

# Lecture 3: Area of Surface

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# Outline for section 1

## 1 Surface (2-d Manifold)

- What is a Surface ?
- A First Look at the Calculus of Area

## 2 Metric Tensor

- Length
- Area

# Outline

## 1 Surface (2-d Manifold)

- What is a Surface ?
- A First Look at the Calculus of Area

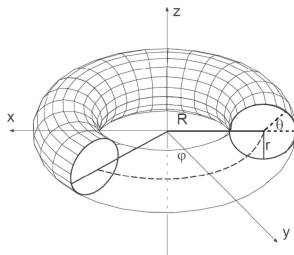
## 2 Metric Tensor

- Length
- Area

# What is a surface in $\mathbb{R}^3$

Some example of surface in  $\mathbb{R}^3$

- Hyperplane:  $z = 0$ .
- Sphere:  $\mathbb{S}^2$ .
- Torus:
- .....



# What is a Surface ?

## Definition (Smooth Surface)

A surface in  $\mathbb{R}^3$  locally looks like a subset  $U \subset \mathbb{R}^2$ . That is there exists a  $C^1$  map  $(u, v) \mapsto (x, y, z)$  such that the Jacob matrix

$$\frac{\partial(x, y, z)}{\partial(u, v)}$$

has rank 2, i.e.  $(x_u, y_u, z_u) \neq \lambda(x_v, y_v, z_v)$ .

# What is a Surface ?

## Examples:

- Hyperplane:  $x = u, y = v, z = 0$ .
- Sphere:  $x = \cos u \cos v, y = \cos u \sin v, z = \sin u$ .
- Torus:  $x = (R + r \cos u) \cos v, y = (R + r \cos u) \sin v, z = \sin u$ .

# What is a Surface ?

Surface (of Microsoft) is not a smooth surface (in maths).

The image shows a Google search results page for the query "surface". The search bar contains the word "surface" and the Google logo is on the left. Below the search bar are navigation options: "Tous", "Images", "Maps", "Actualités", "Shopping", "Plus", "Paramètres", and "Outils". There are also filter buttons for "pro 7", "microsoft", "pro 6", "laptop", "surface laptop", "touch screen intel pentium gold 4gb", and "surface book".

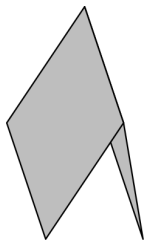
Under the filters, there is a "Commercial" filter. Below that, six product listings are shown, each with an image of the device, its name, specifications, price, and the retailer.

| Product Name                         | Specifications          | Price         | Retailer                      |
|--------------------------------------|-------------------------|---------------|-------------------------------|
| Surface Laptop 3                     | 256GB SSD, i7,...       | 1 599,99 \$US | Microsoft Store               |
| Surface Laptop 3 for Business,...    | 256GB SSD, i7,...       | 2 199,99 \$US | Microsoft Store               |
| Surface Book 2                       | 256GB SSD, i7,...       | 2 199,99 \$US | Microsoft Store (1k+)         |
| Surface Laptop                       | 256GB SSD, i5, 8G...    | 1 109,00 \$US | Reconditionné Microsoft Store |
| Dell Inspiron 27 7000 Series Touc... | 27" monitor, i5, 8G...  | 1 349,00 \$US | Dell                          |
| Microsoft Surfa Studio 2 - all-in-   | 28" monitor, i7, 16G... | 3 576,99 \$US | CDW (76)                      |

URL: <https://www.google.com/fint/fi/about/products?tab=ih>

# What is a Surface ?

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# Outline

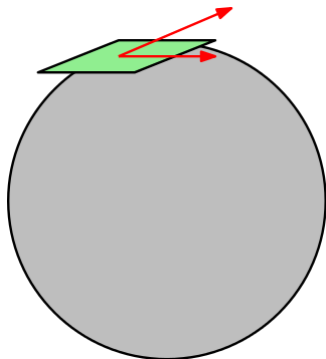
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# Cross Product



$$\text{area} = |(\mathbf{x}_u, y_u, z_u) \times (\mathbf{x}_v, y_v, z_v)| = \det \begin{pmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ x_u & y_u & z_u \\ x_v & y_v & z_v \end{pmatrix}$$

# A First Look at the Calculus of Area

Let  $S$  be a smooth surface,  $S = \sqcup_{i=1}^N \Phi_i$ , where  $\Phi_i : (u, v) \rightarrow (x, y, z)$  is  $C^1$  map, then we have

$$\text{Area}(S) = \sum_{i=1}^N \int |(x_u, y_u, z_u) \times (x_v, y_v, z_v)| \, du \, dv.$$

# Outline for section 2

- 1 Surface (2-d Manifold)
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# Generalization of Surface in $\mathbb{R}^d$

A surface  $M$  in  $\mathbb{R}^d$  locally looks like a subset  $U \subset \mathbb{R}^2$ . That is locally there exists a  $C^1$  map  $X : (u, v) \rightarrow (x_1, x_2, \dots, x_d)$  and  $X_u \neq \lambda X_v$ .

# Metric Tensor

- Tangent plane  $T_pM$  at the position  $p \in M$ :  
 $T_pM = \text{Vec}\{X_u(p), X_v(p)\}$ .
- Metric tensor

$$\begin{aligned}g(p) &= \begin{pmatrix} g_{uu}(p) & g_{uv}(p) \\ g_{uv}(p) & g_{vv}(p) \end{pmatrix} \\ &= \begin{pmatrix} \langle X_u(p), X_u(p) \rangle & \langle X_u(p), X_v(p) \rangle \\ \langle X_u(p), X_v(p) \rangle & \langle X_v(p), X_v(p) \rangle \end{pmatrix}.\end{aligned}$$

- $g(p)$  contains much information.

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# Length of Curve

For a regular curve  $\gamma : [a, b] \rightarrow M, t \mapsto X(u(t), v(t))$ , the length of curve is that

$$|\gamma| = \int_a^b \sqrt{g_{uu}(\gamma(t))(u'(t))^2 + 2g_{uv}(\gamma(t))u'(t)v'(t) + g_{vv}(\gamma(t))(v'(t))^2} dt.$$



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# Area

In fact, what we need is  $|X_u||X_v| \sin \theta$  rather than  $|X_u \times X_v|$ .

We use the formula that  $\langle X_u, X_v \rangle = |X_u||X_v| \cos \theta$ , thus we have

$$\begin{aligned} |X_u||X_v| \sin \theta &= \sqrt{(|X_u||X_v|)^2 - \langle X_u, X_v \rangle^2} \\ &= \sqrt{\det(g)}, \end{aligned}$$

with  $g$  the metric tensor.

$$\text{Area}(M) = \sum_{i=1}^N \int |X_u \times X_v| \, du dv = \sum_{i=1}^N \int_{M \cap \Phi_i} \sqrt{\det(g)}.$$

## Recap: Green's Theorem

### Theorem (Green's Theorem)

Let  $D \subset \mathbb{R}^2$  be a region, with boundary  $\partial D$  is piece-wise smooth, positively oriented, closed and let  $F = Pdx + Qdy$  a  $C^1$  1-form on  $D$ , then we have

$$\int_D dF = \int_{\partial D} F. \quad (2.1)$$

**Question:** Can it also be true for  $D$  a surface?