Name: ____________________ **NetId:** __________

Probability Limit Theorems

Final Exam, Fall 2021

DO NOT OPEN YET

...and wait until the proctor announces that it is time to start.

In the mean time, please write your name and NetID legibly,
and **read the instructions below carefully**.

* Please do not fold or damage the exam papers. After you finish your exam, please put the pages in correct order back into the sleeve.

* There are 6 questions in this exam, the sleeve should contain 8 pieces of paper.

* The scratch paper is included: three last papers are blank. If your solution continues on scratch paper, please clearly indicate it.

* For questions asking to prove a result, the clarity of the mathematical argument will be taken into account in the score.

* Questions formulated in terms of real functions should be answered with real functions.

Good luck!

1. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. Let X be a real-valued random variable, $(X_n)_{n\in\mathbb{N}}$ be a sequence of real-valued random variables in this probability space.

   (Throughout this exercise, you may use elementary properties of measures and measurable functions without proof provided they are clearly stated.)

   (a) i. What does it mean to say $(X_n)_{n\in\mathbb{N}}$ converges to X almost surely ? What does it mean to say X_n converges to X in probability ?

   ii. Show, using Reverse Fatou's Lemma, that if $(X_n)_{n\in\mathbb{N}}$ converges to X almost surely, then $(X_n)_{n\in\mathbb{N}}$ converges to X in probability.

   (b) i. What does it mean to say that random variables $X_n, n \in \mathbb{N}$ are independent ? State the second Borel-Cantelli Lemma on $(\Omega, \mathcal{F}, \mathbb{P})$. [No proof is required.]

   ii. Show, using the first Borel-Cantelli Lemma that if X_n converges to X in probability, then there exists a strictly increasing sequence $n_k \xrightarrow{k\to\infty} \infty$ such that $(X_{n_k})_{k\in\mathbb{N}}$ converges to X almost surely.

   iii. A sequence of real-valued random variables $(X_n)_{n\in\mathbb{N}}$ is said to be completely convergent if

   $$\sum_{n=1}^{\infty} \mathbb{P}[|X_n - X| > \epsilon] < \infty,$$

   for all $\epsilon > 0$. Show that for sequence of independent random variables, complete convergence is equivalent to almost surely convergence.

   iv. Find a sequence of (dependent) random variables which converge almost surely but not completely.
   [Hint: you may choose $X_n = a_n X$ for an adequate random variable X and real sequence $(a_n)_{n\in\mathbb{N}}$.]

2. (a) Let Z be random variable of law Poisson(λ), i.e.

$$\forall k \in \mathbb{N}, \qquad \mathbb{P}[Z = k] = e^{-\lambda}\frac{\lambda^k}{k!}. \tag{1}$$

Calculate $\mathbb{P}[Z = 0]$, $\mathbb{P}[Z = 1]$ and $\mathbb{E}[Z]$.

(b) Compute the characteristic function of Z.

(c) State Lévy's convergence theorem.

(d) Let $(X_{n,i})_{n,i\in\mathbb{N}}$ be independent Bernoulli random variables with

$$\mathbb{P}[X_{n,i} = 1] = p_{n,i}, \qquad \mathbb{P}[X_{n,i} = 0] = 1 - p_{n,i}.$$

Suppose that

- $\sum_{i=1}^n p_{n,i} \xrightarrow{n\to\infty} \lambda \in (0,\infty)$;
- $\max_{1\le i\le n} p_{n,i} \xrightarrow{n\to\infty} 0$.

Let $S_n := \sum_{i=1}^n X_{n,i}$, prove that $S_n \xrightarrow[n\to\infty]{w} Z$, where Z has a law Poisson(λ) defined in (1).

3. Let X_n be a sequence of independent real random variables which converges in probability to the limit X. Show that X is almost surely constant.

4. Let X be an L^1 real random variable, and for $\delta > 0$, set

$$I_X(\delta) = \sup\{\mathbb{E}[|X|\mathbf{1}_A] : A \in \mathcal{F}, \mathbb{P}[A] \leq \delta\}.$$

Using Dominated Convergence Theorem, show that

$$\lim_{\delta \to 0} I_X(\delta) = 0.$$

5. Let X_n be i.i.d. random variables such that

$$\mathbb{P}[X_i = 1] = \mathbb{P}[X_i = -1] = \frac{1}{2},$$

and $S_n := \sum_{i=1}^n X_i$. We also define the natural filtration $\mathcal{F}_n = \sigma((X_i)_{1 \leq i \leq n})$ and a quantity that

$$\forall a \in \mathbb{Z}_+, \tau_a = \min\{n \geq 0 : S_n = -a\}.$$

(a) Check that τ_a is a stopping time with respect to $(\mathcal{F}_n)_{n \in \mathbb{N}}$.

(b) Check that for any $\theta \in \mathbb{R}$, $Y_n := \exp(\theta S_n) \left(\frac{e^\theta + e^{-\theta}}{2}\right)^{-n}$ is a martingale.

(c) Prove that, when $\theta < 0$, the martingale $(Y_{n \wedge \tau_a})_{n \in \mathbb{N}}$ is bounded.

(d) Define

$$A_+ = \left\{ \limsup_{n \to \infty} S_n = +\infty \right\}, \qquad A_- = \left\{ \liminf_{n \to \infty} S_n = -\infty \right\}.$$

i. Prove that $\mathbb{P}[A_+], \mathbb{P}[A_-] \in \{0, 1\}$.

ii. Show, using the Central Limit Theorem that, $\mathbb{P}[A_+ \cup A_-] = 1$.

iii. Conclude that, for all $a \in \mathbb{Z}_+$, $\mathbb{P}[\tau_a < \infty] = 1$.

(e) Prove that, for every $s \in (0, 1)$, one has

$$\mathbb{E}[s^{\tau_a}] = \left(\frac{1 - \sqrt{1 - s^2}}{s}\right)^a.$$

(f) For $a = 1$, use the formula above to compute explicitly the probabilities $\mathbb{P}[\tau_a = 2k - 1]$ for $k \geq 1$.

6. Assume that (f_n), (g_n), f, $g \in L^1(\mathbb{R})$, $f_n \to f$ a.s., $g_n \to g$ a.s., $|f_n| \leq g_n$ a.s. and

$$\int_{\mathbb{R}} g_n d\mu \to \int_{\mathbb{R}} g d\mu.$$

Show that

$$\int_{\mathbb{R}} f_n d\mu \to \int_{\mathbb{R}} f d\mu.$$