp}(u) \subseteq Q_{l_u}$, any $t \geqslant \max\left\{l_u^2, 16\Lambda^2\right\}$, $K \geqslant \sqrt{t}$, and u_t the function associated to the generator $\mathcal{L}$ at time t, then we have the estimate

$$\mathbb{E}_\rho\left[(u_t - \mathcal{A}_K u_t)^2\right] \leqslant C(\Lambda)\exp\left(-\frac{K}{\sqrt{t}}\right)\mathbb{E}_\rho\left[u^2\right]. \tag{4.1}$$

Remark: in application, we usually choose that $K = \gamma \log t \sqrt{t}$, so that we get $\mathbb{E}_\rho\left[(u_t - \mathcal{A}_K u_t)^2\right] \leqslant t^{-\gamma}\mathbb{E}_\rho\left[u^2\right]$.

Proof: multiscale functional - outline (from JLQY)

Let $\alpha_s = \exp\left(\frac{s}{\beta}\right), \beta > 0$ to be fixed. The key is to consider a multiscale functional

$$S_{k,K,\beta}(f) = \alpha_k \mathbb{E}_\rho\left[(\mathcal{A}_k f)^2\right] + \int_k^K \alpha_s \, d\mathbb{E}_\rho\left[(\mathcal{A}_s f)^2\right] + \alpha_K \mathbb{E}_\rho\left[(f - \mathcal{A}_K f)^2\right]$$
$$= \alpha_K \mathbb{E}_\rho\left[f^2\right] - \int_k^K \alpha_s' \mathbb{E}_\rho\left[(\mathcal{A}_s f)^2\right] ds,$$

and put u_t in the place of f and do derivative (non-trivial)

$$\frac{d}{dt} S_{k,K,\beta}(u_t) \leqslant \frac{2\Lambda^2}{\beta^2} S_{k,K,\beta}(u_t),$$

we choose $k = \sqrt{t} \geqslant l_u$ and $\beta = \sqrt{t}$ to obtain

$$\alpha_K \mathbb{E}_\rho\left[(u_t - \mathcal{A}_K u_t)^2\right] \leqslant S_{k,K,\beta}(u_t)$$
$$\leqslant \exp\left(\frac{2\Lambda^2 t}{\beta^2}\right) S_{k,K,\beta}(u_0) = \exp\left(\frac{2\Lambda^2 t}{\beta^2}\right) \alpha_k \mathbb{E}_\rho\left[(u_0)^2\right].$$

Proof: multiscale functional - a hidden trap

- Warning: in the step of $\frac{d}{dt} S_{k,K,\beta}(u_t)$

$$\frac{d}{dt}\mathbb{E}_\rho\left[-(\mathcal{A}_s u_t)^2\right] = \frac{d}{dt}\mathbb{E}_\rho\left[-\left(\mathcal{A}_s u_t\right) u_t\right] = 2\mathbb{E}_\rho\left[\mathcal{A}_s u_t(-\mathcal{L} u_t)\right].$$

but $\mathcal{A}_s u_t \notin \mathcal{D}(\mathcal{E}^{\mathbf{a}})$. If we pretend it is the case, one may have

$$\begin{aligned}
&\mathbb{E}_\rho\left[\mathcal{A}_s u_t(-\mathcal{L} u_t)\right] \\
=&\mathbb{E}_\rho\left[\int_{Q_{s-1}} \nabla(\mathcal{A}_s f)\cdot \mathbf{a}\nabla f \,\mathrm{d}\mu\right] + \mathbb{E}_\rho\left[\int_{Q_s \setminus Q_{s-1}} \nabla(\mathcal{A}_s f)\cdot \mathbf{a}\nabla f \,\mathrm{d}\mu\right] \\
\leqslant&\mathbb{E}_\rho\left[\int_{Q_{s-1}} \nabla f\cdot \mathbf{a}\nabla f \,\mathrm{d}\mu\right] + \Lambda\mathbb{E}_\rho\left[\int_{Q_s \setminus Q_{s-1}} \nabla f\cdot \mathbf{a}\nabla f \,\mathrm{d}\mu\right].
\end{aligned}$$

Proof: multiscale functional - counter example

Example

Let $\eta \in C_c^{\infty}(\mathbb{R}^d)$ be a plateau function:

$$\operatorname{supp}(\eta) \subseteq B_1, 0 \leqslant \eta \leqslant 1, \eta \equiv 1 \text{ in } B_{\frac{1}{2}},$$
$$\eta(x) = \eta(|x|) \text{ decreasing with respect to } |x|.$$

and we define our function $f(\mu) = \left(\int_{\mathbb{R}^d} \eta(x) \, \mathrm{d}\mu(x)\right) \wedge 3$.
We define the level set B_r such that

$$B_r := \left\{ x \in \mathbb{R}^d \,\middle|\, \frac{1}{2} \leqslant \eta(x) \leqslant 1 \right\}.$$

Then, we have $\mathbb{E}_\rho[f|\mathcal{F}_{B_r}] \notin C_c^{\infty}(\mathcal{M}_\delta(\mathbb{R}^d))$.

Proof: multiscale functional - counter example

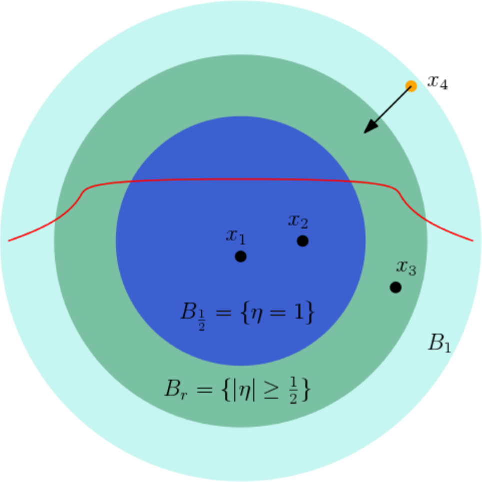


Figure: $f(\mu) = \left(\int_{\mathbb{R}^d} \eta(x)\, \mathrm{d}\mu(x)\right) \wedge 3$, then $\mathbb{E}_\rho[f|\mathcal{F}_{B_r}]$ is not smooth in this example.

Proof: multiscale functional - regularity of $\mathcal{A}_s f$

Recall that $\mathcal{A}_s f = \mathbb{E}_\rho[f|\mathcal{F}_{Q_s}]$, it is both a function and a martingale with respect to $(\mathcal{F}_{Q_s})_{s\geqslant 0}$. Note $\mathscr{M}_s^f := \mathcal{A}_s f$.

Lemma

With probability 1, for any $0 < s < \infty$, there is at most one particle one the boundary ∂Q_s.

Lemma

After a modification, for any $f \in C_c^\infty(\mathcal{M}_\delta(\mathbb{R}^d))$ the process $\left(\mathscr{M}_s^f\right)_{s\geqslant 0}$ is a càdlàg L^2-martingale with finite variation, and the discontinuity point occurs for s such that $\mu(\partial Q_s) = 1$.

Proof: multiscale functional - regularization

- We do some regularization

$$\mathcal{A}_{s,\varepsilon} u_t := \frac{1}{\varepsilon} \int_0^{\varepsilon} \mathcal{A}_{s+r} u_t \, dr.$$

- We study the regularized multiscale functional $S_{k,K,\beta,\varepsilon}(u_t)$

$$S_{k,K,\beta,\varepsilon}(u_t) = \alpha_k \mathbb{E}_\rho \left[(\mathcal{A}_{k,\varepsilon} u_t)^2 \right] + \int_k^K \alpha_s \left(\tfrac{d}{ds} \mathbb{E}_\rho \left[(\mathcal{A}_{s,\varepsilon} u_t)^2 \right] \right) ds + \alpha_K \mathbb{E}_\rho \left[f^2 - (\mathcal{A}_{K,\varepsilon} u_t)^2 \right]$$

- Then we focus on

$$\frac{d}{dt} S_{k,K,\beta,\varepsilon}(u_t) = 2\alpha_K \mathbb{E}_\rho \left[u_t (\mathcal{L} u_t) \right] + \int_k^K 2\alpha_s' \mathbb{E}_\rho \left[\widetilde{\mathcal{A}_{s,\varepsilon}} u_t (-\mathcal{L} u_t) \right] ds,$$

where

$$\widetilde{\mathcal{A}_{s,\varepsilon}} u_t := \frac{2}{\varepsilon^2} \int_0^{\varepsilon} (\varepsilon - r) \mathcal{A}_{s+r} u_t \, \mathrm{d}r.$$

Proof: multiscale functional - regularization

$$
\begin{aligned}
&\mathbb{E}_\rho\left[\widetilde{\mathcal{A}_{s,\varepsilon}} f(-\mathcal{L}f)\right] \\
=&\mathbb{E}_\rho\left[\int_{Q_{s-1}} \nabla(\widetilde{\mathcal{A}_{s,\varepsilon}} f)\cdot \mathbf{a}\nabla f \,\mathrm{d}\mu\right] + \mathbb{E}_\rho\left[\int_{Q_s \backslash Q_{s-1}} \nabla(\widetilde{\mathcal{A}_{s,\varepsilon}} f)\cdot \mathbf{a}\nabla f \,\mathrm{d}\mu\right] \\
&\quad + \mathbb{E}_\rho\left[\int_{Q_{s+\varepsilon} \backslash Q_s} \nabla(\widetilde{\mathcal{A}_{s,\varepsilon}} f)\cdot \mathbf{a}\nabla f \,\mathrm{d}\mu\right] \\
\leqslant&\mathbb{E}_\rho\left[\int_{Q_{s-1}} \nabla f\cdot \mathbf{a}\nabla f \,\mathrm{d}\mu\right] + \Lambda\mathbb{E}_\rho\left[\int_{Q_s \backslash Q_{s-1}} \nabla f\cdot \mathbf{a}\nabla f \,\mathrm{d}\mu\right] \\
&\quad + \left(\frac{\theta\Lambda}{\varepsilon}+\Lambda\right)\mathbb{E}_\rho\left[\int_{Q_{s+\varepsilon} \backslash Q_s^{\circ}} \nabla f\cdot \mathbf{a}\nabla f \,\mathrm{d}\mu\right] + \frac{\Lambda}{2\theta}\frac{d}{ds}\mathbb{E}_\rho\left[(\mathcal{A}_{s,\varepsilon} f)^2\right].
\end{aligned}
$$

The term in red is the one finally contributes to the analysis. One intermediate step will appear a miracle L^2 martingale isometry.

Outline for section 5

Discussion

1. Can we reduce the term logarithm?
2. Can we identify the long time behavior like in the work of zero range model that

$$\mathbb{V}\mathrm{ar}_{\rho}[u_t] = \frac{[\widetilde{u}'(\rho)]^2 \chi(\rho)}{[8\pi\phi'(\rho)t]^{\frac{d}{2}}} + o\left(t^{-\frac{d}{2}}\right).$$

3. Long range interaction ?
4. Diffusion on other manifold rather than $\mathbb{R}^d$?
5. Other type of dynamics?

Thank you for your attention.