PROCESSUS STOCHASTIQUES 29-01-2021

2 hours 30 minutes. No documents allowed.

1. Exercise 1

Part 1.

(1) We have to prove that one can go from any integer i to any integer j in a finite number of steps with positive probability. The chain has a positive probability to from i to 1 in the first step and then make $j-1$ steps from 1 to j. This settles the irreducibility. As for aperiodicity, it is enough to notice that $Q(1,1)>0$.

(2) Let us show first that the chain is recurrent. Let $A := \{\omega, (X_n(\omega))_{n\geqslant 0} \text{ diverges to infinity}\}$. Clearly $A \subseteq \bigcup_{n\geqslant 1}\bigcap_{k\geqslant n}\{X_{k+1}(\omega)-X_k(\omega)=1\}$. The RHS has a probability zero since using the explicit transition probabilities and passing to the limit one gets $\mathbb{P}(\bigcap_{k\geqslant n}\{X_{k+1}(\omega)-X_k(\omega)=1\})=0$.
To show null-recurrence we can compute the invariant measure (which exists and is unique up to a multiplicative constant because the chain is irreducible and recurrent). The chain is then positive recurrent if and only if any invariant measure has finite mass. The invariant measure π has to satisfy for all $k\geqslant 1$, $\pi(k+1)=\frac{k}{k+1}\pi(k)$, then $\pi(k+1)=\frac{1}{k+1}\pi(1)$. This implies that $\sum_{k\geqslant 1}\pi(k)=\infty$ which concludes.
Alternatively, for this chain it is easy to compute $\mathbb{P}_1(T_1=n)$, in fact there is only one *excursion* from 1 to 1 (i.e., only one way to go from 1 to 1 without visiting 1). So $\mathbb{P}_1(T_1=n)=(1/2)(2/3)\cdots((n-2)/(n-1))((n-1)/n)(1/(n+1))=1/(n(n+1))=1/n-1/(n+1)$. Since $\sum_{n=1}^{\infty}\mathbb{P}_1(T_1=n)=1$ the chain is recurrent. Added to that, $\mathbb{E}_1[T_1]=\sum_{n=1}^{\infty}1/(n+1)=\infty$, so the chain is null recurrent.

(3) $\lim_n \mathbb{P}(X_n=2)/\mathbb{P}(X_n=3)=\frac{3}{2}$ but this result requires techniques beyond what we did in the course (sorry! We made a mistake). In any case, $\mathbb{P}(X_n=2)/\mathbb{P}(X_n=3)=(\mathbb{P}(X_{n-1}=1)/(\mathbb{P}(X_{n-2}=1)Q(1,2)/(Q(1,2)Q(2,3))$ so it suffices to show that $\lim_n \mathbb{P}(X_n=1)/\mathbb{P}(X_{n+1}=1)=1$. One way to proceed is proving that $\mathbb{P}(X_n=1)\sim\log n$ (contact us if you want the non elementary details for this).

Part 2.

(4) For exactly the same reasons (i.e. $\mathbb{P}(\bigcap_{k\geqslant n}\{X_{k+1}(\omega)-X_k(\omega)=1\})=0$) this new chain is still recurrent. This time, when computing recurrence relations for an invariant measure, we get $\pi(k+1)=\frac{k}{k+2}\pi(k)$, which gives by recurrence for $k\geqslant 1$, $\pi(k+1)=\frac{2}{(k+1)(k+2)}$. Since $\sum_{k\geqslant 1}\frac{2}{(k+1)(k+2)}=1<\infty$ we deduce that the chain is positive recurrent. On course this result can be achieved also by making explicit the law of T_1 when $X_0=1$. See point (6).

(5) Since the chain is defined on positive integers and starts at the invariant measure π, we have $\mathbb{E}_\pi[X_n]=\mathbb{E}_\pi[X_0]$ for all n (since $X_n\overset{(d)}{=}X_0$). It is then enough to prove that $\mathbb{E}_\pi[X_0]=\infty$. This follows from the fact that $k\mathbb{P}_\pi(X_0=k)\sim\frac{C}{k}$ for some positive constant C as $k\to\infty$.

(6) When the chain starts at 1, we have $\{T_1=n\}=\{X_0=1,X_1=2,X_2=3,\dots X_{n-1}=n,X_n=1\}$. Then using the Markov property one directly gets $\mathbb{P}_1(T_1=n)=(\prod_{k=1}^{n-1}Q(k,k+1))\times Q(n,1)=$

$\frac{4}{n(n+1)(n+2)}$. This allows to deduce that $\mathbb{E}_1[T_1] = \sum_{n\geqslant 1} n\mathbb{P}_1(T_1 = n) = 4\sum_{n\geqslant 2}\frac{1}{n(n+1)} = 4\sum_{n\geqslant 2}(\frac{1}{n} - \frac{1}{n+1}) = 2$. This is coherent with the fact that $\mathbb{E}_1[T_1] = \frac{1}{\mu(1)} = 2$ (one can indeed compute explicitely the invariant measure by normalizing the expression found at question (4)).

(7) Set call τ_j the j^{th} visit to 1 by the chain. We set also $\tau_0 := 0$. By the Strong Markov Property and recurrence, $(\tau_j - \tau_{j-1})_{j=1,2,\dots}$ is an independent sequence and $(\tau_j - \tau_{j-1})_{j=2,3,\dots}$ is IID. By the Law of Large Numbers we have that $\lim_j \tau_j/j = \mathbb{E}_1[\tau_1]$ a.s.. But this means that $\limsup_j(\tau_j - \tau_{j-1})/j = 0$ a.s.. We know that for every n there exists a unique $j_n = j_n(\omega)$ such that $\tau_{j_n-1} < n \leq \tau_{j_n}$ and a.s. $j_n \sim n/\mathbb{E}_1[\tau_1]$. If $X_n > \varepsilon n$ then $\tau_{j_n} - \tau_{j_n-1} > \varepsilon n \sim \varepsilon\mathbb{E}_1[\tau_1]j_n$, but this cannot happen for infinitely many values of n, because $\limsup_j(\tau_j - \tau_{j-1})/j = 0$ a.s..

2. Exercise 2

In the first part we consider the classical urn of Polya: we start with an urn that contains $B_0 > 0$ blue balls and $R_0 > 0$ red balls. At each time step we choose one ball from the urn and we put it back together with $\alpha > 0$ balls of the same color. This defines the the two sequences (R_n) and (B_n) of random variables: of course $R_n + B_n = R_0 + B_0 + \alpha n$. We set $\mathcal{F}_n := \sigma(R_0, R_1, \dots, R_n)$.

(1) The family $(X_n)_{n\geqslant 0}$ is UI if for each $\varepsilon > 0$, one can fine $M > 0$ such that $\sup_n \mathbb{E}[|X_n|; |X_n| > M] \leqslant \varepsilon$. A sequence converges in $\mathbb{L}^1$ iff it is UI and converges in probability.

(2) This is a classical computation. Clearly the process M_n is $\mathcal{F}_n$ adapted and bounded, thus integrable. Let $(T_n)_{n\geqslant 0}$ the random variable denoting which color B or R picked at step n. Then $\mathbb{E}[M_{n+1}|\mathcal{F}_n] = \mathbb{E}[M_{n+1}1_{T_{n+1}=B}|\mathcal{F}_n] + \mathbb{E}[M_{n+1}1_{T_{n+1}=R}|\mathcal{F}_n]$. Then one easliy sees that $M_{n+1}1_{T_{n+1}=B} = \frac{R_n}{R_n+B_n+\alpha}1_{T_{n+1}=B}$ and $M_{n+1}1_{T_{n+1}=R} = \frac{R_n+\alpha}{R_n+B_n+\alpha}1_{T_{n+1}=B}$. We also have that $\mathbb{E}[1_{T_{n+1}=B}|\mathcal{F}_n] = \frac{B_n}{B_n+T_n}$ and $\mathbb{E}[1_{T_{n+1}=R}|\mathcal{F}_n] = \frac{R_n}{B_n+T_n}$. This concludes the martingale property. The martingale is bounded thus UI.

(3) The martingale $(M_n)_{n\geqslant 0}$ is bounded this converges a.s. and in $\mathbb{L}^1$. We have for $p \geqslant n$, $M_n = \mathbb{E}[M_p|\mathcal{F}_n]$. Using the UI hypothesis, one can send $p \to \infty$ and exange limits $M_n = \lim_p \mathbb{E}[M_p|\mathcal{F}_n] = \mathbb{E}[\lim_p M_p|\mathcal{F}_n]$. If M were a.s. constant, so would be $M_1 = \mathbb{E}[M|\mathcal{F}_1]$ which is clearly not the case.

(4) Clearly Y_n is a bounded adapted process, thus the family is UI. Moreover using the same conditionning on the value of the sample at time $n+1$ gives the martingale property.

(5) By the same boundedness argument, Y_n converges to Y, a.s. and in $\mathbb{L}^2$. Moreover $Y(\omega) = \lim_{n\to\infty}\frac{R_n(\omega)}{R_n(\omega)+B_n(\omega)}\frac{R_n(\omega)+\alpha}{R_n(\omega)+B_n(\omega)+\alpha} = \lim_{n\to\infty}\frac{R_n(\omega)}{R_n(\omega)+B_n(\omega)}\frac{R_n(\omega)}{R_n(\omega)+B_n(\omega)} = M(\omega)^2$. Thus one can compute $\mathbb{E}[Y] = \mathbb{E}[Y_0] = \mathbb{E}[M^2]$ and $\mathbb{E}[M] = \mathbb{E}[M_0]$. The net result is

$$\mathrm{Var}(M) = \frac{\alpha R_0 B_0}{(R_0 + B_0)^2(R_0 + B_0 + \alpha)}\,.$$

Since $\mathrm{Var}(M) > 0$, M is not constant.

(6) By conditioning on the events $T_{n+1} = R$ and $T_{n+1} = B$, we get that $R_{n+1} - B_{n+1} = R_n - B_n + \delta(2 \times 1_{T_{n+1}=R} - 1)$. The martingale property then directly comes from the fact that $\mathbb{P}(T_{n+1} = R) = \frac{R_n}{R_n+B_n} = \frac{R_n}{R_0+B_0+n\tau}$ and the independence between T_{n+1} and $\mathcal{F}_n$ To be corrected: not independent!.

(7) Using the relation $R_{n+1} - B_{n+1} = R_n - B_n + \delta(2 \times 1_{T_{n+1}=R} - 1)$, one gets $\mathbb{E}[(R_{n+1} - B_{n+1})^2] = \mathbb{E}[(R_n - B_n)^2] + 2\delta\mathbb{E}[(R_n - B_n)(2\times 1_{T_{n+1}=R} - 1)] + \delta^2$. Since a.s $\delta^2(2\times 1_{T_{n+1}=R} - 1)^2 = \delta^2$. To conclude it is enough to see that $\mathbb{E}[(R_n - B_n)(2 \times 1_{T_{n+1}=R} - 1)] = \mathbb{E}[\mathbb{E}[(R_n - B_n)(2 \times 1_{T_{n+1}=R} - 1)|\mathcal{F}_n]] = \mathbb{E}[(R_n - B_n)\mathbb{E}[(2 \times 1_{T_{n+1}=R} - 1)|\mathcal{F}_n]] = \mathbb{E}[(R_n - B_n)\frac{R_n - B_n}{R_n + B_n}] = \frac{1}{R_0+B_0+n\tau}\mathbb{E}[(R_n - B_n)^2]$.

(8) Set $C_\rho := C_{\rho,R_0+B_0,\tau}$. We have that $\prod_{j=0}^{n-1} a_j \sim C_{2\rho} n^{2\rho}$, thus $\sum_{j=0}^{\infty} \frac{1}{\prod_{k=0}^{j-1} a_k} < \infty$ (since $\rho > \frac{1}{2}$). Moreover $Z_n \sim \frac{R_n - B_n}{C_\rho n^\rho}$ thus $n^{2\rho}\mathbb{E}[Z_n^2] \sim \left[(R_0 - B_0)^2 + \delta^2 \sum_{j=0}^{\infty} \frac{1}{\prod_{k=0}^{j-1} a_k}\right] C_{2\rho} n^{2\rho}$ which allows to conclude.

(9) The martingale $(Z_n)_{n \geqslant 0}$ is bounded in $\mathbb{L}^2$ thus converges a.s. and in $\mathbb{L}^2$ (and in particular in $\mathbb{L}^1$). Thus $\frac{R_n - B_n}{n^\rho} \sim \frac{Z_n}{C_\rho}$ converges a.s. and in $\mathbb{L}^1$.