# STOCHASTIC PROCESSES 29-01-2021

2 hours 30 minutes. No documents allowed.

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## 1. Exercise 1

The two parts of this first exercise are (essentially) independent.

**Part 1.** $X$ is a $Q$-MC on $E = \mathbb{N} = \{1, 2, \ldots\}$ with $Q$ defined by

$$Q(k, k+1) = \frac{k}{k+1} \quad \text{and} \quad Q(k, 1) = \frac{1}{k+1}\,.$$

(1) Explain, in a concise but complete way, why this chain is irreducible and aperiodic.
(2) Show that this chain is null recurrent.
(3) Compute $\lim_{n\to\infty} \mathbb{P}(X_n = 2)/\mathbb{P}(X_n = 3)$.

**Part 2.** $X$ is now a $Q$-MC on $E = \mathbb{N} = \{1, 2, \ldots\}$ with $Q$ defined by

$$Q(k, k+1) = \frac{k}{k+2} \quad \text{and} \quad Q(k, 1) = \frac{2}{k+2}\,.$$

Like in the first part, the chain is irreducible and aperiodic.

(4) Show that this chain is positive recurrent. We recall that $\sum_{k=2}^{\infty} 1/(k(k+1)(k+2)) = 1/12$ (which may or may not be needed).
(5) We call $\pi$ the invariant probability for the chain: show that $\mathbb{E}_\pi[X_n] = \infty$ for all $n$.
(6) With $T_k = \inf\{n = 1, 2, \ldots : X_n = k\}$, compute $\mathbb{P}_1(T_1 = n)$ and (verify with) $\mathbb{E}_1[T_1]$.
(7) ($\star$) We show now that, $\mathbb{P}_\pi$-a.s., $\limsup_n X_n/n = 0$.[1]
   (a) Set $\tau_j$ the $j^{\text{th}}$ visit to 1 by the chain and $\tau_0 := 0$. Explain briefly why $\tau_j$ is a stopping time and (still briefly) why $(\tau_j - \tau_{j-1})_{j=1,2,\ldots}$ is an independent sequence with $(\tau_j - \tau_{j-1})_{j=2,3,\ldots}$ is IID. Explain why the Law of Large Numbers implies $\lim_j (\tau_j - \tau_{j-1})/j = 0$ a.s..
   (b) For $n$ set $j_n = j_n(\omega)$ such that $\tau_{j_n - 1} < n \le \tau_{j_n}$. Prove that a.s. $j_n \sim n/\mathbb{E}_1[\tau_1]$ and then conclude.

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## 2. Exercise 2

In the first part we consider the classical urn of Polya: we start with an urn that contains $R_0 > 0$ red balls and $B_0 > 0$ blue balls. At each time step we choose one ball from the urn and we put it back together with $\alpha > 0$ balls of the same color. Let $R_n$ and $B_n$ the number of red and blue balls at step $n$. So, $(R_{n+1}, B_{n+1}) = (R_n + \alpha, B_n)$ with probability $R_n/(R_n + B_n)$ and $(R_{n+1}, B_{n+1}) = (R_n, B_n + \alpha)$ with probability $B_n/(R_n + B_n)$. This defines the the two sequences $(R_n)$ and $(B_n)$ of random variables: of course $R_n + B_n = R_0 + B_0 + \alpha n$. We set $\mathcal{F}_n := \sigma(R_1, \ldots, R_n)$. We consider non random $(R_0, B_0)$, so we choose $\mathcal{F}_0$ trivial.

(1) Give the definition of Uniform Integrability (UI) for a sequence of random variables. Explain, by stating the appropriate theorems, why if a UI sequence $(X_n)$ converges in law to $X$, then $\lim_n \mathbb{E}[X_n] = \mathbb{E}[X]$.

[1] What is is obvious is that $\limsup_n X_n/n \le 1$ (because $X_n \le X_0 + n$). This fact is of interest because it shows that a stationary MC can be very different from an IID sequence: in the IID case we know that $\mathbb{E}[|X_1|] = \infty$ if and only if $\limsup_n X_n/n = 0$.

(2) Show that $(M_n)_{n=0,1,\dots}$, with $M_n = R_n/(R_n + B_n)$ is an UI martingale.
(3) Explain why $M = \lim_n M_n$ exists a.s. and in $\mathbb{L}^1$. Explain also (by at least sketching the proof given in the course) why $M_n = \mathbb{E}[M|\mathcal{F}_n]$. Deduce that $M$ is not a constant.
(4) Show that $(Y_n)_{n=0,1,\dots}$, with $Y_n = R_n(R_n+\alpha)/((R_n+B_n)(R_n+B_n+\alpha))$, is also an UI martingale
(5) Compute the variance of $M$ and conclude, once again, that $M$ is not a constant.
*Obs.: in case you want to check your result, the law of $M$ is known and for $R_0 = B_0 = \alpha = 1$ it is $U(0,1)$, whose variance is $1/12$.*

Now we *slightly* generalize the model (*urn of Friedman*): when putting back the ball we have chosen together with $\alpha$ balls of the same color, we put also $\beta$ balls of the other color (the general case can be treated, but choose $\beta < \alpha$). We define this way two new sequences $(R_n)$ and $(B_n)$ of random variables: this time $R_n + B_n = R_0 + B_0 + (\alpha + \beta)n$. The filtration we choose is once again defined by $\mathcal{F}_n := \sigma(R_1, \dots, R_n)$. We set

$$\delta := \alpha - \beta, \qquad \tau := \alpha + \beta \quad \text{and} \quad \rho := \frac{\delta}{\tau}.$$

For what follows you may want to use that for $a, b, c > 0$ we have

$$\prod_{j=0}^{n} \left(1 + \frac{a}{b + cj}\right) \overset{n\to\infty}{\sim} C_{a,b,c} n^{a/c}.$$

where[2] $C_{a,b,c} > 0$. As usual, $f(n) \sim g(b)$ means $\lim_n f(n)/g(n) = 1$.

(6) Set $Z_0 := R_0 - B_0$ and

$$Z_n := (R_n - B_n) \Big/ \prod_{j=0}^{n-1} \left(1 + \frac{\delta}{R_0 + B_0 + \tau j}\right),$$

for $n = 1, 2, \dots$. Compute $\mathbb{E}[R_{n+1} - B_{n+1}|\mathcal{F}_n]$ and infer that $(Z_n)$ is a martingale.
(7) Show that

$$\mathbb{E}\left[(R_{n+1} - B_{n+1})^2\right] = \left(1 + \frac{2\delta}{R_0 + B_0 + n\tau}\right) \mathbb{E}\left[(R_n - B_n)^2\right] + \delta^2 =: a_n \mathbb{E}\left[(R_n - B_n)^2\right] + \delta^2,$$

that is (no need to show it, no probability involved)

$$\mathbb{E}\left[(R_n - B_n)^2\right] = \left((R_0 - B_0)^2 + \delta^2 \sum_{j=0}^{n-1} \frac{1}{\prod_{k=0}^{j-1} a_k}\right) \prod_{j=0}^{n-1} a_j,$$

where empty sums should be read as zero, as well as empty products should be read as one.
(8) Show that if $\rho > 1/2$ we have $\sup_n \mathbb{E}[Z_n^2] < \infty$.
(9) Explain why we therefore have (if $\rho > 1/2$) that $\lim_n (R_n - B_n)/n^\rho$ exists a.s. and in $\mathbb{L}^1$.

*Conclusion and curiosities:* let us admit it, the result is extremely surprising. The Friedman urn seems a minor modification of the Polya urn, but point (9) says that $\lim_n R_n/(R_n + B_n) = 1/2$ a.s.. In the Polya urn $\lim_n R_n/(R_n + B_n)$ is a non trivial random variable!

The fact that $\lim_n R_n/(R_n + B_n) = 1/2$ a.s. is true for every $\rho \in (0,1)$ and this general result can be obtained by arguments similar to the one we just used. For more on this: D. A. Freedman, *Bernard Friedman's urn*, Ann. Math. Statist. 36 (1965), 956-970.

[2] Actually, $C_{a,b,c} = \Gamma(b/c)/\Gamma((a+b)/c)$ but you will not need that.