Lecture 3: Area of Surface

Chenlin GU

DMA/ENS, PSL Research University

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Outline for section 1

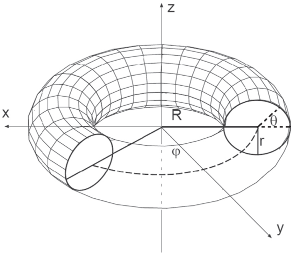
Outline

1. Surface (2-d Manifold)
   - What is a Surface ?
   - A First Look at the Calculus of Area
2. Metric Tensor
   - Length
   - Area

What is a surface in $\mathbb{R}^3$

Some example of surface in $\mathbb{R}^3$

- Hyperplane: $z = 0$.
- Sphere: $\mathbb{S}^2$.
- Torus:
- $\cdots\cdots$.



What is a Surface ?

Definition (Smooth Surface)

A surface in $\mathbb{R}^3$ locally looks like a subset $U \subset \mathbb{R}^2$. That is there exists a C^1 map $(u, v) \mapsto (x, y, z)$ such that the Jacob matrix

$$\frac{\partial(x, y, z)}{\partial(u, v)}$$

has rank 2, i.e. $(x_u, y_u, z_u) \neq \lambda(x_v, y_v, z_v)$.

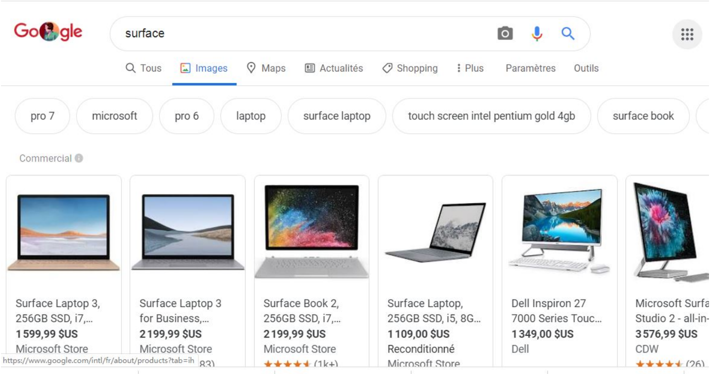
What is a Surface ?

Examples:

- Hyperplane: $x = u, y = v, z = 0$.
- Sphere: $x = \cos u \cos v, y = \cos u \sin v, z = \sin u$.
- Torus: $x = (R + r\cos u)\cos v, y = (R + r\cos u)\sin v, z = \sin u$.

What is a Surface ?

Surface (of Microsoft) is not a smooth surface (in maths).



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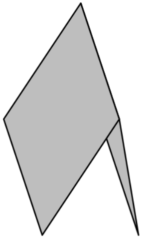
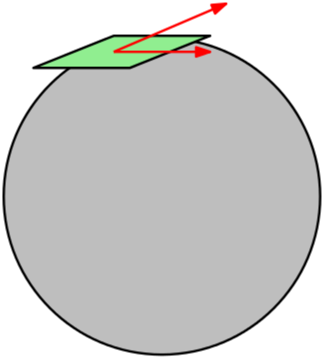


Outline

Cross Product



$$\text{area} = |(x_u, y_u, z_u) \times (x_v, y_v, z_v)| = \det \begin{pmatrix} i & j & k \\ x_u & y_u & z_u \\ x_v & y_v & z_v \end{pmatrix}$$

A First Look at the Calculus of Area

Let S be a smooth surface, $S = \sqcup_{i=1}^{N} \Phi_i$, where $\Phi_i : (u, v) \to (x, y, z)$ is C^1 map, then we have

$$\mathrm{Area}(S) = \sum_{i=1}^{N} \int |(x_u, y_u, z_u) \times (x_v, y_v, z_v)| \, du dv.$$

Outline for section 2

Generalization of Surface in $\mathbb{R}^d$

A surface M in $\mathbb{R}^d$ locally looks like a subset $U \subset \mathbb{R}^2$. That is locally there exists a C^1 map $X : (u, v) \to (x_1, x_2, \cdots x_d)$ and $X_u \neq \lambda X_v$.

Metric Tensor

- Tangent plane T_pM at the position $p \in M$:
  $T_pM = \mathrm{Vec}\{X_u(p), X_v(p)\}$.
- Metric tensor

$$
\begin{aligned}
g(p) &= \begin{pmatrix} g_{uu}(p) & g_{uv}(p) \\ g_{uv}(p) & g_{vv}(p) \end{pmatrix} \\
&= \begin{pmatrix} \langle X_u(p), X_u(p) \rangle & \langle X_u(p), X_v(p) \rangle \\ \langle X_u(p), X_v(p) \rangle & \langle X_v(p), X_v(p) \rangle \end{pmatrix}.
\end{aligned}
$$

- $g(p)$ contains much information.

Outline

Length of Curve

For a regular curve $\gamma : [a, b] \to M, t \mapsto X(u(t), v(t))$, the length of curve is that

$$|\gamma| = \int_a^b \sqrt{g_{uu}(\gamma(t))(u'(t))^2 + 2g_{uv}(\gamma(t))u'(t)v'(t) + g_{vv}(\gamma(t))(v'(t))^2}\, dt.$$

Outline

Area

In fact, what we need is $|X_u||X_v|\sin\theta$ rather than $|X_u \times X_v|$.
We use the formula that $\langle X_u, X_v\rangle = |X_u||X_v|\cos\theta$, thus we have

$$\begin{aligned} |X_u||X_v|\sin\theta &= \sqrt{(|X_u||X_v|)^2 - \langle X_u, X_v\rangle^2} \\ &= \sqrt{\det(g)}, \end{aligned}$$

with g the metric tensor.

$$\mathrm{Area}(M) = \sum_{i=1}^{N} \int |X_u \times X_v| \, du dv = \sum_{i=1}^{N} \int_{M\cap\Phi_i} \sqrt{\det(g)}.$$

Recap: Green's Theorem

Theorem (Green's Theorem)

Let $D \subset \mathbb{R}^2$ be a region, with boundary ∂D is piece-wise smooth, positively oriented, closed and let $F = Pdx + Qdy$ a C^1 1-form on D, then we have

$$\int_D dF = \int_{\partial D} F. \tag{2.1}$$

Question: Can it also be true for D a surface?