MATH-SHU 362

Homework 1: First Order ODE

Due: 09/24/2021

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Throughout the exercises, we always use $y(t)$ to represent the unknown function and t for variable.

Exercise 1. *(3 points) Draw the directional field of* $y' = \sin(t), y' = \cos(t), y' = y\cos(t)$.

Exercise 2. *(3 points) Solve the following initial value problem.*

1. $y' = \sin(5t)$ *for* $y(0) = 2$.
2. $y' = e^t + t$ *for* $y(0) = 0$.
3. $y' = (y-1)(y+1)$ *for* $y(0) = 3$.

Exercise 3. *(3 points) Take* $y' = f(t,y)$, $y(0) = 0$, *where* $f(t,y) > 1$ *for all* t *and* y *. If the solution exists for all* t*, can you say what happens to* $y(t)$ *as* t *goes to positive infinity? Explain.*

Exercise 4. *(3 points) Is it possible to solve the equation* $y' = y\sqrt{|t|}$ *for* $y(0) = 0$ *? Is the solution unique? Justify.*

Exercise 5 (A simple example of fixed point)**.** *(3 points) Let* $f(x) = \frac{x^2+1}{2}$*, and we construct an iteration* $x_{n+1} = f(x_n)$. *Then, for any* $x_0 \in [-1,1]$, *prove that*

1. *This iteration admits a limit that* $\lim_{n\to\infty} x_n = x_*$.
2. *This limit* x_* *does not depend on the initial value.*
3. *Calculate* x_*.

Exercise 6 (On Banach fixed point theorem)**.** *(5 points)*

1. *State the Banach fixed point theorem.*
2. *Explain why this theorem requires a complete metric space.*
3. *Prove this theorem.*
4. *Justify briefly how this theorem is applied to Picard's iteration.*