MATH-SHU 362

Homework 3: Series Solution for ODE

Due: 11/17/2021

Name: *Univ ID:*

Throughout the exercises, we always use $y(x)$ to represent the unknown function and x for variable.

Exercise 1. *Find all singular points of the given equation and determine whether each one is regular or irregular.*

1. $x^2(1-x)y'' + (x-2)y' - 3xy = 0$;
2. $y'' + (\ln|x|)y' + 3xy = 0$.

Exercise 2. *For the following equations, show that the given differential equation has a regular singular point at $x = 0$, and determine the indicial equation, and the roots of the indicial equation.*

1. $xy'' + y' - y = 0$;
2. $x^2y'' + xy' + (x-2)y = 0$.

Exercise 3. *Let $y(x)$ be the solution of Euler equation*

$$x^2y'' + \alpha xy' + \beta y = 0,$$

on $x \in \mathbb{R}^+$ with $\alpha, \beta \in \mathbb{R}$. With a change of variable $t = \ln x$ and $z(t) = y(e^t)$, write down the second order ODE with constant coefficient for function z and variable t.

Exercise 4. *In this question, we prove that there is no series which converges / diverges slowest.*

1. *Prove that for any absolute convergent series $\sum_{n=1}^{\infty} a_n$, there exists another absolute convergent series $\sum_{n=1}^{\infty} b_n$ such that $\lim_{n\to\infty} \frac{b_n}{a_n} = \infty$.*
2. *Prove that for any divergent series $\sum_{n=1}^{\infty} a_n$, there exists another divergent series $\sum_{n=1}^{\infty} b_n$ such that $\lim_{n\to\infty} \frac{b_n}{a_n} = 0$.*

Exercise 5. *In this question, we aim to review the proof of Fuchs' theorem. We consider the equation*

$$y''(x) + p(x)y'(x) + q(x)y(x) = 0, \tag{1}$$

where p, q are analytic that $p(x) = \sum_{n=0}^{\infty} b_n x^n$ with radius of convergence $\rho_1 > 0$, and $q(x) = \sum_{n=0}^{\infty} c_n x^n$ with radius of convergence $\rho_2 > 0$. We start by writing y formally as series $y(x) = \sum_{n=0}^{\infty} a_n x^n$.

1. *Write down $y'(x)$ and $y''(x)$ as series. What is the radius of convergence of y'' if y has a radius of convergence ρ ?*
2. *Identify $y(0)$ and $y'(0)$ using the series $(a_n)_{n\in\mathbb{N}}$*
3. *Establish the recurrence equation of $(a_n)_{n\in\mathbb{N}}$. Express a_n with $\{a_i\}_{0\leqslant i\leqslant n-1}$ and $\{b_j\}_{j\in\mathbb{N}}$ and $\{c_k\}_{k\in\mathbb{N}}$.*
4. *Using induction to prove that, for all $r < \min\{\rho_1, \rho_2\}$, $\{|a_n|r^n\}_{n\in\mathbb{N}}$ is bounded.*