# Concentration for Norms of Infinitely Divisible Vectors With Independent Components

C. Houdré* P. Marchal † P. Reynaud-Bouret‡

June 7, 2006

**Abstract**

We obtain dimension-free concentration inequalities for $L^p$ norms, $p \geq 2$, of infinitely divisible random vectors with independent coordinates. The methods and results extend to some other classes of Lipschitz functions.

## 1 Introduction

Talagrand [T] proved in 1991 an isoperimetric inequality for the product measure $\mu^d$ where $\mu$ is the symmetric exponential measure (i.e. $(e^{-|x|}/2)dx$ on the real line). This inequality was the first one to mix two different norms ($L^1$ and $L^2$) improving some aspects of Gaussian isoperimetry. Rewriting this inequality for Lipschitz functions we get that:

**Theorem 1 (Talagrand)** *Let $X$ be a random vector of $\mathbb{R}^d$ with i.i.d. symmetric exponential components. Let $f$ be a real valued function on $\mathbb{R}^d$ with median $0$, such that*

$$\exists \alpha, \beta > 0, \forall x, y \in \mathbb{R}^d, |f(x) - f(y)| \leq \min(\alpha \|x-y\|_2, \beta \|x-y\|_1).$$

*Then there exists a universal constant $K$ such that*

$$\mathbb{P}\left(f(X) \geq \alpha\sqrt{u} + \beta u\right) \leq \frac{1}{2} e^{-\frac{u}{K}}.$$

*School of Mathematics, Georgia Institute of Technology, Atlanta, GA 30332, USA.
†CNRS and DMA, Ecole Normale Supérieure, 45 rue d'Ulm 75230 Paris Cedex 05, France.
‡CNRS and DMA, Ecole Normale Supérieure, 45 rue d'Ulm 75230 Paris Cedex 05, France.

What is remarkable here, is that this concentration formula is dimension free. For instance, if we apply this result to the euclidean norm, we remark that $\alpha = \beta = 1$ and that the only dependence in the dimension $d$ is through the median itself. This result of Talagrand, which clearly continues to hold for Lipschitz images of the exponential measure, actually holds for any law $\mu$ satisfying a Poincaré inequality (see [BL1]).

We would like here to extend this result to infinitely divisible variables for which exponential variables are a particular example. Let $X \sim ID(\gamma, 0, \nu)$ be an infinitely divisible (ID) vector (without Gaussian component) in $\mathbb{R}^d$, and with characteristic function $\varphi(t) = \mathbb{E}e^{i\langle t,X\rangle}$, $t \in \mathbb{R}^d$ (throughout, $\langle \cdot, \cdot \rangle$ denotes the Euclidean inner product in $\mathbb{R}^d$, while $\|\cdot\|$ is the corresponding Euclidean norm). As well known,

$$\varphi(t) = \exp\left\{ i\langle t, \gamma\rangle + \int_{\mathbb{R}^d} (e^{i\langle t,u\rangle} - 1 - i\langle t, u\rangle \mathbf{1}_{\|u\|\leq 1})\nu(du) \right\}, \tag{1.1}$$

where $\gamma \in \mathbb{R}^d$ and where $\nu \not\equiv 0$ (the Lévy measure) is a positive Borel measure on $\mathbb{R}^d$, without atom at the origin and such that $\int_{\mathbb{R}^d}(1 \wedge \|u\|^2)\nu(du) < +\infty$.

As also well known, $X$ has independent components if and only if $\nu$ is supported on the axes of $\mathbb{R}^d$, i.e.,

$$\nu(dx_1, \ldots, dx_d) = \sum_{k=1}^{d} \delta_0(dx_1)\cdots\delta_0(dx_{k-1})\tilde{\nu}_k(dx_k)\delta_0(dx_{k+1})\cdots\delta_0(dx_d). \tag{1.2}$$

Moreover, the independent components of $X$ have the same law if and only if the one dimensional Lévy measures $\tilde{\nu}_k$ are the same measure denoted by $\tilde{\nu}$.

Some work (see [H]) has already been done for general Lipschitz functions and general ID vectors but some of these results are not dimension free for vectors with iid components. One may even think, in analogy with the results of ....... **(ref a mettre cf truc envoye par christian)**, that such dimension free results do not exist for every Lipschitz functions.

Here we focus first on the particular example of the euclidean norm. Even if we do get dimension free results, the next three results are not as general as we could hope for, even in the case of the euclidean norm of ID vector with i.i.d. components.

**Theorem 2** *Let $X \sim ID(\gamma, 0, \nu)$ have i.i.d. components and be such that $\mathbb{E}e^{t\|X\|} < +\infty$, for* *some* $t > 0$. *Let* $M = \sup\{t > 0 : \mathbb{E}e^{t|X_1|} < +\infty\}$. *Let* $\varepsilon > 0$. *Then, for all* $0 < x < h(M^-)$

$$\mathbb{P}(\|X\| \geq (1+\varepsilon)\mathbb{E}\|X\| + x) \leq e^{-\int_0^x h^{-1}(s)ds}, \tag{1.3}$$

*where the (dimension free) function* $h$ *is given by*

$$h(t) = 8\int_{\mathbb{R}} |u|(e^{t|u|}-1)\tilde{\nu}(du) + \frac{2d}{(\varepsilon\mathbb{E}\|X\|)^2}\int_{\mathbb{R}} |u|^3(e^{t|u|}-1)\tilde{\nu}(du).$$

This leads to q first corollary which is a dimension free extension of the results of [H, R].

**Corollary 1** *Let* $\tilde{\nu}$ *have bounded support with* $R = \inf\{\rho \ : \ \tilde{\nu}(|x| > \rho) = 0\}$, *then*

$$\mathbb{E}\left[e^{\frac{\|X\|}{R}\log\left(\frac{\lambda\|X\|}{R}\right)}\right] < +\infty,$$

*for all* $\lambda$ *such that* $\lambda V^2/R^2 < 1/e$, *where* $V^2 = 8\int_{\mathbb{R}} |u|^2\tilde{\nu}(du)$.

Theorem 2 has still some weak dimension dependency, through the term $\varepsilon\mathbb{E}\|X\|$ (the expectation and the median playing the same role up to some constant). In particular, it does not precisely recover Talagrand's result even for the euclidean norm. Here is another possible result.

**Theorem 3** *With the notation of Theorem 2, let also* $l = -\log\mathbb{E}[e^{-X_1^2}]$. *Let for all* $0 < t < M$,

$$h(t) = \frac{12}{l}\int_{\mathbb{R}} |u|(e^{t|u|}-1)\tilde{\nu}(du) \text{ and}$$

$$g(t) = \left(8 + \frac{12\log(2)}{l}\right)\int_{\mathbb{R}} |u|(e^{t|u|}-1)\tilde{\nu}(du) + \frac{8}{l}\int_{\mathbb{R}} |u|^3(e^{t|u|}-1)\tilde{\nu}(du).$$

*Let* $T$ *be such that for all* $t \leq T$, $tg(t) \leq 1/2$. *Then for all positive* $x$,

$$\mathbb{P}(\|X\| \geq \mathbb{E}\|X\| + x) \leq e^{-\sup_{0\leq t\leq T}\left[tx - \int_0^t 2g(s)ds\right]}. \tag{1.4}$$

This result does recover Talagrand's inequality for the Euclidean norm (up to the value of the constants). Indeed, it is sufficient to take $t = x$ for $x \leq T$ and $t = T$ otherwise. However, the improvement in $x\log x$ obtained in the proof of Corollary 1 for Poisson random variables (and, more generally, ID variables with boundedly supported Lévy measure) disappears.

Our last result recovers Talagrand's inequality and Corollary 1 for Poisson random variables, but requires some technical assumptions on the random variables themselves.

Recall that $X$ can be viewed as $X_1$, the value at time 1 of a Lévy process $(X_t, t \geq 0)$. For every $t \in [0,1]$, write

$$X = Y_t + Z_t \tag{1.5}$$

where $Y_t = X_t$, $Z_t = X_1 - X_t$, so that $Y_t, Z_t$ are independent. If $1 \leq k \leq d$, we write $(Y_k)_t, (Z_k)_t$ for the $k$-th coordinate of $Y_t, Z_t$.

**Theorem 4** *Let $X$ be as in Theorem 2. Then for all positive $x$,*

$$\mathbb{P}(\|X\| - \mathbb{E}\|X\| \geq x) \leq \exp\left(-\int_0^x h^{-1}(s)ds\right), \tag{1.6}$$

*where the (dimension free) function $h$ is given by*

$$h(t) = 64 \int_{\mathbb{R}} \left[\left(1 + \frac{\sqrt{2}|u|}{\sqrt{\underline{m_2}}}\right)^2 + 4\frac{\overline{m_4}}{\underline{m_2}^2}\right] |u|(e^{t|u|} - 1)\tilde{\nu}(du),$$

*and the moments $\overline{m_4}, \underline{m_2}$ are defined as follows:*

- *if $X$ has almost surely positive coordinates, we can take $\underline{m_q} = \overline{m_q} = \mathbb{E}[(X_1)_1^q]$ for $q = 2, 4$.*
- *otherwise, we take*

$$\underline{m_2} = \inf_{t\in[0,1]} \left[\inf\{\mathbb{E}[|(Y_1)_t + (Z_1)_t|^2 \mathbf{1}_{(Z_1)_t\geq 0,(Y_1)_t\geq 0}],\ \mathbb{E}[|(Y_1)_t + (Z_1)_t|^2 \mathbf{1}_{(Z_1)_t\leq 0,(Y_1)_t\leq 0}]\}\right]$$

$$\overline{m_4} = \sup_{t\in[0,1]} \left[\sup\{\mathbb{E}[|(Y_1)_t + (Z_1)_t|^4 \mathbf{1}_{(Z_1)_t\geq 0,(Y_1)_t\geq 0}],\ \mathbb{E}[|(Y_1)_t + (Z_1)_t|^4 \mathbf{1}_{(Z_1)_t\leq 0,(Y_1)_t\leq 0}]\}\right]$$

The drawback of this result is that it gives a trivial bound if $\underline{m_2} = 0$. For instance, if the coordinates of $X$ are not positive almost surely, but if $X_t$ has positive coordinates with probability tending to 1 as $t \to 0$, then $\underline{m_p} = 0$. However, in most "natural" situations where ID random variables occur (this is the case, for instance, when $X$ is symmetric), $\underline{m_2} > 0$, and Theorem 4 gives a nontrivial dimension-free bound.

In fact, we obtain a generalization of Theorem 4 to general $L^p$ norms for $2 \leq p < \infty$, see Theorem 5 in the last section.

**blabla sur le plan on mettra apres**

## 2 The covariance formula and its first applications

The result at the root of every proof in this paper is the following one.

**Proposition 1** *Let $X = (X_1, \dots, X_d) \sim ID(\gamma, 0, \nu)$ have independent components and be such that $\mathbb{E}e^{t\|X\|} < +\infty$, for some $t > 0$. Let $f : \mathbb{R}^d \to \mathbb{R}$ be such that $\mathbb{E}f(X) = 0$, and let there exist $b_k \in \mathbb{R}$, $k = 1, \dots, d$, such that $|f(x + ue_k) - f(x)| \le b_k|u|$, for all $u \in \mathbb{R}$, $x \in \mathbb{R}^d$. Let $M = \sup\left\{t > 0 : \forall\ k = 1, \dots, d, \mathbb{E}e^{tb_k|X_k|} < +\infty\right\}$. Then for all $0 \le t < M$,*

$$\mathbb{E}fe^{tf} \le \int_0^1 \mathbb{E}_z \left[\sum_{k=1}^d \int_{\mathbb{R}} \frac{|f(U + ue_k) - f(U)|^2 + |f(V + ue_k) - f(V)|^2}{2} e^{tf(V)} \left(\frac{e^{tb_k|u|} - 1}{b_k|u|}\right) \tilde{\nu}_k(du)\right] dz,$$

*where the expectation $\mathbb{E}_z$ is with respect to the ID vector, $(U, V)$ in $\mathbb{R}^{2d}$ of parameter $(\gamma, \gamma)$ and with Lévy measure $z\nu_1 + (1 - z)\nu_0$, $0 \le z \le 1$. The measure $\nu_0$ is given by*

$$\nu_0(du, dv) = \nu(du)\delta_0(dv) + \delta_0(du)\nu(dv), u, v \in \mathbb{R}^d,$$

*while $\nu_1$ is the measure $\nu$ supported on the main diagonal of $\mathbb{R}^{2d}$.*

An important feature of this proposition is the fact that the first marginal of $(U, V)$ is $X$ and so is its second marginal.

So in fact a main problem in estimating the right-hand side of the inequality in Proposition 1 will be to uncouple $U$ and $V$, i.e. to split the product $|f(U + ue_k) - f(U)|^2 e^{tf(V)}$ without changing the term $e^{tf(V)}$. To do so, a first attempt could be to use a supremum.

**Corollary 2** *Let $X = (X_1, \dots, X_d) \sim ID(\gamma, 0, \nu)$ have independent components and be such that $\mathbb{E}e^{t\|X\|} < +\infty$, for some $t > 0$. Let $f : \mathbb{R}^d \to \mathbb{R}$, and let there exist $b_k \in \mathbb{R}$, $k = 1, \dots, d$, such that $|f(x + ue_k) - f(x)| \le b_k|u|$, for all $u \in \mathbb{R}$, $x \in \mathbb{R}^d$. Let*

$$h_f(t) = \sup_{x \in \mathbb{R}^d} \sum_{k=1}^d \int_{\mathbb{R}} |f(x + ue_k) - f(x)|^2 \frac{e^{tb_k|u|} - 1}{b_k|u|} \tilde{\nu}_k(du),\ 0 \le t < M,$$

*where $M = \sup\left\{t > 0 : \forall\ k = 1, \dots, d, \mathbb{E}e^{tb_k|X_k|} < +\infty\right\}$. Then*

$$\mathbb{P}(f(X) - \mathbb{E}f(X) \ge x) \le e^{-\int_0^x h_f^{-1}(s)ds}, \tag{2.1}$$

*for all $0 < x < h_f^{-1}(M^-)$.*

**Proof.**[Proposition 1 and Corollary 2]**verifier qu'on abien toutes les hypothese dans ces deux resultats** Below, and throughout, by $f$ Lipschitz with constant $a$ we mean that $|f(x)-f(y)| \leq a\|x-y\|$, for all $x,y \in \mathbb{R}^d$ (the Lipschitz convention stated in [H] also applies). Let us start by recalling the following simple lemma which will be crucial to our approach [HPAS].

**Lemma 1** *Let $X \sim ID(\gamma, 0, \nu)$ be such that $\mathbb{E}\|X\|^2 < +\infty$. Let $f, g : \mathbb{R}^d \to \mathbb{R}$ be Lipschitz functions. Then,*

$$\begin{aligned}&\mathbb{E}f(X)g(X) - \mathbb{E}f(X)\mathbb{E}g(X)\\ &\qquad = \int_0^1 \mathbb{E}_z\left[\int_{\mathbb{R}^d}(f(U+u)-f(U))(g(V+u)-g(V))\nu(du)\right]dz, \qquad (2.2)\end{aligned}$$

*where $\mathbb{E}_z$ is as in Proposition 1*

Then we follow [H]. First, by independence,

$$\begin{aligned}C &= \left\{t>0 : \forall\ k=1,\dots,d, \mathbb{E}e^{tb_k|X_k|} < +\infty\right\}\\ &= \left\{t>0 : \forall\ k=1,\dots,d, \int_{|u|>1} e^{tb_k|u|}\tilde{\nu}_k(du) < +\infty\right\}.\end{aligned}$$

Next, we apply the covariance representation (2.2) to $f$ satisfying the above hypotheses and moreover assumed to be bounded and such that $\mathbb{E}f = 0$. Thus,

$$\begin{aligned}\mathbb{E}fe^{tf} &= \int_0^1 \mathbb{E}_z\left[e^{tf(V)}\sum_{k=1}^d \int_{\mathbb{R}}(f(U+ue_k)-f(U))(e^{t(f(V+ue_k)-f(V))}-1)\tilde{\nu}_k(du)\right]dz\\ &\leq \int_0^1 \mathbb{E}_z\left[e^{tf(V)}\sum_{k=1}^d \int_{\mathbb{R}}|f(U+ue_k)-f(U)||f(V+ue_k)-f(V)|\,\frac{e^{tb_k|u|}-1}{b_k|u|}\,\tilde{\nu}_k(du)\right]dz\\ &\leq \int_0^1 \mathbb{E}_z\left[e^{tf(V)}\sum_{k=1}^d \int_{\mathbb{R}}\frac{|f(U+ue_k)-f(U)|^2+|f(V+ue_k)-f(V)|^2}{2}\left(\frac{e^{tb_k|u|}-1}{b_k|u|}\right)\tilde{\nu}_k(du)\right]dz,\end{aligned}$$

which gives Proposition 1. For Corollary 2, we continue:

$$\mathbb{E}fe^{tf} \leq h_f(t)\mathbb{E}\left[e^{tf}\right],$$

where we have used the "marginal property" mentioned above and since $h_f(t)$ is well defined for $0 \leq t < M$. Integrating this last inequality, applied to $f - \mathbb{E}f$, leads to

$$\mathbb{E}e^{t(f-\mathbb{E}f)} \leq e^{\int_0^t h_f(s)ds}, \qquad 0 \leq t < M, \qquad (2.3)$$

for all $f$ bounded satisfying the hypotheses of the theorem. Fatou's lemma allows to remove the boundedness assumption in (2.3).

To obtain the tail inequality (2.1), the Bienaymé-Chebyshev inequality gives

$$\mathbb{P}(f(X) - \mathbb{E}f(X) \geq x) \leq \exp\left(-\sup_{0<t<M}\left(tx - \int_0^t h_f(s)ds\right)\right) = e^{-\int_0^x h_f^{-1}(s)ds},$$

by standard arguments, e.g., see [H]. ■

In general, this corollary does not provide dimension free results, even if it can improve a bit the results in [H] (we refer the reader to [HR], superseeded by the present paper, on applications of Theorem 1). For particular functions, the above formula can in fact be quite efficient. As a consequence of the previous corollary, we present some almost dimension free results. These results recover, in the case of the euclidean norm, Theorem 2 for a vector with i.i.d. components.

**Corollary 3** *Let $X \sim ID(\gamma, 0, \nu)$ have independent components and be such that $\mathbb{E}e^{t\|X\|} < +\infty$, for some $t > 0$. Let $M = \sup\{t > 0 : \forall\ k = 1, \ldots, d,\ \mathbb{E}e^{t|X_k|} < +\infty\}$. Let $S$ be a subspace of $\mathbb{R}^d$ and $\Pi_S$ the orthogonal projection on $S$. Let $E > 0$. Then, for all $0 < x < h(M^-)$*

$$\mathbb{P}(\|\Pi_S(X)\| \geq \mathbb{E}\|\Pi_S(X)\| + E + x) \leq e^{-\int_0^x h^{-1}(s)ds}, \tag{2.4}$$

*and*

$$\mathbb{P}(\|\Pi_S X\| \leq \mathbb{E}\|\Pi_S(X)\| - E - x) \leq e^{-\int_0^x h^{-1}(s)ds}, \tag{2.5}$$

*where the function $h$ is given by*

$$h(t) = 8\max_{1\leq k\leq d}\int_{\mathbb{R}}|u|(e^{t|u|}-1)\tilde{\nu}_k(du) + \frac{2}{E^2}\sum_{k=1}^d \|\Pi_S(e_k)\|^4\int_{\mathbb{R}}|u|^3(e^{t|u|}-1)\tilde{\nu}_k(du).$$

**Proof.** We apply Corollary 2 to $f(x) = (\|\Pi_S(x)\| - E)^+$. First, it is easily verified that for each $k$, $|f(x+ue_k) - f(x)| \leq |\|\Pi_S(x+ue_k)\| - \|\Pi_S(x)\||\,\mathbf{1}_{A_k}$, where $A_k = \{\|\Pi_S(x+ue_k)\| \geq E$ or $\|\Pi_S(x)\| \geq E\}$. We then have

$$\begin{aligned}|f(x+ue_k) - f(x)| &\leq \frac{|2\langle u\Pi_S(e_k)|\Pi_S(x)\rangle + u^2\|\Pi_S(e_k)\|^2|\mathbf{1}_{A_k}}{\|\Pi_S(x+ue_k)\| + \|\Pi_S(x)\|} \\ &\leq \frac{2|u||\langle\Pi_S(e_k)|\Pi_S(x)\rangle|}{\|\Pi_S(x)\|} + \frac{u^2\|\Pi_S(e_k)\|^2}{E}.\end{aligned} \tag{2.6}$$

Moreover, since $|f(x+ue_k)-f(x)|\le |u|$, we have

$$\begin{aligned}
&\sum_{k=1}^{d}\int_{\mathbb{R}}|f(x+ue_k)-f(x)|^2\frac{e^{tb_k|u|}-1}{b_k|u|}\,\tilde{\nu}_k(du)\\
&\qquad\le\sum_{k=1}^{d}\int_{\mathbb{R}}\left(8u^2\frac{|\langle\Pi_S(e_k)|\Pi_S(x)\rangle|^2}{\|\Pi_S(x)\|^2}+\frac{2u^4\|\Pi_S(e_k)\|^4}{E^2}\right)\left(\frac{e^{t|u|}-1}{|u|}\right)\tilde{\nu}_k(du)\\
&\qquad\le\sum_{k=1}^{d}\int_{\mathbb{R}}\left(8u^2\frac{|\langle e_k|\Pi_S(x)\rangle|^2}{\|\Pi_S(x)\|^2}+\frac{2u^4\|\Pi_S(e_k)\|^4}{E^2}\right)\left(\frac{e^{t|u|}-1}{|u|}\right)\tilde{\nu}_k(du),
\end{aligned}$$

Hence $h_f\le h$. To finish the proof of (2.4) note that $\|\Pi_S(X)\|-E\le(\|\Pi_S(X)\|-E)^+$ and that $\mathbb{E}(\|\Pi_S(X)\|-E)^+\le\mathbb{E}\|\Pi_S(X)\|$. To get the lower bound (2.5), just proceed as above but with the function $f(x)=-(\|\Pi_S(x)\|-E)^+$ and note that $(\|\Pi_S(X)|-E)^+\le\|\Pi_S(X)\|$ and that $\mathbb{E}\|\Pi_S(X)\|-E\le\mathbb{E}(\|\Pi_S(X)\|-E)^+$. ■

**Proof.**[Theorem 2] For $S=\mathbb{R}^d$, one can take $E=\varepsilon\mathbb{E}\|X\|$. This is dimension free in the i.i.d. case since

$$d(\mathbb{E}|X_1|)^2\le(\mathbb{E}\|X\|)^2\le d\mathbb{E}(X_1^2).$$

■

The case of projections is of interest since it can be applied to model selection, in regression, when the error is a centered ID random variable which is no longer normal.

The following version may be easier to use.

**Corollary 4** *Let $X\sim ID(\gamma,0,\nu)$ have i.i.d. centered components and be such that $\mathbb{E}e^{t\|X\|}<+\infty$, for some $t>0$. Let $M=\sup\{t>0:\mathbb{E}e^{t|X_1|}<+\infty\}$. Let $S$ be a subspace of $\mathbb{R}^d$ and let $\Pi_S$ be the orthogonal projection on $S$. Let $\varepsilon>0$. Then, for all $0<x<h(M^-)$*

$$\mathbb{P}(\|\Pi_S(X)\|\ge(1+\varepsilon)\sqrt{\mathbb{E}[\|\Pi_S(X)\|^2]}+x)\le e^{-\int_0^x h^{-1}(s)ds},\tag{2.7}$$

*and*

$$\mathbb{P}(\|\Pi_S\|\le\mathbb{E}\|\Pi_S(X)\|-\varepsilon\sqrt{\mathbb{E}[\|\Pi_S(X)\|^2]}-x)\le e^{-\int_0^x h^{-1}(s)ds},\tag{2.8}$$

*where the (dimension free) function $h$ is given by*

$$h(t) = 8\int_{\mathbb{R}} |u|(e^{t|u|}-1)\tilde{\nu}(du) + \frac{2}{\varepsilon^2 \mathbb{E}X_1^2}\int_{\mathbb{R}} |u|^3(e^{t|u|}-1)\tilde{\nu}(du).$$

**Proof.** First let us take $E = \varepsilon\sqrt{\mathbb{E}[\|\Pi_S(X)\|^2]}$. Then remark that in the centered i.i.d case,

$$\begin{aligned}
\mathbb{E}[\|\Pi_S(X)\|^2] &= \mathbb{E}\left[\sum_{l=1}^{d}\left(\sum_{k=1}^{d} X_k\langle\Pi_S(e_k)|e_l\rangle\right)^2\right] \\
&= \sum_{l=1}^{d}\sum_{k=1}^{d}\mathbb{E}[X_k^2]\langle\Pi_S(e_k)|e_l\rangle^2 \\
&= \mathbb{E}(X_1)^2\sum_{k=1}^{d}\|\Pi_S(e_k)\|^2 \\
&\geq \mathbb{E}(X_1)^2\sum_{k=1}^{d}\|\Pi_S(e_k)\|^4,
\end{aligned}$$

since $\|\Pi_S(e_k)\| \leq \|e_k\| = 1$. ■

Another possible application of Corollary 2 is to the $L^p$-norms. For simplicity, we only state the result for i.i.d. components.

**Corollary 5** *Let $X \sim ID(\gamma, 0, \nu)$ have i.i.d components and be such that $\mathbb{E}e^{t\|X\|} < +\infty$, for some $t > 0$. Let $M = \sup\{t > 0 : \mathbb{E}e^{t|X_1|} < +\infty\}$. Let $p \geq 2$ and $\varepsilon > 0$. Then, for all $0 < x < h(M^-)$*

$$\mathbb{P}(\|X\|_p \geq (1+\varepsilon)\mathbb{E}(\|X\|_p) + x) \leq e^{-\int_0^x h^{-1}(s)ds}, \tag{2.9}$$

*and*

$$\mathbb{P}(\|X\|_p \geq (1-\varepsilon)\mathbb{E}(\|X\|_p) + x) \leq e^{-\int_0^x h^{-1}(s)ds}, \tag{2.10}$$

*where the function $h$ is given by*

$$h(t) = p^2\int_{\mathbb{R}}\left(1 + \frac{|u|d^{1/(2p-2)}}{\varepsilon\mathbb{E}(\|X\|_p)}\right)^{2p-2}|u|\left(e^{t|u|}-1\right)\tilde{\nu}(du).$$

**Proof.** We apply Corollary 2 to $f(x) = (\|x\|_p - \varepsilon\mathbb{E}(\|X\|_p))^+$. First, it is easily verified that for each $k$, $|f(x+ue_k) - f(x)| \leq |\|x+ue_k\|_p - N_p(x)|\,\mathbf{1}_{A_k}$, where $A_k = \{\|x+ue_k\|_p \geq \varepsilon\mathbb{E}(\|X\|_p)$ or $\|x\|_p \geq \varepsilon\mathbb{E}(\|X\|_p)\}$. Since

$$\forall a, b \geq 0, |a-b| \leq \frac{|a^p - b^p|}{\sup(a,b)^{p-1}},$$

we have

$$|f(x+ue_k)-f(x)| \leq \frac{||X_k+u|^p-|X_k|^p|}{\sup(\|X\|_p, \|X+ue_k\|_p)^{p-1}}\mathbf{1}_{A_k}. \tag{2.11}$$

But since $x \mapsto x^p$ is convex, one has:

$$||X_k+u|^p-|X_k|^p| \leq |(|X_k|+|u|)^p-|X_k|^p|\,.$$

Combining this with the fact that

$$\forall x \geq 0, (1+x)^p-1 \leq px(1+x)^{p-1},$$

implies that

$$\begin{aligned}|f(x+ue_k)-f(x)| &\leq \frac{p|u|(|X_k|+|u|)^{p-1}}{\sup(\|X\|_p, \|X+ue_k\|_p)^{p-1}}\mathbf{1}_{A_k}\\ &\leq \frac{p|u|(|X_k|+|u|)^{p-1}}{\sup(\|X\|_p, \varepsilon\mathbb{E}(\|X\|_p))^{p-1}}.\end{aligned}$$

Moreover, since $|f(x+ue_k)-f(x)| \leq |u|$, we have

$$\begin{aligned}&\sum_{k=1}^{d}\int_{\mathbb{R}}|f(x+ue_k)-f(x)|^2\frac{e^{tb_k|u|}-1}{b_k|u|}\,\tilde{\nu}_k(du)\\ &\qquad\leq p^2\int_{\mathbb{R}}\frac{\||X|+|u|I\|_{2p-2}^{2p-2}}{\sup(\|X\|_p, \varepsilon\mathbb{E}(\|X\|_p))^{2p-2}}|u|(e^{t|u|}-1)\tilde{\nu}(du)\\ &\qquad\leq p^2\int_{\mathbb{R}}\left(\frac{\|X\|_{2p-2}+|u|\|I\|_{2p-2}}{\sup(\|X\|_p, \varepsilon\mathbb{E}(\|X\|_p))}\right)^{2p-2}|u|(e^{t|u|}-1)\tilde{\nu}(du),\end{aligned}$$

where $I=(1,...,1)\in\mathbb{R}^d$. Since $p \geq 2$, $2p-2 \geq p$ and $\|X\|_{2p-2} \leq \|X\|_p$, which implies that

$$\begin{aligned}&\sum_{k=1}^{d}\int_{\mathbb{R}}|f(x+ue_k)-f(x)|^2\frac{e^{tb_k|u|}-1}{b_k|u|}\,\tilde{\nu}_k(du)\\ &\qquad\leq p^2\int_{\mathbb{R}}\left(1+|u|\frac{d^{1/(2p-2)}}{\varepsilon\mathbb{E}(\|X\|_p)}\right)^{2p-2}|u|(e^{t|u|}-1)\tilde{\nu}(du).\end{aligned}$$

The proof is complete. ■

Again, the result just obtained is dimension free since $\mathbb{E}(\|X\|_p) \geq \mathbb{E}(|X_1|)d^{1/p}$.

# 3 Using Young's inequality

Another method to uncouple $U$ and $V$ in Proposition 1 is to use the following inequality, which is a particular instance of the Young inequality.

**Lemma 2** *Let $\lambda > 0$, and let $X$ and $Y$ be random variables such that the expectations, below, exist. Then,*

$$\mathbb{E}[Xe^{\lambda Y}] \leq \mathbb{E}[Ye^{\lambda Y}] + \frac{\log \mathbb{E}[e^{\lambda X}]}{\lambda}\mathbb{E}[e^{\lambda Y}] - \frac{\log \mathbb{E}[e^{\lambda Y}]}{\lambda}\mathbb{E}[e^{\lambda Y}]. \tag{3.1}$$

**Proof.** Indeed, if

$$dQ = \frac{e^{\lambda Y}}{\mathbb{E}[e^{\lambda Y}]} d\mathbb{P},$$

then by Jensen's inequality

$$\lambda \mathbb{E}_Q(X - Y) \leq \log \mathbb{E}_Q(e^{\lambda(X-Y)}).$$

■

This leads to the following result.

**Corollary 6** *Let $X = (X_1, \ldots, X_d) \sim ID(\gamma, 0, \nu)$ have i.i.d. components and be such that $\mathbb{E}e^{t\|X\|} < +\infty$, for some $t > 0$. Let $f : \mathbb{R}^d \to \mathbb{R}$ such that $\mathbb{E}f(X) = 0$ and let there exist $b \in \mathbb{R}$, such that for all $k$ $|f(x+ue_k) - f(x)| \leq b|u|$, for all $u \in \mathbb{R}$, $x \in \mathbb{R}^d$. Assume moreover that for all $u \in \mathbb{R}$, there exists a function $C_u$ verifying*

$$\sum_{k=1}^{d} \int_{\mathbb{R}} |f(X + ue_k) - f(X)|^2 \leq u^2 C_u(X),$$

*where $C_u$ is such that there exists $\lambda(u,t) > 0$ satisfying $\mathbb{E}[e^{\lambda(u,t)C_u(X)}] < \infty$. Then for all $t$ such that every quantity is well defined, one has*

$$(1 - h(t))\mathbb{E}[fe^{tf}] \leq g(t)\mathbb{E}[e^{tf}],$$

*where*

$$h(t) = \int_{\mathbb{R}} \frac{t}{\lambda(u,t)} |u| \frac{e^{tb|u|} - 1}{b} \tilde{\nu}(du),$$

*and*

$$g(t) = \int_{\mathbb{R}} \frac{\ln(\phi(u,t))}{\lambda(u,t)} |u| \frac{e^{tb|u|}-1}{b} \tilde{\nu}(du),$$

*and where* $\phi(t,u) = \mathbb{E}[e^{\lambda(u,t)C_u(X)}]$.

**Proof.** Applying Proposition 1 to $f$, the above assumptions entail that

$$\mathbb{E} f e^{tf} \leq \int_0^1 \mathbb{E}_z \left[ \int_{\mathbb{R}} \frac{C_u(U)+C_u(V)}{2} e^{tf(V)} \left( |u| \frac{e^{tb|u|}-1}{b} \right) \tilde{\nu}(du) \right] dz.$$

Next, apply Lemma 2 to $\lambda(u,t)Y = tf(V)$ and to $X = C_u(U)$ or $X = C_u(V)$. Since $Y$ has zero mean, one can ignore the last term in (3.1). This leads to

$$\mathbb{E} f e^{tf} \leq \int_{\mathbb{R}} \left[ \mathbb{E}\left( \frac{t}{\lambda(u,t)} f e^{tf} \right) + \frac{\ln(\phi(u,t))}{\lambda(u,t)} \mathbb{E}(e^{tf}) \right] |u| \frac{e^{tb|u|}-1}{b} \tilde{\nu}(du),$$

which concludes the proof. ■

This inequality is non trivial only when $h(t) < 1$, one of its applications is to the euclidean norm.

**Lemma 3** *For* $\alpha > 0$ *let* $\ell_\alpha = -\ln \mathbb{E}[e^{-\alpha X_1^2}]$. *Then, for all* $\lambda, v, \alpha > 0$, *such that* $\ell_\alpha \geq \lambda/v$,

$$\mathbb{E}\left( \exp\left( \frac{\lambda d}{\|X\|^2 + v} \right) \right) \leq 1 + \exp\left( \frac{\alpha\lambda}{\ell_\alpha - \frac{\lambda}{v}} \right).$$

**Proof.** Let $\varepsilon > 0$ which we will choose later. Let $a = \exp\left( \frac{\lambda d}{\varepsilon d \mathbb{E}(X_1^2) + v} \right)$ and $b = \exp\left( \frac{\lambda d}{v} \right)$. Then

$$\begin{aligned}
\mathbb{E}\left( \exp\left( \frac{\lambda d}{\|X\|^2 + v} \right) \right) &= \int_0^b \mathbb{P}\left( \exp\left( \frac{\lambda d}{\|X\|^2 + v} \right) \geq t \right) dt \\
&\leq a + \int_a^b \mathbb{P}\left( -\|X\|^2 \geq v - \frac{\lambda d}{\ln t} \right) dt \\
&\leq a + \int_a^b \mathbb{E}\left[ e^{-\alpha X_1^2} \right]^d e^{-\alpha v + \alpha \frac{\lambda d}{\ln t}} dt \\
&\leq a + e^{-d\ell_\alpha + \alpha \varepsilon d \mathbb{E}(X_1^2)} (b - a) \\
&\leq a + e^{d(\varepsilon \alpha \mathbb{E}(X_1)^2 + (\lambda/v) - \ell_\alpha)}.
\end{aligned}$$

Taking $\varepsilon$ such that $\varepsilon \alpha \mathbb{E}(X_1)^2 + (\lambda/v) - \ell_\alpha = 0$, leads to

$$\mathbb{E}\left( \exp\left( \frac{\lambda d}{\|X\|^2 + v} \right) \right) \leq a + 1 \leq 1 + \exp\left( \frac{\alpha\lambda}{\ell_\alpha - \frac{\lambda}{v}} \right).$$

■

*Rmq : avant on avait pris $\alpha = 1$, je ne sais pas trop quoi en faire pour l'instant.*

**Lemma 4** *There exists $c_1, c_2, c_3$ positive constants such that for all $x \in \mathbb{R}^d$ and $u \in \mathbb{R}$*

$$\sum_{k=1}^{d} |\|x + ue_k| \ - \|x\||^2 \leq u^2 \left( c_1 + \frac{c_2 d u^2}{\|x\| + c_3 u^2} \right).$$

**Proof.** The proof is similar to an argument used in the proof of Corollary 3.

$$\sum_{k=1}^{d} |\|x + ue_k| \ - \|x\||^2 \leq \sum_{k=1}^{d} \left( \frac{2ux_k + u^2}{\|x + ue_k\| + \|x\|} \right)^2.$$

But for all positive $\varepsilon$

$$\|x + ue_k\| = \sum_{j \neq k} x_j^2 + (x_k + u)^2 = \|x\|^2 + 2ux_k + u^2 \geq \|x\|^2 - \varepsilon x_k^2 + (1 - \varepsilon^{-1})u^2.$$

Consequently

$$\|x + ue_k\| + \|x\|^2 \geq (2 - \varepsilon)\|x\|^2 + (1 - \varepsilon^{-1})u^2.$$

Taking $\varepsilon = 3/2$ leads to the result. ■

With the help of the previous lemma, we now get: **Proof. [Theorem 3]**
Again we want to apply Corollary 6 to $f(X) = \|X\| - \mathbb{E}\|X\|$. With the notations of Lemma 4,

$$C_u(X) = c_1 + + \frac{c_2 d u^2}{\|x\| + c_3 u^2},$$

works. Then, one has to compute $\ln(\phi(u,t))$. But,

$$\ln(\phi(u,t)) = c_1 \lambda(u,t) + \ln \left( \mathbb{E} \left[ \exp \left( \frac{\lambda(u,t) c_2 u^2 d}{\|X\|^2 + c_3 u^2} \right) \right] \right),$$

and so by Lemma 3, it follows that

$$\ln(\phi(u,t)) = c_1 \lambda(u,t) + \ln \left( 1 + \exp \left( \frac{\alpha \lambda(u,t) c_2 u^2}{\ell_\alpha - \lambda(u,t) c_2 / c_3} \right) \right),$$

for all $\alpha$ such that $\ell_\alpha > \lambda(u,t) c_2 / c_3$. *Ici je ne sais pas comment me servir de $\ell_\alpha$ vraiment jusqu'au bout.*

Let us fix $\alpha = 1$ and $\lambda(u,t) = c_3 l/(2c_2)$. Then

$$\ln(\phi(u,t)) \leq c_1\lambda(u,t) + \ln(2) + c_3 u^2.$$

which leads to the result by classical arguments. ■

*reprendre a ce stade les constantes propres .... remarque si $X_1^2$ suit une poisson($\theta$) $\ell_\alpha = \theta(1-e^{-\alpha})$ donc ca tend vers 1 et c'est pas ca qui va aider la convergence ....*

# 4 A result for $L^p$-norms

We state and prove in this section the following generalization of Theorem 4:

**Theorem 5** *Let $X$ be as in Theorem 2 and let $2 \leq p < \infty$. Then for all positive $x$,*

$$\mathbb{P}(\|X\|_p - \mathbb{E}\|X\|_p \geq x) \leq \exp\left(-\int_0^x h_p^{-1}(s)ds\right), \tag{4.1}$$

*where the (dimension free) function $h_p$ is given by*

$$h_p(t) = 2^{4p-4}p^2 \int_{\mathbb{R}} \left[\left(1 + \frac{2^{1/p}|u|}{\underline{m_p}^{1/p}}\right)^{2p-2} + 2^{2p-2}\frac{\overline{m_{2p}}}{\underline{m_p}^2}\right] |u|(e^{t|u|}-1)\tilde{\nu}(du),$$

*and the moments $\overline{m_{2p}}, \underline{m_p}$ are defined as follows:*

- *if $X$ has almost surely positive coordinates, we can take $\underline{m_q} = \overline{m_q} = \mathbb{E}[(X_1)_1^q]$ for $q = p, 2p$.*
- *otherwise, we take*

$$\underline{m_p} = \inf_{t\in[0,1]} \left[\inf\{\mathbb{E}[|(Y_1)_t + (Z_1)_t|^p \mathbf{1}_{(Z_1)_t\geq 0,(Y_1)_t\geq 0}],\ \mathbb{E}[|(Y_1)_t + (Z_1)_t|^p \mathbf{1}_{(Z_1)_t\leq 0,(Y_1)_t\leq 0}]\}\right]$$

$$\overline{m_{2p}} = \sup_{t\in[0,1]} \left[\sup\{\mathbb{E}[|(Y_1)_t + (Z_1)_t|^{2p} \mathbf{1}_{(Z_1)_t\geq 0,(Y_1)_t\geq 0}],\ \mathbb{E}[|(Y_1)_t + (Z_1)_t|^{2p} \mathbf{1}_{(Z_1)_t\leq 0,(Y_1)_t\leq 0}]\}\right]$$

*When $1 \leq p < 2$, an inequality similar to (4.1) holds, where $h_p$ is now replaced by the following function $h_{p,d}$, which is not dimension-free:*

$$h_{p,d}(t) = 2^{4p-4}p^2 \int_{\mathbb{R}} \left[\left(d^{\frac{1}{2p-2}-\frac{1}{p}} + \frac{2^{1/p}|u|}{\underline{m_p}^{1/p}}\right)^{2p-2} + 2^{2p-2}\frac{\overline{m_{2p}}}{\underline{m_p}^2}\right] |u|(e^{t|u|}-1)\tilde{\nu}(du).$$

Again, we need the condition $\underline{m}_p > 0$ in order to get a non-trivial bound. However, this condition is satisfied in most natural cases. Let us now proceed to the proof of Theorem 5.

**Proof.**

First, using the notation of (1.5), Lemma 1 can be rewritten as

$$\begin{aligned}
&\mathbb{E}f(X)g(X) - \mathbb{E}f(X)\mathbb{E}g(X) \\
&= \int_0^1 \mathbb{E}\left[\sum_k \int_{\mathbb{R}} (f(Y_t + Z_t + ue_k) - f(Y_t + Z_t))(g(\widetilde{Y}_t + Z_t + ue_k) - g(\widetilde{Y}_t + Z_t))\tilde{\nu}(du)\right] dt
\end{aligned} \tag{4.2}$$

where $\widetilde{Y}_t$ is an independent copy of $Y_t$ and $f, g$ are Lipschitz. Indeed, recall the definition of $\nu_0$ and $\nu_1$ in Proposition 1 and remark that for every $t \in [0,1]$, $(Z_t, Z_t)$ is the ID random variable with Lévy measure $t\nu_1$ while $(Y_t, \widetilde{Y}_t)$ is the ID random variable with Lévy measure $t\nu_0$. Of course, by approximation, (4.2) remains true if $f, g$ are locally Lipschitz and

$$\int_0^1 \mathbb{E}\left[\sum_k \int_{\mathbb{R}} |f(Y_t + Z_t + ue_k) - f(Y_t + Z_t)| \left|g(\widetilde{Y}_t + Z_t + ue_k) - g(\widetilde{Y}_t + Z_t)\right| \tilde{\nu}(du)\right] dt < \infty.$$

Let $2 \leq p < \infty$, $\lambda > 0$. We want to apply (4.2) to the functions $f(X) = \|X\|_p$ and $g(X) = e^{\lambda \|X\|_p}$.

Using the same inequalities as in the beginning of the proof of Corollary 5, together with the fact that for $x, u \in \mathbb{R}$, $\sup(|x+u|, |x|) \geq (|x| + |u|)/4$, we obtain, for $X \in \mathbb{R}^d$, $u \in \mathbb{R}$ and $1 \leq k \leq d$,

$$|\|X + ue_k\|_p - \|X\|_p| \leq 2^{2p-2} p |u| \left(\frac{|X_k| + |u|}{(\|X\|_p^p + |u|^p)^{1/p}}\right)^{p-1}.$$

Similarly, if $\lambda > 0$, using the fact that $(e^x - 1)/x$ is an increasing function and that $|\|X + ue_k\|_p - \|X\|_p| \leq |u|$, we get

$$|\exp(\lambda\|X + ue_k\|_p) - \exp(\lambda\|X\|_p)| \leq 2^{2p-2} p (e^{\lambda|u|} - 1) \left(\frac{|X_k| + |u|}{(\|X\|_p^p + |u|^p)^{1/p}}\right)^{p-1} \exp(\lambda\|X\|_p).$$

Now fix $t \in [0,1]$ and consider that $X_i = (Y_i)_t + z_i$ where the $(Y_i)_t$ are iid random variables as in (1.5) and the $z_i$ are deterministic. Then

$$\sum_k \mathbb{E}_{Y_t} \left| \|X + ue_k\|_p - \|X\|_p \right| \mathbb{E}_{Y_t} \left| \exp(\lambda \|X + ue_k\|_p) - \exp(\lambda \|X\|_p) \right|$$
$$\leq 2^{4p-4} p^2 |u| (e^{\lambda|u|} - 1) \mathbb{E}_{Y_t, \widetilde{Y}_t} \sum_i \left| \frac{|(\widetilde{Y}_k)_t + z_k| + |u|}{(\|\widetilde{Y}_t + z\|_p^p + |u|^p)^{1/p}} \right|^{p-1} \left| \frac{|(Y_k)_t + z_k| + |u|}{(\|Y_t + z\|_p^p + |u|^p)^{1/p}} \right|^{p-1} \exp(\lambda \|Y_t + z\|_p)$$

where the $(\widetilde{Y}_i)_t$ are i.i.d. copies of the $(Y_i)_t$. (We write $\mathbb{E}_{Y_t}, \mathbb{E}_{Y_t, \widetilde{Y}_t}$ to make precise which are the random variables and which are the parameters). Cauchy-Schwarz and the independence of the $(\widetilde{Y}_i)_t$, $(Y_i)_t$ lead to,

$$\sum_k \mathbb{E}_{Y_t} \left| \|X + ue_k\|_p - \|X\|_p \right| \mathbb{E}_{Y_t} \left| \exp(\lambda \|X + ue_k\|_p) - \exp(\lambda \|X\|_p) \right|$$
$$\leq 2^{4p-4} p^2 |u| (e^{\lambda|u|} - 1) \mathbb{E}_{\widetilde{Y}_t} \left( \sum_k \left| \frac{|(\widetilde{Y}_k)_t + z_k| + |u|}{(\|\widetilde{Y}_t + z\|_p^p + |u|^p)^{1/p}} \right|^{2p-2} \right)^{1/2}$$
$$\times \mathbb{E}_{Y_t} \left[ \left( \sum_k \left| \frac{|(Y_k)_t + z_k| + |u|}{(\|Y_t + z\|_p^p + |u|^p)^{1/p}} \right|^{2p-2} \right)^{1/2} \exp(\lambda \|Y_t + z\|_p) \right].$$

Denote $I = (1, 1, \ldots, 1) \in \mathbb{R}^d$ and remark that

$$\begin{aligned} \sum_{k=1}^d (|(Y_k)_t + z_k| + |u|)^{2p-2} &= \|Y_t + z + uI\|_{2p-2}^{2p-2} \\ &\leq (\|Y_t + z\|_{2p-2} + \|uI\|_{2p-2})^{2p-2} \\ &\leq (\|Y_t + z\|_{2p-2} + |u| d^{1/(2p-2)})^{2p-2}, \end{aligned}$$

whence

$$\sum_k \left| \frac{|(Y_k)_t + z_k| + |u|}{(\|Y_t + z\|_p^p + |u|^p)^{1/p}} \right|^{2p-2} \leq \left( 1 + \frac{|u| d^{1/(2p-2)}}{(\|Y_t + z\|_p^p + |u|^p)^{1/p}} \right)^{2p-2}, \tag{4.3}$$

where $\|X\|_{2p-2} \leq \|X\|_p$ since $p \geq 2$. If we now write

$$N_{p,t} := (\|Y_t + z\|_p^p + |u|^p)^{1/p} \geq \|X\|_p,$$

then

$$\sum_k \left| \frac{|(Y_k)_t + z_k| + |u|}{(\|Y_t + z\|_p^p + |u|^p)^{1/p}} \right|^{2p-2} \leq \left(1 + \frac{|u| d^{1/(2p-2)}}{N_{p,t}}\right)^{2p-2},$$

and thus

$$\begin{aligned}
&\sum_k \mathbb{E}_{Y_t} \left| \|Y_t + z + ue_k\|_p - \|Y_t + z\|_p \right| \mathbb{E}_{Y_t} \left| \exp(\lambda \|Y_t + z + ue_k\|_p) - \exp(\lambda \|Y_t + z\|_p) \right| \\
&\leq 2^{4p-4} p^2 |u| (e^{\lambda |u|} - 1) \mathbb{E}_{Y_t} \left[ \sqrt{Q_p(N_{p,t}/|u| d^{1/(2p-2)})} \right] \mathbb{E}_{Y_t} \left[ \sqrt{Q_p(N_{p,t}/|u| d^{1/(2p-2)})} \exp(\lambda \|Y_t + z\|_p) \right],
\end{aligned}$$

where $Q_p(x) = (1 + 1/x)^{2p-2}$. Recall that if $T$ is a positive random variable, $A$ is a positive increasing function and $B$ is a positive decreasing function, then

$$\mathbb{E}[A(T)B(T)] \leq \mathbb{E}[A(T)]\mathbb{E}[B(T)]. \tag{4.4}$$

(A proof of this inequality is provided below). Applying it to $T = \|X\|_p$, $A(x) = \exp(\lambda x)$ and $B(x) = \sqrt{Q_p((|u|^p + x^p)^{1/p}/|u| d^{1/(2p-2)})}$ gives:

$$\begin{aligned}
&\sum_i \mathbb{E}_{Y_t} \left| \|X + ue_k\|_p - \|X\|_p \right| \mathbb{E}_{Y_t} \left| \exp(\lambda \|X + ue_k\|_p) - \exp(\lambda \|X\|_p) \right| \\
&\leq 2^{4p-4} p^2 |u| (e^{\lambda |u|} - 1) \left( \mathbb{E}_{Y_t} \left[ \sqrt{Q_p(N_{p,t}/|u| d^{1/(2p-2)})} \right] \right)^2 \mathbb{E}_{Y_t} \left[ [\exp(\lambda \|Y_t + z\|_p)] \right].
\end{aligned}$$

Now consider that $z$ is no longer a parameter but a random variable of the form $Z_t$ as in (1.5). Then we have:

$$\begin{aligned}
&\int_{\mathbb{R}^d} \mathbb{P}(Z_t \in dz) \sum_i \mathbb{E}_{Y_t} \left| \|X + ue_k\|_p - \|X\|_p \right| \mathbb{E}_{Y_t} \left| \exp(\lambda \|X + ue_k\|_p) - \exp(\lambda \|X\|_p) \right| \\
&\leq 2^{4p-4} p^2 |u| (e^{\lambda |u|} - 1) \int_{\mathbb{R}^d} \mathbb{P}(Z_t \in dz) \left( \mathbb{E}_{Y_t} \left[ \sqrt{Q_p(N_{p,t}/|u| d^{1/(2p-2)})} \right] \right)^2 \mathbb{E}_{Y_t} [\exp(\lambda \|Y_t + z\|_p)] \\
&\leq 2^{4p-4} p^2 |u| (e^{\lambda |u|} - 1) \int_{\mathbb{R}^d} \mathbb{P}(Z_t \in dz) \mathbb{E}_{Y_t} [Q_p(N_{p,t}/|u| d^{1/(2p-2)})] \mathbb{E}_{Y_t} [\exp(\lambda \|Y_t + z\|_p)].
\end{aligned}$$

For $y, y', z \in \mathbb{R}^d$ put

$$M_p(|u|, y, y', z) = [|u|^p + \sum_i \mathbf{1}_{sgn(y_i) = sgn(y'_i) = sgn(z_i)} |z_i + y'_i|^p]^{1/p}.$$

Then since $M_p < N_{p,t}$ and $Q_p$ is decreasing,

$$\begin{aligned}
&\int_{\mathbb{R}^d} \mathbb{P}(Z_t \in dz)\mathbb{E}_{Y_t}[Q_p(N_{p,t}/|u|d^{1/(2p-2)})]\mathbb{E}_{Y_t}[\exp(\lambda\|Y_t + z\|_p)] \\
&\leq \int_{\mathbb{R}^d} \mathbb{P}(Y_t \in dy) \int_{\mathbb{R}^d} \mathbb{P}(\widetilde{Y}_t \in dy') \int_{\mathbb{R}^d} \mathbb{P}(Z_t \in dz) \\
&\quad Q_p(M_p(|u|, y, y', z)/|u|d^{1/(2p-2)}) \exp(\lambda\|y + z\|_p).
\end{aligned}$$

We now use the following generalization of (4.4). Let $T$ be a random variable in $\mathbb{R}_+^d$ with i.i.d. components and let $A, B : \mathbb{R}_+^d \to \mathbb{R}_+$ be two functions. For every $i \leq d$ and every $x \in \mathbb{R}_+^{d-1}$, define the functions $A_{x,i}, B_{x,i} : \mathbb{R}_+ \to \mathbb{R}_+$ by

$$A_{x,i}(t) = A(x_1, \ldots, x_{i-1}, t, x_{i+1}, \ldots, x_d)$$

$$B_{x,i}(t) = (x_1, \ldots, x_{i-1}, t, x_{i+1}, \ldots, x_d)$$

Assume that for every $i \leq d$ and every $x \in \mathbb{R}_+^{d-1}$, one of the two functions $A_{x,i}, B_{x,i}$ is decreasing and the other one is increasing. Then

$$\mathbb{E}[A(T)B(T)] \leq \mathbb{E}[A(T)]\mathbb{E}[B(T)]. \tag{4.5}$$

The proof of (4.5) is obtained by induction on $d$, applying (4.4) repeatedly.

Applying (4.5) to the functions $A(z) = Q_p(M_p(|u|, y, y', z)/|u|d^{1/(2p-2)})$ and $B(z) = \exp(\lambda\|y + z\|_p)$ we get:

$$\begin{aligned}
&\int_{\mathbb{R}^d} \mathbb{P}(Z_t \in dz) Q_p(M_p(|u|, y, y', z)/|u|d^{1/(2p-2)}) \exp(\lambda\|y + z\|_p) \\
&\leq \int_{\mathbb{R}^d} \mathbb{P}(Z_t \in dz) Q_p(M_p(|u|, y, y', z)/|u|d^{1/(2p-2)}) \\
&\quad \times \int_{\mathbb{R}^d} \mathbb{P}(Z_t \in dz) \exp(\lambda\|y + z\|_p),
\end{aligned}$$

and so integrating in $y, y'$ further gives

$$\begin{aligned}
&\int_{\mathbb{R}^d} \mathbb{P}(Z_t \in dz)\mathbb{E}_{Y_t}[Q_p(N_{p,t}/|u|d^{1/(2p-2)})]\mathbb{E}_{Y_t}[\exp(\lambda\|Y_t + z\|_p)] \\
&\leq \left( \sup_{y \in \mathbb{R}^d} \mathbb{E}_{\widetilde{Y}_t, Z_t}[Q_p(M_p(|u|, y, \widetilde{Y}_t, Z_t)/|u|d^{1/(2p-2)})] \right) \mathbb{E}[\exp(\lambda\|X\|_p)].
\end{aligned}$$

Next, we want to lower bound

$$\sup_{y\in\mathbb{R}^d} \mathbb{E}_{\widetilde{Y}_t,Z_t}[Q_p(M_p(|u|,y,\widetilde{Y}_t,Z_t)/|u|d^{1/(2p-2)})].$$

Let $V_p$ denote the inverse of $Q_p$:

$$V_p(x)=\inf\{s>0,Q_p(s)<x\}.$$

Then,

$$\mathbb{P}(Q_p(M_p(|u|,y,\widetilde{Y}_t)/|u|d^{1/(2p-2)})\geq s)=\mathbb{P}(M_p(|u|,y,\widetilde{Y}_t)/|u|d^{1/(2p-2)}\leq V_p(s)).$$

This last probability is zero if $s\geq Q_p(1/d^{1/(2p-2)})$.

Suppose now that $y$ has $k$ positive coordinates and $d-k$ negative coordinates. Let $I_+$ be the set of $i$ such that $y_i>0$ and $I_-$ be the set of $i$ such that $y_i<0$. Denote

$$m_q^+(t)=\mathbb{E}(\mathbf{1}_{(Y_1)_t\geq 0,(Z_1)_t\geq 0}|(Y_1)_t+(Z_1)_t|^q),$$

$$m_q^-(t)=\mathbb{E}(\mathbf{1}_{(Y_1)_t\leq 0,(Z_1)_t\leq 0}|(Y_1)_t+(Z_1)_t|^q)$$

and

$$\overline{m_q}(t)=\sup(m_q^+(t),m_q^-(t)),$$

$$\underline{m_q}(t)=\inf(m_q^+(R),m_q^-(t)).$$

Then we have

$$\begin{aligned}
&\mathbb{P}(M_p(|u|,y,\widetilde{Y}_t)/|u|d^{1/(2p-2)}\leq V_p(s))\\
&=\mathbb{P}\left(\sum_{i\in I_+}\mathbf{1}_{(Y_i)_t\geq 0,(Z_i)_t\geq 0}|(Y_i)_t+(Z_i)_t|^p+\sum_{i\in I_-}\mathbf{1}_{(Y_i)_t\leq 0,(Z_i)_t\leq 0}|(Y_i)_t+(Z_i)_t|^p\leq d^{p/(2p-2)}|u|^pV_p(s)^p-|u|^p\right)\\
&\leq\frac{km_{2p}^+(t)+(d-k)m_{2p}^-(t)}{(d^{p/(2p-2)}|u|^pV_p(s)^p-\||u|^p-km_p^+(t)-(d-k)m_p^-(t))^2}.
\end{aligned}$$

In particular, if $s\geq Q_p(d^{(p-2)/p(2p-2)}\underline{m_p}(t))^{1/p}/2^{1/p}|u|)$, then

$$\mathbb{P}(Q_p(M_p(|u|,y,\widetilde{Y}_t)/|u|d^{1/(2p-2)})\geq s)\leq\frac{4\overline{m_{2p}}(t)}{d(\underline{m_p}(t))^2}.$$

It thus follows that

$$
\begin{aligned}
&\mathbb{E}_{Y_t}(Q_p(M_p(|u|,y,\widetilde{Y}_t)/|u|d^{1/(2p-2)}))\\
&=\int_0^{Q_p(1/d^{1/(2p-2)})}\mathbb{P}(Q_p(/M_p(|u|,y,\widetilde{Y}_t)/|u|d^{1/(2p-2)})\geq s)ds\\
&\leq\int_0^{Q_p(d^{(p-2)/p(2p-2)}\underline{m_p}(t))^{1/p}/2^{1/p}|u|)}\mathbb{P}(Q_p(M_p(|u|,y,\widetilde{Y}_t)/|u|d^{1/(2p-2)})\geq s)ds\\
&\quad+\int_{Q_p(d^{(p-2)/p(2p-2)}\underline{m_p}(t))^{1/p}/2^{1/p}|u|)}^{Q_p(1/d^{1/(2p-2)})}\mathbb{P}(Q_p(M_p(|u|,y,\widetilde{Y}_t)/|u|d^{1/(2p-2)})\geq s)ds\\
&\leq Q_p(d^{(p-2)/p(2p-2)}\underline{m_p}(t))^{1/p}/2^{1/p}|u|)+Q_p(1/d^{1/(2p-2)})\frac{4\overline{m_{2p}}(t)}{d(\underline{m_p}(t))^2}\\
&\leq Q_p(\underline{m_p}(t))^{1/p}/2^{1/p}|u|)+Q_p(1/d^{1/(2p-2)})\frac{4\overline{m_{2p}}(t)}{d(\underline{m_p}(t))^2}.
\end{aligned}
$$

Observe too that
$$
\frac{Q_p(1/d^{1/(2p-2)})}{d}=\frac{1}{d}\left(1+d^{1/(2p-2)}\right)^{2p-2}\leq 2^{2p-2},
$$
and therefore,

$$
\begin{aligned}
&\int_{\mathbb{R}^d}\mathbb{P}(Z_t\in dz)\sum_i\mathbb{E}_{Y_t}\left|\|X+ue_k\|_p-\|X\|_p\right|\mathbb{E}_{Y_t}|\exp(\lambda\|X+ue_ky\|_p)-\exp(\lambda\|X\|_p)|\\
&\leq 2^{4p-4}p^2|u|(e^{\lambda|u|}-1)\left[Q_p(\underline{m_p}(t))^{1/p}/2^{1/p}|u|)+2^{2p-2}\frac{\overline{m_{2p}}(t)}{(\underline{m_p}(t))^2}\right]\mathbb{E}[\exp(\lambda\|X\|_p)].
\end{aligned}
$$

Note that when $X$ has almost surely positive coordinates, we can apply (4.5) directly, without replacing $N_{p,t}$ with $M_p$. In that case we obtain the same inequality with

$$
\underline{m_p}(t)=\mathbb{E}[X^p]
$$

and

$$
\overline{m_{2p}}(t)=\mathbb{E}[X^{2p}].
$$

So defining $\underline{m_p},\overline{m_{2p}}$ as in Theorem 4 and putting

$$
C_p(\lambda,|u|)=2^{4p-4}p^2|u|(e^{\lambda|u|}-1)\left[Q_p((\underline{m_p})^{1/p}/2^{1/p}|u|)+2^{2p-2}\frac{\overline{m_{2p}}}{(\underline{m_p})^2}\right],
$$

we obtain, using the covariance formula

$$|\mathbb{E}(\|X\|_p \exp(\lambda\|X\|_p) - \mathbb{E}\|X\|_p \mathbb{E}(\exp(\lambda\|X\|_p))|$$
$$\leq \int_{-\infty}^{\infty} C_p(\lambda, |u|)\mathbb{E}\left[\exp(\lambda\|X\|_p\right] \tilde{\nu}(du).$$

In other words,

$$\left| \frac{\mathbb{E}[\|X\|_p \exp(\lambda\|X\|_p)]}{\mathbb{E}(\exp(\lambda\|X\|_p))} - \mathbb{E}\|X\|_p \right| \leq \int_{-\infty}^{\infty} C_p(\lambda, |u|)\tilde{\nu}(du) := F_p(\lambda). \tag{4.6}$$

Integrating both sides of (4.6) yields

$$|\log[\mathbb{E}(\exp(\lambda\|X\|_p))] - \lambda\mathbb{E}\|X\|_p| \quad \leq \quad \int_0^\lambda F_p(\mu)d\mu,$$

from which Theorem 4 follows.

If $1 \leq p \leq 2$, (4.3) can be replaced by

$$\sum_k \left| \frac{|(Y_k)_t + z_k| + |u|}{(\|Y_t + z\|_p^p + |u|^p)^{1/p}} \right|^{2p-2} \leq \left( d^{\frac{1}{2p-2} - \frac{1}{p}} + \frac{|u| d^{1/(2p-2)}}{(\|Y_t + z\|_p^p + |u|^p)^{1/p}} \right)^{2p-2},$$

and the rest of the proof goes likewise.

**A correlation inequality**

For the sake of completeness, we prove (4.4), although this result should be easy to find elsewhere. Suppose first that $T$ is absolutely continuous. Let $S$ be the size-biased version of $A(T)$ : $S$ is defined by the fact that for every bounded, measurable function $f$,

$$\mathbb{E}f(S) = \frac{\mathbb{E}[f(T)A(T)]}{\mathbb{E}A(T)}.$$

It is easy to check that $S$ is a well-defined random variable. Moreover, we claim that $S$ is stochastically greater than $T$: for every $x > 0$,

$$\mathbb{P}(S > x) \geq \mathbb{P}(T > x).$$

To prove this, write

$$\mathbb{P}(S > x) = \frac{\mathbb{E}[A(T)\mathbf{1}_{T>x}]}{\mathbb{E}A(T)},$$

so that our claim is equivalent to:

$$\frac{\mathbb{E}[A(T)\mathbf{1}_{T>x}]}{\mathbb{P}(T>x)} \geq \mathbb{E}A(T).$$

Let $\tau_x : \mathbb{R}_+ \to \mathbb{R}_+$ be the increasing function such that for every $y > x$,

$$\mathbb{P}(\tau_x(T) > y) = \frac{\mathbb{P}(T>y)}{\mathbb{P}(T>x)}.$$

$\tau_x$ is simply the transport of mass from the law of $T$ to the conditional law of $T$ given $T > x$, and $\tau_x$ exists since $T$ is absolutely continuous. In particular, since for every $y > 0$, $\mathbb{P}(\tau_x(T) > y) \geq \mathbb{P}(T > y)$, we have, for every $t > 0$, $\tau_x(t) \geq t$ and consequently, since $A$ is increasing, $A(\tau_x(t)) \geq A(t)$. Therefore,

$$\frac{\mathbb{E}[A(T)\mathbf{1}_{T>x}]}{\mathbb{P}(T>x)} = \mathbb{E}[A(\tau_x(T))] \geq \mathbb{E}[A(T)],$$

which proves our claim.

It follows that if $B$ is decreasing, then $B(S)$ is stochastically smaller than $B(T)$, whence

$$\mathbb{E}B(S) \leq \mathbb{E}B(T),$$

which proves (4.4) in the absolutely continuous case. The general case follows by passage to the limit.

# References


[BL1] Bobkov, S. G. and Ledoux, M. (1997). Poincaré inequalities and Talagrand's concentration phenomenon for the exponential measure. *Probab. Theory Related Fields* **107** 383–400.

[BL2] Bobkov, S. G. and Ledoux, M. (1998). On modified logarithmic Sobolev inequalities for Bernoulli and Poisson measures. *J. Funct. Anal.* **156** 347–365.

[HPAS] Houdré, C., Pérez-Abreu, V. and Surgailis, D. (1998). Interpolation, correlation identities and inequalities for infinitely divisible variables. *J. Fourier Anal. Appl.* **4** 651–668.

[H] Houdré, C. (2002). Remarks on deviation inequalities for functions of infinitely divisible random vectors. *Ann. Probab.* **30** 1223–1237.

[HM] Houdré, C. and Marchal P. (2002). On the concentration of measure phenomenon for stable and related random vectors. To appear: *Ann. Probab.*

[HR] Houdré, C. and Reynaud-Bouret P.

[MS] Milman V.D. and Schechtman G. (1986) Asymptotic theory of finite-dimensional normed spaces. *Lecture Notes in Math.* **1200** Springer, Berlin.

[RB] Reynaud-Bouret, P. (2003). Adaptive estimation of the intensity of inhomogeneous Poisson processes via concentration inequalities. *Probab. Theory Related Fields* **126** 103–153.

[R] Rosiński, J. Remarks on strong exponential integrability of vector–valued random series and triangular arrays. *Ann. Prob.* **23** (1996), 464–473.

[T] Talagrand, M. (1991). A new isoperimetric inequality for product measure, and the concentration of measure phenomenon. Israel Seminar (GAFA). *Lecture Notes in Math.* **1469** 91–124. Springer, Berlin.