

CONVERGENCE OF THE TWO-DIMENSIONAL DYNAMIC ISING-KAC MODEL TO Φ_2^4

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ABSTRACT. The Ising-Kac model is a variant of the ferromagnetic Ising model in which each spin variable interacts with all spins in a neighbourhood of radius γ^{-1} for $\gamma \ll 1$ around its base point. We study the Glauber dynamics for this model on a discrete two-dimensional torus $\mathbb{Z}^2/(2N+1)\mathbb{Z}^2$, for a system size $N \gg \gamma^{-1}$ and for an inverse temperature close to the critical value of the mean field model. We show that the suitably rescaled coarse-grained spin field converges in distribution to the solution of a non-linear stochastic partial differential equation.

This equation is the dynamic version of the Φ_2^4 quantum field theory, which is formally given by a reaction diffusion equation driven by an additive space-time white noise. It is well-known that in two spatial dimensions, such equations are distribution valued and a *Wick renormalisation* has to be performed in order to define the non-linear term. Formally, this renormalisation corresponds to adding an infinite mass term to the equation. We show that this need for renormalisation for the limiting equation is reflected in the discrete system by a shift of the critical temperature away from its mean field value.

1. INTRODUCTION

The aim of this article is to show the convergence of a rescaled discrete spin system to the solution of a stochastic partial differential equation formally given by

$$\partial_t X(t, x) = \Delta X(t, x) - \frac{1}{3} X^3(t, x) + AX(t, x) + \sqrt{2} \xi(t, x). \quad (1.1)$$

Here $x \in \mathbb{T}^2$ takes values in the two-dimensional torus, ξ denotes a space-time white noise, and $A \in \mathbb{R}$ is a real parameter.

The particle system that we consider is an Ising-Kac model evolving according to the Glauber dynamics. This model is similar to the usual ferromagnetic Ising model. The difference is that every spin variable interacts with all other spin variables in a large ball of radius γ^{-1} around its base point. We consider this model on a discrete two-dimensional torus $\mathbb{Z}^2/(2N+1)\mathbb{Z}^2$, for $N \gg \gamma^{-1}$. We then study the random fluctuations of a coarse-grained and suitably rescaled magnetisation field X_γ in the limit $\gamma \rightarrow 0$, for inverse temperature close to the critical temperature of the mean field model. Our main result, Theorem 2.1, states that under suitable assumptions on the initial configuration, these fields X_γ converge in law to the solution of (1.1). A similar result in one spatial dimension was shown in the nineties in [6, 22]. Our two-dimensional result was conjectured in [24].

The two-dimensional situation is more difficult than the one-dimensional case, because the solution theory for (1.1) is more intricate. Indeed, it is well-known that in dimension higher than one, equation (1.1) does not make sense as it stands. We

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will recall below in Section 3 that the solution X to (1.1) is a distribution-valued process. For each fixed t , the regularity properties of X are identical to those of the Gaussian free field. In this regularity class, it is a priori not possible to give a consistent interpretation to the nonlinear term X^3 . In order to even give a meaning to (1.1), a renormalisation procedure has to be performed. Formally, this procedure corresponds to adding an *infinite mass term* to (1.1), i.e. (1.1) is formally replaced by

$$\partial_t X(t, x) = \Delta X(t, x) - \frac{1}{3}(X^3(t, x) - 3\infty \times X(t, x)) + AX(t, x) + \sqrt{2}\xi(t, x),$$

where “ ∞ ” denotes a constant that diverges to infinity in the renormalisation procedure (see Section 3 for a more precise discussion).

A similar renormalisation was performed for the equilibrium state of (1.1) in the seventies, in the context of constructive quantum field theory (see [25] and the references therein). This equilibrium state is given by a measure on the space of distributions over \mathbb{T}^2 (or \mathbb{R}^2) which is formally described by

$$\frac{1}{\mathcal{Z}} \exp\left(-\int \frac{1}{12}(X^4(x) - 6\infty \times X(x)^2 + 3\infty) + \frac{1}{2}AX(x)^2 dx\right) \nu(dX), \quad (1.2)$$

where ν is the law of a Gaussian free field. The measure (1.2) is usually called the Φ_2^4 model, and we will therefore refer to the solution of (1.1) as the *dynamic* Φ_2^4 model. Connections between the measure (1.2) and the Ising model are well known: In [29] the measure (1.2) was obtained as a limit of spin models with continuous spins on a lattice (essentially the natural discretisation of (1.2)). Then in [51] the one point distribution appearing in this approximation, i.e. the measure on \mathbb{R} whose density is proportional to $\exp(-ax^4 + bx^2)dx$ was shown to arise as a limit of mean field Ising models. Furthermore, in [26] the authors showed that the ϕ_2^4 model admits a phase transition, just like the two-dimensional Ising model, the parameter A playing the role of the inverse temperature. We hope that our dynamic result will allow to also derive the invariant measure (1.2) directly as a scaling limit of a Kac-Ising model in equilibrium.

The construction of the dynamic Φ_2^4 model was a challenge for stochastic analysts for many years. Notable contributions include [46, 38, 2, 40]. Our analysis builds on the fundamental work of da Prato and Debussche [17], in which strong solutions for (1.1) were constructed for the first time.

Much more recently, there was great progress in the theory of singular SPDEs, in particular with [31, 32]. In these papers, Hairer developed a theory of *regularity structures* which allows to perform similar renormalisation procedures for much more singular equations such as the three-dimensional version of (1.1) or the Kardar-Parisi-Zhang (KPZ) equation. Parallel to [31, 32], another fruitful approach to give a meaning to such equations was developed in [28, 11]. One of the motivations for these works is to develop a technique to show that fluctuations of non-linear particle systems are governed by such an equation. The present article establishes such a result in this framework for the first time. One interesting feature of our result is that it gives a natural interpretation for the infinite renormalisation constant as a shift of critical temperature away from its mean field value (as was already predicted in [10, 24]).

The study of the KPZ equation has recently witnessed tremendous developments (see [50, 14, 15] and references therein). In their seminal paper [5], Bertini and Giacomin proved that the (suitably rescaled height process associated with the)

weakly asymmetric exclusion process converges to the KPZ equation. This result relies on two crucial properties: (1) that the KPZ equation can be transformed into a *linear* equation (the stochastic heat equation with multiplicative noise) via a Cole-Hopf transform; and (2) that the discrete system itself allows for a microscopic analogue of the Cole-Hopf transform [23]. The result can be extended to other particle systems as long as some form of microscopic Cole-Hopf transform is available [20]. However, for more general models, the question is still open, although notable results in this direction were obtained in [27].

For the asymmetric exclusion process to converge to the KPZ equation, it is essential to tune down the asymmetry parameter to zero while taking the scaling limit (hence the name *weakly* asymmetric). This procedure enables to keep the system away from the scale-invariant *KPZ fixed point*. This KPZ fixed point remains partly elusive [16]. Proving that a discrete system converges to the KPZ fixed point has so far only been possible (in a limited sense) by relying on the special algebraic properties of integrable models, see [4, 37, 47] for early works, and [15] and references therein for more recent developments.

We would like to underline the analogy between these observations and the situation with the two-dimensional Ising, Ising-Kac and Φ^4 models. The scaling limit of the (static) two-dimensional Ising model with nearest-neighbour interactions is now well understood, see [52, 8, 12, 13] and references therein. We may call this limit the (static, critical) *continuous Ising model*. Our replacement of nearest-neighbour by long-range, Kac-type interactions does *not* simply serve as a technical convenience. It also plays the role that the tunable asymmetry has for the convergence of the weakly asymmetric exclusion process discussed above. That is, it enables to keep the model away from the (scale-invariant) continuous Ising model. In order to best see that this limit is qualitatively different from the continuous Ising model (and its near-critical analogues), we point out that the probability for the field averaged over the torus to be above a large value x decays roughly like $\exp(-x^{16})$ for the continuous Ising model [9], while one can check that this probability decays roughly like $\exp(-x^4)$ for the measure in (1.2), as it does for the Curie-Weiss model [21, Theorem V.9.5]. It is expected that the measure (1.2) with critical A converges, under a suitable scaling, to the continuous Ising model.

1.1. Structure of the article. In Section 2, we define our model and give a precise statement of our main results. In Section 3, we describe briefly the solution theory for the limiting dynamics, following essentially the strategy of [17]. Sections 4 to 6 contain the core of our article. There, we introduce a suitable linearisation of the rescaled particle model and prove the convergence of this linearisation and several non-linear functionals thereof to the continuum model. In Section 7, we analyse the nonlinear evolution and complete the proof of our main theorem. Finally, in Section 8, we derive some additional bounds on an approximation to the heat semigroup that are referred to freely throughout the paper. In several appendices, we provide background material on Besov spaces, martingale inequalities and the martingale characterisation of the solution to the stochastic heat equation.

1.2. Notation. Throughout the paper, C will denote a generic constant that can change at every occurrence. We sometimes write $C(a, b, \dots)$ if the exact value of C depends on the quantities a, b, \dots . For $x \in \mathbb{R}^d$, we write $|x| = \sqrt{x_1^2 + \dots + x_d^2}$ for the Euclidean norm of x . For $x \in \mathbb{R}^d$, and $r > 0$, $B(x, r) = \{y \in \mathbb{R}^d : |x - y| < r\}$ is

the Euclidean ball of radius r around x . For $a, b \in \mathbb{R}$, we write $a \wedge b$ and $a \vee b$ to denote the minimum and the maximum of $\{a, b\}$. We denote by $\mathbb{N} = \{1, 2, 3, \dots\}$ the set of natural numbers and by $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. We also write $\mathbb{R}_+ = [0, \infty)$.

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2. SETTING AND MAIN RESULT

For $N \geq 1$, let $\Lambda_N = \mathbb{Z}^2 / (2N+1)\mathbb{Z}^2$ be the two-dimensional discrete torus which we identify with the set $\{-N, -(N-1), \dots, N\}^2$. Denote by $\Sigma_N = \{-1, +1\}^{\Lambda_N}$ the set of spin configurations on Λ_N . We will always denote spin configurations by $\sigma = (\sigma(k), k \in \Lambda_N)$.

Let $\mathfrak{K}: \mathbb{R}^2 \rightarrow [0, 1]$ be a \mathcal{C}^2 function with compact support contained in $B(0, 3)$, the Euclidean ball of radius 3 around 0 in \mathbb{R}^2 . We assume that \mathfrak{K} is invariant under rotations and satisfies

$$\int_{\mathbb{R}^2} \mathfrak{K}(x) dx = 1, \quad \int_{\mathbb{R}^2} \mathfrak{K}(x) |x|^2 dx = 4. \quad (2.1)$$

For $0 < \gamma < \frac{1}{3}$, let $\kappa_\gamma: \Lambda_N \rightarrow [0, \infty)$ be defined as $\kappa_\gamma(0) = 0$ and

$$\kappa_\gamma(k) = c_{\gamma,1} \gamma^2 \mathfrak{K}(\gamma k) \quad k \neq 0, \quad (2.2)$$

where $c_{\gamma,1}^{-1} = \sum_{k \in \Lambda_N^*} \gamma^2 \mathfrak{K}(\gamma k)$ and $\Lambda_N^* = \Lambda_N \setminus \{0\}$.

For any $\sigma \in \Sigma_N$, we introduce the locally averaged field

$$h_\gamma(\sigma, k) := \sum_{j \in \Lambda_N} \kappa_\gamma(k-j) \sigma(j) =: \kappa_\gamma \star \sigma(k), \quad (2.3)$$

and the Hamiltonian

$$\mathcal{H}_\gamma(\sigma) := -\frac{1}{2} \sum_{k,j \in \Lambda_N} \kappa_\gamma(k-j) \sigma(j) \sigma(k) = -\frac{1}{2} \sum_{k \in \Lambda_N} \sigma(k) h_\gamma(\sigma, k). \quad (2.4)$$

In both (2.3) and (2.4), subtraction on Λ_N is to be understood with periodic boundary conditions. Throughout this article we will always assume that $N \gg \gamma^{-1}$, so that the assumption $\kappa_\gamma(0) = 0$ implies that there is no self-interaction in (2.4).

For any inverse temperature $\beta > 0$, we define the Gibbs measure λ_γ on Σ_N as

$$\lambda_\gamma(\sigma) := \frac{1}{\mathcal{Z}_\gamma} \exp\left(-\beta \mathcal{H}_\gamma(\sigma)\right),$$

where

$$\mathcal{Z}_\gamma := \sum_{\sigma \in \Sigma_N} \exp\left(-\beta \mathcal{H}_\gamma(\sigma)\right)$$

denotes the normalisation constant that makes λ_γ a probability measure. On Σ_N , we study the Markov process given by the generator

$$\mathcal{L}_\gamma f(\sigma) = \sum_{j \in \Lambda_N} c_\gamma(\sigma, j) (f(\sigma^j) - f(\sigma)), \quad (2.5)$$

acting on functions $f: \Sigma_N \rightarrow \mathbb{R}$. Here $\sigma^j \in \Sigma_N$ is the spin configuration that coincides with σ except for a flipped spin at position j . As jump rates $c_\gamma(\sigma, j)$ we choose those of the Glauber dynamics,

$$c_\gamma(\sigma, j) := \frac{\lambda_\gamma(\sigma^j)}{\lambda_\gamma(\sigma) + \lambda_\gamma(\sigma^j)}. \quad (2.6)$$

It is clear that these jump rates are reversible with respect to the measure λ_γ . Since $\kappa_\gamma(0) = 0$, the local mean field $h_\gamma(\sigma, j)$ does not depend on $\sigma(j)$. Using also the fact that $\sigma(j) \in \{-1, 1\}$, we can conveniently rewrite the jump rates as

$$\begin{aligned} c_\gamma(\sigma, j) &= \frac{e^{-\sigma(j)\beta h_\gamma(\sigma, j)}}{e^{\beta h_\gamma(\sigma, j)} + e^{-\beta h_\gamma(\sigma, j)}} \\ &= \frac{1}{2} \left(1 - \sigma(j) \tanh(\beta h_\gamma(\sigma, j)) \right). \end{aligned} \quad (2.7)$$

We write $(\sigma(t))_{t \geq 0}$ for the (pure jump) Markov process on Σ_N thus defined, with the notation $\sigma(t) = (\sigma(t, k))_{k \in \Lambda_N}$. With a slight abuse of notation, we let

$$h_\gamma(t, k) = h_\gamma(\sigma(t), k). \quad (2.8)$$

The aim of this article is to describe the critical fluctuations of the local mean field h_γ as defined in (2.8), and to derive a non-linear SPDE for a suitably rescaled version of it. To this end we write, for $t \geq 0$ and $k \in \Lambda_N$,

$$h_\gamma(t, k) = h_\gamma(0, k) + \int_0^t \mathcal{L}_\gamma h_\gamma(s, k) ds + m_\gamma(t, k), \quad (2.9)$$

where the process $m_\gamma(\cdot, k)$ is a martingale. Observing that for any $\sigma \in \Sigma_N$ and for any $j, k \in \Lambda_N$, we have $h_\gamma(\sigma^j, k) - h_\gamma(\sigma, k) = -2\kappa_\gamma(k - j)\sigma(j)$, we get from (2.5) and (2.7)

$$\begin{aligned} \mathcal{L}_\gamma h_\gamma(\sigma, k) &= -h_\gamma(\sigma, k) + \kappa_\gamma \star \tanh(\beta h_\gamma(\sigma, k)) \\ &= \left(\kappa_\gamma \star h_\gamma(\sigma, k) - h_\gamma(\sigma, k) \right) + (\beta - 1) \kappa_\gamma \star h_\gamma(\sigma, k) \\ &\quad - \frac{\beta^3}{3} \left(\kappa_\gamma \star h_\gamma^3(\sigma, k) \right) + \dots, \end{aligned} \quad (2.10)$$

where we have used the Taylor expansion $\tanh(\beta h) = \beta h - \frac{1}{3}(\beta h)^3 + \dots$.

The predictable quadratic covariations (see Appendix C) of the martingales $m_\gamma(\cdot, k)$ are given by

$$\langle m_\gamma(\cdot, k), m_\gamma(\cdot, j) \rangle_t = 4 \int_0^t \sum_{\ell \in \Lambda_N} \kappa_\gamma(k - \ell) \kappa_\gamma(j - \ell) c_\gamma(\sigma(s), \ell) ds. \quad (2.11)$$

Furthermore, the jumps of $m_\gamma(\cdot, k)$ coincide with those of $h_\gamma(\cdot, k)$. In particular, if for some $\ell \in \Lambda_N$ the spin $\sigma(\ell)$ changes sign, then $m_\gamma(\cdot, k)$ has a jump of $-2\sigma(\ell)\kappa_\gamma(k - \ell)$.

2.1. Rescaled dynamics. For any $0 < \gamma < 1$ let $N = N(\gamma)$ be the microscopic system size determined below (in (2.16)). Then set $\varepsilon = \frac{2}{2N+1}$. Every *microscopic* point $k \in \Lambda_N$ can be identified with $x = \varepsilon k \in \Lambda_\varepsilon = \{x = (x_1, x_2) \in \varepsilon\mathbb{Z}^2: -1 < x_1, x_2 < 1\}$. We view Λ_ε as a discretisation of the continuous two-dimensional

torus \mathbb{T}^2 identified with $[-1, 1]^2$. For suitable scaling factors $\alpha, \delta > 0$ and inverse temperature β to be determined below we set

$$X_\gamma(t, x) = \frac{1}{\delta} h_\gamma\left(\frac{t}{\alpha}, \frac{x}{\varepsilon}\right) \quad x \in \Lambda_\varepsilon, t \geq 0. \quad (2.12)$$

In these *macroscopic coordinates*, the evolution equation (2.9) (together with (2.10)) reads

$$\begin{aligned} X_\gamma(t, x) = & X_\gamma(0, x) + \int_0^t \left(\frac{\varepsilon^2}{\gamma^2} \frac{1}{\alpha} \tilde{\Delta}_\gamma X_\gamma(s, x) + \frac{(\beta-1)}{\alpha} K_\gamma \star_\varepsilon X_\gamma(s, x) \right. \\ & \left. - \frac{\beta^3}{3} \frac{\delta^2}{\alpha} K_\gamma \star_\varepsilon X_\gamma^3(s, x) + K_\gamma \star_\varepsilon E_\gamma(s, x) \right) ds + M_\gamma(t, x), \end{aligned} \quad (2.13)$$

for $x \in \Lambda_\varepsilon$. Here we have set $K_\gamma(x) = \varepsilon^{-2} \kappa_\gamma(\varepsilon^{-1}x) = c_{\gamma,1} \frac{\gamma^2}{\varepsilon^2} \mathfrak{K}\left(\frac{\gamma}{\varepsilon}x\right)$ (the second equality being valid for $x \neq 0$). The convolution \star_ε on Λ_ε is defined through $X \star_\varepsilon Y(x) = \sum_{z \in \Lambda_\varepsilon} \varepsilon^2 X(x-z)Y(z)$ (where subtraction on Λ_ε is to be understood with periodic boundary conditions on $[-1, 1]^2$) and $\tilde{\Delta}_\gamma X = \frac{\gamma^2}{\varepsilon^2} (K_\gamma \star_\varepsilon X - X)$ (so that $\tilde{\Delta}_\gamma$ scales like the continuous Laplacian). The rescaled martingale is defined as $M_\gamma(t, x) := \frac{1}{\delta} m_\gamma\left(\frac{t}{\alpha}, \frac{x}{\varepsilon}\right)$. Finally, the error term $E_\gamma(t, x)$ (implicit in (2.10)) is given by

$$E_\gamma(t, \cdot) = \frac{1}{\delta \alpha} \left(\tanh(\beta \delta X_\gamma(t, \cdot)) - \beta \delta X_\gamma(t, \cdot) + \frac{(\beta \delta)^3}{3} X_\gamma(t, \cdot)^3 \right). \quad (2.14)$$

In these coordinates, the expression (2.11) for the quadratic variation becomes

$$\begin{aligned} & \langle M_\gamma(\cdot, x), M_\gamma(\cdot, y) \rangle_t \\ &= 4 \frac{\varepsilon^2}{\delta^2 \alpha} \int_0^t \sum_{z \in \Lambda_\varepsilon} \varepsilon^2 K_\gamma(x-z) K_\gamma(y-z) C_\gamma(s, z) ds, \end{aligned} \quad (2.15)$$

where $C_\gamma(s, z) := c_\gamma(\sigma(s/\alpha), z/\varepsilon)$. In these macroscopic coordinates, a spin flip at the microscopic position $k = \varepsilon^{-1}y$ causes a jump of $-2\sigma(\varepsilon^{-1}y)\delta^{-1}\varepsilon^2 K_\gamma(y-x)$ for the martingale $M_\gamma(\cdot, x)$.

The scaling of the approximated Laplacian, the term $K_\gamma \star_\varepsilon X_\gamma^3$ and the quadratic variation in (2.15) suggest that in order to see non-trivial behaviour for these terms, we need to impose $1 \approx \frac{\varepsilon^2}{\gamma^2} \frac{1}{\alpha} \approx \frac{\delta^2}{\alpha} \approx \frac{\varepsilon^2}{\delta^2 \alpha}$. Hence, from now on we set

$$N = \lfloor \gamma^{-2} \rfloor, \quad \varepsilon = \frac{2}{2N+1}, \quad \alpha = \gamma^2, \quad \delta = \gamma. \quad (2.16)$$

For later reference, we note that this implies that for $0 < \gamma < \frac{1}{3}$ we have

$$\varepsilon = \gamma^2 c_{\gamma,2} \quad \text{with} \quad (1 - \gamma^2) \leq c_{\gamma,2} \leq (1 + \gamma^2). \quad (2.17)$$

Under these assumptions, the leading order term in the expansion of the error term (2.14) scales like $\delta^4 \alpha^{-1} = \gamma^2$. Hence it seems reasonable to suspect that it will disappear in the limit. In order to prevent the (essentially irrelevant) factor $c_{\gamma,2}$ from appearing in too many formulas, we define

$$\Delta_\gamma := c_{\gamma,2}^2 \tilde{\Delta}_\gamma = \frac{\varepsilon^2}{\gamma^2} \frac{1}{\alpha} \tilde{\Delta}_\gamma. \quad (2.18)$$

At first sight, (2.13) suggests that β should be so close to one that $(\beta - 1)/\alpha = O(1)$. It was already observed in [10] (for the equilibrium system) that this naive guess is incorrect. Instead, we will always assume that

$$(\beta - 1) = \alpha(\mathfrak{c}_\gamma + A), \quad (2.19)$$

where $A \in \mathbb{R}$ is fixed. The extra term \mathfrak{c}_γ reflects the fact that the limiting equation has to be renormalised (see Section 3 for a detailed explanation). Its precise value is given below in (2.20), but we mention right away that the difference between \mathfrak{c}_γ and

$$\sum_{\substack{\omega \in \mathbb{Z}^2 \\ 0 < |\omega| < \gamma^{-1}}} \frac{1}{4\pi^2 |\omega|^2}$$

remains bounded as γ goes to 0. In particular, \mathfrak{c}_γ diverges logarithmically as γ goes to 0.

2.2. Main result. For any metric space \mathcal{S} , we denote by $\mathcal{D}(\mathbb{R}_+, \mathcal{S})$ the space of \mathcal{S} valued cadlag function endowed with the Skorokhod topology (see [7] for a discussion of this topology). For any $\nu > 0$ we denote by $\mathcal{C}^{-\nu}$ the Besov space $B_{\infty, \infty}^{-\nu}$ discussed in Appendix B.

We denote by X the solution of the renormalised limiting SPDE (1.1), discussed in Section 3 for a fixed initial datum $X^0 \in \mathcal{C}^{-\nu}$. This process X is continuous taking values in $\mathcal{C}^{-\nu}$.

Assume that for $\gamma > 0$, the spin configuration at time 0 is given by $\sigma_\gamma(0, k)$, $k \in \Lambda_N$, and define for $x \in \Lambda_\varepsilon$

$$X_\gamma^0(x) = \delta^{-1} \sum_{y \in \Lambda_\varepsilon} \varepsilon^2 K_\gamma(x - y) \sigma_\gamma(0, \varepsilon^{-1}y).$$

We extend X_γ^0 to a function on all of \mathbb{T}^2 as a trigonometric polynomial of degree $\leq N$ still denoted by X_γ^0 ; a precise definition and some properties of this extension can be found on Appendix A. Let $X_\gamma(t, x)$, $t \geq 0$, $x \in \Lambda_\varepsilon^2$ be defined by (2.12) and extend $X_\gamma(t, \cdot)$ to $x \in \mathbb{T}^2$ as a trigonometric polynomial of degree $\leq N$, still denoted by X_γ .

Theorem 2.1. *Assume that the scaling assumptions (2.16), (2.17) and (2.19) hold, where the precise value of \mathfrak{c}_γ is given by*

$$\mathfrak{c}_\gamma = \frac{1}{4} \sum_{\substack{\omega \in \{-N, \dots, N\}^2 \\ \omega \neq 0}} \frac{|\hat{K}_\gamma(\omega)|^2}{\gamma^{-2}(1 - \hat{K}_\gamma(\omega))}. \quad (2.20)$$

Assume also that X_γ^0 converges to X^0 in $\mathcal{C}^{-\nu}$ for $\nu > 0$ small enough and that X^0, X_γ^0 are uniformly bounded in $\mathcal{C}^{-\nu+\kappa}$ for an arbitrarily small $\kappa > 0$. Then X_γ converges in law to X with respect to the topology of $\mathcal{D}(\mathbb{R}_+, \mathcal{C}^{-\nu})$.

Remark 2.2. In principle one can perform the analysis that leads to (2.16) in any spatial dimension n . Indeed, the only necessary change is to replace the ε^2 terms appearing in (2.15) by ε^n , so that one wishes to impose $1 \approx \frac{\varepsilon^2}{\gamma^2} \frac{1}{\alpha} \approx \frac{\delta^2}{\alpha} \approx \frac{\varepsilon^n}{\delta^2 \alpha}$. In this way, one obtains the scaling relation

$$\varepsilon \approx \gamma^{\frac{4}{4-n}}, \quad \alpha \approx \gamma^{\frac{2n}{4-n}}, \quad \delta \approx \gamma^{\frac{n}{4-n}}. \quad (2.21)$$

This relation was already obtained in [24] and for $n = 1$ it is indeed the scaling used by [6, 22]. For $n = 3$, we expect that it is possible to combine the techniques

developed in this article with the theory developed in [32, 11] to get a convergence result to the dynamic Φ_3^4 model. For $n = 4$, relation (2.21) cannot be satisfied. This corresponds exactly to the fact that the Φ_4^4 model fails to satisfy the *subcriticality condition* in [32] and indeed, a limiting theory is not expected to exist for $n \geq 4$ (see [1]).

Remark 2.3. We stress that our analysis is purely based on the dynamics of the system; we do not rely on properties of the invariant measure λ_γ .

Remark 2.4. A simple analysis (see e.g. [19, Chapter 2.1]) of the trapezoidal approximation $c_{\gamma,1}^{-1} = \sum_{k \in \Lambda_N} \gamma^2 \mathfrak{K}(\gamma k)$ to $\int \mathfrak{K}(x) dx = 1$ shows that under our C^2 assumption on \mathfrak{K} , we have $|c_{\gamma,1} - 1| \leq C\gamma^2$. If we were to replace $c_{\gamma,1}$ by 1 in (2.2), this error term of the form $C\gamma^2$ would not disappear in our scaling, but would produce an $O(1)$ contribution to the “mass” A in the limiting equation. But this effect could be removed under slightly different assumptions: if we assumed that \mathfrak{K} is C^∞ and in addition removed the assumption $\kappa_\gamma(0) = 0$, then by the Euler-Maclaurin formula (see [19, Chapter 2.9]) we could get an arbitrary polynomial rate of convergence. Under these modified assumptions, setting $c_{\gamma,1} = 1$ would not change the result.

Remark 2.5. The condition $\kappa_\gamma(0) = 0$ in the definition of h_γ is very convenient, because it allows to rewrite the jump rates defined in (2.6) as a function of h_γ in a clean way (2.7). However, at some places it causes some minor technical problems: much of our analysis is performed in Fourier space, and due to this condition the Fourier transform $\hat{K}_\gamma(\omega)$ of K_γ decays at most like $\frac{C}{|\gamma\omega|^2}$ for large ω (see Lemma 8.2 below), whereas without this condition (and with a stronger regularity assumption on \mathfrak{K}) one could obtain $\frac{C_m}{|\gamma\omega|^m}$ for any $m \geq 1$. Fortunately, this only produces some irrelevant logarithmic error terms.

Remark 2.6. In order to state our result, we have made two assumptions that may seem debatable. On the one hand, we have chosen to define the coarse-graining h_γ in terms of the same kernel κ_γ that determines the interaction. On the other hand, we have extended the field X_γ as a trigonometric polynomial.

The reader will see below that the first choice is necessary in order to get control on Fourier modes ω satisfying $\gamma^{-1} \ll |\omega| \ll \gamma^{-2}$. The second choice is convenient but essentially irrelevant.

A posteriori, it is not too hard to show that even without any coarse-graining, the evolution

$$\mathbf{X}_\gamma(t, \varphi) = \sum_{x \in \Lambda_\varepsilon} \varepsilon^2 \varphi(x) \delta^{-1} \sigma(t/\alpha, x/\varepsilon),$$

viewed as an evolution in the space of distributions $\mathcal{S}'(\mathbb{T}^2)$ converges in law to the same limit, but only with respect to the weaker topology of $\mathcal{D}(\mathbb{R}_+, \mathcal{S}'(\mathbb{T}^2))$.

Remark 2.7. Of course, there are many choices other than (2.7) for a rate c_γ to define a Markov process on Σ_N that is reversible with respect to λ_γ . The Glauber dynamics is a standard choice, but it would be possible to extend our proof to a more general jump rate. We make strong use of the fact that $c_\gamma(\sigma, k)$ is a function of $h_\gamma(\sigma, k)$, and of the specific form of the Taylor expansion around $h_\gamma(\sigma, k) = 0$ (see (2.10)). We furthermore use the fact that c_γ is bounded by 1.

3. ANALYSIS OF THE LIMITING SPDE

As stated above, a well-posedness theory for the limiting equation (1.1) was provided in [17]. More precisely, in that article local in time existence and uniqueness for arbitrary $C^{-\nu}$ initial data and for $\nu > 0$ small enough was shown (recall the definition of the Besov space $C^{-\nu}$ in Appendix B). Furthermore, it was shown, using an idea due to Bourgain, that global well-posedness holds for *almost every* initial datum with respect to the invariant measure. In our companion paper [42], we extend these results to show *global* well-posedness for *every* initial datum in $C^{-\nu}$ for $\nu > 0$ small enough. In [42], we also show how to extend these arguments from the two-dimensional torus to the full space \mathbb{R}^2 , but this extension is not relevant for the present article. The purpose of this section is to review the relevant results from [17, 42]. We give some ideas of the proofs where it helps the reader's understanding of our argument in the more complicated discussion of the discrete system that follows.

As it stands, it is not clear at first sight how equation (1.1) should be interpreted. Indeed, the space-time white noise is a quite irregular distribution, and it is well-known that the regularising property of the heat semigroup is not enough to turn the solution of the linearised equation (the one obtained by removing the X^3 term from (1.1)) into a function. We will see below that this linearised solution takes values in all of the distributional Besov spaces of negative regularity, but not of positive regularity. We cannot expect the solution X to the non-linear problem to be more regular, and hence it is not clear how to interpret the non-linear term X^3 .

A naive approach consists in approximating solutions to (1.1) by solutions to a regularised equation. More precisely, let X_ε be the solution to the stochastic PDE

$$dX_\varepsilon = \left(\Delta X_\varepsilon - \frac{1}{3} X_\varepsilon^3 + A X_\varepsilon \right) dt + \sqrt{2} dW_\varepsilon. \quad (3.1)$$

Here $W_\varepsilon(t, x) = \frac{1}{4} \sum_{|\omega| < \varepsilon^{-1}} e^{i\pi\omega \cdot x} \hat{W}(\omega, t)$ is a spatially regularised cylindrical Wiener process. For every $\omega \in \mathbb{Z}^2$ the process $\hat{W}(\omega, t)$ is a complex valued Brownian motion with $\mathbb{E}|\hat{W}(\omega, t)|^2 = 4t$. These Brownian motions are independent except for the constraint $\hat{W}(\omega, t) = \overline{\hat{W}(-\omega, t)}$ for all $\omega \in \mathbb{Z}^2$ and $t \geq 0$. We choose to regularise the noise by considering a cut-off in Fourier space, but this choice of regularisation is inessential. A solution to equation (3.1) for fixed value of $\varepsilon > 0$ can be constructed with standard methods, see e.g. [18, 30, 48]. It seems natural to study the behaviour of these regularised solutions as ε goes to 0. Unfortunately, letting ε go to zero yields a trivial result. Indeed, it is shown in [33] that X_ε converges to zero in probability (in a space of distributions).

In order to obtain a non-trivial result, the approximations (3.1) have to be modified. Indeed, it is shown in [17] that if instead of (3.1), we consider

$$dX_\varepsilon = \left(\Delta X_\varepsilon - \left(\frac{1}{3} X_\varepsilon^3 - \mathfrak{c}_\varepsilon X_\varepsilon \right) + A X_\varepsilon \right) dt + \sqrt{2} dW_\varepsilon, \quad (3.2)$$

for a particular choice of constant \mathfrak{c}_ε , then a non-trivial limit can be obtained as ε goes to zero. Similar to (2.20), the precise value of \mathfrak{c}_ε is given by

$$\mathfrak{c}_\varepsilon = \sum_{0 < |\omega| < \varepsilon^{-1}} \frac{1}{4\pi^2 |\omega|^2}. \quad (3.3)$$

In particular, the constants \mathfrak{c}_ε diverge logarithmically as $\varepsilon \rightarrow 0$. Note however, that our choice of \mathfrak{c}_ε is not universal - other choices would lead to a non-trivial limit

as long as the difference to our choice remains bounded as ε goes to zero. See Remark 3.3 below for more on this.

As a first step to show this convergence, equation (3.2) is linearised. Let Z_ε be the unique mild solution to

$$\begin{aligned} dZ_\varepsilon(t, x) &= \Delta Z_\varepsilon(t, x) dt + \sqrt{2} dW_\varepsilon(t, x) \\ Z_\varepsilon(0, x) &= 0, \end{aligned} \quad (3.4)$$

i.e. $Z_\varepsilon(t, \cdot) = \sqrt{2} \int_0^t P_{t-s} dW_\varepsilon(s, \cdot)$, where $P_t = e^{\Delta t}$ is the solution operator of the heat equation on the torus \mathbb{T}^2 . Then the renormalisation is performed on the level of this linearised equation. We explain this procedure in some detail, because a similar study of a linearised version of (2.13) constitutes the core of our argument.

We start by recalling that the Hermite polynomials $H_n = H_n(X, T)$ are defined recursively by setting

$$\begin{cases} H_0 = 1, \\ H_n = XH_{n-1} - T \partial_X H_{n-1} \end{cases} \quad (n \in \mathbb{N}), \quad (3.5)$$

so that $H_1 = X$, $H_2 = X^2 - T$, $H_3 = X^3 - 3XT$, etc. One can check by induction that

$$\partial_X H_n = nH_{n-1} \quad (3.6)$$

and

$$\partial_T H_n = -\frac{n(n-1)}{2} H_{n-2} \quad (3.7)$$

(the identities are even valid for $n \in \{0, 1\}$, for an arbitrary interpretation of H_{-1} and H_{-2}).

Now, for any fixed $\varepsilon > 0$, Z_ε is a random continuous function, and there is no ambiguity in the definition of Z_ε^n , but for $n \geq 2$ these random functions Z_ε^n fail to converge to random distributions as ε goes to zero. If, however, Z_ε^n are replaced by the Hermite polynomials

$$Z_\varepsilon^{:n:}(t, x) := H_n(Z_\varepsilon(t, x), \mathbf{c}_\varepsilon(t))$$

for

$$\begin{aligned} \mathbf{c}_\varepsilon(t) &= \mathbb{E}[Z_\varepsilon(t, 0)^2] = \frac{1}{2} \sum_{|\omega| < \varepsilon^{-1}} \int_0^t \exp(-2r\pi^2|\omega|^2) dr \\ &= \frac{t}{2} + \sum_{0 < |\omega| < \varepsilon^{-1}} \frac{1}{4\pi^2|\omega|^2} \left(1 - \exp(-2t\pi^2|\omega|^2)\right), \end{aligned} \quad (3.8)$$

then a non-trivial limit is obtained. Note that $\mathbf{c}_\varepsilon = \lim_{t \rightarrow \infty} (\mathbf{c}_\varepsilon(t) - \frac{t}{2})$, where the term $\frac{t}{2}$ comes from the summand for $\omega = 0$ in (3.8) which does not converge as $t \rightarrow \infty$. Furthermore, for every fixed $t > 0$ the difference $|\mathbf{c}_\varepsilon - \mathbf{c}_\varepsilon(t)|$ is uniformly bounded in ε .

The following result, essentially [17, Lemma 3.2], summarises this convergence.

Proposition 3.1. *For every $T > 0$ and every $\nu > 0$, the stochastic processes Z_ε and $Z_\varepsilon^{:n:}$ for $n \geq 2$ converge almost surely and in every stochastic L^p space with respect to the metric of $\mathcal{C}([0, T], \mathcal{C}^{-\nu})$. We denote the limiting processes by Z and $Z^{:n:}$.*

We outline an argument for Proposition 3.1 which is inspired by the treatment of a (more complicated) renormalisation procedure in [32, Section 10]. We start with

an alternative representation of the $Z_\varepsilon^{n:}$. As explained above we have $Z_\varepsilon(t, \cdot) = \sqrt{2} \int_0^t P_{t-r} dW_\varepsilon(r)$. It will be useful to introduce the processes

$$R_{\varepsilon,t}(s, x) = R_{\varepsilon,t}^{1:}(s, x) = \sqrt{2} \int_0^s P_{t-r} dW_\varepsilon(r, x),$$

defined for $s \leq t$, and to define recursively

$$R_{\varepsilon,t}^{n:}(s, x) = n \int_{r=0}^s R_{\varepsilon,t}^{n-1:}(r, x) dR_{\varepsilon,t}(r, x).$$

It can be checked easily using Itô's formula and the relations (3.6) and (3.7) that for any $\varepsilon > 0$, we have

$$R_{\varepsilon,t}^{n:}(t, x) = Z_\varepsilon^{n:}(t, x).$$

For any smooth function $f: \mathbb{T}^2 \rightarrow \mathbb{R}$, the expectation

$$\mathbb{E} Z_\varepsilon^{n:}(t, f)^2 := \mathbb{E} \left(\int_{\mathbb{T}^2} Z_\varepsilon^{n:}(t, x) f(x) dx \right)^2$$

can now be calculated explicitly via Itô's isometry, and we obtain¹

$$\mathbb{E} Z_\varepsilon^{n:}(t, f)^2 = n! \frac{2^n}{4^n} \int_{[0,t]^n} \sum_{|\omega_i| \leq \varepsilon^{-1}} |\hat{f}(\omega_1 + \dots + \omega_n)|^2 \prod_{j=1}^n |\hat{P}_{t-r_j}(\omega_j)|^2 dr,$$

where $dr = dr_1 \dots dr_n$ and where $\hat{P}_t(\omega) = \exp(-t\pi^2|\omega|^2)$ denotes the Fourier transform of the heat kernel on the torus. This quantity converges as ε goes to zero, and the limit can be expressed as

$$n! 2^n \int_{[0,t]^n} \int_{(\mathbb{T}^2)^n} \left(\int_{\mathbb{T}^2} f(x) \prod_{j=1}^n P_{t-r_j}(x - z_j) dx \right)^2 dz dr,$$

where $dz = dz_1 \dots dz_n$. A crucial observation now uses the Gaussian structure of the noise and the fact that $Z_\varepsilon^{n:}(t, f)$ is a random variable in the n -th homogenous Wiener chaos over this Gaussian noise (see [45, Chapter 1] for a definition and properties of Gaussian Wiener chaos). According to Nelson's estimate (see [44] or [45, Chapter 1.5]) the estimate on $\mathbb{E} Z_\varepsilon^{n:}(t, f)^2$ can be turned into an equivalent estimate on $\mathbb{E} Z_\varepsilon^{n:}(t, f)^p$. Then one can specialise this bound to $f = \eta_k(u - \cdot)$ (defined in (B.8)), and apply Proposition B.4 to obtain bounds on $\mathbb{E} \|Z_\varepsilon^{n:}(t, \cdot)\|_{\mathcal{C}^{-\nu}}^p$ that are uniform in ε .

For continuity in time, one can modify this argument to get bounds on $\mathbb{E} \|Z_\varepsilon^{n:}(t, \cdot) - Z_\varepsilon^{n:}(s, \cdot)\|_{\mathcal{C}^{-\nu}}^p$, and then apply the Kolmogorov criterion.

In Sections 4 and 5, we will perform a similar argument for a linearised version of the evolution equation (2.13). One obstacle we need to overcome, is that without the

¹To derive this formula we introduce the Fourier coefficients $\hat{R}_{\varepsilon,t}(s, \omega)$, $\hat{R}_{\varepsilon,t}^{n:}(s, \omega)$ ($\omega \in \mathbb{Z}^2$) of $R_{\varepsilon,t}(s, \cdot)$ and $R_{\varepsilon,t}^{n:}(s, \cdot)$ respectively, and observe that

$$\begin{aligned} \hat{R}_{\varepsilon,t}(s, \omega) &= \sqrt{2} \int_{r=0}^t \hat{P}_{t-r}(\omega) d\hat{W}(\omega, r) \quad (|\omega| \leq \varepsilon^{-1}), \\ \hat{R}_{\varepsilon,t}^{n:}(s, \omega) &= n\sqrt{2} \int_{r=0}^s \frac{1}{4} \sum_{\omega_1 + \omega_2 = \omega} \hat{R}_{\varepsilon,t}^{n-1:}(r, \omega_1) \hat{P}_{t-r}(\omega_2) d\hat{W}(\omega_2, r). \end{aligned}$$

For instance,

$$\int_{\mathbb{T}^2} R_{\varepsilon,t}(s, x) f(x) dx = \frac{\sqrt{2}}{4} \sum_{|\omega| \leq \varepsilon^{-1}} \hat{f}(\omega) \int_{r=0}^s \hat{P}_{t-r}(\omega) d\hat{W}(\omega, r).$$

Gaussian structure of the noise, Nelson's estimate is not available. In Lemma 4.1, we replace it by a suitable version of the Burkholder-Davis-Gundy inequality. The price we have to pay is that various error terms caused by the jumps need to be controlled.

It is useful to note that for fixed values of s and t , $s < t$, the processes $R_{\varepsilon,t}^{n:}(s, \cdot)$ actually converge in nicer spaces than the $C^{-\nu}$. Indeed $R_{\varepsilon,t}(s, \cdot) = P_{t-s}Z_{\varepsilon}(s, \cdot)$, and by the convergence of $Z_{\varepsilon}(s)$ in $C^{-\nu}$ and standard regularising properties of the heat semigroup (see e.g. (8.22) and the discussion following it) $R_{\varepsilon,t}(s, x)$ converges to

$$R_t(s, x) := \sqrt{2} \int_{r=0}^s P_{t-r} dW(r, x),$$

in C^k for every $k \in \mathbb{N}$. In the same way, $R_{t,\varepsilon}^{n:}$ converges to

$$R_t^{n:}(s, x) = n \int_{r=0}^s R_t^{n-1:}(r, x) dR_t(r, x) = H_n(R_t(s, x), \langle R_t(\cdot, x) \rangle_s). \quad (3.9)$$

Here the quadratic variation of the continuous martingale $s \mapsto R_t(s, x)$ for $s < t$ is given by

$$\begin{aligned} \langle R_t(\cdot, x) \rangle_s &= 2 \int_0^s P_{2(t-r)}(0) dr \\ &= \frac{1}{2} \sum_{\omega \in \mathbb{Z}^2} \int_0^s \exp(-2(t-r)\pi^2|\omega|^2) dr, \end{aligned}$$

where $P_t(x)$ is the heat kernel associated with the semigroup P_t .

Finally, note that the convergence of $R_t^{n:}(t, \cdot)$ to $Z^{n:}(t, \cdot)$ in $C^{-\nu}$ can be quantified. Using an argument in the same spirit as the one sketched above, one can see that for all $\nu > 0$, $0 \leq \lambda \leq 1$, $p \geq 2$ and $T > 0$, there exists $C = C(\nu, \lambda, p, T)$ such that

$$\mathbb{E} \| Z^{n:}(t, \cdot) - R_t^{n:}(s, \cdot) \|_{C^{-\nu-\lambda}}^p \leq C |t-s|^{\frac{\lambda p}{2}} \quad (3.10)$$

for all $0 \leq s \leq t \leq T$. A similar bound in the more complicated discrete situation is derived below in (4.20).

In order to study the convergence of the non-linear equations (3.2), we study the remainder $v_{\varepsilon} := X_{\varepsilon} - Z_{\varepsilon}$. For $\varepsilon > 0$, we observe that v_{ε} is a solution to the random partial differential equation

$$\begin{aligned} \partial_t v_{\varepsilon}(t, x) &= \Delta v_{\varepsilon} - \left(\frac{1}{3} (v_{\varepsilon} + Z_{\varepsilon})^3 - \mathfrak{c}_{\varepsilon} (v_{\varepsilon} + Z_{\varepsilon}) \right) + A(v_{\varepsilon} + Z_{\varepsilon}) \\ &= \Delta v_{\varepsilon} - \left(v_{\varepsilon}^3 + 3v_{\varepsilon}^2 Z_{\varepsilon} + 3v_{\varepsilon} Z_{\varepsilon}^{2:} + Z_{\varepsilon}^{3:} \right) + A_{\varepsilon}(t)(v_{\varepsilon} + Z_{\varepsilon}), \end{aligned} \quad (3.11)$$

where we have set $A_{\varepsilon}(t) := A + \mathfrak{c}_{\varepsilon} - \mathfrak{c}_{\varepsilon}(t)$. Note that the noise term dW_{ε} has disappeared from equation (3.11). Note furthermore that in the second line, we have rewritten the right-hand side in terms of the processes Z_{ε} , $Z_{\varepsilon}^{2:}$, and $Z_{\varepsilon}^{3:}$, which converge to a non-trivial limit as ε goes to zero. This is possible due to the relation

$$H_n(z + v, c) = \sum_{k=0}^n \binom{n}{k} H_k(z, c) v^{n-k},$$

which holds for arbitrary $z, v \in \mathbb{R}$ and $c > 0$.

Equation (3.11) can be treated as a normal PDE, without taking into account stochastic cancellations or stochastic integrals. The argument in [17] is concluded by observing that (at least for small times) the solutions of (3.11) are stable under approximation of the functions Z_{ε} , $Z_{\varepsilon}^{2:}$, and $Z_{\varepsilon}^{3:}$ in $\mathcal{C}([0, \infty), C^{-\nu})$ as well as the limit of A_{ε} as $\varepsilon \rightarrow 0$. In this way, local in time solutions are obtained in [17].

The authors then show that these solutions do not blow up in finite time for almost every initial datum with respect to the invariant measure. For our purposes, it is slightly more convenient to have global existence for *every* initial datum in $\mathcal{C}^{-\nu}$ for $\nu > 0$ small enough, and we show this in the forthcoming article [42]. In order to state the main result, it is necessary to briefly discuss the role of the initial datum $X^0 \in \mathcal{C}^{-\nu}$.

There are essentially two possibilities – either equation (3.4) is started with X^0 , in which case the initial datum for (3.11) is zero; or the linear equation is started with zero, and (3.11) is started with X^0 . The first option turns out to be slightly more convenient. Hence, we define $Y(t) := P_t X^0$ and set

$$\begin{aligned}\tilde{Z}(t, \cdot) &= Y(t, \cdot) + Z(t, \cdot), \\ \tilde{Z}^{:2:}(t, \cdot) &= Z^{:2:}(t, \cdot) + 2Y(t, \cdot)Z(t, \cdot) + Y(t, \cdot)^2, \\ \tilde{Z}^{:3:}(t, \cdot) &= Z^{:3:}(t, \cdot) + 3Y(t, \cdot)Z(t, \cdot)^{:2:} + 3Y(t, \cdot)^2 Z(t, \cdot) + Y(t, \cdot)^3.\end{aligned}\quad (3.12)$$

Notice that by the regularisation property of the heat semigroup (see (8.22)), for any $t > 0$, $Y(t, \cdot)$ is actually a smooth function and, in particular, the products appearing in these expressions are well-defined. More precisely, for every $\beta > \nu$, there exists a constant $C = C(\nu, \beta)$ such that for every $t > 0$, we get $\|Y(t, \cdot)\|_{\mathcal{C}^\beta} \leq Ct^{-\frac{\nu+\beta}{2}} \|X^0\|_{\mathcal{C}^{-\nu}}$, and hence $\|Y(t, \cdot)^2\|_{\mathcal{C}^{-\nu}} \leq Ct^{-\frac{\nu+\beta}{2}} \|X^0\|_{\mathcal{C}^{-\nu}}^2$, $\|Y(t, \cdot)^2\|_{\mathcal{C}^\beta} \leq Ct^{-(\nu+\beta)} \|X^0\|_{\mathcal{C}^{-\nu}}^2$ and $\|Y^3(t, \cdot)\|_{\mathcal{C}^{-\nu}} \leq Ct^{-(\nu+\beta)} \|X^0\|_{\mathcal{C}^{-\nu}}^3$, where we use the fact that \mathcal{C}^β is an algebra for $\beta > 0$ as well as the multiplicative inequality, Lemma B.5. Using Lemma B.5 once more, we can conclude that for every $T > 0$, there exists a *random* constant C_0 (depending on $T, \nu, \beta, \|X^0\|_{\mathcal{C}^{-\nu}}$ and on the particular realisation of $Z, Z^{:2:}$, and $Z^{:3:}$) such that

$$\begin{aligned}\sup_{0 \leq t \leq T} \|\tilde{Z}(t, \cdot)\|_{\mathcal{C}^{-\nu}} &\leq C_0, & \sup_{0 \leq t \leq T} t^{\frac{\beta+\nu}{2}} \|\tilde{Z}^{:2:}(t, \cdot)\|_{\mathcal{C}^{-\nu}} &\leq C_0, \\ \sup_{0 \leq t \leq T} t^{\beta+\nu} \|\tilde{Z}^{:3:}(t, \cdot)\|_{\mathcal{C}^{-\nu}} &\leq C_0.\end{aligned}\quad (3.13)$$

After this preliminary discussion, we are now ready to state the main existence result. For $\tilde{Z}, \tilde{Z}^{:2:}$, and $\tilde{Z}^{:3:}$ satisfying (3.13), consider the problem.

$$\begin{aligned}\partial_t v &= \Delta v - \frac{1}{3}(v^3 + 3\tilde{Z}v^2 + 3\tilde{Z}^{:2:}v + \tilde{Z}^{:3:}) + A(t)(\tilde{Z} + v), \\ v(0, \cdot) &= 0,\end{aligned}\quad (3.14)$$

where

$$A(t) := A + \lim_{\varepsilon \rightarrow 0} (\mathbf{c}_\varepsilon - \mathbf{c}_\varepsilon(t)) = A - \frac{t}{2} + \sum_{\omega \in \mathbb{Z}^2 \setminus \{0\}} \frac{e^{-2t\pi^2|\omega|^2}}{4\pi^2|\omega|^2}.\quad (3.15)$$

Note that $A(t)$ only diverges logarithmically in t as t goes to 0, and in particular any power of $A(t)$ is integrable at 0. The following theorem is essentially [42, Theorem 6.1]. (The continuity of the solution map is not stated explicitly there, but it is contained in the method of proof).

Theorem 3.2. *For $\nu > 0$ small enough, fix an initial datum $X^0 \in \mathcal{C}^{-\nu}$. For $(Z, Z^{:2:}, Z^{:3:}) \in (L^\infty([0, T], \mathcal{C}^{-\nu}))^3$, let $(\tilde{Z}, \tilde{Z}^{:2:}, \tilde{Z}^{:3:})$ be defined as in (3.12). Let $\mathcal{S}_T(Z, Z^{:2:}, Z^{:3:})$ denote the solution v on $[0, T]$ of the PDE (3.14). Then*

for any $\kappa > 0$, the mapping \mathcal{S}_T is Lipschitz continuous on bounded sets from $(L^\infty([0, T], C^{-\nu}))^3$ to $\mathcal{C}([0, T], C^{2-\nu-\kappa}(\mathbb{T}^2))$.

Remark 3.3. The choice of two different time dependent normalisation constants \mathfrak{c}_ε and $\mathfrak{c}_\varepsilon(t)$ may not seem particularly elegant. Indeed, in [17, 32], all processes are renormalised with time independent constants. This is possible because in those papers, the processes Z_ε and $Z_\varepsilon^{:n:}$ are replaced by similar processes that are *stationary* in t . This can be done by adding a linear damping term and moving the initial condition in (3.4) to $t = -\infty$ (as in [17]), or by cutting off the heat kernel P_t outside of a ball (as in [32]). In the discrete setting below, however, the choices for the linearised process presented here seem most convenient.

This discussion shows as well that there is no canonical choice of the renormalisation constant \mathfrak{c}_ε and hence no canonical value of the constant A . The different procedures in [17] and [32] yield different choices of \mathfrak{c}_ε . The difference between these two constants remains bounded as ε goes to zero, but it does not disappear in the limit.

4. BOUNDS FOR THE LINEARISED SYSTEM

We now come back to the study of the discrete system. By Duhamel's principle and using the scaling relations (2.16) and (2.19), the evolution equation (2.13) can be rewritten as

$$\begin{aligned} X_\gamma(t, \cdot) = & P_t^\gamma X_\gamma^0 + \int_0^t P_{t-r}^\gamma K_\gamma \star_\varepsilon \left(-\frac{\beta^3}{3} X_\gamma^3(r, \cdot) + (\mathfrak{c}_\gamma + A) X_\gamma(r, \cdot) \right. \\ & \left. + E_\gamma(r, \cdot) \right) dr + \int_{r=0}^t P_{t-r}^\gamma dM_\gamma(r, \cdot) \quad \text{on } \Lambda_\varepsilon, \end{aligned} \quad (4.1)$$

where we use the convention $\int_0^t = \int_{(0,t]}$, and we denote by $P_t^\gamma = e^{\Delta_\gamma t}$ the semigroup generated by Δ_γ . For every $t \geq 0$, the operator P_t^γ acts on a function $Y: \Lambda_\varepsilon \rightarrow \mathbb{R}$ by convolution with a kernel, also denoted by P_t^γ . This kernel is characterised by its Fourier transform (defined as in (A.1))

$$\hat{P}_t^\gamma(\omega) = \exp\left(t\gamma^{-2}(\hat{K}_\gamma(\omega) - 1)\right) \quad \text{if } \omega \in \{-N, \dots, N\}^2. \quad (4.2)$$

Viewing P_t^γ as a Fourier multiplication operator (with $\hat{P}_t^\gamma(\omega) = 0$ if $\omega \notin \{-N, \dots, N\}^2$) enables to make sense of $P_t^\gamma f$ for every $f: \mathbb{T}^2 \rightarrow \mathbb{R}$. Further properties of the operator P_t^γ are summarised in Lemmas 8.3 and 8.4 as well as Corollary 8.7.

As explained above for the continuous equation, a crucial step in studying the limiting behaviour of X_γ consists of the analysis of a linearised evolution. For $x \in \Lambda_\varepsilon$, we denote by

$$Z_\gamma(t, x) = \int_{r=0}^t P_{t-r}^\gamma dM_\gamma(r, x)$$

the stochastic convolution appearing on the right-hand side of (4.1). The process Z_γ is the solution to the linear stochastic equation

$$\begin{aligned} dZ_\gamma(t, x) &= \Delta_\gamma Z_\gamma(t, x) dt + dM_\gamma(t, x) \\ Z_\gamma(0, x) &= 0, \end{aligned} \quad (4.3)$$

for $x \in \Lambda_\varepsilon$, $t \geq 0$. It will be convenient to work with the following family of approximations to $Z_\gamma(t, x)$. For $s \leq t$, we introduce

$$R_{\gamma,t}(s, x) := \int_{r=0}^s P_{t-r}^\gamma dM_\gamma(r, x).$$

As explained above (see the discussion following (A.2)), we extend $R_{\gamma,t}(s, \cdot): \Lambda_\varepsilon \rightarrow \mathbb{R}$ and $Z_\gamma(t, \cdot): \Lambda_\varepsilon \rightarrow \mathbb{R}$ to functions on all of \mathbb{T}^2 by trigonometric polynomials of degree $\leq N$. Note that for any t and any $x \in \mathbb{T}^2$, the process $R_{\gamma,t}(\cdot, x)$ is a martingale and $R_{\gamma,t}(t, \cdot) = Z_\gamma(t, \cdot)$.

As in the case of the continuous process, it is not enough to control Z_γ , the solution of the linearised evolution: we also need to control additional non-linear functions thereof. We introduce recursively the following quantities: for a fixed $t \geq 0$ and $x \in \mathbb{T}^2$, we set $R_{\gamma,t}^{1:}(s, x) = R_{\gamma,t}(s, x)$. For $n \geq 2$, $t \geq 0$ and $x \in \Lambda_\varepsilon$, we set

$$R_{\gamma,t}^{n:}(s, x) = n \int_{r=0}^s R_{\gamma,t}^{n-1:}(r^-, x) dR_{\gamma,t}(r, x). \quad (4.4)$$

We use the notation $R_{\gamma,t}^{n-1:}(r^-, x)$ to denote the left limit of $R_{\gamma,t}^{n-1:}(\cdot, x)$ at r . This definition ensures that $(R_{\gamma,t}^{n:}(s, x))_{0 \leq s \leq t}$ is a martingale. To define an extension of $R_{\gamma,t}^{n:}(s, \cdot)$ to arguments $x \in \mathbb{T}^2 \setminus \Lambda_\varepsilon$ for $n \geq 2$, it is advisable *not* to extend it by a trigonometric polynomial of degree $\leq N$. Indeed, products are not well captured by this extension. It is more natural to define the extension recursively through its Fourier series

$$\hat{R}_{\gamma,t}^{n:}(s, \omega) := n \int_{r=0}^s \frac{1}{4} \sum_{\tilde{\omega} \in \mathbb{Z}^2} \hat{R}_{\gamma,t}^{n-1:}(r^-, \omega - \tilde{\omega}) d\hat{R}_{\gamma,t}(r, \tilde{\omega}), \quad (4.5)$$

and set $R_{\gamma,t}^{n:}(s, x) := \frac{1}{4} \sum_{\omega \in \mathbb{Z}^2} \hat{R}_{\gamma,t}^{n:}(s, \omega) e^{i\pi\omega \cdot x}$. This definition coincides with (4.4) on Λ_ε , and for every $n \geq 2$ the function $R_{\gamma,t}^{n:}(s, \cdot): \mathbb{T}^2 \rightarrow \mathbb{R}$ is a trigonometric polynomial of degree $\leq nN$. For any $n \geq 2$ and for $t \geq 0$, $x \in \mathbb{T}^2$ we define

$$Z_\gamma^{n:}(t, x) := R_{\gamma,t}^{n:}(t, x). \quad (4.6)$$

The main objective of this section is to prove uniform bounds on the Besov norms of the processes $Z_\gamma^{n:}$ and $R_{\gamma,t}^{n:}$. These bounds are stated in Proposition 4.2.

As a first step, we derive a general bound on p -th moments of iterated stochastic integrals. We start by introducing some more notation: Let $F: [0, \infty)^n \times \Lambda_\varepsilon^n \times \Omega \rightarrow \mathbb{R}$ be adapted and left continuous in each of the n “time” variables. By adapted, we mean that if $s_1, \dots, s_n \leq t$, then for all y_1, \dots, y_n , the random variable $F(s_1, \dots, s_n, y_1, \dots, y_n)$ is measurable with respect to the sigma algebra generated by X_γ up to time t . We recursively define iterated integrals $\mathfrak{J}_n F(t)$ as follows. For $n = 1$, we set $\mathfrak{J}_1 F(t) = \int_{r=0}^t \sum_{y \in \Lambda_\varepsilon} \varepsilon^2 F(r, y) dM(r, y)$. For $n \geq 2$, we set

$$\mathfrak{J}_n F(t) := \int_{r_1=0}^t \sum_{y_1 \in \Lambda_\varepsilon} \varepsilon^2 \mathfrak{J}_{n-1} F^{(r_1, y_1)}(r_1^-) dM(r_1, y_1),$$

where $F^{(r_1, y_1)}: [0, \infty)^{n-1} \times \Lambda_\varepsilon^{n-1}$ is defined as

$$F^{(r_1, y_1)}(r_2, \dots, r_n, y_2, \dots, y_n) = F(r_1, \dots, r_n, y_1, \dots, y_n).$$

As above (but somewhat abusively here), we denote by $\mathfrak{J}_{n-1} F^{(r_1, y_1)}(r_1^-)$ the left limit of $r_1 \mapsto \mathfrak{J}_{n-1} F^{(r_1, y_1)}(r_1)$. With this definition, for every n and F , the process

$t \mapsto \mathfrak{J}_n F(t)$ is a martingale. Finally, given any F as above and $1 \leq \ell \leq n$, we define

$$F_\ell(r_1, \dots, r_n, z_1, \dots, z_\ell, y_{\ell+1}, \dots, y_n) \quad (4.7)$$

$$:= \sum_{y_1, \dots, y_\ell \in \Lambda_\varepsilon^\ell} \varepsilon^{2\ell} F(r_1, \dots, r_n, y_1, \dots, y_n) \prod_{i=1}^{\ell} K_\gamma(y_i - z_i), \quad (4.8)$$

i.e. F is convolved with the kernel K_γ in the first ℓ spatial arguments. In the sequel, when we write $\mathfrak{J}_{n-\ell} F_\ell(r_1, \dots, r_\ell; z_1, \dots, z_\ell)$ this means that the iterated stochastic integral is taken with respect to the variables $r_{\ell+1}, \dots, r_n$ and $y_{\ell+1}, \dots, y_n$ and evaluated at time $t = r_\ell$ (the variables $r_1, \dots, r_\ell, z_1, \dots, z_\ell$ are just treated as fixed parameters). Using the definition (4.4), it is easy to see that for $x \in \Lambda_\varepsilon$ and $0 \leq s \leq t$, we can write $R_{\gamma,t}^{n;}(s, x) = \mathfrak{J}_n F(s)$ for

$$F(r_1, \dots, r_n, y_1, \dots, y_n) = n! \prod_{i=1}^n P_{t-r_i}^\gamma(x - y_i). \quad (4.9)$$

Furthermore, note that for every n , the mapping $F \mapsto \mathfrak{J}_n F$ is linear in F .

Lemma 4.1. *Let $n \geq 1$ and let $F: [0, \infty)^n \times \Lambda_\varepsilon^n \rightarrow \mathbb{R}$ and F_ℓ for $1 \leq \ell \leq n$ be deterministic and left-continuous in each time variable. Then for any $p \geq 2$, there exists a constant $C = C(n, p)$ such that*

$$\left(\mathbb{E} \sup_{0 \leq r \leq t} |\mathfrak{J}_n F(r)|^p \right)^{\frac{2}{p}} \leq C \int_{r_1=0}^t \dots \int_{r_n=0}^{r_{n-1}} \sum_{\mathbf{z} \in \Lambda_\varepsilon^n} \varepsilon^{2n} F_n(\mathbf{r}, \mathbf{z})^2 d\mathbf{r} + \text{Err}(n), \quad (4.10)$$

where we use the short-hand $\mathbf{r} = (r_1, \dots, r_n)$, and $d\mathbf{r} = dr_n \dots dr_2 dr_1$. The error term $\text{Err}(n)$ is given by

$$\begin{aligned} \text{Err}(n) = & C \varepsilon^4 \delta^{-2} \sum_{\ell=1}^n \int_{r_1=0}^t \dots \int_{r_{\ell-1}=0}^{r_{\ell-2}} \sum_{z_1, \dots, z_{\ell-1} \in \Lambda_\varepsilon} \varepsilon^{2(\ell-1)} \\ & \left(\mathbb{E} \sup_{\substack{0 \leq r_\ell \leq r_{\ell-1} \\ z_\ell \in \Lambda_\varepsilon}} |\mathfrak{J}_{n-\ell} F_\ell(\mathbf{r}_\ell, \mathbf{z}_\ell)|^p \right)^{\frac{2}{p}} d\mathbf{r}_{\ell-1}, \end{aligned} \quad (4.11)$$

where $\mathbf{r}_\ell = (r_1, \dots, r_\ell)$, $\mathbf{z}_\ell = (z_1, \dots, z_\ell)$, $d\mathbf{r}_{\ell-1} = dr_{\ell-1} \dots dr_2 dr_1$ (there is no variable to integrate for $\ell = 1$), and $r_0 = t$.

Proof. We proceed by induction. Let us consider the case $n = 1$ first. In order to apply the Burkholder-Davis-Gundy inequality (Lemma C.1), we need to bound the quadratic variation as well as the size of the jumps of the martingale $\mathfrak{J}_1 F(t)$. The

quadratic variation is given by

$$\begin{aligned}
& \langle \mathfrak{J}_1 F \rangle_t \\
&= \int_{r=0}^t \sum_{\substack{y \in \Lambda_\varepsilon \\ \bar{y} \in \Lambda_\varepsilon}} \varepsilon^4 F(r, y) F(r, \bar{y}) d\langle M_\gamma(\cdot, y), M_\gamma(\cdot, \bar{y}) \rangle_r \\
&= 4c_{\gamma,2}^2 \int_0^t \sum_{\substack{y \in \Lambda_\varepsilon \\ \bar{y} \in \Lambda_\varepsilon}} \varepsilon^4 F(r, y) F(r, \bar{y}) \sum_{z \in \Lambda_\varepsilon} \varepsilon^2 K_\gamma(y-z) K_\gamma(\bar{y}-z) C_\gamma(r, z) dr \\
&= 4c_{\gamma,2}^2 \int_0^t \sum_{z \in \Lambda_\varepsilon} \varepsilon^2 \left(\sum_{y \in \Lambda_\varepsilon} \varepsilon^2 F(r, y) K_\gamma(y-z) \right)^2 C_\gamma(r, z) dr \\
&\leq 4c_{\gamma,2}^2 \int_0^t \sum_{z \in \Lambda_\varepsilon} \varepsilon^2 F_1(r, z)^2 dr. \tag{4.12}
\end{aligned}$$

Here we have used (2.15) for the second equality and the deterministic estimate $0 \leq C_\gamma(r, z) \leq 1$ in the last inequality. Let us now turn to the jumps. We had seen above that a jump of the spin $\sigma(k)$ at microscopic position $k = \varepsilon^{-1}z$ causes a jump of size $2\delta^{-1}\varepsilon^2 K_\gamma(y-z)$ for M_γ . With probability one, two such jumps never occur at the same time, so that we only have to estimate the impact of such an event on $\mathfrak{J}_1 F$. If such an event takes place at (macroscopic) time r , then the martingale $\mathfrak{J}_1 F$ has a jump of absolute value

$$2\varepsilon^2 \delta^{-1} \left| \sum_{y \in \Lambda_\varepsilon} \varepsilon^2 F(r, y) K_\gamma(y-z) \right| = 2\varepsilon^2 \delta^{-1} |F_1(r, z)|. \tag{4.13}$$

Hence, the Burkholder-Davis-Gundy inequality implies that for every $p > 0$, we have

$$\left(\mathbb{E} \sup_{0 \leq r \leq t} |\mathfrak{J}_1 F(r)|^p \right)^{\frac{2}{p}} \leq C(p) \left(\int_0^t \sum_{z \in \Lambda_\varepsilon} \varepsilon^2 F_1(r, z)^2 dr + \varepsilon^4 \delta^{-2} \sup_{\substack{0 \leq r \leq t \\ z \in \Lambda_\varepsilon}} F_1(r, z)^2 \right),$$

which is what we wanted.

Let us now assume that (4.10) is established for $n-1$. In order to bound moments of the martingale $\mathfrak{J}_n F(t)$, we bound again the quadratic variation and the size of the jumps. For the jumps, we can see as in (4.13) that a jump of $\sigma(k)$ at location $k = \varepsilon^{-1}z_1$ at time r_1 causes a jump of $\mathfrak{J}_n F$ of absolute value

$$2\varepsilon^2 \delta^{-1} \left| \sum_{y_1 \in \Lambda_\varepsilon} \varepsilon^2 \mathfrak{J}_{n-1} F^{(r_1, y_1)}(r_1^-) K_\gamma(y_1 - z_1) \right|.$$

Recalling that

$$\sum_{y_1 \in \Lambda_\varepsilon} \varepsilon^2 F^{(r_1, y_1)}(r_2, \dots, r_n, y_2, \dots, y_n) K_\gamma(y_1 - z_1) = F_1(r_1, \dots, r_n, z_1, y_2, \dots, y_n)$$

and that $F \mapsto \mathfrak{J}_{n-1} F$ is linear, we can rewrite the quantity above as

$$2\varepsilon^2 \delta^{-1} \left| \sum_{y_1 \in \Lambda_\varepsilon} \varepsilon^2 \mathfrak{J}_{n-1} F_1(r_1^-, z_1) \right|.$$

Hence, the corresponding error term in the bound for $\left(\mathbb{E} \sup_{0 \leq r \leq t} |\mathfrak{J}_n F(r)|^p\right)^{\frac{2}{p}}$ takes the form

$$C(p) \varepsilon^4 \delta^{-2} \left(\mathbb{E} \sup_{\substack{0 \leq r_1 \leq t \\ z_1 \in \Lambda_\varepsilon}} |\mathfrak{J}_{n-1} F_1(r_1, z_1)|^p \right)^{\frac{2}{p}}, \quad (4.14)$$

which is precisely the term corresponding to $\ell = 1$ in (4.11).

For the quadratic variation of $\mathfrak{J}_n F(t)$, we get as above in (4.12) that

$$\begin{aligned} & \langle \mathfrak{J}_n F \rangle_t \\ &= \int_{r_1=0}^t \sum_{\substack{y_1 \in \Lambda_\varepsilon \\ \bar{y}_1 \in \Lambda_\varepsilon}} \varepsilon^4 \mathfrak{J}_{n-1} F^{(r_1, y_1)}(r_1) \mathfrak{J}_{n-1} F^{(r_1, \bar{y}_1)}(r_1) d\langle M_\gamma(\cdot, y_1), M_\gamma(\cdot, \bar{y}_1) \rangle_{r_1} \\ &\leq 4c_{\gamma,2}^2 \int_0^t \sum_{z_1 \in \Lambda_\varepsilon} \varepsilon^2 \left(\sum_{y_1 \in \Lambda_\varepsilon} \varepsilon^2 \mathfrak{J}_{n-1} F^{(r_1, y_1)}(r_1) K_\gamma(y_1 - z_1) \right)^2 dr_1 \\ &= 4c_{\gamma,2}^2 \int_0^t \sum_{z_1 \in \Lambda_\varepsilon} \varepsilon^2 (\mathfrak{J}_{n-1} F_1(r_1, z_1))^2 dr_1. \end{aligned}$$

In the first equality above, we have used the fact that $F^{(r_1, y_1)}(r_1^-) = F^{(r_1, y_1)}(r_1)$ for Lebesgue almost every r_1 . Then Minkowski's inequality (for the exponent $\frac{p}{2} \geq 1$) implies that

$$\left(\mathbb{E} \langle \mathfrak{J}_n F \rangle_t^{\frac{p}{2}} \right)^{\frac{2}{p}} \leq 4c_{\gamma,2}^2 \int_0^t \sum_{z_1 \in \Lambda_\varepsilon} \varepsilon^2 \left(\mathbb{E} |\mathfrak{J}_{n-1} F_1(r_1, z_1)|^p \right)^{\frac{2}{p}} dr_1. \quad (4.15)$$

But the induction hypothesis implies that for some $C = C(n, p)$, for every $r_1 \geq 0$ and $z_1 \in \Lambda_\varepsilon$, we have

$$\begin{aligned} \left(\mathbb{E} |\mathfrak{J}_{n-1} F_1(r_1, z_1)|^p \right)^{\frac{2}{p}} &\leq C \int_{r_2=0}^{r_1} \dots \int_{r_n=0}^{r_{n-1}} \sum_{z_2, \dots, z_n \in \Lambda_\varepsilon} \varepsilon^{2(n-1)} F_n(\mathbf{r}, \mathbf{z})^2 d\mathbf{r}' \\ &\quad + \text{Err}(r_1, z_1), \end{aligned} \quad (4.16)$$

where $d\mathbf{r}' = dr_n \dots dr_2$, and where the error term is given by

$$\begin{aligned} \text{Err}(r_1, z_1) &= C \varepsilon^4 \delta^{-2} \sum_{\ell=2}^n \int_{r_2=0}^{r_1} \dots \int_{r_{\ell-1}=0}^{r_{\ell-2}} \sum_{z_2, \dots, z_{\ell-1} \in \Lambda_\varepsilon} \varepsilon^{2(\ell-2)} \\ &\quad \left(\mathbb{E} \sup_{\substack{0 \leq r_\ell \leq r_{\ell-1} \\ z_\ell \in \Lambda_\varepsilon}} |\mathfrak{J}_{n-\ell} F_\ell(\mathbf{r}_\ell, \mathbf{z}_\ell)|^p \right)^{\frac{2}{p}} d\mathbf{r}'_\ell, \end{aligned} \quad (4.17)$$

with $d\mathbf{r}'_\ell = dr_\ell \dots dr_2$. We then obtain the desired estimate by plugging (4.16) and (4.17) into (4.15). \square

With Lemma 4.1 in hand, we now proceed to derive bounds on Z_γ^n .

Proposition 4.2. *There exists a constant $\gamma_0 > 0$ such that the following holds. For every $n \in \mathbb{N}$, $p \geq 1$, $\nu > 0$, $T > 0$, $0 \leq \lambda \leq \frac{1}{2}$ and $0 < \kappa \leq 1$, there exists a constant*

$C = C(n, p, \nu, T, \lambda, \kappa)$ such that for every $0 \leq s \leq t \leq T$ and $0 < \gamma < \gamma_0$,

$$\mathbb{E} \sup_{0 \leq r \leq t} \|R_{\gamma,t}^{n:}(r, \cdot)\|_{\mathcal{C}^{-\nu-2\lambda}}^p \leq C t^{\lambda p} + C \gamma^{p(1-\kappa)}, \quad (4.18)$$

$$\mathbb{E} \sup_{0 \leq r \leq t} \|R_{\gamma,t}^{n:}(r, \cdot) - R_{\gamma,s}^{n:}(r \wedge s, \cdot)\|_{\mathcal{C}^{-\nu-2\lambda}}^p \leq C |t-s|^{\lambda p} + C \gamma^{p(1-\kappa)}, \quad (4.19)$$

$$\mathbb{E} \sup_{0 \leq r \leq t} \|R_{\gamma,t}^{n:}(r, \cdot) - R_{\gamma,t}^{n:}(r \wedge s, \cdot)\|_{\mathcal{C}^{-\nu-2\lambda}}^p \leq C |t-s|^{\lambda p} + C \gamma^{p(1-\kappa)}. \quad (4.20)$$

Remark 4.3. In particular, the bounds (4.18) – (4.20) imply that (under the same conditions on p, ν, λ, κ) we have

$$\begin{aligned} \mathbb{E} \|Z_\gamma^{n:}(t, \cdot)\|_{\mathcal{C}^{-\nu-2\lambda}}^p &\leq C t^{\lambda p} + C \gamma^{p(1-\kappa)}, \\ \mathbb{E} \|Z_\gamma^{n:}(t, \cdot) - Z_\gamma^{n:}(s, \cdot)\|_{\mathcal{C}^{-\nu-2\lambda}}^p &\leq C |t-s|^{\lambda p} + C \gamma^{p(1-\kappa)}, \\ \mathbb{E} \|Z_\gamma^{n:}(t, \cdot) - R_{\gamma,t}^{n:}(s, \cdot)\|_{\mathcal{C}^{-\nu-2\lambda}}^p &\leq C |t-s|^{\lambda p} + C \gamma^{p(1-\kappa)}. \end{aligned}$$

These weaker bounds are the key ingredient for the proof of tightness in Proposition 5.4 and for the proof of convergence in law in Theorem 6.2 below.

Proof. Recalling that $R_{\gamma,0}^{n:}(0, \cdot) = 0$, we see that the bound (4.18) is contained in (4.19), so that it suffices to show (4.19) and (4.20). Furthermore, note that by the monotonicity in p of stochastic L^p norms, it is sufficient to prove these bounds for p large enough.

For any smooth function $f: \mathbb{T}^2 \rightarrow \mathbb{R}$ and any n , we write

$$R_{\gamma,t}^{n:}(s, f) := \int_{\mathbb{T}^2} R_{\gamma,t}^{n:}(s, x) f(x) dx,$$

and similarly, $Z_\gamma^{n:}(t, f) := R_{\gamma,t}^{n:}(t, f)$. Note that in general, neither f nor $R_{\gamma,t}^{n:}$ (defined for all $x \in \mathbb{T}^2$ as in (4.5)) are trigonometric polynomials of degree $\leq N$, so that this integral does not coincide exactly with its Riemann sum approximation on Λ_ε .

In order to obtain (4.19) and (4.20), we derive bounds on

$$\mathbb{E} \sup_{0 \leq r \leq t} |R_{\gamma,t}^{n:}(r, f) - R_{\gamma,s}^{n:}(r \wedge s, f)|^p \quad \text{and} \quad \mathbb{E} \sup_{0 \leq r \leq t} |R_{\gamma,t}^{n:}(r, f) - R_{\gamma,t}^{n:}(r \wedge s, f)|^p$$

for an arbitrary smooth test function $f: \mathbb{T}^2 \rightarrow \mathbb{R}$ and for an arbitrary $p \geq 2$. Later, we will specialise to $f(x) = \eta_k(u-x)$ (defined in (B.8)) for some $u \in \mathbb{T}^2$ and $k \geq -1$ to apply Proposition B.4.

A simple recursion based on (4.5) shows that for $0 \leq s \leq t$, we have $R_{\gamma,t}^{n:}(s, f) = \mathfrak{J}_n F^t(s)$ where for $\mathbf{y} = (y_1, \dots, y_n)$, $\mathbf{r} = (r_1, \dots, r_n)$, and $\boldsymbol{\omega} = (\omega_1, \dots, \omega_n)$,

$$\begin{aligned} F^t(\mathbf{y}, \mathbf{r}) &= \frac{n!}{4^n} \sum_{\boldsymbol{\omega} \in (\mathbb{Z}^2)^n} \overline{\hat{f}(\omega_1 + \dots + \omega_n)} \prod_{j=1}^n \hat{P}_{t-r_j}^\gamma(\omega_j) e^{-i\pi \omega_j \cdot y_j} \\ &= n! \int_{\mathbb{T}^2} f(x) \prod_{j=1}^n P_{t-r_j}^\gamma(x - y_j) dx. \end{aligned}$$

Here, each $P_{t-r_j}^\gamma$ is viewed as a function on all of \mathbb{T}^2 . Note that the kernel $\prod_{j=1}^n P_{t-r_j}^\gamma(x - y_j)$ is a trigonometric polynomial of degree $\leq nN$ in the x variable and a trigonometric polynomial of degree $\leq N$ in each y_j coordinate.

By linearity of the operator \mathfrak{J}_n , we get for any $0 \leq s \leq t$ and $0 \leq r \leq t$ that

$$\begin{aligned} R_{\gamma,t}^{:n:}(r, f) - R_{\gamma,s}^{:n:}(r \wedge s, f) &= \mathfrak{J}_n(F^t - F^s)(r), \\ R_{\gamma,t}^{:n:}(r, f) - R_{\gamma,t}^{:n:}(r \wedge s, f) &= \mathfrak{J}_n(F^t \mathbf{1}_{r_1 \in [s,t]})(r). \end{aligned}$$

Here we use the convention to set $P_{s-r_j}^\gamma(x - y_j) = 0$ for $r_j > s$.

Hence, by Lemma 4.1 there exists $C = C(n, p)$ such that

$$\begin{aligned} & (\mathbb{E} \sup_{0 \leq r \leq t} |R_{\gamma,t}^{:n:}(r, f) - R_{\gamma,s}^{:n:}(r, f)|^p)^{\frac{2}{p}} \\ & \leq \frac{C}{n!} \int_{r_1, \dots, r_n=0}^t \sum_{\mathbf{z} \in \Lambda_\varepsilon^n} \varepsilon^{2n} (F_n^t(\mathbf{r}, \mathbf{z}) - F_n^s(\mathbf{r}, \mathbf{z}))^2 d\mathbf{r} + \text{Err}, \end{aligned} \quad (4.21)$$

and

$$\begin{aligned} & (\mathbb{E} \sup_{0 \leq r \leq t} |R_{\gamma,t}^{:n:}(r, f) - R_{\gamma,t}^{:n:}(r \wedge s, f)|^p)^{\frac{2}{p}} \\ & \leq \frac{C}{(n-1)!} \int_{r_1=s}^t \int_{r_2, \dots, r_n=0}^{r_1} \sum_{\mathbf{z} \in \Lambda_\varepsilon^n} \varepsilon^{2n} F_n^t(\mathbf{r}, \mathbf{z})^2 d\mathbf{r} + \text{Err}' . \end{aligned} \quad (4.22)$$

Here F_n^t denotes the convolution of F^t with the kernel K_γ in all n spatial arguments as defined in (4.7). In (4.21) and (4.22), we have used the symmetry of the kernels F_n^t and F_n^s in their time arguments to replace the integrals over the simplices $0 \leq r_n \leq r_{n-1} \leq \dots \leq r_1 \leq t$ and $0 \leq r_n \leq \dots \leq r_1$ by integrals over $[0, t]^n$ and $[0, r_1]^{n-1}$. The precise form of the error terms Err and Err' is discussed below.

We start by bounding the first term on the right-hand side of (4.21). Using Parseval's identity (A.3) in each of the z_j summations, we get for any fixed \mathbf{r}

$$\begin{aligned} & \sum_{\mathbf{z} \in \Lambda_\varepsilon^n} \varepsilon^{2n} (F_n^t(\mathbf{r}, \mathbf{z}) - F_n^s(\mathbf{r}, \mathbf{z}))^2 \\ & = \frac{(n!)^2}{4^n} \sum_{\boldsymbol{\omega} \in (\mathbb{Z}^2)^n} |\hat{f}(\omega_1 + \dots + \omega_n)|^2 \left(\prod_{j=1}^n \hat{P}_{t-r_j}^\gamma \hat{K}_\gamma(\omega_j) - \prod_{j=1}^n \hat{P}_{s-r_j}^\gamma \hat{K}_\gamma(\omega_j) \right)^2, \end{aligned}$$

where as above we write $\boldsymbol{\omega} = (\omega_1, \dots, \omega_n)$. For fixed $\boldsymbol{\omega}$ and \mathbf{r} we bound

$$\begin{aligned} & \left(\prod_{j=1}^n \hat{P}_{t-r_j}^\gamma \hat{K}_\gamma(\omega_j) - \prod_{j=1}^n \hat{P}_{s-r_j}^\gamma \hat{K}_\gamma(\omega_j) \right)^2 \\ & \leq n \sum_{k=1}^n \left(\prod_{j=1}^{k-1} (\hat{P}_{t-r_j}^\gamma \hat{K}_\gamma(\omega_j))^2 (\hat{P}_{t-r_k}^\gamma \hat{K}_\gamma(\omega_k) - \hat{P}_{s-r_k}^\gamma \hat{K}_\gamma(\omega_k))^2 \right. \\ & \quad \left. \times \prod_{j=k+1}^n (\hat{P}_{s-r_j}^\gamma \hat{K}_\gamma(\omega_j))^2 \right). \end{aligned} \quad (4.23)$$

We perform the \mathbf{r} -integrations for fixed value of k and $\boldsymbol{\omega}$ for each term on the right-hand side of (4.23) separately, recalling that for $\omega \in \{-N, \dots, N\}^2$ and for any $t \geq 0$ we have $\hat{P}_t^\gamma(\omega) = \exp(-t\gamma^{-2}(1 - \hat{K}_\gamma(\omega)))$ according to (4.2). For the integrals over r_j for $j \neq k$ we use the elementary estimate

$$\int_0^t e^{-(t-r)2\ell} dr \leq e \int_0^\infty e^{-r(2\ell + \frac{1}{t})} dr = \frac{e}{\frac{1}{t} + 2\ell}, \quad (4.24)$$

for $\ell = \gamma^{-2}(1 - \hat{K}_\gamma(\omega_j)) \geq 0$. We split the integral over r_k into an integral over $[0, s]$ and an integral over $[s, t]$. Then for the same choice of ℓ we use the bounds

$$\begin{aligned} \int_0^s (e^{-(s-r)\ell} - e^{-(t-r)\ell})^2 dr &= (1 - e^{-(t-s)\ell})^2 \int_0^s e^{-(s-r)2\ell} dr \\ &\leq \ell \int_0^{t-s} e^{-r\ell} dr \frac{1}{2\ell} \\ &\leq \frac{e}{\frac{1}{t-s} + 2\ell}, \end{aligned}$$

as well as

$$\int_s^t e^{-2(t-r)\ell} dr \leq \frac{e}{\frac{1}{t-s} + 2\ell}. \quad (4.25)$$

In this way we obtain for every $k \in \{1, \dots, n\}$

$$\begin{aligned} &\int_{[0,t]^n} \prod_{j=1}^{k-1} (\hat{P}_{t-r_j}^\gamma \hat{K}_\gamma(\omega_j))^2 (\hat{P}_{t-r_k}^\gamma \hat{K}_\gamma(\omega_k) - \hat{P}_{s-r_k}^\gamma \hat{K}_\gamma(\omega_k))^2 \\ &\quad \times \prod_{j=k+1}^n (\hat{P}_{s-r_j}^\gamma \hat{K}_\gamma(\omega_j))^2 dr \\ &\leq e^n \prod_{j=1}^{k-1} \frac{|\hat{K}_\gamma(\omega_j)|^2}{\frac{1}{t} + 2\gamma^{-2}(1 - \hat{K}_\gamma(\omega_j))} \frac{|\hat{K}_\gamma(\omega_k)|^2}{\frac{1}{t-s} + 2\gamma^{-2}(1 - \hat{K}_\gamma(\omega_k))} \\ &\quad \times \prod_{j=k+1}^n \frac{|\hat{K}_\gamma(\omega_j)|^2}{\frac{1}{s} + 2\gamma^{-2}(1 - \hat{K}_\gamma(\omega_j))}. \end{aligned}$$

Here we have again made use of the convention $\hat{P}_{s-r_j}^\gamma(\omega_j) = 0$ for $r_j > s$. We only make this bound worse, if we replace the $\frac{1}{s}$ appearing in last line of this expression by $\frac{1}{t}$. Plugging this back into (4.21), summing over ω and using the invariance of this expression under changing the value of $k \in \{1, \dots, n\}$, we obtain

$$\begin{aligned} &\frac{1}{n!} \int_{r_1, \dots, r_n=0}^t \sum_{\mathbf{z} \in \Lambda_{\mathbb{E}}^n} \varepsilon^{2n} (F_n^t(\mathbf{r}, \mathbf{z}) - F_n^s(\mathbf{r}, \mathbf{z}))^2 d\mathbf{r} \\ &\leq e^n n! n^2 \frac{1}{4^n} \sum_{\omega \in (\mathbb{Z}^2)^n} |\hat{f}(\omega_1 + \dots + \omega_n)|^2 \frac{|\hat{K}_\gamma(\omega_1)|^2}{\frac{1}{t-s} + 2\gamma^{-2}(1 - \hat{K}_\gamma(\omega_1))} \\ &\quad \times \prod_{j=2}^n \frac{|\hat{K}_\gamma(\omega_j)|^2}{\frac{1}{t} + 2\gamma^{-2}(1 - \hat{K}_\gamma(\omega_j))}. \quad (4.26) \end{aligned}$$

The corresponding calculation for the integral in (4.22) is very similar (but slightly simpler). After passing to spatial Fourier variables and performing the integrations

over r_2, \dots, r_n using (4.24), we get

$$\begin{aligned} & \frac{1}{(n-1)!} \int_{r_1=s}^t \int_{r_2, \dots, r_n=0}^{r_1} \sum_{\mathbf{z} \in \Lambda_\varepsilon^n} \varepsilon^{2n} F_n^t(\mathbf{r}, \mathbf{z})^2 d\mathbf{r} \\ & \leq e^{n-1} n! n \frac{1}{4^n} \sum_{\boldsymbol{\omega} \in (\mathbb{Z}^2)^n} |\hat{f}(\omega_1 + \dots + \omega_n)|^2 \int_{r_1=s}^t (\hat{P}_{t-r_1}^\gamma \hat{K}_\gamma(\omega_1))^2 \\ & \quad \times \prod_{j=2}^n \frac{|\hat{K}_\gamma(\omega_j)|^2}{\frac{1}{r_1} + 2\gamma^{-2}(1 - \hat{K}_\gamma(\omega_j))} dr_1. \end{aligned}$$

As above, we only make this bound worse, if we replace the expression $\frac{1}{r_1}$ appearing in the last line by $\frac{1}{t}$. Then we can perform the dr_1 integral using (4.25). In this way we get the same upper bound (up to an inessential factor n) (4.26) for the integrals appearing on the right-hand side of (4.21) and (4.22).

Now we specialise to $f(x) = \eta_k(u - x)$ for some $u \in \mathbb{T}^2$ and $k \geq -1$. Note that according to (B.9), for this choice of f , we have $R_{\gamma,t}^{n_i}(s, f) = \delta_k R_{\gamma,t}^{n_i}(s, u)$. For this f , we recall from (B.7) that $\hat{f}(\omega) = \chi_k(\omega) e^{-i\pi u \cdot \omega}$. In particular, $|\hat{f}(\omega)| \leq 1$ for all $\omega \in \mathbb{Z}^2$ and $\hat{f}(\omega) = 0$ for $|\omega| \notin I_k$, where $I_k = 2^k[3/4, 8/3]$ for $k \geq 0$ and $I_{-1} = [0, 4/3]$. Summarising, we can conclude that for every $n \geq 1$ and $p \geq 2$, there exists $C = C(n, p)$ such that for all $0 \leq s \leq t$, $k \geq -1$ and $u \in \mathbb{T}^2$,

$$\begin{aligned} & \left(\mathbb{E} \sup_{0 \leq r \leq t} |\delta_k R_{\gamma,t}^{n_i}(r, u) - \delta_k R_{\gamma,s}^{n_i}(r \wedge s, u)|^p \right)^{\frac{2}{p}} \\ & \leq C \sum_{\boldsymbol{\omega} \in I_k} \frac{|\hat{K}_\gamma(\omega_1)|^2}{\frac{1}{t-s} + 2\gamma^{-2}(1 - \hat{K}_\gamma(\omega_1))} \prod_{j=2}^n \frac{|\hat{K}_\gamma(\omega_j)|^2}{\frac{1}{t} + 2\gamma^{-2}(1 - \hat{K}_\gamma(\omega_j))} + \text{Err}, \quad (4.27) \end{aligned}$$

where, for $\boldsymbol{\omega} = (\omega_1, \dots, \omega_n) \in (\mathbb{Z}^2)^n$, we write $\Sigma \boldsymbol{\omega} = \sum_{j=1}^n \omega_j$. In the same way, we have

$$\begin{aligned} & \left(\mathbb{E} \sup_{0 \leq r \leq t} |\delta_k R_{\gamma,t}^{n_i}(r, u) - \delta_k R_{\gamma,t}^{n_i}(r \wedge s, u)|^p \right)^{\frac{2}{p}} \\ & \leq C \sum_{\boldsymbol{\omega} \in I_k} \frac{|\hat{K}_\gamma(\omega_1)|^2}{\frac{1}{t-s} + 2\gamma^{-2}(1 - \hat{K}_\gamma(\omega_1))} \prod_{j=2}^n \frac{|\hat{K}_\gamma(\omega_j)|^2}{\frac{1}{t} + 2\gamma^{-2}(1 - \hat{K}_\gamma(\omega_j))} + \text{Err}' . \quad (4.28) \end{aligned}$$

We defer the analysis of the sum appearing on the right-hand side of both (4.27) and (4.28) to Lemma 4.4 below, and proceed to analyse the error terms Err and Err' , going back to the setting of an arbitrary smooth $f : \mathbb{T}^2 \rightarrow \mathbb{R}$ for the moment. The term Err on the right-hand side of (4.21) is given by

$$\begin{aligned} \text{Err} & = C \varepsilon^4 \delta^{-2} \sum_{\ell=1}^n \int_{r_1=0}^t \cdots \int_{r_{\ell-1}=0}^{r_{\ell-2}} \sum_{z_1, \dots, z_{\ell-1} \in \Lambda_\varepsilon} \varepsilon^{2(\ell-1)} \\ & \quad \left(\mathbb{E} \sup_{\substack{0 \leq r_\ell \leq r_{\ell-1} \\ z_\ell \in \Lambda_\varepsilon}} |\mathfrak{J}_{n-\ell}(F_\ell^t - F_\ell^s)(\mathbf{r}_\ell, \mathbf{z}_\ell)|^p \right)^{\frac{2}{p}} d\mathbf{r}_{\ell-1}, \quad (4.29) \end{aligned}$$

where as above $\mathbf{r}_\ell = (r_1, \dots, r_\ell)$, $\mathbf{z}_\ell = (z_1, \dots, z_\ell)$, and $d\mathbf{r}_{\ell-1} = dr_{\ell-1} \cdots dr_2 dr_1$. Here, for any $1 \leq \ell \leq n$ and for fixed $\mathbf{r}_\ell, \mathbf{z}_\ell$, we have

$$\begin{aligned}
& |\mathfrak{J}_{n-\ell}(F_\ell^t - F_\ell^s)(\mathbf{r}_\ell, \mathbf{z}_\ell)|^p \\
&= \left| \int_{\mathbb{T}^2} f(x) \left(R_{\gamma,t}^{n-\ell}(\mathbf{r}_\ell, x) \prod_{j=1}^{\ell} P_{t-r_j}^\gamma \star K_\gamma(x - z_j) \right. \right. \\
&\quad \left. \left. - R_{\gamma,s}^{n-\ell}(\mathbf{r}_\ell, x) \prod_{j=1}^{\ell} P_{s-r_j}^\gamma \star K_\gamma(x - z_j) \right) dx \right|^p \\
&\leq p \|R_{\gamma,t}^{n-\ell}(\mathbf{r}_\ell, \cdot)\|_{L^\infty}^p \left(\int_{\mathbb{T}^2} |f(x)| \prod_{j=1}^{\ell} |P_{t-r_j}^\gamma \star K_\gamma(x - z_j)| dx \right)^p \\
&\quad + p \|R_{\gamma,s}^{n-\ell}(\mathbf{r}_\ell, \cdot)\|_{L^\infty}^p \left(\int_{\mathbb{T}^2} |f(x)| \prod_{j=1}^{\ell} |P_{s-r_j}^\gamma \star K_\gamma(x - z_j)| dx \right)^p, \quad (4.30)
\end{aligned}$$

where we use the convention $R_{\gamma,t}^{0:} = 1$ and $R_{\gamma,s}^{n-\ell}(\mathbf{r}_\ell, x) = 0$ for $r_\ell > s$. By Lemma B.3 and the definition of $R_{\gamma,t}^{n-\ell}$ as a trigonometric polynomial of degree $\leq (n-\ell)N \leq nN$, for every $\kappa' > 0$ there exists a constant $C = C(n, \kappa')$ such that

$$\|R_{\gamma,t}^{n-\ell}(\mathbf{r}_\ell, \cdot)\|_{L^\infty} \leq C \gamma^{-2\kappa'} \|R_{\gamma,t}^{n-\ell}(\mathbf{r}_\ell, \cdot)\|_{C^{-\kappa'}}.$$

Hence, plugging (4.30) back into (4.29) and applying Minkowski's inequality, we get

$$\begin{aligned}
\text{Err} &\leq C(n, p, \kappa') \varepsilon^4 \delta^{-2} \gamma^{-4\kappa'} \\
&\sum_{\ell=1}^n \left[\left(\mathbb{E} \sup_{0 \leq r \leq t} \|R_{\gamma,t}^{n-\ell}(r, \cdot)\|_{C^{-\kappa'}}^p + \mathbb{E} \sup_{0 \leq r \leq s} \|R_{\gamma,s}^{n-\ell}(r, \cdot)\|_{C^{-\kappa'}}^p \right)^{\frac{2}{p}} \right. \\
&\quad \times \int_{r_1=0}^t \cdots \int_{r_{\ell-1}=0}^{r_{\ell-2}} \sum_{z_1, \dots, z_{\ell-1} \in \Lambda_\varepsilon} \varepsilon^{2(\ell-1)} \\
&\quad \left. \sup_{\substack{0 \leq r_\ell \leq r_{\ell-1} \\ z_\ell \in \Lambda_\varepsilon}} \left(\int_{\mathbb{T}^2} |f(x)| \prod_{j=1}^{\ell} |P_{t-r_j}^\gamma \star K_\gamma(x - z_j)| dx \right)^2 d\mathbf{r}_{\ell-1} \right]. \quad (4.31)
\end{aligned}$$

We turn to bound Err' . In the same way,

$$\begin{aligned}
\text{Err}' &= C\varepsilon^4\delta^{-2}\left(\mathbb{E}\sup_{\substack{s\leq r_n\leq t \\ z_1\in\Lambda_\varepsilon}}|\mathfrak{J}_{n-1}F_\ell^t(r_1, z_1)|^p\right)^{\frac{2}{p}} \\
&\quad + C\varepsilon^4\delta^{-2}\sum_{\ell=2}^n\int_{r_1=s}^t\cdots\int_{r_{\ell-1}=0}^{r_{\ell-2}}\sum_{z_1,\dots,z_{\ell-1}\in\Lambda_\varepsilon}\varepsilon^{2(\ell-1)} \\
&\quad\quad\quad\left(\mathbb{E}\sup_{\substack{0\leq r_\ell\leq r_{\ell-1} \\ z_\ell\in\Lambda_\varepsilon}}|\mathfrak{J}_{n-\ell}F_\ell^t(\mathbf{r}_\ell, \mathbf{z}_\ell)|^p\right)^{\frac{2}{p}}d\mathbf{r}_{\ell-1} \\
&\leq C(n, p, \kappa')\varepsilon^4\delta^{-2}\gamma^{-4\kappa'}\sum_{\ell=1}^n\left[\left(\mathbb{E}\sup_{0\leq r\leq t}\|R_{\gamma, t}^{n-\ell}(r, \cdot)\|_{\mathcal{C}^{-\kappa'}}^p\right)^{\frac{2}{p}}\right. \\
&\quad\times\int_{r_1=0}^t\cdots\int_{r_{\ell-1}=0}^{r_{\ell-2}}\sum_{z_1,\dots,z_{\ell-1}\in\Lambda_\varepsilon}\varepsilon^{2(\ell-1)} \\
&\quad\quad\quad\left.\sup_{\substack{0\leq r_\ell\leq r_{\ell-1} \\ z_\ell\in\Lambda_\varepsilon}}\left(\int_{\mathbb{T}^2}|f(x)|\prod_{j=1}^{\ell}|P_{t-r_j}^\gamma\star K_\gamma(x-z_j)|dx\right)^2d\mathbf{r}_{\ell-1}\right]. \quad (4.32)
\end{aligned}$$

The integral appearing both on the right-hand side of (4.31) and (4.32) specialised to the case $f(x) = \eta_k(u-x)$ for some $k \geq -1$ and $u \in \mathbb{T}^2$ is bounded in Lemma 4.5.

Actually, in the case $n = 1$, we obtain a slightly better bound, because the only stochastic process $R_{\gamma, t}^{n-\ell}$ appearing on the right-hand side of (4.31) and (4.32) is $R_{\gamma, t}^{0:} = 1$. Hence, the embedding from $\mathcal{C}^{-\kappa'} \rightarrow L^\infty$ is unnecessary, and we do not need to introduce the factor $\gamma^{-4\kappa'}$.

Finally, summarising our calculations (4.27) and (4.31) as well as the bounds derived in Lemmas 4.4 and 4.5, we can conclude that for every $n \geq 1$, $p \geq 2$, $T \geq 0$, $\lambda \in [0, 1]$, and $\nu > 0$, there exists a constant $C = C(n, p, T, \lambda, \nu)$ such that for all $0 < \gamma < \gamma_0$, $0 \leq s \leq t \leq T$, $u \in \mathbb{T}^2$, and every $k \geq -1$,

$$\begin{aligned}
&\left(\mathbb{E}\sup_{0\leq r\leq t}|\delta_k R_{\gamma, t}^{n:}(r, u) - \delta_k R_{\gamma, s}^{n:}(r \wedge s, u)|^p\right)^{\frac{2}{p}} \leq C|t-s|^\lambda 2^{2k\lambda}(k+2)^n \\
&\quad + C\gamma^{2-4\nu}(\log(\gamma^{-1}))^{2+6(n-1)}\max_{\substack{\ell=0,\dots,n-1 \\ \tau=s,t}}\left(\mathbb{E}\sup_{0\leq r\leq\tau}\|R_{\gamma, \tau}^{\ell:}(r, \cdot)\|_{\mathcal{C}^{-\nu}}^p\right)^{\frac{2}{p}}, \quad (4.33)
\end{aligned}$$

where as above we use the convention $R_{\gamma, t}^{0:} = 1$. We can bound

$$\left(\mathbb{E}\sup_{0\leq r\leq t}|\delta_k R_{\gamma, t}^{n:}(r, u) - \delta_k R_{\gamma, t}^{n:}(r \wedge s, u)|^p\right)^{\frac{2}{p}}$$

by exactly the same quantity, with the only difference that the max in the second line only needs to be taken with respect to $\tau = t$. The desired bounds (4.19) and (4.20) now follow easily by induction, as we now explain. The arguments are identical for both bounds, so we restrict ourselves to (4.19).

For $n = 1$, the bound (4.33) reduces to

$$\begin{aligned}
&\left(\mathbb{E}\sup_{0\leq r\leq t}|\delta_k R_{\gamma, t}(r, u) - \delta_k R_{\gamma, s}(r \wedge s, u)|^p\right)^{\frac{2}{p}} \\
&\leq C|t-s|^\lambda 2^{2k\lambda}(k+2) + C\gamma^2(\log(\gamma^{-1}))^2,
\end{aligned}$$

so that (4.19) for $n = 1$ and p large enough (choosing $p > \frac{2}{\nu}$ suffices) follows from Proposition B.4. To pass from $n - 1$ to n , we observe that for every fixed $\kappa' > 0$, the inductive hypothesis implies that

$$\max_{\substack{\ell=0, \dots, n-1 \\ \tau=s, t}} \left(\mathbb{E} \sup_{0 \leq r \leq \tau} \|R_{\gamma, \tau}^{\ell:}(r, \cdot)\|_{\mathcal{C}^{-\kappa'}}^p \right)^{\frac{2}{p}} \quad (4.34)$$

is uniformly bounded for $t \leq T$. Hence (4.33) turns into

$$\begin{aligned} & \left(\mathbb{E} \sup_{0 \leq r \leq t} |\delta_k R_{\gamma, t}^{n:}(r, u) - \delta_k R_{\gamma, s}^{n:}(r \wedge s, u)|^p \right)^{\frac{2}{p}} \\ & \leq C |t - s|^\lambda 2^{2k\lambda} (k + 2)^n + C \gamma^{2-4\kappa'} (\log(\gamma^{-1}))^{2+6(n-1)}, \end{aligned}$$

for $C = C(n, p, T, \lambda, \kappa')$. By choosing $\kappa' = \frac{\kappa}{4}$ and applying Proposition B.4 for p large enough, (4.19) follows for arbitrary $n \in \mathbb{N}$. \square

Lemma 4.4. *Let $n \in \mathbb{N}$ and $\underline{c} < \bar{c}$ be fixed. Let $0 < \gamma < \gamma_0$, where γ_0 is the constant appearing in Lemma 8.2. For any $k \geq 0$ let $I_k = 2^k[\underline{c}, \bar{c}]$, and let $I_{-1} = [0, 2^{-1}\bar{c}]$. Then for every $T > 0$ and $\lambda \in [0, 1]$, there exists a constant $C = C(n, \underline{c}, \bar{c}, \lambda, T)$ such that for all $k \geq -1$, and $0 \leq s < t \leq T$, we have*

$$\begin{aligned} & \sum_{\omega \in I_k} \frac{|\hat{K}_\gamma(\omega_1)|^2}{\frac{1}{t-s} + 2\gamma^{-2}(1 - \hat{K}_\gamma(\omega_1))} \prod_{j=2}^n \frac{|\hat{K}_\gamma(\omega_j)|^2}{\frac{1}{t} + 2\gamma^{-2}(1 - \hat{K}_\gamma(\omega_j))} \\ & \leq C |t - s|^\lambda 2^{2k\lambda} (k + 2)^{n-1}, \end{aligned} \quad (4.35)$$

where for any $\omega = (\omega_1, \dots, \omega_n) \in (\{-N, \dots, N\}^2)^n$, we use the short-hand notation $\sum \omega = \sum_{j=1}^n \omega_j$.

Proof. We assume that $0 < \gamma < \gamma_0$ where γ_0 is the constant appearing in Lemma 8.2. We only need to consider those ω with $\omega_j \in \{-N, \dots, N\}^2$ for all j because all other summands vanish.

For $|\omega_j| \leq \gamma^{-1}$, we can use (8.7) and (8.13) to bound

$$\frac{|\hat{K}_\gamma(\omega_j)|^2}{\frac{1}{t} + 2\gamma^{-2}(1 - \hat{K}_\gamma(\omega_j))} \leq \frac{1}{\frac{1}{t} + \frac{2}{C_1}|\omega_j|^2} \leq C(T) \left(\frac{1}{1 + |\omega_j|^2} \wedge t \right).$$

For $|\omega_j| > \gamma^{-1}$, we get essentially the same bound using (8.7), (8.10) and (8.13):

$$\frac{|\hat{K}_\gamma(\omega_j)|^2}{\frac{1}{t} + 2\gamma^{-2}(1 - \hat{K}_\gamma(\omega_j))} \leq \frac{C}{|\gamma\omega_j|^2} \frac{1}{\frac{1}{t} + \frac{2}{C_1}\gamma^{-2}} \leq C \left(\frac{1}{1 + |\omega_j|^2} \wedge t \right). \quad (4.36)$$

To bound the sum over these terms we claim that for any $\omega \in \mathbb{Z}^2$, $0 \leq \lambda \leq 1$ and $r > 0$, we have the following bound

$$\begin{aligned} G^{(n)}(\omega) & := \sum_{\substack{\omega \in (\mathbb{Z}^2)^n \\ \sum \omega = \omega}} \left(\frac{1}{1 + |\omega_1|^2} \wedge r \right) \prod_{j=2}^n \frac{1}{1 + |\omega_j|^2} \\ & \leq C(n, \lambda) r^\lambda \frac{1}{1 + |\omega|^2(1-\lambda)} \log(1 + |\omega|)^{n-1}. \end{aligned} \quad (4.37)$$

We show (4.37) by induction. For $n = 1$, it follows from an easy interpolation. For $n \geq 2$, we observe that for any $\omega \in \mathbb{Z}^2$,

$$G^{(n)}(\omega) = \sum_{\omega_1 + \omega_2 = \omega} G^{(n-1)}(\omega_1) \frac{1}{1 + |\omega_2|^2}.$$

To bound this sum, we split its index set $\mathcal{A}(\omega) := \{(\omega_1, \omega_2) \in (\mathbb{Z}^2)^2 : \omega_1 + \omega_2 = \omega\}$ into the following three sets

$$\begin{aligned} \mathcal{A}_1(\omega) &:= \{(\omega_1, \omega_2) \in \mathcal{A} : |\omega_2| \leq \tfrac{1}{2}|\omega|\}, \\ \mathcal{A}_2(\omega) &:= \{(\omega_1, \omega_2) \in \mathcal{A} : |\omega_2| > \tfrac{1}{2}|\omega|, \text{ and } |\omega_1| \leq 3|\omega|\}, \\ \mathcal{A}_3(\omega) &:= \{(\omega_1, \omega_2) \in \mathcal{A} : |\omega_1| > 3|\omega|\}. \end{aligned}$$

On $\mathcal{A}_1(\omega)$, we have by the triangle inequality that $|\omega_1| \geq \frac{1}{2}|\omega|$, so that we can bound

$$\begin{aligned} &\sum_{\mathcal{A}_1(\omega)} G^{(n-1)}(\omega_1) \frac{1}{1 + |\omega_2|^2} \\ &\leq C \frac{r^\lambda}{1 + |\frac{1}{2}\omega|^2(1-\lambda)} \log(1 + \tfrac{1}{2}|\omega|)^{n-2} \sum_{|\omega_2| \leq \frac{1}{2}|\omega|} \frac{1}{1 + |\omega_2|^2} \\ &\leq C \frac{r^\lambda}{1 + |\omega|^2(1-\lambda)} \log(1 + |\omega|)^{n-1}. \end{aligned}$$

For the sum over $\mathcal{A}_2(\omega)$ we write

$$\begin{aligned} \sum_{\mathcal{A}_2(\omega)} G^{(n-1)}(\omega_1) \frac{1}{1 + |\omega_2|^2} &\leq C \frac{1}{1 + \frac{1}{4}|\omega|^2} \sum_{|\omega_1| \leq 3|\omega|} \frac{r^\lambda}{1 + |\omega_1|^2(1-\lambda)} \log(1 + |\omega_1|)^{n-2} \\ &\leq C \frac{r^\lambda}{1 + |\omega|^2(1-\lambda)} \log(1 + |\omega|)^{n-1}. \end{aligned}$$

Actually, the extra logarithm on the right-hand side of this bound is only necessary in the case $\lambda = 0$, but we do not optimise this. For the sum over $\mathcal{A}_3(\omega)$, we observe that on \mathcal{A}_3 we have $|\omega_2| \geq \frac{1}{2}|\omega_1|$ to write

$$\begin{aligned} \sum_{\mathcal{A}_3(\omega)} G^{(n-1)}(\omega_1) \frac{1}{1 + |\omega_2|^2} &\leq C \sum_{|\omega_1| \geq 3|\omega|} \frac{r^\lambda}{1 + |\omega_1|^2(1-\lambda)} \log(1 + |\omega_1|)^{n-2} \frac{1}{1 + \frac{1}{4}|\omega_1|^2} \\ &\leq C \frac{r^\lambda}{1 + |\omega|^2(1-\lambda)} \log(1 + |\omega|)^{n-1}, \end{aligned}$$

which finishes the proof of (4.37). Summing $G^{(n)}(\omega)$ over all ω with $|\omega| \in I_k$ establishes (4.35). \square

Lemma 4.5. *There exists $\gamma_0 > 0$ such for any $T > 0$, there exists a constant $C = C(T)$ such that for any $\ell \in \mathbb{N}$, $0 \leq t \leq T$, $k \geq -1$, $0 < \gamma < \gamma_0$ and $u \in \mathbb{T}^2$, we*

have

$$\begin{aligned} & \int_{r_1=0}^t \cdots \int_{r_{\ell-1}=0}^{r_{\ell-2}} \sum_{z_1, \dots, z_{\ell-1} \in \Lambda_\varepsilon} \varepsilon^{2(\ell-1)} \\ & \sup_{\substack{0 \leq r_\ell \leq r_{\ell-1} \\ z_\ell \in \Lambda_\varepsilon}} \left(\int_{\mathbb{T}^2} |\eta_k(u-x)| \prod_{j=1}^{\ell} |P_{t-r_j}^\gamma \star K_\gamma(x-z_j)| dx \right)^2 d\mathbf{r}_{\ell-1} \\ & \leq C \gamma^{-4} (\log(\gamma^{-1}))^{2+6(\ell-1)}. \end{aligned} \quad (4.38)$$

Proof. Throughout the calculation we make heavy use of the pointwise bounds on $P_t^\gamma \star K_\gamma$ derived in Lemma 8.3. From now on we choose γ_0 as the constant appearing in this lemma, and we assume that $0 < \gamma < \gamma_0$. Furthermore, for notational simplicity we assume that $u = 0$, but the argument and the choice of constants are independent of this.

We start by fixing $r_1 > r_2 > \dots > r_{\ell-1} > 0$ and $z_1, \dots, z_{\ell-1} \in \Lambda_\varepsilon$ and bounding

$$\begin{aligned} & \sup_{\substack{0 \leq r_\ell \leq r_{\ell-1} \\ z_\ell \in \Lambda_\varepsilon}} \left(\int_{\mathbb{T}^2} |\eta_k(x)| \prod_{j=1}^{\ell} |P_{t-r_j}^\gamma \star K_\gamma(x-z_j)| dx \right)^2 \\ & \leq \sup_{0 \leq r \leq T} \|P_r^\gamma \star K_\gamma\|_{L^\infty(\mathbb{T}^2)}^2 \left(\int_{\mathbb{T}^2} |\eta_k(x)| \prod_{j=1}^{\ell-1} |P_{t-r_j}^\gamma \star K_\gamma(x-z_j)| dx \right)^2. \end{aligned}$$

Lemma 8.3 implies that for $0 \leq r_\ell \leq T$ we have

$$\|P_r^\gamma \star K_\gamma\|_{L^\infty(\mathbb{T}^2)}^2 \leq C(T) \gamma^{-4} (\log(\gamma^{-1}))^2.$$

Hence we can write

$$\begin{aligned} & \int_{r_1=0}^t \cdots \int_{r_{\ell-1}=0}^{r_{\ell-2}} \sum_{z_1, \dots, z_{\ell-1} \in \Lambda_\varepsilon} \varepsilon^{2(\ell-1)} \\ & \sup_{\substack{0 \leq r_\ell \leq r_{\ell-1} \\ z_\ell \in \Lambda_\varepsilon}} \left(\int_{\mathbb{T}^2} |\eta_k(x)| \prod_{j=1}^{\ell} |P_{t-r_j}^\gamma \star K_\gamma(x-z_j)| dx \right)^2 d\mathbf{r}_{\ell-1} \\ & \leq C \gamma^{-4} (\log(\gamma^{-1}))^2 \int_{\mathbb{T}^2} \int_{\mathbb{T}^2} |\eta_k(x_1)| |\eta_k(x_2)| \mathcal{K}(x_1, x_2) dx_1 dx_2, \end{aligned} \quad (4.39)$$

where

$$\begin{aligned} & \mathcal{K}(x_1, x_2) \\ & = \int_{r_1=0}^t \cdots \int_{r_{\ell-1}=0}^{r_{\ell-2}} \sum_{z_1, \dots, z_{\ell-1} \in \Lambda_\varepsilon} \varepsilon^{2(\ell-1)} \\ & \quad \prod_{j=1}^{\ell-1} |P_{t-r_j}^\gamma \star K_\gamma(x_1-z_j)| |P_{t-r_j}^\gamma \star K_\gamma(x_2-z_j)| d\mathbf{r}_{\ell-1} \\ & = \frac{1}{(\ell-1)!} \left(\int_0^t \sum_{z \in \Lambda_\varepsilon} \varepsilon^2 |P_r^\gamma \star K_\gamma(x_1-z)| |P_r^\gamma \star K_\gamma(x_2-z)| dr \right)^{\ell-1}. \end{aligned}$$

Here, in the second line we have used the symmetry of the integrand in the time variables to replace the integral over the simplex $r_{\ell-1} \leq r_{\ell-1} \leq \dots \leq r_1 \leq t$ by an integral over $[0, t]^{\ell-1}$. We claim that (up to a power of $\log(\gamma^{-1})$) the convolution

$\sum_{z \in \Lambda_\varepsilon} \varepsilon^2 |P_r^\gamma \star K_\gamma(x_1 - z)| |P_r^\gamma \star K_\gamma(x_2 - z)|$ satisfies the same bounds (8.16) and (8.17) as $P_r^\gamma \star K_\gamma$. Indeed, we get for all $x_1, x_2 \in \mathbb{T}$ that

$$\begin{aligned} & \sum_{z \in \Lambda_\varepsilon} \varepsilon^2 |P_r^\gamma \star K_\gamma(x_1 - z)| |P_r^\gamma \star K_\gamma(x_2 - z)| \\ & \leq \|P_r^\gamma \star K_\gamma(x_1 - \cdot)\|_{L^\infty(\Lambda_\varepsilon)} \|P_r^\gamma \star K_\gamma(x_2 - \cdot)\|_{L^1(\Lambda_\varepsilon)} \end{aligned}$$

with the obvious conventions $\|f\|_{L^1(\Lambda_\varepsilon)} := \sum_{z \in \Lambda_\varepsilon} \varepsilon^2 |f(z)|$ and $\|f\|_{L^\infty(\Lambda_\varepsilon)} := \sup_{z \in \Lambda_\varepsilon} |f(z)|$ for any function $f: \Lambda_\varepsilon \rightarrow \mathbb{R}$. According to (8.16), we have uniformly over $x \in \mathbb{T}^2$ and $0 \leq r \leq T$ that $\|P_r^\gamma \star K_\gamma(x - \cdot)\|_{L^\infty(\Lambda_\varepsilon)} \leq C(\log(\gamma^{-1}))^2 (t^{-1} \wedge \gamma^{-2})$. On the other hand, (8.16) and (8.17) imply that

$$\begin{aligned} & \|P_r^\gamma \star K_\gamma(x - \cdot)\|_{L^1(\Lambda_\varepsilon)} \\ & \leq C \log(\gamma^{-1})^2 \sum_{z \in \Lambda_\varepsilon} \varepsilon^2 \left(\frac{1}{|x - z|^2} \wedge \gamma^{-1} \right) \leq C \log(\gamma^{-1})^3. \quad (4.40) \end{aligned}$$

Combining these bounds, we get

$$\sum_{z \in \Lambda_\varepsilon} \varepsilon^2 |P_r^\gamma \star K_\gamma(x_1 - z)| |P_r^\gamma \star K_\gamma(x_2 - z)| \leq C(\log(\gamma^{-1}))^5 (t^{-1} \wedge \gamma^{-2}),$$

for a constant C that is uniform in $0 \leq r \leq T$ and $0 < \gamma < \gamma_0$.

Integrating this bound over r for any fixed $x_1, x_2 \in \mathbb{T}^2$ brings

$$\begin{aligned} & \int_0^t \sum_{z \in \Lambda_\varepsilon} \varepsilon^2 |P_r^\gamma \star K_\gamma(x_1 - z)| |P_r^\gamma \star K_\gamma(x_2 - z)| dr \\ & \leq C(\log(\gamma^{-1}))^5 \left(\int_0^{\gamma^2} \gamma^{-2} dr + \int_{\gamma^2}^t r^{-1} dr \right) \\ & \leq C(\log(\gamma^{-1}))^6. \end{aligned}$$

Plugging this back into (4.39) leads to

$$\begin{aligned} & \int_{r_1=0}^t \cdots \int_{r_{\ell-1}=0}^{r_{\ell-2}} \sum_{z_1, \dots, z_{\ell-1} \in \Lambda_\varepsilon} \varepsilon^{2(\ell-1)} \\ & \sup_{\substack{0 \leq r_\ell \leq r_{\ell-1} \\ z_\ell \in \Lambda_\varepsilon}} \left(\int_{\mathbb{T}^2} |\eta_k(x)| \prod_{j=1}^{\ell} |P_{t-r_j}^\gamma \star K_\gamma(x - z_j)| dx \right)^2 dr_{\ell-1} \\ & \leq C \gamma^{-4} (\log(\gamma^{-1}))^{2+6(\ell-1)} \|\eta_k\|_{L^1(\mathbb{T}^2)}^2. \end{aligned}$$

Recalling that according to Lemma B.1, $\|\eta_k(x_1)\|_{L^1(\mathbb{T}^2)}^2 \leq C$ uniformly in $k \geq -1$, we get the desired conclusion (4.38). \square

For technical reasons, below in Lemma 7.4 we will need an additional bound on Z_γ that states that the very high frequencies of Z_γ are actually much smaller than predicted by Proposition 4.2. We define

$$Z_\gamma^{\text{high}}(t, x) = \sum_{2^k \geq \frac{2^{-2}}{10}} \delta_k Z_\gamma(t, x).$$

Lemma 4.6. *There exists a constant $\gamma_0 > 0$ such that any $p \geq 1$, $T > 0$, and $\kappa > 0$, there exists a constant $C = C(p, T, \kappa)$ such that for all $0 \leq s \leq t \leq T$ and $0 < \gamma < \gamma_0$,*

$$\mathbb{E} \|Z_\gamma^{\text{high}}(t, \cdot)\|_{L^\infty}^p \leq C \gamma^{p(1-\kappa)}. \quad (4.41)$$

Proof. In the proof of Proposition 4.2, we had already seen in (4.27) that for all $k \geq 0$, we have

$$\left(\mathbb{E} |\delta_k Z_\gamma(t, u)|^p\right)^{\frac{2}{p}} \leq C \sum_{|\omega| \in 2^k \left[\frac{3}{4}, \frac{8}{3}\right]} \frac{|\hat{K}_\gamma(\omega)|^2}{\frac{1}{t} + 2\gamma^{-2}(1 - \hat{K}_\gamma(\omega))} + \text{Err}. \quad (4.42)$$

According to (4.33), we have

$$\text{Err} \leq C \gamma^2 (\log(\gamma^{-1}))^2.$$

In order to bound the first expression in (4.42), recall that we are only interested in k with $2^k \geq \frac{\gamma^{-2}}{10}$. In particular, for γ small enough all frequencies appearing in this sum are much larger than γ^{-1} . For such ω , the bounds derived in Lemma 4.4 are suboptimal. Indeed, the bound (4.36) can be improved to

$$\frac{|\hat{K}_\gamma(\omega_j)|^2}{\frac{1}{t} + 2\gamma^{-2}(1 - \hat{K}_\gamma(\omega_j))} \leq \frac{C}{|\gamma \omega_j|^4} \frac{1}{\frac{1}{t} + \frac{2}{C_1} \gamma^{-2}} \leq C \gamma^2 \left(\frac{1}{1 + |\omega_j|^2} \wedge t \right), \quad (4.43)$$

where we have used the fact that $|\omega| \geq \frac{\gamma^{-2}}{10} \frac{3}{4}$. Following the rest of the argument as before, we see that

$$\mathbb{E} \|Z_\gamma^{\text{high}}(t, \cdot)\|_{C^{-\nu}}^p \leq C \gamma^p + C \gamma^p \log(\gamma^{-1})^p,$$

and the desired bound follows from Lemma B.3. \square

5. TIGHTNESS FOR THE LINEARISED SYSTEM

In this section, we continue the discussion of the processes $Z_\gamma^{n:}$ and $R_{\gamma,t}^{n:}$ defined in (4.5) and (4.6). The first main result is Proposition 5.3 which states that $R_{\gamma,t}^{n:}$ can be approximately written as a Hermite polynomial applied to $R_{\gamma,t}$. In Proposition 5.4, we combine this result with the bounds obtained in Proposition 4.2 to show tightness for the family $Z_\gamma^{n:}$ in an appropriate space.

We start by comparing the quadratic variation $\langle R_{\gamma,t}(\cdot, x) \rangle_t$ to the bracket process $[R_{\gamma,t}(\cdot, x)]_t$ (see Appendix C for the different notions of quadratic variation for a martingale with jumps). Implicitly in the proof of Lemma 4.1, we have already seen that for $x \in \Lambda_\varepsilon$ and $0 \leq s \leq t$, we have

$$\langle R_{\gamma,t}(\cdot, x) \rangle_s = 4c_{\gamma,2}^2 \int_0^s \sum_{z \in \Lambda_\varepsilon} \varepsilon^2 (P_{t-r}^\gamma \star_\varepsilon K_\gamma)^2(x-z) C_\gamma(r, z) dr. \quad (5.1)$$

Lemma 5.1. *For $x \in \Lambda_\varepsilon$, let*

$$Q_{\gamma,t}(s, x) = [R_{\gamma,t}(\cdot, x)]_s - \langle R_{\gamma,t}(\cdot, x) \rangle_s, \quad (5.2)$$

where $s \mapsto [R_{\gamma,t}(\cdot, x)]_s$ is the bracket process of the martingale $R_{\gamma,t}(\cdot, x)$. Let $\gamma_0 > 0$ be the constant appearing in Lemma 8.2. For any $t \geq 0$, $\kappa > 0$ and $1 \leq p < +\infty$, there exists $C = C(t, \kappa, p)$ such that for $0 < \gamma < \gamma_0$,

$$\mathbb{E} \sup_{x \in \Lambda_\varepsilon} \sup_{0 \leq s \leq t} |Q_{\gamma,t}(s, x)|^p \leq C \gamma^{p(1-\kappa)}.$$

Proof. By monotonicity of stochastic L^p norms, it suffices to show the statement for p large enough. We wish to apply the Burkholder-Davis-Gundy inequality to the martingale $Q_{\gamma,t}(\cdot, x)$ for a fixed $x \in \Lambda_\varepsilon$ and in order to do so, we need to estimate the quadratic variation and the jumps of this martingale.

Since the martingale $R_{\gamma,t}(\cdot, x)$ is of finite total variation, its bracket process is simply

$$[R_{\gamma,t}(\cdot, x)]_s = \sum_{0 < r \leq s} (\Delta_r R_{\gamma,t}(\cdot, x))^2.$$

Moreover, as the process $s \mapsto \langle R_{\gamma,t}(\cdot, x) \rangle_s$ is continuous, the jumps of $Q_{\gamma,t}(\cdot, x)$ are identical to those of $s \mapsto [R_{\gamma,t}(\cdot, x)]_s$.

Recall that an update of the spin $\sigma(k)$ at microscopic position $k = \varepsilon^{-1}z$ causes a jump of size $2\delta^{-1}\varepsilon^2 K_\gamma(y - z)$ for $M_\gamma(y)$. If such an event takes place at the macroscopic time s , then the martingale $R_{\gamma,t}(\cdot, x)$ has a jump at time s of absolute value

$$2\varepsilon^2\delta^{-1}(P_{t-s}^\gamma \star_\varepsilon K_\gamma)(x - z) \leq 2\varepsilon^2\delta^{-1}\|K_\gamma\|_{L^\infty(\Lambda_\varepsilon)} \leq 3\gamma, \quad (5.3)$$

where we have used the fact that $\|P_t^\gamma\|_{L^1(\Lambda_\varepsilon)} = 1$, cf. (8.15). Under the same circumstance, the jump of the martingale $Q_{\gamma,t}(\cdot, x)$ at time s is

$$(2\varepsilon^2\delta^{-1}(P_{t-s}^\gamma \star_\varepsilon K_\gamma)(x - z))^2 \leq (3\gamma)^2.$$

This gives us the required estimate on the jumps of $Q_{\gamma,t}(\cdot, x)$. We now turn to its quadratic variation. It is the same as the quadratic variation of the process $s \mapsto [R_{\gamma,t}(\cdot, x)]_s$, hence,

$$\begin{aligned} \langle Q_{\gamma,t}(\cdot, x) \rangle_t &= \int_0^t \sum_{z \in \Lambda_\varepsilon} \frac{C_\gamma(s, z)}{\alpha} (2\varepsilon^2\delta^{-1}(P_{t-s}^\gamma \star_\varepsilon K_\gamma)(x - z))^4 ds \\ &\leq \frac{16\varepsilon^6}{\alpha\delta^4} \int_0^t \sum_{z \in \Lambda_\varepsilon} \varepsilon^2 (P_{t-s}^\gamma \star_\varepsilon K_\gamma)^4(z) ds, \end{aligned}$$

where we used the fact that $C_\gamma(s, z) \leq 1$. Using again (5.3) in the form

$$\|P_{t-s}^\gamma \star_\varepsilon K_\gamma\|_{L^\infty(\Lambda_\varepsilon)} \leq 2\frac{\gamma^2}{\varepsilon^2}$$

yields

$$\langle Q_{\gamma,t}(\cdot, x) \rangle_t \leq 2^6 \frac{\varepsilon^2 \gamma^4}{\alpha \delta^4} \int_0^t \sum_{z \in \Lambda_\varepsilon} \varepsilon^2 (P_{t-s}^\gamma \star_\varepsilon K_\gamma)^2(z) ds. \quad (5.4)$$

We now use $(\varepsilon^2 \gamma^2 / \alpha \delta^4) \leq 2$ and (4.24) to obtain

$$\langle Q_{\gamma,t}(\cdot, x) \rangle_t \leq 2^7 e \gamma^2 \sum_{\omega \in \{-N, \dots, N\}^2} \frac{|\hat{K}_\gamma(\omega)|^2}{t^{-1} + 2\gamma^{-2}(1 - \hat{K}_\gamma(\omega))}. \quad (5.5)$$

For $0 < \gamma < \gamma_0$, the bounds (8.10) and (8.13) imply that the sum over $|\omega| \geq \gamma^{-1}$ is bounded uniformly. By (8.7) and (8.13), the sum over $|\omega| \leq C\gamma^{-1}$ is smaller than $C(t) \log(\gamma^{-1})$. We have shown the deterministic bound

$$\langle Q_{\gamma,t}(\cdot, x) \rangle_t \leq C\gamma^2 \log(\gamma^{-1}). \quad (5.6)$$

By the Burkholder-Davis-Gundy inequality (Lemma C.1), it follows that for all $p > 0$,

$$\mathbb{E} \sup_{0 \leq s \leq t} |Q_{\gamma,t}(s, x)|^p \leq C\gamma^p \log^{p/2}(\gamma^{-1}),$$

where $C = C(p, t)$, and in particular the constant does not depend on $x \in \Lambda_\varepsilon$. The conclusion is then obtained using the observation that

$$\mathbb{E} \sup_{x \in \Lambda_\varepsilon} \sup_{0 \leq s \leq t} |Q_{\gamma,t}(s, x)|^p \leq \sum_{x \in \Lambda_\varepsilon} \mathbb{E} \sup_{0 \leq s \leq t} |Q_{\gamma,t}(s, x)|^p, \quad (5.7)$$

and choosing p sufficiently large. \square

Lemma 5.2. *Let $\gamma_0 > 0$ be the constant appearing in Lemma 8.2. For any $t \geq 0$ and $1 \leq p < +\infty$, there exists $C = C(t, p) > 0$ such that for every $0 < \gamma < \gamma_0$,*

$$\left(\mathbb{E} \sup_{x \in \Lambda_\varepsilon} \left| \sum_{0 \leq s \leq t} (\Delta_s R_{\gamma,t}(\cdot, x))^2 \right|^p \right)^{1/p} \leq C \log(\gamma^{-1}).$$

Proof. We observe that

$$\sum_{0 \leq s \leq t} (\Delta_s R_{\gamma,t}(\cdot, x))^2 = Q_{\gamma,t}(t, x) + \langle R_{\gamma,t}(\cdot, x) \rangle_t.$$

By Lemma 5.1, it thus suffices to show that

$$\sup_{x \in \Lambda_\varepsilon} \langle R_{\gamma,t}(\cdot, x) \rangle_t \leq C \log(\gamma^{-1}). \quad (5.8)$$

We learn from (5.1) (as when passing from (5.4) to (5.5)) that

$$\langle R_{\gamma,t}(\cdot, x) \rangle_t \leq 4c_{\gamma,2}^2 \sum_{\omega \in \{-N, \dots, N\}^2} \frac{|\hat{K}_\gamma(\omega)|^2}{t^{-1} + 2\gamma^{-2}(1 - \hat{K}_\gamma(\omega))}.$$

We obtain (5.8) arguing in the same way as from (5.5) to (5.6). \square

We are now ready to prove that the $R_{\gamma,t}^{n:}$ can approximately be represented as a Hermite polynomial applied to $R_{\gamma,t}$. Recall the recursive definition (3.5) of the Hermite polynomials $H_n = H_n(X, T)$ as well as the identities (3.6) and (3.7) for their derivatives.

We aim to bound the quantity

$$E_{\gamma,t}^{n:}(s, x) := H_n(R_{\gamma,t}(s, x), [R_{\gamma,t}(\cdot, x)]_s) - R_{\gamma,t}^{n:}(s, x), \quad (5.9)$$

for any $x \in \mathbb{T}^2$. Here, we view $[R_{\gamma,t}(\cdot, x)]_s$ as defined on all of \mathbb{T}^2 , by extending it as a trigonometric polynomial of degree $\leq N$. Recall that according to (4.6), $R_{\gamma,t}^{n:}(t, x) = Z_\gamma^{n:}(t, x)$.

Proposition 5.3 ($Z_\gamma^{n:}$ as a Hermite polynomial). *Let γ_0 be the constant appearing in Lemma 8.2. Then for any $n \in \mathbb{N}$, $\kappa > 0$, $t > 0$ and $1 \leq p < \infty$, there exists $C = C(n, p, t, \kappa) > 0$ such that for every $0 < \gamma < \gamma_0$,*

$$\left(\mathbb{E} \sup_{x \in \mathbb{T}^2} \sup_{0 \leq s \leq t} |E_{\gamma,t}^{n:}(s, x)|^p \right)^{1/p} \leq C \gamma^{1-\kappa}.$$

Proof. We start by reducing the bound on the spatial supremum over $x \in \mathbb{T}^2$ to the pointwise bounds for x in a grid. According to the definition, for every n the function $E_{\gamma,t}^{n:}$ is a trigonometric polynomial of degree $\leq nN$. For $n = 1$, Lemma B.6 implies that we can control the supremum over $x \in \mathbb{T}^2$ by the supremum over $x \in \Lambda_\varepsilon$ at the price of loosing an arbitrarily small power of ε . For $n \geq 2$, this lemma does not apply directly, but we can circumvent this problem by refining the grid. Indeed, set $\Lambda_\varepsilon^{(n)} = \{x \in \frac{\varepsilon}{n} \mathbb{Z}^2: -1 < x_1, x_2 \leq 1\}$. Then Lemma B.6 applies (the fact that the number of grid points in this lemma is an odd multiple

of the dimension is just for convenience of notation) and we can conclude that $\sup_{x \in \mathbb{T}^2} |E_{\gamma,t}^{n:}(s, x)| \leq C(\kappa) \varepsilon^{-\kappa} \sup_{x \in \Lambda_\varepsilon^{(n)}} |E_{\gamma,t}^{n:}(s, x)|$. Finally, as in (5.7) we can reduce the bound on the supremum over $x \in \Lambda_\varepsilon^{(n)}$ to bounds on a single point $x \in \Lambda_\varepsilon^{(n)}$. We now proceed to derive such a bound.

The proof proceeds by induction on n . For $n = 1$, it is obviously true since $E_{\gamma,t}^{1:} = 0$. We now assume $n \geq 2$.

To begin with, we observe that for any n , there exists C such that for every $x, t \in \mathbb{R}$ and every $|h| \leq 1, |s| \leq 1$,

$$\begin{aligned} \left| H_n(x+h, t+s) - H_n(x, t) - \partial_X H_n(x, t) h - \frac{1}{2} \partial_X^2 H_n(x, t) h^2 - \partial_T H_n(x, t) s \right| \\ \leq C \left(|x|^{n-2} + |t|^{(n-2)/2} + 1 \right) (|h|^3 + |s|^2). \end{aligned} \quad (5.10)$$

We fix $x \in \Lambda_\varepsilon^{(n)}$ and use the shorthand notation

$$\underline{R}(s) = (R_{\gamma,t}(s, x), [R_{\gamma,t}(\cdot, x)]_s).$$

By Itô's formula (see Lemma C.1 and Remark C.3),

$$\begin{aligned} H_n(\underline{R}(s)) &= \int_{r=0}^s \partial_T H_n(\underline{R}(r^-)) d[R_{\gamma,t}(\cdot, x)]_r \\ &\quad + \int_{r=0}^s \partial_X H_n(\underline{R}(r^-)) dR_{\gamma,t}(r, x) \\ &\quad + \frac{1}{2} \int_{r=0}^s \partial_X^2 H_n(\underline{R}(r^-)) d[R_{\gamma,t}(\cdot, x)]_r \\ &\quad + \text{Err}(s, x), \end{aligned} \quad (5.11)$$

where $\text{Err}(s, x)$ is the error term caused by the jumps,

$$\begin{aligned} \text{Err}(s, x) &= \sum_{0 < r \leq s} \left\{ \Delta_r H_n(\underline{R}) - \partial_T H_n(\underline{R}(r^-)) \Delta_r [R_{\gamma,t}(\cdot, x)] \right. \\ &\quad \left. - \partial_X H_n(\underline{R}(r^-)) \Delta_r R_{\gamma,t}(\cdot, x) - \frac{1}{2} \partial_X^2 H_n(\underline{R}(r^-)) (\Delta_r R_{\gamma,t}(\cdot, x))^2 \right\}. \end{aligned}$$

We have seen in (5.3) that uniformly over r and $x \in \Lambda_\varepsilon$, we have $|\Delta_r R_{\gamma,t}(\cdot, x)| \leq 3\gamma$. Together with Lemma B.6, this implies that

$$\sup_{0 \leq r \leq t} \sup_{x \in \Lambda_\varepsilon^{(n)}} |\Delta_r R_{\gamma,t}(\cdot, x)| \leq C\gamma^{1-\kappa} \quad (5.12)$$

for any given $\kappa > 0$, where $C = C(\kappa)$. Therefore,

$$\Delta_r [R_{\gamma,t}(\cdot, x)] = (\Delta_r R_{\gamma,t}(\cdot, x))^2 \leq (C\gamma^{1-\kappa})^2.$$

As a consequence, we can apply the estimate in (5.10) on the error term, to get

$$\begin{aligned} |\text{Err}(s, x)| &\leq C \left(\sup_{0 \leq r \leq s} |R_{\gamma,t}(r, x)|^{n-2} + \sup_{0 \leq r \leq s} [R_{\gamma,t}(\cdot, x)]_r^{(n-2)/2} + 1 \right) \\ &\quad \times \sum_{0 < r \leq s} \left\{ (\Delta_r R_{\gamma,t}(\cdot, x))^3 + (\Delta_r R_{\gamma,t}(\cdot, x))^4 \right\}. \end{aligned}$$

In view of (5.12), it is clear that the summand $(\Delta_r R_{\gamma,t}(\cdot, x))^4$ can be neglected. From (5.12) and Lemma 5.2, we obtain that for every $1 \leq p < +\infty$, there exists C

such that for every $s \leq t$,

$$\left(\mathbb{E} \sup_{x \in \Lambda_\varepsilon} \left| \sum_{0 \leq r \leq s} (\Delta_r R_{\gamma,t}(\cdot, x))^3 \right|^p \right)^{1/p} \leq C \gamma^{1-\kappa} \log(\gamma^{-1}),$$

and using Lemma B.6, we can replace the supremum over $x \in \Lambda_\varepsilon$ by a supremum over $x \in \Lambda_\varepsilon^{(n)}$ in this bound (at the price of changing the exact value of the arbitrarily small κ).

By the bounds derived in Lemma 4.1, Proposition 4.2 and Lemma 5.2, we also get that for every $\kappa > 0$, there exists $C > 0$ such that for every $s \leq t$ and $x \in \Lambda_\varepsilon^{(n)}$,

$$\left(\mathbb{E} \left| \sup_{0 \leq r \leq s} |R_{\gamma,t}(r, x)|^{n-2} + \sup_{0 \leq r \leq s} [R_{\gamma,t}(\cdot, x)]_r^{(n-2)/2} + 1 \right|^p \right)^{1/p} \leq C \gamma^{-\kappa}.$$

It follows from these observations that for every $\kappa > 0$ and $1 \leq p < +\infty$, there exists C such that uniformly over $x \in \Lambda_\varepsilon$,

$$\left(\mathbb{E} \sup_{s \leq t} |\text{Err}(s, x)|^p \right)^{1/p} \leq C \gamma^{1-\kappa}. \quad (5.13)$$

Going back to the relation in (5.11), we can use (3.6) and (3.7) to see that $\partial_T H_n + \partial_X^2 H_n / 2 = 0$, so the first and third integrals in (5.11) cancel out. Using (3.6) again, we arrive at

$$H_n(\underline{R}(s)) = n \int_{r=0}^s H_{n-1}(\underline{R}(r^-)) dR_{\gamma,t}(r, x) + \text{Err}(s, x).$$

In view of the definition (4.4) of $R_{\gamma,t}^{n:}(s, x)$ (which remains valid for $x \in \Lambda_\varepsilon^{(n)}$), we can rewrite this as

$$E_{\gamma,t}^{n:}(s, x) = n \int_{r=0}^s E_{\gamma,t}^{n-1:}(r^-, x) dR_{\gamma,t}(r, x) + \text{Err}(s, x). \quad (5.14)$$

Assuming that the Proposition is true for the index $n-1$, we want to prove that it holds for the index n . In fact, it suffices to prove that for every $\kappa > 0$, every $t > 0$ and every p sufficiently large, there exists $C > 0$ such that uniformly over $x \in \Lambda_\varepsilon$,

$$\left(\mathbb{E} \sup_{0 \leq s \leq t} |E_{\gamma,t}^{n:}(s, x)|^p \right)^{1/p} \leq C \gamma^{1-\kappa}, \quad (5.15)$$

since we can later on argue as in (5.7) to conclude. The error term in (5.14) will not cause any trouble by (5.13). There remains to consider the integral in the right-hand side of (5.14). Since this integral is a martingale as s varies, we can use the Burkholder-Davis-Gundy inequality, provided that we can estimate its quadratic variation and its maximal jump size. The quadratic variation at time t is bounded by

$$n^2 \sup_{s \leq t} |E_{\gamma,t}^{n-1:}(s, x)|^2 \langle R_{\gamma,t}(\cdot, x) \rangle_t,$$

with $\langle R_{\gamma,t}(\cdot, x) \rangle_t \leq C \log(\gamma^{-1})$ by (5.8). The maximal jump size is bounded by

$$n \sup_{s \leq t} |E_{\gamma,t}^{n-1:}(s, x)| \sup_{s \leq t} |\Delta_s R_{\gamma,t}(\cdot, x)|,$$

and we already saw that $\sup_{s \leq t} |\Delta_s R_{\gamma,t}(\cdot, x)| \leq C \gamma^{1-\kappa}$. The induction hypothesis and the Burkholder-Davis-Gundy inequality thus lead to (5.15), and the proof is complete. \square

Finally, we are ready to conclude and to prove the tightness of the processes $Z_\gamma^{:n:}$. Before we state the result, recall that for any separable metric space \mathcal{S} , we denote by $\mathcal{D}(\mathbb{R}_+, \mathcal{S})$ the space of cadlag functions on \mathbb{R}_+ taking values in \mathcal{S} , endowed with the Skorokhod topology (see [7, Chapters 16 and 18]). Recall in particular that according to [7, Theorem 16.4], a family of processes is tight on $\mathcal{D}(\mathbb{R}_+, \mathcal{S})$ as soon as their restrictions to all compact time intervals are tight.

Proposition 5.4. *For any fixed $n \in \mathbb{N}$ and any $\nu > 0$, the family $\{Z_\gamma^{:n:}, \gamma \in (0, \frac{1}{3})\}$ is tight on $\mathcal{D}(\mathbb{R}_+, \mathcal{C}^{-\nu})$. Any weak limit is supported on $\mathcal{C}(\mathbb{R}_+, \mathcal{C}^{-\nu})$. Furthermore, for any $p \geq 1$ and $T > 0$, we have*

$$\sup_{\gamma \in (0, \frac{1}{3})} \mathbb{E} \sup_{0 \leq t \leq T} \|Z_\gamma^{:n:}(t, \cdot)\|_{\mathcal{C}^{-\nu}}^p < \infty. \quad (5.16)$$

Proof. We can restrict ourselves to consider $0 < \gamma < \gamma_0$, where γ_0 is the constant appearing in Lemma 8.2 and Proposition 4.2. We fix a $T > 0$ and show tightness in $\mathcal{D}([0, T], \mathcal{C}^{-\nu})$.

Our strategy is similar to that of [43]. Let $m \in \mathbb{N}$ be fixed below. The estimate (4.19) implies that for all $s \neq t \in \gamma^m \mathbb{N}_0$, all $p \geq 1$, $\nu' > 0$, $\lambda \leq \frac{1}{2m}$, and $n \in \mathbb{N}$, we have

$$\mathbb{E} \|Z_\gamma^{:n:}(t, \cdot) - Z_\gamma^{:n:}(s, \cdot)\|_{\mathcal{C}^{-\nu'-2\lambda}}^p \leq C \left(|t - s|^\lambda + \gamma^{\frac{1}{2}} \right)^p \leq C |t - s|^{\lambda p}, \quad (5.17)$$

where $C = C(n, p, \nu', T, \lambda)$. We now define the following continuous interpolation for $Z_\gamma^{:n:}$: set $\tilde{Z}_\gamma^{:n:}(t, \cdot) = Z_\gamma^{:n:}(t, \cdot)$ for $t \in \gamma^m \mathbb{N}_0$, and interpolate linearly between these points. It is easy to check that $\tilde{Z}_\gamma^{:n:}$ satisfies

$$\mathbb{E} \|\tilde{Z}_\gamma^{:n:}(t, \cdot) - \tilde{Z}_\gamma^{:n:}(s, \cdot)\|_{\mathcal{C}^{-\nu'-2\lambda}}^p \leq C |t - s|^{\lambda p},$$

for all values of $s, t \in [0, T]$ and hence, the Kolomogorov criterion implies the desired properties when $Z_\gamma^{:n:}$ is replaced by $\tilde{Z}_\gamma^{:n:}$.

We claim that for any $\kappa > 0$ and $p \geq 1$, we have

$$\mathbb{E} \sup_{0 \leq t \leq T} \sup_{x \in \mathbb{T}^2} |\tilde{Z}_\gamma^{:n:}(t, x) - Z_\gamma^{:n:}(t, x)|^p \leq C(n, p, T, \kappa) \gamma^{(1-\kappa)p}. \quad (5.18)$$

Once we have established this bound, the proof is complete.

By monotonicity of L^p -norms, it is sufficient to establish (5.18) for large p . We treat the case $n = 1$ first. In view of Lemma B.6, it suffices to establish (5.18) with the supremum over $x \in \mathbb{T}^2$ replaced by the supremum over $x \in \Lambda_\varepsilon$. We fix an interval $I_k = [k\gamma^m, (k+1)\gamma^m]$ for some $k \in \mathbb{N}_0$ and an $x \in \Lambda_\varepsilon$, and we start with the estimate

$$\sup_{t \in I_k} |\tilde{Z}_\gamma(t, x) - Z_\gamma(t, x)| \leq 2 \sup_{t \in I_k} |Z_\gamma(t, x) - Z_\gamma(k\gamma^m, x)|.$$

Using the definition (4.3) of Z_γ , the definition of M_γ just above (2.14), as well as (2.9), we get for any $t \in I_k$

$$\begin{aligned} & Z_\gamma(t, x) - Z_\gamma(k\gamma^m, x) \\ &= \int_{k\gamma^m}^t \Delta_\gamma Z_\gamma(s, x) ds + (M_\gamma(t, x) - M_\gamma(k\gamma^m, x)) \\ &= \int_{k\gamma^m}^t \Delta_\gamma Z_\gamma(s, x) ds - \frac{1}{\delta} \left(\int_{\frac{k\gamma^m}{\alpha}}^{\frac{t}{\alpha}} \mathcal{L}_\gamma h_\gamma(s, \frac{x}{\varepsilon}) ds \right) \\ & \quad + \frac{1}{\delta} \left(h_\gamma\left(\frac{t}{\alpha}, \frac{x}{\varepsilon}\right) - h_\gamma\left(\frac{k\gamma^m}{\alpha}, \frac{x}{\varepsilon}\right) \right). \end{aligned} \quad (5.19)$$

We bound the terms on the right-hand side of (5.19) one by one: using the definition of Δ_γ , we get for the first term that

$$\begin{aligned} \sup_{x \in \Lambda_\varepsilon} \sup_{t \in I_k} \left| \int_{k\gamma^m}^t \Delta_\gamma Z_\gamma(s, x) ds \right| &\leq \frac{C}{\gamma^2} \int_{I_k} \|Z_\gamma(s, \cdot)\|_{L^\infty(\mathbb{T}^2)} ds \\ &\leq \frac{C(\kappa')}{\gamma^{2+2\kappa'}} \int_{I_k} \|Z_\gamma(s, \cdot)\|_{C^{-\kappa'}} ds. \end{aligned}$$

In the second inequality we have used Lemma B.3 for an arbitrary $\kappa' > 0$. Hence we get for any $p \geq 1$

$$\begin{aligned} & \mathbb{E} \sup_{k \leq T\gamma^{-m}} \sup_{x \in \Lambda_\varepsilon} \sup_{t \in I_k} \left| \int_{k\gamma^m}^t \Delta_\gamma Z_\gamma(s, x) ds \right|^p \\ & \leq \sum_{k \leq T\gamma^{-m}} \mathbb{E} \sup_{x \in \Lambda_\varepsilon} \sup_{t \in I_k} \left| \int_{k\gamma^m}^t \Delta_\gamma Z_\gamma(s, x) ds \right|^p \\ & \leq C(\kappa', p) \sum_{k \leq T\gamma^{-m}} \gamma^{-(2+2\kappa')p} \mathbb{E} \left(\int_{I_k} \|Z_\gamma(s, \cdot)\|_{C^{-\kappa'}} ds \right)^p \\ & \leq C(\kappa', p, T) \gamma^{-m} \gamma^{-(2+2\kappa')p} \gamma^{mp} \sup_{0 \leq t \leq T+\gamma^m} \mathbb{E} \|Z_\gamma(s, \cdot)\|_{C^{-\kappa'}}^p. \end{aligned}$$

By (4.18), the supremum on the right-hand side of this expression is bounded by a constant depending on T, κ' and p , so that the whole expression can be bounded by

$$C(\kappa', p, T) \gamma^{p(m(1-\frac{1}{p})-(2+2\kappa'))}.$$

Choosing $m \geq 3$, κ' small enough and p large enough we can obtain any exponent of the form $p(1-\kappa)$ for γ ($\kappa' < \frac{\kappa}{4}$ and $p > \frac{3}{2\kappa}$ suffice).

For the second term on the right-hand side of (5.19), we use the deterministic estimate $|\mathcal{L}_\gamma h_\gamma(s, k)| \leq 2$ which holds for any $k \in \Lambda_N$ and any time s to get for any $x \in \Lambda_\varepsilon$ and $k \leq T\gamma^{-m}$ and any $t \in I_k$

$$\frac{1}{\delta} \left(\int_{\frac{k\gamma^m}{\alpha}}^{\frac{t}{\alpha}} \mathcal{L}_\gamma h_\gamma(s, \frac{x}{\varepsilon}) ds \right) \leq 2 \frac{\gamma^m}{\alpha \delta} \leq 2\gamma^{m-3}.$$

Hence this term satisfies the estimate (5.18) as soon as $m \geq 4$.

Let us turn to the third term on the right-hand side of (5.19). The process $h_\gamma(\cdot, \frac{x}{\varepsilon})$ only evolves by jumps. Let us recall that a jump-event at position $j \in \Lambda_N$ at time $s \in [\frac{k\gamma^m}{\alpha}, \frac{t}{\alpha}]$ causes a jump of magnitude $2\kappa_\gamma(\frac{x}{\varepsilon}, j) \leq 3\gamma^2$ for $h_\gamma(\cdot, \frac{x}{\varepsilon})$. Hence, we

have

$$\sup_{x \in \Lambda_\varepsilon} \sup_{t \in I_k} \frac{1}{\delta} \left(h_\gamma \left(\frac{t}{\alpha}, \frac{x}{\varepsilon} \right) - h_\gamma \left(\frac{k\gamma^m}{\alpha}, \frac{x}{\varepsilon} \right) \right) \leq 3\gamma J_k ,$$

where J_k is the total number of jumps at all locations $j \in \Lambda_N$ during the time interval $\left[\frac{k\gamma^m}{\alpha}, \frac{(k+1)\gamma^m}{\alpha} \right]$. According to (2.7), the jump rate at any given location is always bounded by 1, so the total jump rate is bounded by $|\Lambda_\varepsilon|$. This implies that for every k , the random variable J_k is stochastically dominated by $\text{Poi}(\lambda)$, a Poisson random variable with mean $\lambda = \gamma^m \alpha^{-1} |\Lambda_\varepsilon| \leq C\gamma^{m-6}$. We impose $m > 6$, so that this rate goes to zero. We note that

$$\mathbb{E} \sup_{k \leq T\gamma^{-m}} J_k^p \leq \sum_{k \leq T\gamma^{-m}} \mathbb{E} J_k^p \leq T\gamma^{-m} \mathbb{E} \text{Poi}(\lambda)^p .$$

Since $\mathbb{E} [\text{Poi}(\lambda)^p] \leq C(p)\gamma^{m-6}$, we arrive at

$$\gamma^p \mathbb{E} \left[\sup_{k \leq T\gamma^{-m}} J_k^p \right] \leq C(p, T)\gamma^{p-6} , \quad (5.20)$$

and as above we can obtain the exponent $p(1 - \kappa)$ by choosing $p > \frac{6}{\kappa}$. Summarising these calculations and invoking Lemma B.6, we see that for any $\kappa > 0$, $m > 6$ and $p > \frac{6}{\kappa}$ there exists a constant $C = C(p, T, \kappa)$ such that

$$\mathbb{E} \sup_{k \leq T\gamma^{-m}} \sup_{x \in \mathbb{T}^2} \sup_{t \in I_k} |Z_\gamma(t, x) - Z_\gamma(k\gamma^m, x)|^p \leq C\gamma^{p(1-\kappa)} . \quad (5.21)$$

which establishes (5.18) in the case $n = 1$.

We now proceed to prove (5.18) in the case $n \geq 2$. According to Proposition 5.3, it suffices to show

$$\mathbb{E} \sup_{0 \leq t \leq T} \sup_{x \in \mathbb{T}^2} |H_n(\underline{Z}(t, x)) - \widetilde{H}_n(\underline{Z})(t, x)|^p \leq C\gamma^{(1-\kappa)p} .$$

where $\underline{Z}(t, x) := (Z_\gamma(t, x), [R_{\gamma,t}(\cdot, x)]_t)$, and $\widetilde{H}_n(\underline{Z})$ denotes the process obtained by evaluating $H_n(\underline{Z}(t, x))$ at points $t \in \gamma^m \mathbb{N}$ exactly and taking the linear interpolation in between. It is easy to see that for fixed $x \in \mathbb{T}^2$ and $t \in I_k$ we have

$$\begin{aligned} & |H_n(\underline{Z}(t, x)) - \widetilde{H}_n(\underline{Z})(t, x)| \\ & \leq C(n) \left(1 + \sup_{t \in I_k} |Z_\gamma(t, x)|^n + \sup_{t \in I_k} [R_{\gamma,t}(\cdot, x)]_t^{(n/2)} \right) \\ & \quad \times \left(\sup_{t \in I_k} |Z_\gamma(t, x) - Z_\gamma(k\gamma^m, x)| + \sup_{t \in I_k} |[R_{\gamma,t}(\cdot, x)]_t - [R_{\gamma,k\gamma^m}(\cdot, x)]_{k\gamma^m}| \right) . \end{aligned} \quad (5.22)$$

We have already established (5.16) for $n = 1$, which yields a bound on all moments of $\sup_{t \in I_k} |Z_\gamma(t, x)|^n$ as well as (5.21), so that it remains to bound the terms involving the bracket process. We start with the last term on the right-hand side of (5.22). For a fixed k , we write $s = k\gamma^m$ and get for any $t \in I_k$

$$\begin{aligned} & |[R_{\gamma,t}(\cdot, x)]_t - [R_{\gamma,s}(\cdot, x)]_s| \\ & \leq \sum_{0 \leq r \leq s} |(\Delta_r R_{\gamma,t}(\cdot, x))^2 - (\Delta_r R_{\gamma,s}(\cdot, x))^2| + \sum_{s \leq r \leq t} \Delta_r (R_{\gamma,t}(\cdot, x))^2 . \end{aligned} \quad (5.23)$$

Recall, that a jump of spin $\sigma(k)$ for $k = \varepsilon^{-1}z$ at time r causes a jump of absolute value $2\varepsilon^2 \delta^{-1} P_{t-r}^\gamma \star_\varepsilon K_\gamma(x - z)$ for $R_{\gamma,t}(\cdot, x)$, so that the first term on the right-hand

side of (5.23) can be bounded by

$$\begin{aligned} & \sum_{0 \leq r \leq s} \left| \Delta_r (R_{\gamma,t}(\cdot, x))^2 - (\Delta_r R_{\gamma,s}(\cdot, x))^2 \right| \\ & \leq 4 \frac{\varepsilon^4}{\delta^2} J \sup_{z \in \Lambda_\varepsilon} \sup_{0 \leq r \leq s} \left| (P_{t-r}^\gamma \star_\varepsilon K_\gamma(z))^2 - (P_{s-r}^\gamma \star_\varepsilon K_\gamma(z))^2 \right|, \end{aligned}$$

where J is the total number of jumps at any point in Λ_N and any point in time before T in macroscopic time, which corresponds to T/α in microscopic units. As the jump rate at any given point is always bounded by 1, J is stochastically dominated by a Poisson variable $\text{Poi}(\lambda)$ with mean $\lambda \leq CT\varepsilon^{-2}\alpha^{-1} \leq CT\gamma^{-6}$, which implies in particular that for every $p \geq 1$,

$$\mathbb{E}J^p \leq C(p, T)\gamma^{-6p}.$$

On the other hand, for any $0 \leq r \leq s$ and $z \in \Lambda_\varepsilon$, we can write

$$\begin{aligned} & \left| (P_{t-r}^\gamma \star_\varepsilon K_\gamma(z))^2 - (P_{s-r}^\gamma \star_\varepsilon K_\gamma(z))^2 \right| \\ & \leq \left| P_{t-r}^\gamma \star_\varepsilon K_\gamma(z) + P_{s-r}^\gamma \star_\varepsilon K_\gamma(z) \right| \left| P_{t-r}^\gamma \star_\varepsilon K_\gamma(z) - P_{s-r}^\gamma \star_\varepsilon K_\gamma(z) \right| \\ & \leq C\gamma^{-2} \log(\gamma^{-1}) \left| P_{t-r}^\gamma \star_\varepsilon K_\gamma(z) - P_{s-r}^\gamma \star_\varepsilon K_\gamma(z) \right|, \end{aligned}$$

where we have made use of (8.16) in the last line. We continue by bounding brutally

$$\begin{aligned} & \left| P_{t-r}^\gamma \star_\varepsilon K_\gamma(z) - P_{s-r}^\gamma \star_\varepsilon K_\gamma(z) \right| \\ & = \left| \frac{1}{4} \sum_{\omega \in \{-N, \dots, N\}^2} e^{i\pi\omega \cdot z} \hat{K}_\gamma(\omega) \left(e^{\frac{t-r}{\gamma^2}(\hat{K}_\gamma(\omega)-1)} - e^{\frac{s-r}{\gamma^2}(\hat{K}_\gamma(\omega)-1)} \right) \right| \\ & \leq C\gamma^{-4} \frac{t-s}{\gamma^2}, \end{aligned}$$

where we have used the bound $|\hat{K}_\gamma(\omega)| \leq 1$ twice. Summarising these bounds, we get for any p

$$\begin{aligned} & \mathbb{E} \sup_{k \leq T\gamma^{-m}} \sup_{x \in \Lambda_\varepsilon} \sup_{t \in I_k} \left(\sum_{0 \leq r \leq s} \left| \Delta_r (R_{\gamma,t}(\cdot, x))^2 - (\Delta_r R_{\gamma,k\gamma^m}(\cdot, x))^2 \right| \right)^p \\ & \leq C(p, T)\gamma^{mp} \gamma^{-8p} \log(\gamma^{-1})^p, \end{aligned}$$

which is bounded by $C(p, T)\gamma^{p(2-\kappa)}$ as soon as $m \geq 10$.

For the second term on the right-hand side of (5.23), we write using (5.3)

$$\sum_{k\gamma^m \leq r \leq (k+1)\gamma^m} \Delta_r (R_{\gamma,t}(\cdot, x))^2 \leq C\gamma^2 J_k,$$

where as above J_k is the total number of jumps at all locations $j \in \Lambda_N$ during the time interval $\left[\frac{k\gamma^m}{\alpha}, \frac{(k+1)\gamma^m}{\alpha} \right]$. Repeating the argument that leads to (5.20), we get that for $m > 6$

$$\gamma^{2p} \mathbb{E} \sup_{k \leq T\gamma^{-m}} J_k^p \leq C(p, T)\gamma^{2p-6}.$$

Summarising these calculations, we get that for $m \geq 10$ and $p > \frac{6}{\kappa}$, there exists $C = C(p, T, \kappa)$ such that

$$\mathbb{E} \sup_{k \leq T\gamma^{-m}} \sup_{x \in \Lambda_\varepsilon} \sup_{t \in I_k} \left| [R_{\gamma,t}(\cdot, x)]_t - [R_{\gamma,k\gamma^m}(\cdot, x)]_{k\gamma^m} \right|^p \leq C\gamma^{p(2-\kappa)}. \quad (5.24)$$

Finally, going back to (5.22), it remains to bound

$$\begin{aligned} & \mathbb{E} \left(\sup_{k \leq T\gamma^{-m}} \sup_{x \in \Lambda_\varepsilon} \sup_{t \in I_k} [R_{\gamma,t}(\cdot, x)]_t^{(n/2)} \right)^p \\ & \leq C \sum_{k \leq T\gamma^{-m}} \mathbb{E} \left\| [R_{\gamma, k\gamma^m}(\cdot, \cdot)]_{k\gamma^m}^{(n/2)} \right\|_{L^\infty(\Lambda_\varepsilon)}^p \\ & \quad + C \mathbb{E} \sup_{k \leq T\gamma^{-m}} \sup_{x \in \Lambda_\varepsilon} \sup_{t \in I_k} \left| [R_{\gamma,t}(\cdot, x)]_t - [R_{\gamma, k\gamma^m}(\cdot, x)]_{k\gamma^m} \right|^p. \end{aligned}$$

In view of (5.24), the second term can be neglected. Using Lemma 5.2, we see that the first term is bounded by $C(p, T)\gamma^{-m} \log(\gamma^{-1})^{mp}$. Plugging all of this back into (5.22), invoking Lemma B.6 once more, and using Hölder's inequality (with a large exponent on the terms in the last line of (5.22)) we get for any $\kappa > 0$ and p large enough

$$\mathbb{E} \sup_{x \in \Lambda_\varepsilon} \sup_{0 \leq t \leq T} |H_n(\underline{Z}(t, x)) - \widetilde{H}_n(\underline{Z})(t, x)|^p \leq C(n, p, T, \kappa) \gamma^{p(1-\kappa)},$$

and the proof is complete. \square

6. CONVERGENCE IN LAW OF THE LINEARISED SYSTEM

The aim of this section is to prove the convergence in law of Z_γ and the approximate Wick powers $Z_\gamma^{:n:}$ to the solution of the stochastic heat equation and its Wick powers. We will only be able to show the convergence in law of Z_γ and $Z_\gamma^{:n:}$ up to a stopping time that depends on X_γ (the “non-linear” dynamics), which we do not control for now. For a fixed $\nu \in (0, \frac{1}{2})$, any $m > 1$ and $0 < \gamma < 1$, we set

$$\tau_{\gamma, m} := \inf \{ t \geq 0 : \|X_\gamma(t, \cdot)\|_{C^{-\nu}} \geq m \}. \quad (6.1)$$

The following Theorem 6.1 states, roughly speaking, that Z_γ converges to Z (the solution of the stochastic heat equation introduced in Section 3) “until $\tau_{\gamma, m}$ ”. In order to state this properly, we introduce a different extension of Z_γ beyond $\tau_{\gamma, m}$. We start by modifying the microscopic jump process σ for times $t \geq \tau_{\gamma, m}$. Indeed, for $k \in \Lambda_N$ and for $t \geq 0$, define

$$\sigma_{\gamma, m}(t, k) := \begin{cases} \sigma(t, k) & \text{if } t < \frac{\tau_{\gamma, m}}{\alpha}, \\ \sigma'_{\gamma, m}(t, k) & \text{otherwise.} \end{cases}$$

Here $\sigma'_{\gamma, m}$ is a spin system with $\sigma'_{\gamma, m}(\tau_{\gamma, m}/\alpha, k) = \sigma(\tau_{\gamma, m}/\alpha, k)$ and with jumps occurring for every $t > \tau_{\gamma, m}/\alpha$ and every $k \in \Lambda_N$ at rate $\frac{1}{2}$, independently from σ . We now construct processes $M_{\gamma, m}$ and $Z_{\gamma, m}$ following exactly the construction of M_γ and Z_γ with σ replaced by $\sigma_{\gamma, m}$. For $t \geq 0$ and $k \in \Lambda_N$ we set $h_{\gamma, m}(t, k) = (\sigma_{\gamma, m}(t, \cdot) \star \kappa_\gamma)(k)$ and as in (2.9) we set

$$m_{\gamma, m}(t, k) := h_{\gamma, m}(t, k) - h_{\gamma, m}(0, k) - \int_0^t \mathcal{L}_{\gamma, m}^s h_{\gamma, m}(s, k) ds, \quad (6.2)$$

where $\mathcal{L}_{\gamma, m}^s$ is defined as in (2.5) with c_γ replaced by

$$c_{\gamma, m}^s = \begin{cases} c_\gamma & \text{if } s < \frac{\tau_{\gamma, m}}{\alpha}, \\ \frac{1}{2} & \text{otherwise} \end{cases}, \quad (6.3)$$

(in other words, $\mathcal{L}_{\gamma, m}^s$ is the infinitesimal generator of $\sigma'_{\gamma, m}$). Finally, let $M_{\gamma, m}(t, x) := \frac{1}{\delta} m_{\gamma, m}\left(\frac{t}{\alpha}, \frac{x}{\varepsilon}\right)$. The processes $M_{\gamma, m}(\cdot, x)$ are martingales with quadratic variations

given by (2.15) with C_γ replaced by

$$C_{\gamma,m}(s, z) := c_{\gamma,m}^s(\sigma_{\gamma,m}(s/\alpha), z/\varepsilon). \quad (6.4)$$

Let $R_{\gamma,t,m}^{n:}$ and $Z_{\gamma,m}^{n:}$ be defined as iterated stochastic integrals just as $R_{\gamma,t}^{n:}$ and $Z_{\gamma,t}^{n:}$ in (4.4), (4.5), and (4.6), but with M_γ replaced by $M_{\gamma,m}$. It is clear that for $s \leq \tau_{\gamma,m}$, we have $R_{\gamma,t,m}^{n:}(s, \cdot) = R_{\gamma,t}^{n:}(s, \cdot)$. Furthermore, the main results of the previous two sections, i.e. Proposition 4.2, Lemma 5.1, Proposition 5.3, as well as Proposition 5.4 hold true unchanged if $R_{\gamma,t}^{n:}$ and $Z_{\gamma,m}^{n:}$ are replaced by $R_{\gamma,t,m}^{n:}$ and $Z_{\gamma,m}^{n:}$. Indeed, the only property used in these proofs concerning the jump rate C_γ is the fact that it is bounded by 1, and this remains true for $C_{\gamma,m}$.

Theorem 6.1 (Convergence of Z_γ). *Let $\nu \in (0, 1/2)$ and $m > 1$. As γ tends to 0, the processes $Z_{\gamma,m}$ converge in law to Z with respect to the Skorokhod topology on $\mathcal{D}(\mathbb{R}_+, \mathcal{C}^{-\nu})$, where Z is defined in Proposition 3.1.*

Proof. As in equation (4.3) on \mathbb{T}^2 , we have that

$$Z_{\gamma,m}(t, x) = \int_0^t \Delta_\gamma Z_{\gamma,m}(s, x) ds + M_{\gamma,m}(t, x). \quad (6.5)$$

As explained above, Proposition 5.4 also applies to the family of processes $Z_{\gamma,m}$. Therefore, for any fixed $m > 1$ the family $(Z_{\gamma,m})_{\gamma \in (0, \frac{1}{3})}$ is tight for the Skorokhod topology on $\mathcal{D}(\mathbb{R}_+, \mathcal{C}^{-\nu})$. Let \bar{Z} be a sub-sequential limit of $Z_{\gamma,m}$ as γ tends to 0. It suffices to show that the law of \bar{Z} is that of Z . In order to see this, we appeal to the martingale characterization of this law, as recalled in Theorem D.1 below. By Proposition 5.4, \bar{Z} must take values in $\mathcal{C}(\mathbb{R}_+, \mathcal{C}^{-\nu})$.

Let $\phi \in \mathcal{C}^\infty(\mathbb{T}^2)$. We define

$$\mathcal{M}_{\gamma,\phi}(t) = (Z_{\gamma,m}(t), \phi) - \int_0^t (Z_{\gamma,m}(s), \Delta_\gamma \phi) ds,$$

where $(f, g) = \int f(x)g(x) dx$. More generally, when f is a distribution and g a smooth test function, we write (f, g) to denote the evaluation of f on g . As $(f, \Delta_\gamma g) = (\Delta_\gamma f, g)$, the process $\mathcal{M}_{\gamma,\phi}$ is a martingale.

Let us first see that for any $s \geq 0$,

$$|(Z_{\gamma,m}(s), \Delta_\gamma \phi) - (Z_{\gamma,m}(s), \Delta \phi)| \leq C\gamma^{2-2\nu} \|Z_{\gamma,m}(s)\|_{\mathcal{C}^{-\nu}}, \quad (6.6)$$

for a constant $C = C(\phi)$. Indeed, according to the definitions of Δ_γ and K_γ (see the discussion below (2.13) as well as (2.18))

$$\begin{aligned} \Delta_\gamma \phi(x) &= c_{\gamma,1} \frac{1}{\gamma^2} \sum_{z \in \gamma \mathbb{Z}^2} \gamma^2 \mathfrak{K}(z) \left(\phi(x + \frac{\varepsilon}{\gamma} z) - \phi(x) \right) \\ &= c_{\gamma,1} \frac{1}{\gamma^2} \sum_{z \in \gamma \mathbb{Z}^2} \gamma^2 \mathfrak{K}(z) \left(\sum_{j=1,2} \partial_j \phi(x) \frac{\varepsilon}{\gamma} z_j + \frac{1}{2} \sum_{j_1, j_2=1,2} \partial_{j_1} \partial_{j_2} \phi(x) \frac{\varepsilon^2}{\gamma^2} z_{j_1} z_{j_2} \right. \\ &\quad \left. + \frac{1}{6} \sum_{j_1, j_2, j_3=1,2} \partial_{j_1} \partial_{j_2} \partial_{j_3} \phi(x) \frac{\varepsilon^3}{\gamma^3} z_{j_1} z_{j_2} z_{j_3} \right) + \text{Err}. \end{aligned}$$

The error term Err is readily seen to be bounded by $C\gamma^2$ uniformly in x . Furthermore, by the symmetry of \mathfrak{K} , all of the sums involving odd powers of z_j vanish. The only remaining contribution is

$$\Delta \phi \frac{1}{2} c_{\gamma,1} \sum_{z \in \gamma \mathbb{Z}^2} \gamma^2 \mathfrak{K}(z) z_1^2,$$

where we used symmetry of the kernel again. The Riemann sum converges to the integral $\int \mathfrak{K}(z) z_1^2 dz = 2$ and the error is bounded by $C\gamma^2$ (see Remark 2.4). Therefore, we get

$$\| \Delta_\gamma \phi - \Delta \phi \|_{L^\infty} \leq C\gamma^2.$$

The estimate (6.6) thus follows from Lemma B.3 and the fact that the Fourier coefficients of Z_γ with frequency larger than $N \leq \gamma^{-2}$ vanish.

For any $L > 0$ and $z \in \mathcal{D}(\mathbb{R}_+, \mathcal{C}^{-\nu})$, we define

$$T_L(z) = \inf \{ t \geq 0 : \|z\|_{\mathcal{C}^{-\nu}} > L \}$$

and for any $t \geq 0$,

$$\mathcal{M}_{z,\phi}(t) = (z(t), \phi) - \int_0^t (z(s), \Delta \phi) ds.$$

The first condition from Theorem D.1 that needs to be checked is that $\mathcal{M}_{\bar{Z},\phi}$ is a local martingale. For some $s \geq 0$, we give ourselves a bounded continuous function $F : \mathcal{D}(\mathbb{R}_+, \mathcal{C}^{-\nu}) \rightarrow \mathbb{R}$ that is measurable with respect to the σ -algebra over $\mathcal{D}([0, s], \mathcal{C}^{-\nu})$, and we consider, for $t \geq s$,

$$\mathcal{G}_{L,t}(z) = (\mathcal{M}_{z,\phi}(t \wedge T_L(z)) - \mathcal{M}_{z,\phi}(s \wedge T_L(z))) F(z),$$

where we slightly abuse notation by writing $\mathcal{M}_{z,\phi}(t \wedge T_L(z))$ to denote the process that is equal to $\mathcal{M}_{z,\phi}(t)$ if $t < T_L(z)$, and is equal to the left limit of $\mathcal{M}_{z,\phi}$ at $T_L(z)$ otherwise. Let us define

$$\text{Loc} = \{ L > 0 : \mathbb{P} [\| \bar{Z} \|_{\mathcal{C}^{-\nu}} \text{ has a local maximum at height } L] > 0 \}.$$

Noting that

$$\text{Loc} \subseteq \bigcup_{\substack{n, m \in \mathbb{N} \\ s \in \mathbb{Q}_+}} \left\{ L > 0 : \mathbb{P} \left[\sup_{t: |t-s| \leq 1/m} \| \bar{Z}(t) \|_{\mathcal{C}^{-\nu}} = L \right] \geq \frac{1}{n} \right\},$$

we see that this set is countable. For $L \notin \text{Loc}$, the process \bar{Z} belongs a.s. to the set of continuity points of the mapping

$$\begin{cases} \mathcal{D}(\mathbb{R}_+, \mathcal{C}^{-\nu}) & \rightarrow \mathbb{R} \\ z & \mapsto T_L(z). \end{cases}$$

Similarly, \bar{Z} belongs a.s. to the set of continuity points of the mapping

$$\begin{cases} \mathcal{D}(\mathbb{R}_+, \mathcal{C}^{-\nu}) & \rightarrow \mathbb{R} \\ z & \mapsto \mathcal{G}_{L,t}(z). \end{cases}$$

By the continuous mapping theorem, $\mathcal{G}_{L,t}(Z_{\gamma,m})$ thus converges in law to $\mathcal{G}_{L,t}(\bar{Z})$ along the sub-sequence $\gamma \rightarrow 0$. Since $z \mapsto \mathcal{G}_{L,t}(z)$ is uniformly bounded, the expectations converge as well, i.e. $\mathbb{E}[\mathcal{G}_{L,t}(Z_{\gamma,m})]$ converges to $\mathbb{E}[\mathcal{G}_{L,t}(\bar{Z})]$ along the sub-sequence $\gamma \rightarrow 0$. Moreover, $\mathcal{M}_{Z_{\gamma,m},\phi}(t \wedge T_L(Z_{\gamma,m}))$ is very close to $\mathcal{M}_{\gamma,\phi}(t \wedge T_L(Z_{\gamma,m}))$, as shown by (6.6) (and where we use the same abuse of notation on $t \wedge T_L(Z_{\gamma,m})$). More precisely,

$$\begin{aligned} \mathbb{E} \left[\left| \mathcal{G}_{L,t}(Z_{\gamma,m}) - (\mathcal{M}_{\gamma,\phi}(t \wedge T_L(Z_{\gamma,m})) - \mathcal{M}_{\gamma,\phi}(s \wedge T_L(Z_{\gamma,m}))) F(Z_{\gamma,m}) \right| \right] \\ \leq C(\phi) t \gamma^{2-2\nu} L \|F\|_{L^\infty}. \end{aligned}$$

By the martingale property of $\mathcal{M}_{\gamma,\phi}(t)$, we have

$$\mathbb{E} \left[(\mathcal{M}_{\gamma,\phi}(t \wedge T_L(Z_{\gamma,m})) - \mathcal{M}_{\gamma,\phi}(s \wedge T_L(Z_{\gamma,m}))) F(Z_{\gamma,m}) \right] = 0,$$

so $\mathbb{E}[\mathcal{G}_{L,t}(Z_{\gamma,m})]$ tends to 0 as γ tends to 0. This implies that $\mathbb{E}[\mathcal{G}_{L,t}(\bar{Z})] = 0$, and thus that $(\mathcal{M}_{\bar{Z},\phi}(t \wedge T_L(\bar{Z})))_{t \geq 0}$ is a martingale. Since the set Loc is countable, we can choose a sequence $L_n \notin \text{Loc}$ that goes to infinity with n . For such a sequence, $T_{L_n}(\bar{Z})$ tends to infinity a.s. as n tends to infinity. We have thus proved that $\mathcal{M}_{\bar{Z},\phi}$ is a local martingale, which is what we wanted.

For $z \in \mathcal{D}(\mathbb{R}_+, \mathcal{C}^{-\nu})$, let

$$\Gamma_z(t) = (\mathcal{M}_{z,\phi}(t))^2 - 2t\|\phi\|_{L^2}^2.$$

In order to verify the assumptions of Theorem D.1, there remains to see that $\Gamma_{\bar{Z}}$ is a local martingale. The reasoning is similar, except that unlike the first part, the argument relies on the presence of the stopping time $\tau_{\gamma,m}$. Let us see how. Recalling $M_{\gamma,m}$ from (6.5), we have

$$\mathcal{M}_{\gamma,\phi}(t) = (M_{\gamma,m}(t), \phi),$$

which coincides with $(M_\gamma(t), \phi)$ if $t \leq \tau_{\gamma,m}$. If we assume further that ϕ is a trigonometric polynomial of degree $K < \infty$ (i.e. $\phi(x) = \frac{1}{4} \sum_{|\omega_j| \leq K} \hat{\phi}(\omega) e^{i\pi\omega \cdot x}$), then by (A.3), we have the identity

$$(M_{\gamma,m}(t), \phi) = \sum_{x \in \Lambda_\varepsilon} \varepsilon^2 M_{\gamma,m}(t, x) \phi(x)$$

for $\gamma^{-2} \geq K$. By (2.15), the predictable quadratic variation of $\mathcal{M}_{\gamma,\phi}$ is given by

$$\langle \mathcal{M}_{\gamma,\phi} \rangle_t = 4c_{\gamma,2}^2 \int_0^t \sum_{x,y,z \in \Lambda_\varepsilon} \varepsilon^6 \phi(x)\phi(y) K_\gamma(x-z) K_\gamma(y-z) C_{\gamma,m}(s, z) ds$$

with $C_{\gamma,m}$ as in (6.4). The central point is the observation that for $t < \tau_{\gamma,m}$,

$$\left| C_{\gamma,m}(t, z) - \frac{1}{2} \right| = \left| C_\gamma(t, z) - \frac{1}{2} \right| \leq C(\nu) \gamma^{1-2\nu} \|X_\gamma(t)\|_{\mathcal{C}^{-\nu}}, \quad (6.7)$$

whereas for $t > \tau_{\gamma,m}$, we have $C_{\gamma,m} = \frac{1}{2}$ by definition.

Indeed, to see (6.7) we note that by (2.7) and (2.12),

$$\left| C_\gamma(t, z) - \frac{1}{2} \right| \leq \beta |h_\gamma(\sigma(t/\alpha), z/\varepsilon)| = \beta \delta |X_\gamma(t, z)|.$$

Moreover, since the Fourier coefficients of X_γ with frequency larger than γ^{-2} vanish, we obtain from Lemma B.3 that

$$\|X_\gamma(t)\|_{L^\infty} \leq C \gamma^{-2\nu} \|X_\gamma(t)\|_{\mathcal{C}^{-\nu}}.$$

Recalling that $\delta = \gamma$ and $\beta \leq 2$ for γ sufficiently small, we obtain (6.7). From this, we deduce that the quadratic variation of $\mathcal{M}_{\gamma,\phi}$ at time t is close to

$$2\varepsilon^6 \int_0^t \sum_{x,y,z \in \Lambda_\varepsilon} \phi(x)\phi(y) K_\gamma(x-z) K_\gamma(y-z) ds$$

up to an error controlled by $\gamma^{1-2\nu}$. It is then clear that this tends to $2t\|\phi\|_{L^2}^2$ as γ tends to 0. We obtain that the martingale

$$\mathcal{M}_{\gamma,\phi}^2(t) - \langle \mathcal{M}_{\gamma,\phi} \rangle_t$$

is close to $\Gamma_{Z_{\gamma,m}}(t)$ up to an error that vanishes as γ tends to 0. We can now proceed as in the first part to obtain that $\Gamma_{\bar{Z}}$ is a local martingale. Recall that we have assumed that ϕ is a trigonometric polynomial. But by Remark D.4, this is sufficient to characterize the law of \bar{Z} , and the proof is complete. \square

We can now prove the convergence in law of the iterated integrals as well. As above, for the moment we can only prove this “before the stopping time $\tau_{\gamma, \mathfrak{m}}$ ”.

Recall the definitions of the processes $R_t^{n:}$ and $Z^{n:}$ in Section 3. Furthermore, recall that for $x \in \mathbb{T}^2$, the process $s \mapsto R_t(s, x) = R_t^{1:}(s, x)$, defined for $s < t$, is a continuous martingale with quadratic variation given by

$$\langle R_t(\cdot, x) \rangle_s = \frac{1}{2} \sum_{\omega \in \mathbb{Z}^2} \int_0^s \exp(2(t-r)\pi^2|\omega|^2) dr, \quad (6.8)$$

and that for $s < t$ and $x \in \mathbb{T}^2$, we have the exact identity

$$R_t^{n:}(s, x) = H_n(R_t(s, x), \langle R_t(\cdot, x) \rangle_s). \quad (6.9)$$

Theorem 6.2 (Convergence of $Z_\gamma^{n:}$). *For every $\mathfrak{m} \in \mathbb{N}$ and $n \in \mathbb{N}$, the processes $(Z_{\gamma, \mathfrak{m}}^{1:}, \dots, Z_{\gamma, \mathfrak{m}}^{n:})$ (defined as in the beginning of this section) converge (jointly) in law to $(Z^{1:}, \dots, Z^{n:})$ with respect to the topology of $\mathcal{D}(\mathbb{R}_+, \mathcal{C}^{-\nu})^n$.*

Proof. As explained at the beginning of this section, the results from Sections 4 and 5 remain true if $R_{\gamma, t}^{n:}$ and $Z_\gamma^{n:}$ are replaced by $R_{\gamma, t, \mathfrak{m}}^{n:}$ and $Z_{\gamma, \mathfrak{m}}^{n:}$. In particular, by Proposition 5.4, the family of processes $(Z_{\gamma, \mathfrak{m}}^{n:})_{\gamma \in (0, \frac{1}{3})}$ is tight with respect to the topology of $\mathcal{D}(\mathbb{R}_+, \mathcal{C}^{-\nu})$, for every given n . This implies immediately that for every n , the family of vectors $(Z_{\gamma, \mathfrak{m}}^{1:}, \dots, Z_{\gamma, \mathfrak{m}}^{n:})$, $\gamma \in (0, \frac{1}{3})$ is tight with respect to the topology of $\mathcal{D}(\mathbb{R}_+, \mathcal{C}^{-\nu})^n$. It remains to check the convergence of the finite dimensional distributions.

From now on, we use the shorthand notations $\mathbf{Z}_\gamma = (Z_{\gamma, \mathfrak{m}}^{1:}, \dots, Z_{\gamma, \mathfrak{m}}^{n:})$ and $\mathbf{Z} = (Z^{1:}, \dots, Z^{n:})$ to denote the random vectors of interest. It will be useful to also use $\mathbf{R}_{\gamma, t} = (R_{\gamma, t, \mathfrak{m}}^{1:}, \dots, R_{\gamma, t, \mathfrak{m}}^{n:})$ and $\mathbf{R}_t = (R_t^{1:}, \dots, R_t^{n:})$. We fix a $K \in \mathbb{N}$ and times $t_1 < t_2 < \dots < t_K$. Furthermore, let $F: (\mathcal{C}^{-\nu})^{n \times K} \rightarrow \mathbb{R}$ be bounded and uniformly continuous (with respect to the product metric on $(\mathcal{C}^{-\nu})^{n \times K}$). We need to show that

$$\lim_{\gamma \rightarrow 0} |\mathbb{E} F(\mathbf{Z}_\gamma(t_1), \dots, \mathbf{Z}_\gamma(t_K)) - \mathbb{E} F(\mathbf{Z}(t_1), \dots, \mathbf{Z}(t_K))| = 0.$$

To this end, fix $s_1 < t_1, \dots, s_K < t_K$ and write

$$\begin{aligned} & |\mathbb{E} F(\mathbf{Z}_\gamma(t_1), \dots, \mathbf{Z}_\gamma(t_K)) - \mathbb{E} F(\mathbf{Z}(t_1), \dots, \mathbf{Z}(t_K))| \\ & \leq \mathbb{E} |F(\mathbf{Z}_\gamma(t_1), \dots, \mathbf{Z}_\gamma(t_K)) - F(\mathbf{R}_{\gamma, t_1}(s_1), \dots, \mathbf{R}_{\gamma, t_K}(s_K))| \\ & \quad + |\mathbb{E} F(\mathbf{R}_{\gamma, t_1}(s_1), \dots, \mathbf{R}_{\gamma, t_K}(s_K)) - \mathbb{E} F(\mathbf{R}_t(s_1), \dots, \mathbf{R}_t(s_K))| \\ & \quad + \mathbb{E} |F(\mathbf{R}_t(s_1), \dots, \mathbf{R}_t(s_K)) - F(\mathbf{Z}(t_1), \dots, \mathbf{Z}(t_K))|. \end{aligned} \quad (6.10)$$

Recall the estimates (3.10) and (4.20) that allow us to control all moments of $\|\mathbf{Z}_\gamma(t_i) - \mathbf{R}_{\gamma, t_i}(s_i)\|_{(\mathcal{C}^{-\nu})^n}$ uniformly in γ . Since F is uniformly continuous, we can thus make the first and last terms on the right-hand side of (6.10) small uniformly in γ by choosing $|t_i - s_i|$ small enough. Therefore, in order to conclude, it suffices to show the convergence in law of the vector $(\mathbf{R}_{\gamma, t_1}(s_1), \dots, \mathbf{R}_{\gamma, t_K}(s_K))$ to $(\mathbf{R}_t(s_1), \dots, \mathbf{R}_t(s_K))$ for fixed values of $s_i < t_i$.

We will show the stronger statement that this convergence in law holds with respect to the topology of $(L^\infty)^{n \times K}$. By Proposition 5.3, it suffices to show that

$$H_\ell(R_{\gamma, t_i, \mathfrak{m}}(s_i, x), [R_{\gamma, t_i, \mathfrak{m}}(\cdot, x)]_{s_i}) \quad \ell = 1, \dots, n, \quad i = 1, \dots, K$$

converges in law to $(\mathbf{R}_t(s_1), \dots, \mathbf{R}_t(s_K))$ in L^∞ . By (6.9) and Lemma 5.1, it suffices to show the two convergences

$$(R_{\gamma, t_1, m}(s_1), \dots, R_{\gamma, t_K, m}(s_K)) \xrightarrow[\gamma \rightarrow 0]{(\text{law})} (R_{t_1}(s_1), \dots, R_{t_K}(s_K)), \quad (6.11)$$

and

$$(\langle R_{\gamma, t_1, m}(\cdot, \cdot) \rangle_{s_1}, \dots, \langle R_{\gamma, t_K, m}(\cdot, \cdot) \rangle_{s_K}) \xrightarrow[\gamma \rightarrow 0]{(\text{law})} (\langle R_{t_1}(\cdot, \cdot) \rangle_{s_1}, \dots, \langle R_{t_K}(\cdot, \cdot) \rangle_{s_K}), \quad (6.12)$$

both being understood for the $(L^\infty)^K$ topology. (The joint convergence in law of (6.11) and (6.12) would follow immediately since the right-hand side of (6.12) is deterministic.) As for (6.11), note that $R_{\gamma, t, m}(s) = P_{t-s}^\gamma Z_{\gamma, m}(s)$. We learn from Corollary 8.7 and Proposition 4.2 that for $i = 1, \dots, K$,

$$\|(P_{t_i-s_i}^\gamma - P_{t_i-s_i}) Z_{\gamma, m}(s_i)\|_{L^\infty} \xrightarrow[\gamma \rightarrow 0]{} 0,$$

almost surely. It thus suffices to check the convergence of $P_{t_i-s_i} Z_\gamma(s_i)$ to $R_{t_i}(s_i) = P_{t_i-s_i} Z(s_i)$. The estimate (8.22) ensures that the mapping

$$\begin{cases} C^{-\nu} & \rightarrow L^\infty \\ Z & \mapsto P_{t-s} Z \end{cases}$$

is continuous, so the convergence in (6.11) follows from Theorem 6.1 and the continuous mapping theorem.

Turning now to (6.12), we learn from (5.1) that for $x \in \Lambda_\varepsilon$,

$$\langle R_{\gamma, t_i, m}(\cdot, x) \rangle_{s_i} = 4c_{\gamma, 2}^2 \int_0^{s_i} \sum_{z \in \Lambda_\varepsilon} \varepsilon^2 (P_{t_i-r}^\gamma \star_\varepsilon K_\gamma)^2(z-x) C_{\gamma, m}(r, z) dr,$$

where $C_{\gamma, m}(s, z)$ satisfies (6.7). It is thus clear that up to an additive error that is bounded by $C\gamma^{1-2\nu}$ uniformly over $x \in \Lambda_\varepsilon$, the quadratic variation $\langle R_{\gamma, t_i, m}(\cdot, x) \rangle_{s_i}$ is given by

$$2 \int_0^{s_i} \sum_{z \in \Lambda_\varepsilon} \varepsilon^2 (P_{t_i-r}^\gamma \star_\varepsilon K_\gamma)^2(z-x) dr$$

(we also used the bound (2.17) which implies that $|c_{\gamma, 2}^2 - 1| \leq C\gamma^2$ and the calculation (5.5) – (5.6) which yields a logarithmic bound on the sum). The latter quantity is equal to

$$\frac{1}{2} \int_0^{s_i} \sum_{\omega \in \{-N, \dots, N\}^2} \exp(-2(t_i - r)\gamma^{-2}(1 - \hat{K}_\gamma(\omega))) |\hat{K}_\gamma(\omega)|^2 dr. \quad (6.13)$$

By (8.4) and (8.13), for any fixed $|\omega| \leq \gamma^{-1}$, we get

$$\begin{aligned} & \left| \exp(-2(t_i - r)\gamma^{-2}(1 - \hat{K}_\gamma(\omega))) |\hat{K}_\gamma(\omega)|^2 - \exp(-2(t_i - r)\pi^2|\omega|^2) \right| \\ & \leq C \exp\left(-2(t_i - r)\frac{|\omega|^2}{C_1}\right) ((t_i - r)\gamma|\omega|^3 + \gamma^2|\omega|^2) \end{aligned} \quad (6.14)$$

From the elementary bound

$$\int_0^s e^{-(t-r)\ell} dr \leq s e^{-(t-s)\ell},$$

we can conclude that after integrating the bound (6.14) over $[0, s_i]$ and summing over $|\omega| \leq \gamma^{-1}$, we obtain a quantity that is bounded by $C(s_i, t_i)\gamma$. In the same

way (using (8.13) once more), we obtain that the sum arising in (6.13) restricted to indices $|\omega| > \gamma^{-1}$ is smaller than

$$\leq s_i \sum_{\substack{\omega \in \{-N, \dots, N\}^2 \\ |\omega| > \gamma^{-1}}} \exp\left(-2(t_i - s_i) \frac{1}{C_1 \gamma^2}\right) \leq C s_i \gamma^{-2} \exp\left(-2(t_i - s_i) \frac{1}{C_1 \gamma^2}\right),$$

and a similar bound holds true for the limiting quantity $\exp(-2(t_i - r)\pi^2|\omega|^2)$. Hence we can conclude that uniformly over $x \in \Lambda_\varepsilon$ we get the deterministic bound

$$|\langle R_{\gamma, t_i, m}(\cdot, x) \rangle_{s_i} - \langle R_{t_i}(\cdot, x) \rangle_{s_i}| \leq C(s_i, t_i) \gamma^{1-2\nu}.$$

It only remains to refer to Lemma B.6 to conclude that the convergence (6.12) also holds for the extensions to arbitrary $x \in \mathbb{T}^2$. \square

7. ANALYSIS OF THE NON-LINEAR EQUATION

In this section, we summarise the calculations of the previous sections and prove our main result, Theorem 2.1. Throughout this section, $\nu > 0$ will be assumed to be small enough and fixed. We furthermore fix $X^0 \in \mathcal{C}^{-\nu}$. We denote by $X \in C(\mathbb{R}_+, \mathcal{C}^{-\nu})$ the solution of the renormalised limiting evolution equation with initial datum X^0 as constructed in Section 3. Throughout the section, we make use of the bounds on K_γ and P_t^γ collected in Section 8.

Recall that the rescaled field $X_\gamma(t, x)$ was defined in (2.12). Recall furthermore that it satisfies the evolution equation (2.13), or equivalently, its mild form (4.1). For the reader's convenience, we repeat here that (4.1) states

$$\begin{aligned} X_\gamma(t, \cdot) = & P_t^\gamma X_\gamma^0 + \int_0^t P_{t-s}^\gamma K_\gamma \star_\varepsilon \left(-\frac{\beta^3}{3} X_\gamma^3(s, \cdot) + (\mathfrak{c}_\gamma + A) X_\gamma(s, \cdot) \right. \\ & \left. + E_\gamma(s, \cdot) \right) ds + Z_\gamma(t, \cdot) \quad \text{on } \Lambda_\varepsilon. \end{aligned} \quad (7.1)$$

This equation is only valid on the grid points $x \in \Lambda_\varepsilon$ because the extension by trigonometric polynomials does not commute with cubing X_γ . Therefore, our first step consists of deriving an equation that holds for all $x \in \mathbb{T}^2$.

The problem is only caused by large ω in Fourier space. Indeed, as long as $|\omega_i| < \frac{N}{3}$, we have

$$\text{Ext}[(e^{i\pi\omega \cdot x})^3] = [\text{Ext } e^{i\pi\omega \cdot x}]^3, \quad (7.2)$$

where as before, we have used Ext to denote the extension operator for a function $\Lambda_\varepsilon \rightarrow \mathbb{R}$ to a function $\mathbb{T}^2 \rightarrow \mathbb{R}$ by trigonometric polynomials of degree $\leq N$. In order to control the error caused by the extension, we will use the notation introduced above Lemma 4.6 and write

$$X_\gamma^{\text{high}} = \sum_{2^k \geq \frac{\gamma^{-2}}{10}} \delta_k X_\gamma \quad \text{and} \quad X_\gamma^{\text{low}} = \sum_{2^k < \frac{\gamma^{-2}}{10}} \delta_k X_\gamma, \quad (7.3)$$

so that $X_\gamma = X_\gamma^{\text{high}} + X_\gamma^{\text{low}}$. It is convenient to already collect some error terms in the first lemma, and to this end we discuss some notation. For $x \in \Lambda_\varepsilon$, we set $\mathfrak{c}_\gamma(s, x) = [R_{\gamma, s}(\cdot, x)]_s$, and we extend this to all $x \in \mathbb{T}^2$ as a trigonometric polynomial. Recall that in (3.15) we had defined

$$A(s) := A - \frac{s}{2} + \sum_{\omega \in \mathbb{Z}^2 \setminus \{0\}} \frac{e^{-2s\pi^2|\omega|^2}}{4\pi^2|\omega|^2}.$$

Finally, recall that $Q_{\gamma,s}$ was defined in Lemma 5.1.

With this notation at hand, we are ready to state the following result.

Lemma 7.1. *For every $t \geq 0$, we have on \mathbb{T}^2*

$$\begin{aligned} X_\gamma(t, \cdot) = & P_t^\gamma X_\gamma^0 + \int_0^t P_{t-s}^\gamma K_\gamma \star \left(-\frac{1}{3} X_\gamma^3(s, \cdot) + (\mathbf{c}_\gamma(s, \cdot) + A(s)) X_\gamma(s, \cdot) \right. \\ & \left. + \text{Err}^{(1)}(s, \cdot) \right) ds + Z_\gamma(t, \cdot). \end{aligned} \quad (7.4)$$

For every $T > 0$ and $\kappa > 0$, there exists $C = C(T, \kappa, \nu)$ such that the error term satisfies for all $0 \leq s \leq T$ and $0 < \gamma < \gamma_0$

$$\begin{aligned} \|\text{Err}^{(1)}(s, \cdot)\|_{L^\infty(\mathbb{T}^2)} \leq & C \gamma^{-10\nu - \kappa} (\|X_\gamma(s, \cdot)\|_{C^{-\nu}}^5 + 1) \\ & \times \left(\gamma^{\frac{2}{3}} s^{-\frac{1}{3}} + \|X_\gamma^{\text{high}}(s, \cdot)\|_{L^\infty(\mathbb{T}^2)} + \|Q_{\gamma,s}(s, \cdot)\|_{L^\infty(\Lambda_\varepsilon)} \right). \end{aligned} \quad (7.5)$$

Proof. We get from (7.1) that

$$\text{Err}^{(1)} = \text{err}^{(1)} + \text{err}^{(2)} + \text{err}^{(3)},$$

where

$$\begin{aligned} \text{err}^{(1)}(s, \cdot) &= E_\gamma(s, \cdot) - \frac{1}{3}(\beta^3 - 1)\text{Ext}(X_\gamma^3(s, \cdot)), \\ \text{err}^{(2)}(s, \cdot) &= -\frac{1}{3} \left(\text{Ext}(X_\gamma^3(s, \cdot)) - (\text{Ext} X_\gamma(s, \cdot))^3 \right), \\ \text{err}^{(3)}(s, \cdot) &= (\mathbf{c}_\gamma + A - \mathbf{c}_\gamma(s, \cdot) - A(s)) X_\gamma(s, \cdot). \end{aligned}$$

The first term is easily bounded using the definition of E_γ in (2.14) and the assumption (2.19) and (2.20) on β . On Λ_ε we get

$$\|\text{err}^{(1)}(s, \cdot)\|_{L^\infty(\Lambda_\varepsilon)} \leq C \gamma^2 \|X_\gamma(s, \cdot)\|_{L^\infty(\Lambda_\varepsilon)}^5 + C \gamma^2 \log(\gamma^{-1}) \|X_\gamma(s, \cdot)\|_{L^\infty(\Lambda_\varepsilon)}^3.$$

Using Lemma B.6 this bound can easily be extended to $x \in \mathbb{T}^2$ at the expense of loosing an arbitrarily small power of γ . Furthermore, as we have already seen at several places, we can use Lemma B.3 and the fact that $\hat{X}_\gamma(\omega) = 0$ for $\omega \notin \{-N, \dots, N\}^2$ to bound the L^∞ norm by the $C^{-\nu}$ norm. In this way we get for any arbitrary small $\kappa > 0$

$$\|\text{err}^{(1)}(s, \cdot)\|_{L^\infty(\mathbb{T}^2)} \leq C(\kappa, \nu) \gamma^{2-\kappa-10\nu} (\|X_\gamma(s, \cdot)\|_{C^{-\nu}}^5 + 1).$$

For the second term, we get on \mathbb{T}^2

$$\begin{aligned} & \text{Ext}(X_\gamma^3) - (\text{Ext} X_\gamma)^3 \\ &= \left[\text{Ext}((X_\gamma^{\text{low}})^3) - (\text{Ext} X_\gamma^{\text{low}})^3 \right] \\ & \quad + \text{Ext} \left[X_\gamma^{\text{high}} \left(3(X_\gamma^{\text{low}})^2 + 3X_\gamma^{\text{low}} X_\gamma^{\text{high}} + (X_\gamma^{\text{high}})^2 \right) \right] \\ & \quad - \left[\text{Ext} X_\gamma^{\text{high}} \left(3(\text{Ext} X_\gamma^{\text{low}})^2 + 3\text{Ext} X_\gamma^{\text{low}} \text{Ext} X_\gamma^{\text{high}} + (\text{Ext} X_\gamma^{\text{high}})^2 \right) \right]. \end{aligned}$$

According to (7.2), the term $\left[\text{Ext}((X_\gamma^{\text{low}})^3) - (\text{Ext} X_\gamma^{\text{low}})^3 \right]$ vanishes. Using Lemma B.6, the remaining terms can easily be bounded, so that we obtain

$$\begin{aligned} \|\text{err}^{(2)}(s, \cdot)\|_{L^\infty(\mathbb{T}^2)} \leq & C(\kappa) \gamma^{-\kappa} \|X_\gamma^{\text{high}}(s, \cdot)\|_{L^\infty(\mathbb{T}^2)} \\ & \times \left(\|X_\gamma^{\text{low}}(s, \cdot)\|_{L^\infty(\mathbb{T}^2)}^2 + \|X_\gamma^{\text{high}}(s, \cdot)\|_{L^\infty(\mathbb{T}^2)}^2 \right). \end{aligned} \quad (7.6)$$

Finally, we bound the terms appearing in the bracket on the right-hand side of (7.6) by

$$\begin{aligned} \|X_\gamma^{\text{low}}(s, \cdot)\|_{L^\infty(\mathbb{T}^2)} &\leq \sum_{2^k < \frac{\gamma^{-2}}{10}} \|\delta_k X_\gamma(s, \cdot)\|_{L^\infty(\mathbb{T}^2)} \leq \sum_{2^k < \frac{\gamma^{-2}}{10}} 2^{-k\nu} \|X_\gamma(s, \cdot)\|_{C^{-\nu}} \\ &\leq C(\nu) \gamma^{-2\nu} \|X_\gamma(s, \cdot)\|_{C^{-\nu}}, \end{aligned}$$

and in the same way

$$\|X_\gamma^{\text{high}}(s, \cdot)\|_{L^\infty(\mathbb{T}^2)} \leq \sum_{2^k \geq \frac{\gamma^{-2}}{10}} \|\delta_k X_\gamma(s, \cdot)\|_{L^\infty(\mathbb{T}^2)} \leq C \gamma^{-2\nu} \|X_\gamma(s, \cdot)\|_{C^{-\nu}}.$$

Note that the number of terms appearing in the last sum is uniformly bounded in γ . We will see below that this last bound does not capture the true behaviour of X_γ^{high} , but it is sufficient for this particular part of the estimate (7.6).

In order to bound $\text{err}^{(3)}$, we recall that the precise value of \mathfrak{c}_γ was defined in (2.20). Then we get for $x \in \mathbb{T}^2$

$$\begin{aligned} &\mathfrak{c}_\gamma + A - \mathfrak{c}_\gamma(s, x) - A(s) \\ &= \sum_{\substack{\omega \in \{-N, \dots, N\}^2 \\ \omega \neq 0}} \frac{|\hat{K}_\gamma(\omega)|^2}{4\gamma^{-2}(1 - \hat{K}_\gamma(\omega))} - [R_{\gamma, s}(\cdot, x)]_s + \frac{s}{2} - \sum_{\substack{\omega \in \mathbb{Z}^2 \\ \omega \neq 0}} \frac{\exp(-2s\pi^2|\omega|^2)}{4\pi^2|\omega|^2}. \end{aligned}$$

Recall from (5.2) that for $x \in \Lambda_\varepsilon$, $[R_{\gamma, r}(\cdot, x)]_r = \langle R_{\gamma, r}(\cdot, x) \rangle_r + Q_{\gamma, r}(s, x)$. Furthermore, according to (5.1) we get for $x \in \Lambda_\varepsilon$

$$\begin{aligned} &\langle R_{\gamma, s}(\cdot, x) \rangle_s \\ &= 4c_{\gamma, 2}^2 \int_0^s \sum_{z \in \Lambda_\varepsilon} \varepsilon^2 (P_{s-r}^\gamma \star_\varepsilon K_\gamma)^2(x-z) C_\gamma(r, z) dr \\ &= 2 \int_0^s \sum_{z \in \Lambda_\varepsilon} \varepsilon^2 (P_{s-r}^\gamma \star_\varepsilon K_\gamma)^2(x-z) dr + \text{err}^{(4)}(s, x) \\ &= \frac{1}{2} \int_0^s \sum_{\omega \in \{-N, \dots, N\}^2} \exp\left(-\frac{2r}{\gamma^2}(1 - \hat{K}_\gamma(\omega))\right) |\hat{K}_\gamma(\omega)|^2 dr + \text{err}^{(4)}(s, x) \\ &= \frac{s}{2} + \sum_{\substack{\omega \in \{-N, \dots, N\}^2 \\ \omega \neq 0}} \frac{|\hat{K}_\gamma(\omega)|^2}{4\gamma^{-2}(1 - \hat{K}_\gamma(\omega))} \\ &\quad \times \left[1 - \exp\left(-\frac{2s}{\gamma^2}(1 - \hat{K}_\gamma(\omega))\right) \right] + \text{err}^{(4)}(s, x), \end{aligned}$$

where for any $\kappa > 0$

$$\begin{aligned} \text{err}^{(4)}(s, x) &= 4(c_{\gamma, 2}^2 - 1) \int_0^s \sum_{z \in \Lambda_\varepsilon} \varepsilon^2 (P_{s-r}^\gamma \star_\varepsilon K_\gamma)^2(x-z) C_\gamma(r, z) dr \\ &\quad + 4 \int_0^s \sum_{z \in \Lambda_\varepsilon} \varepsilon^2 (P_{s-r}^\gamma \star_\varepsilon K_\gamma)^2(x-z) \left(C_\gamma(r, z) - \frac{1}{2}\right) dr \\ &\leq C(T, \kappa) \gamma^{2-\kappa} + C(T, \kappa, \nu) \gamma^{1-2\nu-\kappa} \|X_\gamma(t)\|_{C^{-\nu}}, \end{aligned}$$

using (2.17) and (6.7) as well as the fact that $\int_0^s \sum_{z \in \Lambda_\varepsilon} \varepsilon^2 (P_{s-r}^\gamma \star_\varepsilon K_\gamma)^2(x-z) dr$ is bounded by $C(T) \log(\gamma^{-1}) \leq C(T, \kappa) \gamma^{-\kappa}$.

Summarising these calculations, we get

$$\begin{aligned} & \mathbf{c}_\gamma + A - \mathbf{c}_\gamma(s, \cdot) - A(s) + Q_{\gamma,s}(s, x) + \text{err}^{(4)}(s, x) \\ &= \sum_{\substack{\omega \in \{-N, \dots, N\}^2 \\ \omega \neq 0}} \frac{|\hat{K}_\gamma(\omega)|^2}{4\gamma^{-2}(1 - \hat{K}_\gamma(\omega))} \exp\left(-\frac{2s}{\gamma^2}(1 - \hat{K}_\gamma(\omega))\right) \\ & \quad - \sum_{\substack{\omega \in \mathbb{Z}^2 \\ \omega \neq 0}} \frac{\exp(-2s\pi^2|\omega|^2)}{4\pi^2|\omega|^2}. \end{aligned} \quad (7.7)$$

We bound the difference between the sums over $|\omega| < \gamma^{-1}$ term by term. We write

$$\begin{aligned} & \left| \frac{|\hat{K}_\gamma(\omega)|^2}{4\gamma^{-2}(1 - \hat{K}_\gamma(\omega))} \exp\left(-\frac{2s}{\gamma^2}(1 - \hat{K}_\gamma(\omega))\right) - \frac{\exp(-2s\pi^2|\omega|^2)}{4\pi^2|\omega|^2} \right| \\ & \leq \exp(-2s\pi^2|\omega|^2) \left| \frac{|\hat{K}_\gamma(\omega)|^2}{4\gamma^{-2}(1 - \hat{K}_\gamma(\omega))} - \frac{1}{4\pi^2|\omega|^2} \right| \\ & \quad + \frac{|\hat{K}_\gamma(\omega)|^2}{4\gamma^{-2}(1 - \hat{K}_\gamma(\omega))} \left| \exp\left(-\frac{2s}{\gamma^2}(1 - \hat{K}_\gamma(\omega))\right) - \exp(-2s\pi^2|\omega|^2) \right|. \end{aligned} \quad (7.8)$$

Using Lemma 8.1, we can bound the expression on the right-hand side of (7.8) for $|\omega| < \gamma^{-1}$ by

$$C e^{-2s\pi^2|\omega|^2} \frac{\gamma}{|\omega|} + C e^{-\frac{s|\omega|^2}{C_1}} \frac{1}{|\omega|^2} s \gamma |\omega|^3 \leq C \gamma \frac{1}{|\omega|^{1+\frac{2}{3}}} s^{-\frac{1}{3}}.$$

So, summing over $0 < |\omega| < \gamma^{-1}$ gives a contribution which is bounded by $C\gamma^{\frac{2}{3}}s^{-\frac{1}{3}}$. Of course, the choice of the exponents here is essentially arbitrary – we could have obtained any bound of the type $\gamma^\lambda s^{-\frac{\lambda}{2}}$ for $0 \leq \lambda < 1$ – but this choice is convenient.

For $|\omega| \geq \gamma^{-1}$, the terms in the sum involving \hat{K}_γ are not a good approximation to the corresponding limiting terms, so we simply bound each sum separately. For the first one, using (8.10) and (8.13),

$$\begin{aligned} & \sum_{\substack{\omega \in \{-N, \dots, N\}^2 \\ |\omega| \geq \gamma^{-1}}} \frac{|\hat{K}_\gamma(\omega)|^2}{4\gamma^{-2}(1 - \hat{K}_\gamma(\omega))} \exp\left(-\frac{2s}{\gamma^2}(1 - \hat{K}_\gamma(\omega))\right) \\ & \leq C \sum_{\substack{\omega \in \{-N, \dots, N\}^2 \\ |\omega| \geq \gamma^{-1}}} \frac{\gamma^2}{|\gamma\omega|^4} \exp\left(-\frac{2s}{C_1\gamma^2}\right) \leq C\gamma^{\frac{2}{3}}s^{-\frac{1}{3}} \sum_{\substack{\omega \in \{-N, \dots, N\}^2 \\ |\omega| \geq \gamma^{-1}}} \frac{1}{|\omega|^2} \leq C\gamma^{\frac{2}{3}}s^{-\frac{1}{3}}, \end{aligned}$$

while for the second one,

$$\sum_{\substack{\omega \in \mathbb{Z}^2 \\ \omega \geq \gamma^{-1}}} \frac{\exp(-2s\pi^2|\omega|^2)}{4\pi^2|\omega|^2} \leq C(\kappa) s^{-\frac{1}{3}} \gamma^{\frac{2}{3}-\kappa}.$$

Summarising our calculations, we get the desired estimate on $\text{Err}^{(1)}$. \square

For $m \geq 1$, recall the definition of the stopping time $\tau_{\gamma,m}$ in (6.1) and the definition of the processes $Z_{\gamma,m}$ (see the discussion following (6.3)). It will be useful to

introduce an auxiliary process $X_{\gamma,m}(t, x)$, $t \geq 0$, $x \in \mathbb{T}^2$ as the solution of (7.4) with Z_γ replaced by $Z_{\gamma,m}$ and $\text{Err}^{(1)}$ replaced by

$$\text{Err}_m^{(1)}(s) := \begin{cases} \text{Err}^{(1)}(s) & \text{if } s \leq \tau_{\gamma,m}, \\ 0 & \text{else.} \end{cases} \quad (7.9)$$

The existence and uniqueness of this process for fixed $\gamma > 0$ follows by an elementary ODE argument which we omit. Of course, $X_{\gamma,m}(t, \cdot)$ and $X_\gamma(t, \cdot)$ coincide on the event $\{t < \tau_{\gamma,m}\}$. Note that according to (7.5), by choosing $\nu > 0$ and $\kappa > 0$ small enough, for $s \leq \tau_{\gamma,m}$ we have the deterministic bound

$$\begin{aligned} & \|\text{Err}_m^{(1)}(s, \cdot)\|_{L^\infty(\mathbb{T}^2)} \\ & \leq C(T, m) \left(\gamma^{\frac{1}{2}} s^{-\frac{1}{3}} + \gamma^{-\frac{1}{6}} \|X_\gamma^{\text{high}}(s, \cdot)\|_{L^\infty(\mathbb{T}^2)} + \gamma^{-\frac{1}{6}} \|Q_{\gamma,s}(s, \cdot)\|_{L^\infty(\Lambda_\varepsilon)} \right). \end{aligned} \quad (7.10)$$

We will require yet another process that approximates X_γ . For any $0 \leq t \leq T$, we define

$$\bar{X}_{\gamma,m}(t, \cdot) = P_t X^0 + Z_{\gamma,m}(t, \cdot) + \mathcal{S}_T(Z_{\gamma,m}, Z_{\gamma,m}^{:2:}, Z_{\gamma,m}^{:3:})(t, \cdot),$$

where \mathcal{S}_T is the solution operator to the continuous problem, defined in Theorem 3.2. Recall that the definition of the operator \mathcal{S}_T involves the choice of initial datum X^0 , which we keep fixed. The following lemma is an immediate consequence of the main results of Sections 3 and 6. (The initial datum of the limiting evolution X is $X^0 \in \mathcal{C}^{-\nu}$.)

Lemma 7.2. *For any $T > 0$, let $F: \mathcal{D}([0, T], \mathcal{C}^{-\nu}) \rightarrow \mathbb{R}$ be uniformly continuous and bounded. We have*

$$\lim_{\gamma \rightarrow 0} \left| \mathbb{E}[F(\bar{X}_{\gamma,m})] - \mathbb{E}[F(X)] \right| = 0.$$

Proof. For $t \in [0, T]$, we can write

$$X(t, \cdot) = P_t X^0(\cdot) + Z(t, \cdot) + \mathcal{S}_T(Z, Z^{:2:}, Z^{:3:})(t, \cdot).$$

In Theorem 6.2, it was shown that the triple $(Z_{\gamma,m}, Z_{\gamma,m}^{:2:}, Z_{\gamma,m}^{:3:})$ converges in law with respect to the topology of $\mathcal{D}([0, T], \mathcal{C}^{-\nu})^3$ to the triple $(Z, Z^{:2:}, Z^{:3:})$. As the limit is continuous, we can conclude that the convergence also holds with respect to the metric of $L^\infty([0, T], \mathcal{C}^{-\nu})^3$.

On the other hand, Theorem 3.2 ensures that the operator \mathcal{S}_T is uniformly continuous on bounded sets from $L^\infty([0, T], \mathcal{C}^{-\nu})^3$ to, say, $\mathcal{C}([0, T], \mathcal{C}^1(\mathbb{T}^2))$. Note that the evolution $Y(t) = P_t X^0$ is in $\mathcal{C}([0, T], \mathcal{C}^{-\nu})$ by properties of the heat semigroup P_t . In particular, the mapping that sends $(Z, Z^{:2:}, Z^{:3:})$ to

$$P_t X^0 + Z(t, \cdot) + \mathcal{S}_T(Z, Z^{:2:}, Z^{:3:})(t, \cdot)$$

is continuous from $L^\infty([0, T], \mathcal{C}^{-\nu})^3$ to $\mathcal{D}([0, T], \mathcal{C}^{-\nu})$. This implies the desired result. \square

It remains to bound the difference between $\bar{X}_{\gamma,m}$ and X_γ (or $X_{\gamma,m}$). Note that these two processes are naturally defined on the same probability space, so that we can derive almost sure error bounds. For most of the remainder of this section we

collect bounds to show that for every fixed $m \geq 1$, $T > 0$ and bounded uniformly continuous function $F: \mathcal{D}([0, T], \mathcal{C}^{-\nu}) \rightarrow \mathbb{R}$, we have

$$\limsup_{\gamma \rightarrow 0} \mathbb{E} |F(\overline{X}_{\gamma, m}) - F(X_{\gamma, m})| = 0. \quad (7.11)$$

Once (7.11) is established, it can be combined with Lemma 7.2, implying the convergence in law of $X_{\gamma, m}$ to X for every fixed $m \geq 1$. The stopping time $\tau_{\gamma, m}$ can then be removed by a soft argument that will be explained in the proof of our main result, Theorem 2.1, at the end of the section.

We start by treating the initial datum. We set $Y_\gamma(t, \cdot) = P_t^\gamma X_\gamma^0$ on \mathbb{T}^2 . Lemma 8.4 and the uniform bound on $\|X_\gamma^0\|_{\mathcal{C}^{-\nu}}$ imply that for $\kappa > 0$ and $\beta > 0$, we have

$$\|Y_\gamma(t, \cdot)\|_{\mathcal{C}^{\beta-\kappa}} t^{\nu+\beta} \leq C(T, \beta, \kappa), \quad (7.12)$$

uniformly in γ and in $t \in [0, T]$. This is slightly worse than the bound available for the continuous equation, where we have uniform control on $\|Y(t, \cdot)\|_{\mathcal{C}^\beta} t^{\frac{\nu+\beta}{2}}$ for all values of $\nu, \beta > 0$. The bound is weaker due to the bad behaviour of P_t^γ on high Fourier modes observed in Lemma 8.4, but sufficient for our needs.

We set

$$\begin{aligned} v_{\gamma, m}(t, x) &:= X_{\gamma, m}(t, x) - Z_{\gamma, m}(t, x) - Y_\gamma(t, x) & t \geq 0, x \in \mathbb{T}^2, \\ \bar{v}_{\gamma, m}(t, x) &:= \overline{X}_{\gamma, m}(t, x) - Z_{\gamma, m}(t, x) - Y(t, x) & t \geq 0, x \in \mathbb{T}^2. \end{aligned} \quad (7.13)$$

So the desired bound (7.11) follows as soon as we have established bounds on $\sup_{0 \leq t \leq T} \|Y(t, \cdot) - Y_\gamma(t, \cdot)\|_{\mathcal{C}^{-\nu}}$ as well as $\sup_{0 \leq t \leq T} \|\bar{v}_{\gamma, m}(t, \cdot) - v_{\gamma, m}(t, \cdot)\|_{\mathcal{C}^{\frac{1}{2}}}$ for fixed T and m .

The first of these bounds is established in the following Lemma.

Lemma 7.3. *For every $T > 0$ we have*

$$\sup_{0 \leq t \leq T} \|Y(t, \cdot) - Y_\gamma(t, \cdot)\|_{\mathcal{C}^{-\nu}} \rightarrow 0 \quad (7.14)$$

Proof. For any $t \geq 0$ we can write

$$\|Y(t, \cdot) - Y_\gamma(t, \cdot)\|_{\mathcal{C}^{-\nu}} \leq \|P_t(X^0 - X_\gamma^0)\|_{\mathcal{C}^{-\nu}} + \|(P_t - P_t^\gamma)X_\gamma^0\|_{\mathcal{C}^{-\nu}}. \quad (7.15)$$

The operator P_t is bounded on $\mathcal{C}^{-\nu}$ uniformly in t , so that the first term is bounded by $C\|X^0 - X_\gamma^0\|_{\mathcal{C}^{-\nu}} \rightarrow 0$. For the second term, we use the assumption that states that the X_γ^0 are uniformly bounded in $\mathcal{C}^{-\nu+\kappa}$ for some small $\kappa > 0$, as well as the fact that $\|P_t^\gamma - P_t\|_{\mathcal{C}^{-\nu+\kappa} \rightarrow \mathcal{C}^{-\nu}} \leq C(T, \nu, \kappa)\gamma^{\frac{\kappa}{2}}$ according to (8.34). So (7.14) is established. \square

We now turn to the terms $v_{\gamma, m}$ and $\bar{v}_{\gamma, m}$. As in Section 3, we have to include the initial conditions in the renormalised powers of $Z_{\gamma, m}$. We define

$$\begin{aligned} \tilde{Z}_{\gamma, m}(t, \cdot) &= Y_\gamma(t, \cdot) + Z_{\gamma, m}(t, \cdot), \\ \tilde{Z}_{\gamma, m}^{:2:}(t, \cdot) &:= Z_{\gamma, m}^{:2:}(t, \cdot) + 2Y_\gamma(t, \cdot)Z_{\gamma, m}(t, \cdot) + Y_\gamma(t, \cdot)^2, \\ \tilde{Z}_{\gamma, m}^{:3:}(t, \cdot) &:= Z_{\gamma, m}^{:3:}(t, \cdot) + 3Y_\gamma(t, \cdot)Z_{\gamma, m}^{:2:}(t, \cdot) + 3Y_\gamma(t, \cdot)^2Z_{\gamma, m}(t, \cdot) + Y_\gamma(t, \cdot)^3. \end{aligned} \quad (7.16)$$

Furthermore, we define $\bar{Z}_{\gamma, m}$, $\bar{Z}_{\gamma, m}^{:2:}$ and $\bar{Z}_{\gamma, m}^{:3:}$ in exactly the same way as we defined $\tilde{Z}_{\gamma, m}$, $\tilde{Z}_{\gamma, m}^{:2:}$, $\tilde{Z}_{\gamma, m}^{:3:}$ in (7.16), with the only difference that all occurrences of $Y_\gamma(t, \cdot)$ are replaced by $Y(t, \cdot) = P_t X^0$.

With these notations in place, we can observe that $\bar{v}_{\gamma,m} = \mathcal{S}_T(Z_{\gamma,m}, Z_{\gamma,m}^{2:}, Z_{\gamma,m}^{3:})$ satisfies

$$\bar{v}_{\gamma,m}(t, \cdot) = - \int_0^t P_{t-s} \bar{\Psi}_{\gamma,m}(s) ds, \quad (7.17)$$

where we have set

$$\begin{aligned} \bar{\Psi}_{\gamma,m}(s) := & \frac{1}{3} \left(\bar{v}_{\gamma,m}^3(s) + 3\bar{Z}_{\gamma,m}(s) \bar{v}_{\gamma,m}^2(s) + 3\bar{Z}_{\gamma,m}^{2:}(s) \bar{v}_{\gamma,m}(s) + \bar{Z}_{\gamma,m}^{3:}(s) \right) \\ & - A(s) \left(\bar{v}_{\gamma,m}(s) + \bar{Z}_{\gamma,m}^{1:}(s) \right). \end{aligned} \quad (7.18)$$

In the next lemma, we establish a similar expression for $v_{\gamma,m}$. To this end, recall the definition of $E_{\gamma,t}^{n:}$ in (5.9). We denote by $E_{\gamma,t,m}^{n:}$ the corresponding quantity with $R_{\gamma,t}, R_{\gamma,t}^{n:}$ replaced by $R_{\gamma,t,m}, R_{\gamma,t,m}^{n:}$. As discussed above, Proposition 5.3 holds as well (in fact, with the same constant) with $E_{\gamma,t}^{n:}$ replaced by $E_{\gamma,t,m}^{n:}$.

Lemma 7.4. *For all $\gamma > 0$ and $t \geq 0$ we have on \mathbb{T}^2*

$$v_{\gamma,m}(t, \cdot) = - \int_0^t P_{t-s}^\gamma K_\gamma \star \left(\Psi_{\gamma,m}(s) + \text{Err}_m^{(1)} + \text{Err}_m^{(2)}(s, \cdot) \right) ds, \quad (7.19)$$

where

$$\begin{aligned} \Psi_{\gamma,m}(s) := & \frac{1}{3} \left(v_{\gamma,m}^3(s) + 3\tilde{Z}_{\gamma,m}(s) v_{\gamma,m}^2(s) + 3\tilde{Z}_{\gamma,m}^{2:}(s) v_{\gamma,m}(s) + \tilde{Z}_{\gamma,m}^{3:}(s) \right) \\ & - A(s) \left(v_{\gamma,m}(s) + \tilde{Z}_{\gamma,m}(s) \right). \end{aligned} \quad (7.20)$$

The error term $\text{Err}_m^{(1)}$ was bounded in (7.10), while for $\text{Err}_m^{(2)}$, we have for $0 \leq s \leq T$ and for $0 < \gamma < \gamma_0$

$$\begin{aligned} \|\text{Err}_m^{(2)}(s, \cdot)\|_{L^\infty(\mathbb{T}^2)} \\ \leq C(T, \nu, \kappa) \left(\|E_{\gamma,s,m}^{3:}\|_{L^\infty(\mathbb{T}^2)} + s^{-\nu-\kappa} \|E_{\gamma,s,m}^{2:}\|_{L^\infty(\mathbb{T}^2)} \right). \end{aligned} \quad (7.21)$$

Proof. Using (7.4), we see that $v_{\gamma,m}$ satisfies for any $t \geq 0$ on \mathbb{T}^2

$$\begin{aligned} v_{\gamma,m}(t, \cdot) = & \int_0^t P_{t-s}^\gamma K_\gamma \star_\varepsilon \left(-\frac{1}{3} (v_{\gamma,m} + \tilde{Z}_{\gamma,m})^3(s, \cdot) \right. \\ & \left. + (\mathbf{c}_\gamma(s, \cdot) + A(s))(v_{\gamma,m} + \tilde{Z}_{\gamma,m})(s, \cdot) + \text{Err}_m^{(1)}(s, \cdot) \right) ds. \end{aligned}$$

The term $\text{Err}_m^{(1)}(s, x)$ was already controlled in (7.10). For the rest, we write

$$\begin{aligned} & \frac{1}{3} (v_{\gamma,m} + \tilde{Z}_{\gamma,m})^3 - \mathbf{c}_\gamma(\cdot, \cdot) (v_{\gamma,m} + \tilde{Z}_{\gamma,m}) \\ & = \frac{1}{3} v_{\gamma,m}^3 + v_{\gamma,m}^2 \tilde{Z}_{\gamma,m} + v_{\gamma,m} \tilde{Z}_{\gamma,m}^{2:} + \frac{1}{3} \tilde{Z}_{\gamma,m}^{3:} + \text{err}^{(1)} + \text{err}^{(2)}, \end{aligned}$$

where

$$\begin{aligned} \text{err}^{(1)}(s, \cdot) & = v_{\gamma,m}(s, \cdot) \left(\tilde{Z}_{\gamma,m}^2(s, \cdot) - \mathbf{c}_\gamma(s, \cdot) - \tilde{Z}_{\gamma,m}^{2:}(s, \cdot) \right), \\ \text{err}^{(2)}(s, \cdot) & = \frac{1}{3} \left(\tilde{Z}_{\gamma,m}^3(s, \cdot) - 3\mathbf{c}_\gamma(s, \cdot) \tilde{Z}_{\gamma,m}(s, \cdot) - \tilde{Z}_{\gamma,m}^{3:}(s, \cdot) \right). \end{aligned}$$

To control $\text{err}^{(1)}$, we write using (7.16)

$$\begin{aligned} & \tilde{Z}_{\gamma,m}^2(s, \cdot) - \mathfrak{c}_\gamma(s, \cdot) - \tilde{Z}_{\gamma,m}^{:2:}(s, \cdot) \\ &= (Z_{\gamma,m}^2 + 2Y_\gamma Z_{\gamma,m} + Y_\gamma^2)(s, \cdot) - \mathfrak{c}_\gamma(s, \cdot) - (Z_{\gamma,m}^{:2:} + 2Y_\gamma Z_{\gamma,m} + Y_\gamma^2)(s, \cdot) \\ &= Z_{\gamma,m}^2(s, \cdot) - \mathfrak{c}_\gamma(s, \cdot) - Z_{\gamma,m}^{:2:}(s, \cdot) \\ &= E_{\gamma,s,m}^{:2:}(s, \cdot). \end{aligned}$$

For $\text{err}^{(2)}$, we write using (7.16) once more (and dropping the arguments to improve readability)

$$\begin{aligned} & (\tilde{Z}_{\gamma,m}^3 - 3\mathfrak{c}_\gamma \tilde{Z}_{\gamma,m} - \tilde{Z}_{\gamma,m}^{:3:}) \\ &= (Z_{\gamma,m}^3 + 3Y_\gamma Z_{\gamma,m}^2 + 3Y_\gamma^2 Z_{\gamma,m} + Y_\gamma^3) - 3\mathfrak{c}_\gamma(Z_{\gamma,m} + Y_\gamma) \\ &\quad - (Z_{\gamma,m}^{:3:} + 3Y_\gamma Z_{\gamma,m}^{:2:} + 3Y_\gamma^2 Z_{\gamma,m} + Y_\gamma^3) \\ &= (Z_{\gamma,m}^3 - 3\mathfrak{c}_\gamma Z_{\gamma,m} - Z_{\gamma,m}^{:3:}) + 3Y_\gamma(Z_{\gamma,m}^2 - \mathfrak{c}_\gamma - Z_{\gamma,m}^{:2:}) \\ &= E_{\gamma,s,m}^{:3:} + 3Y_\gamma E_{\gamma,s,m}^{:2:}. \end{aligned}$$

By assumption, X_γ^0 is bounded in $\mathcal{C}^{-\nu}$. The bound (7.12) thus ensures that for any $\kappa > 0$, we have a uniform bound on $s^{\nu+\kappa} \|Y_\gamma\|_{L^\infty(\mathbb{T}^2)}$, from which the desired bound follows. \square

In the next lemma, we combine the expressions (7.17) and (7.19) to derive a bound on $\sup_{0 \leq t \leq T} \|\bar{v}_{\gamma,m}(t, \cdot) - v_{\gamma,m}(t, \cdot)\|_{\mathcal{C}^{\frac{1}{2}}}$.

Lemma 7.5. *Let $\kappa > 0$ be the constant appearing in the boundedness assumption in Theorem 2.1 and let $\nu > 0$ be small enough. For every $0 \leq t \leq T$ and $0 < \gamma < \gamma_0$, we have*

$$\begin{aligned} \|\bar{v}_{\gamma,m}(t, \cdot) - v_{\gamma,m}(t, \cdot)\|_{\mathcal{C}^{\frac{1}{2}}} &\leq \bar{C}_1 \int_0^t (t-s)^{-\frac{1}{3}} s^{-\frac{1}{6}} \|\bar{v}_{\gamma,m}(s, \cdot) - v_{\gamma,m}(s, \cdot)\|_{\mathcal{C}^{\frac{1}{2}}} ds \\ &\quad + \bar{C}_1 (\gamma^{\frac{\kappa}{2}} + \|X_\gamma^0 - X^0\|_{\mathcal{C}^{-\nu}}) + \text{Err}^{(3)}(t), \end{aligned} \quad (7.22)$$

where the constant \bar{C}_1 depends on $\nu, \kappa, T, \|X^0\|_{\mathcal{C}^{-\nu+\kappa}}, \|X_\gamma^0\|_{\mathcal{C}^{-\nu+\kappa}}$ as well as the random quantities $\sup_{0 \leq s \leq T} \|\bar{v}_{\gamma,m}(s, \cdot)\|_{\mathcal{C}^{\frac{1}{2}}}, \sup_{0 \leq s \leq T} \|v_{\gamma,m}(s, \cdot)\|_{\mathcal{C}^{\frac{1}{2}}}, \sup_{0 \leq s \leq T} \|Z_{\gamma,m}(s, \cdot)\|_{\mathcal{C}^{-\nu}}, \sup_{0 \leq s \leq T} \|Z_{\gamma,m}^{:2:}(s, \cdot)\|_{\mathcal{C}^{-\nu}}, \sup_{0 \leq s \leq T} \|Z_{\gamma,m}^{:3:}(s, \cdot)\|_{\mathcal{C}^{-\nu}}$.

The error term $\text{Err}^{(3)}$ satisfies for every $T \geq 0, p \geq 2$ and $0 < \lambda < \frac{5}{6}$

$$\mathbb{E} \sup_{0 \leq t \leq T} |\text{Err}^{(3)}(t)|^p \leq \bar{C}_2 \gamma^{\lambda p}, \quad (7.23)$$

for a constant $\bar{C}_2 = \bar{C}_2(p, T, \lambda)$.

Proof. For any $t \geq 0$ and $\gamma > 0$, we get combining (7.17) and (7.19)

$$\begin{aligned} \bar{v}_{\gamma,m}(t, \cdot) - v_{\gamma,m}(t, \cdot) &= - \int_0^t (P_{t-s} - P_{t-s}^\gamma \star K_\gamma) \bar{\Psi}_{\gamma,m}(s) ds \\ &\quad - \int_0^t P_{t-s}^\gamma \star K_\gamma \star (\bar{\Psi}_{\gamma,m}(s) - \Psi_{\gamma,m}(s)) ds \\ &\quad + \int_0^t P_{t-s}^\gamma \star K_\gamma \star (\text{Err}_m^{(1)}(s, \cdot) + \text{Err}_m^{(2)}(s, \cdot)) ds, \end{aligned} \quad (7.24)$$

where $\bar{\Psi}_{\gamma,m}(s)$ was defined in (7.18) and $\Psi_{\gamma,m}(s)$ was defined in (7.20). Using the multiplicative inequality, Lemma B.5, we can bound for every $s \geq 0$

$$\begin{aligned} \|\bar{\Psi}_{\gamma,m}(s)\|_{C^{-\nu}} &\leq \|\bar{v}_{\gamma,m}(s, \cdot)\|_{C^{\frac{1}{2}}}^3 + C(\nu) \|\bar{Z}_{\gamma,m}(s, \cdot)\|_{C^{-\nu}} \|\bar{v}_{\gamma,m}(s, \cdot)\|_{C^{\frac{1}{2}}}^2 \\ &\quad + C(\nu) \|\bar{Z}_{\gamma,m}^{:2:}(s, \cdot)\|_{C^{-\nu}} \|\bar{v}_{\gamma,m}(s, \cdot)\|_{C^{\frac{1}{2}}} + \|\bar{Z}_{\gamma,m}^{:3:}(s, \cdot)\|_{C^{-\nu}} \\ &\quad + A(s) (\|\bar{v}_{\gamma,m}(s, \cdot)\|_{C^{\frac{1}{2}}} + \|\bar{Z}_{\gamma,m}(s, \cdot)\|_{C^{-\nu}}), \end{aligned} \quad (7.25)$$

where we have chosen $\nu > 0$ small enough to assure $\nu < \frac{1}{2}$. Recalling the definition of $\bar{Z}_{\gamma,m}$, $\bar{Z}_{\gamma,m}^{:2:}$ and $\bar{Z}_{\gamma,m}^{:3:}$ just below (7.16) we get for $s \geq 0$

$$\|\bar{Z}_{\gamma,m}(s, \cdot)\|_{C^{-\nu}} \leq \|Z_{\gamma,m}(s, \cdot)\|_{C^{-\nu}} + \|Y(s, \cdot)\|_{C^{-\nu}} \leq \|Z_{\gamma,m}(s, \cdot)\|_{C^{-\nu}} + C \|X^0\|_{C^{-\nu}}$$

and for any $\bar{\kappa} > 0$

$$\begin{aligned} \|\bar{Z}_{\gamma,m}^{:2:}\|_{C^{-\nu}} &\leq \|Z_{\gamma,m}^{:2:}\|_{C^{-\nu}} + C(\nu, \bar{\kappa}) (\|Z_{\gamma,m}\|_{C^{-\nu}} \|Y\|_{C^{\nu+\bar{\kappa}}} + \|Y\|_{C^{\nu+\bar{\kappa}}} \|Y\|_{C^{-\nu}}) \\ \|\bar{Z}_{\gamma,m}^{:3:}\|_{C^{-\nu}} &\leq \|Z_{\gamma,m}^{:3:}\|_{C^{-\nu}} + C(\nu, \bar{\kappa}) (\|Z_{\gamma,m}^{:2:}\|_{C^{-\nu}} \|Y\|_{C^{\nu+\bar{\kappa}}} + \|Z_{\gamma,m}\|_{C^{-\nu}} \|Y\|_{C^{\nu+\bar{\kappa}}}^2 \\ &\quad + \|Y\|_{C^{\nu+\bar{\kappa}}}^2 \|Y\|_{C^{-\nu}}), \end{aligned}$$

where we have omitted the arguments (s, \cdot) for all functions in the inequality. The regularity bound (8.22) for the heat semigroup P_t implies that for any $\lambda \geq -\nu$

$$\|Y(t)\|_{C^\lambda} \leq C(\lambda) t^{-\frac{\lambda+\nu}{2}} \|X^0\|_{C^{-\nu}}. \quad (7.26)$$

Furthermore, the definition (3.15) of $A(s)$ shows that $|A(s)| \leq C \log(s^{-1})$. Combining these bounds, we get the following (brutal) bound uniformly over $0 \leq s \leq T$

$$\begin{aligned} \|\bar{\Psi}_{\gamma,m}(s)\|_{C^{-\nu}} &\leq C(\nu, T) (\|\bar{v}_{\gamma,m}(s, \cdot)\|_{C^{\frac{1}{2}}}^3 + 1) s^{-\frac{1}{4}} (\|X_\gamma^0\|_{C^{-\nu}}^3 + 1) \\ &\quad (\|Z_{\gamma,m}(s)\|_{C^{-\nu}} + \|Z_{\gamma,m}^{:2:}(s)\|_{C^{-\nu}} + \|Z_{\gamma,m}^{:3:}(s)\|_{C^{-\nu}}), \end{aligned} \quad (7.27)$$

where we have assumed that ν and $\bar{\kappa}$ are small enough to ensure that the exponent on s in the term $\|Y(s)\|_{C^{\nu+\bar{\kappa}}}^2 \leq C(T, \nu, \bar{\kappa}) s^{-(2\nu+\bar{\kappa})} \|X^0\|_{C^\nu}^2$ satisfies $2\nu + \bar{\kappa} \leq \frac{1}{4}$.

Combining Lemma 8.4 and Corollary 8.7, we observe that for any $\lambda < \frac{3}{4} - \frac{\nu}{2}$ and $\bar{\kappa} > 0$, there exists a constant $C = C(T, \lambda, \bar{\kappa})$ such that uniformly over $0 \leq t \leq T$,

$$\|(P_t - P_t^\gamma \star K_\gamma)\|_{C^{-\nu} \rightarrow C^{\frac{1}{2}}} \leq C \gamma^\lambda t^{-\lambda - \frac{1}{4} - \frac{\nu}{2} - \bar{\kappa}}. \quad (7.28)$$

Indeed, a combination of (8.23), (8.25) and (8.28) in conjunction with the standard regularity estimate for the heat semigroup (8.22) yields that for any $0 \leq \beta \leq 2$ and $\bar{\kappa} > 0$, we have uniformly over $0 \leq t \leq T$

$$\|(P_t - P_t^\gamma \star K_\gamma)\|_{C^{-\nu} \rightarrow C^{-\nu+\beta-\bar{\kappa}}} \leq C(\beta, \bar{\kappa}, T) t^{-\frac{\beta}{2}}.$$

Interpolating between this bound and (8.35) yields (7.28).

For any $\bar{\kappa} > 0$ and $\lambda > 0$ small enough, we can thus bound the first term on the right-hand side of (7.24) uniformly over $0 \leq s \leq T$

$$\begin{aligned} & \left\| \int_0^t (P_{t-s} - P_{t-s}^\gamma \star K_\gamma) \bar{\Psi}_{\gamma, \mathfrak{m}}(s) ds \right\|_{\mathcal{C}^{\frac{1}{2}}} \\ & \leq C(\bar{\kappa}, \lambda, T) \int_0^t (t-s)^{-\frac{\nu}{2} - \frac{1}{4} - \bar{\kappa}} (t-s)^{-\frac{\lambda}{2} - \bar{\kappa}} \gamma^\lambda \|\bar{\Psi}_{\gamma, \mathfrak{m}}(s)\|_{\mathcal{C}^{-\nu}} ds \\ & \leq C t^{\frac{1}{2} - \frac{\nu+\lambda}{2} - 2\bar{\kappa}} \gamma^\lambda, \end{aligned} \quad (7.29)$$

where the constant C on the right-hand side depends on $\bar{\kappa}$, λ , T , $\|X_\gamma\|_{\mathcal{C}^{-\nu}}$ but also on the random quantities $\sup_{0 \leq s \leq T} \|\bar{v}_{\gamma, \mathfrak{m}}(s, \cdot)\|_{\mathcal{C}^{\frac{1}{2}}}$, $\sup_{0 \leq s \leq T} \|Z_{\gamma, \mathfrak{m}}(s, \cdot)\|_{\mathcal{C}^{-\nu}}$, $\sup_{0 \leq s \leq T} \|Z_{\gamma, \mathfrak{m}}^{:2:}(s, \cdot)\|_{\mathcal{C}^{-\nu}}$ and $\sup_{0 \leq s \leq T} \|Z_{\gamma, \mathfrak{m}}^{:3:}(s, \cdot)\|_{\mathcal{C}^{-\nu}}$. In the last inequality in (7.29) we have chosen $\nu, \lambda, \bar{\kappa}$ small enough to ensure that the quantity on the right-hand side of the integral in the second line is integrable at as $s \rightarrow t$. By choosing ν and $\bar{\kappa}$ small enough we can still choose $\lambda = \frac{1}{2}$. Then the right-hand side of (7.29) is bounded by $C\gamma^{\frac{1}{2}}$ where C depends on $T, \|X_\gamma\|_{\mathcal{C}^{-\nu}}$ as well as all the random quantities listed above.

As we will now see, the second term on the right-hand side of (7.24) can be treated in a similar way. Using again the multiplicative inequality, Lemma B.5, as well as the definition of $\tilde{Z}_{\gamma, \mathfrak{m}}, \tilde{Z}_{\gamma, \mathfrak{m}}^{:2:}, \tilde{Z}_{\gamma, \mathfrak{m}}^{:3:}, \bar{Z}_{\gamma, \mathfrak{m}}, \bar{Z}_{\gamma, \mathfrak{m}}^{:2:}$ and $\bar{Z}_{\gamma, \mathfrak{m}}^{:3:}$ in (7.16) and below, we get for all $s \geq 0$ and any $\bar{\kappa} > 0$ that

$$\begin{aligned} & \|\bar{\Psi}_{\gamma, \mathfrak{m}}(s) - \Psi_{\gamma, \mathfrak{m}}(s)\|_{\mathcal{C}^{-\nu}} \\ & \leq C(\nu, \bar{\kappa}, T) \left(\|\bar{v}_{\gamma, \mathfrak{m}}(s, \cdot) - v_{\gamma, \mathfrak{m}}(s, \cdot)\|_{\mathcal{C}^{\frac{1}{2}}} + \|Y(s, \cdot) - Y_\gamma(s, \cdot)\|_{\mathcal{C}^{-\nu}} \right) \\ & \quad \times (1 + A(s)) \left(\|\bar{v}_{\gamma, \mathfrak{m}}(s, \cdot)\|_{\mathcal{C}^{\frac{1}{2}}}^2 + \|v_{\gamma, \mathfrak{m}}(s, \cdot)\|_{\mathcal{C}^{\frac{1}{2}}}^2 + \|Z_{\gamma, \mathfrak{m}}(s, \cdot)\|_{\mathcal{C}^{-\nu}}^2 + \|Z_{\gamma, \mathfrak{m}}^{:2:}(s, \cdot)\|_{\mathcal{C}^{-\nu}} \right. \\ & \quad \left. + \|Y(s, \cdot)\|_{\mathcal{C}^{\nu+\bar{\kappa}}}^2 + \|Y_\gamma(s, \cdot)\|_{\mathcal{C}^{\nu+\bar{\kappa}}}^2 + 1 \right). \end{aligned}$$

Arguing as in the proof of Lemma 7.3, we see that

$$\|Y(s, \cdot) - Y_\gamma(s, \cdot)\|_{\mathcal{C}^{-\nu}} \leq C \|X^0 - X_\gamma^0\|_{\mathcal{C}^{-\nu}} + C(\nu, \kappa, T) \gamma^{\frac{\kappa}{2}},$$

where $\kappa > 0$ is the constant appearing in the boundedness condition in the statement of our main result, Theorem 2.1. Furthermore, using (7.12) (or Lemma 8.4), we get for any $\bar{\kappa} > 0$

$$\begin{aligned} \|Y(s, \cdot)\|_{\mathcal{C}^{\nu+\bar{\kappa}}}^2 & \leq C(\nu, \bar{\kappa}) s^{-2\nu-\bar{\kappa}} \|X^0\|_{\mathcal{C}^{-\nu}}^2, \\ \|Y_\gamma(s, \cdot)\|_{\mathcal{C}^{\nu+\bar{\kappa}}}^2 & \leq C(\nu, \bar{\kappa}, T) s^{-4\nu-3\bar{\kappa}} \|X_\gamma^0\|_{\mathcal{C}^{-\nu}}^2. \end{aligned}$$

(As stated above, the bounds for the approximated heat semigroup are weaker than the bounds in the continuous equation.)

This estimate, together with (8.28), gives the following bound on the second term on the right-hand side of (7.24), for any $\bar{\kappa} > 0$:

$$\begin{aligned} & \left\| \int_0^t P_{t-s}^\gamma K_\gamma \star (\bar{\Psi}_{\gamma, \mathfrak{m}}(s) - \Psi_{\gamma, \mathfrak{m}}(s)) ds \right\|_{\mathcal{C}^{\frac{1}{2}}} \\ & \leq C(\nu, \bar{\kappa}, T) \int_0^t (t-s)^{-\frac{1}{4} - \frac{\nu}{2} - \bar{\kappa}} \|\bar{\Psi}_{\gamma, \mathfrak{m}}(s) - \Psi_{\gamma, \mathfrak{m}}(s)\|_{\mathcal{C}^{-\nu}} ds \\ & \leq C \int_0^t (t-s)^{-\frac{1}{4} - \frac{\nu}{2} - \bar{\kappa}} s^{-4\nu-3\bar{\kappa}} \\ & \quad \left(\|\bar{v}_{\gamma, \mathfrak{m}}(s) - v_{\gamma, \mathfrak{m}}(s)\|_{\mathcal{C}^{\frac{1}{2}}} + \|X^0 - X_\gamma^0\|_{\mathcal{C}^{-\nu}} + \gamma^{\frac{\kappa}{2}} \right) ds. \end{aligned} \quad (7.30)$$

for a constant C that depends on $\nu, \bar{\kappa}, \kappa, T, \|X^0\|_{\mathcal{C}^{-\nu+\kappa}}, \|X_\gamma^0\|_{\mathcal{C}^{-\nu+\kappa}}$ as well as the random quantities $\sup_{0 \leq s \leq T} \|\bar{v}_{\gamma, \mathbf{m}}(s, \cdot)\|_{\mathcal{C}^{\frac{1}{2}}}, \sup_{0 \leq s \leq T} \|v_{\gamma, \mathbf{m}}(s, \cdot)\|_{\mathcal{C}^{\frac{1}{2}}}, \sup_{0 \leq s \leq T} \|Z_{\gamma, \mathbf{m}}(s, \cdot)\|_{\mathcal{C}^{-\nu}}, \sup_{0 \leq s \leq T} \|Z_{\gamma, \mathbf{m}}^{2:}(s, \cdot)\|_{\mathcal{C}^{-\nu}}$. Here we have absorbed the logarithmic divergence of $A(s)$ as $s \rightarrow 0$ into the term $s^{-4\nu-3\bar{\kappa}}$ by choosing a slightly larger $\bar{\kappa}$. If $\nu, \bar{\kappa} > 0$ are small enough, the integral in the last line of (7.30) can be bounded by

$$C \int_0^t (t-s)^{-\frac{1}{3}} s^{-\frac{1}{6}} \|\bar{v}_{\gamma, \mathbf{m}}(s) - v_{\gamma, \mathbf{m}}(s)\|_{\mathcal{C}^{\frac{1}{2}}} ds + C \|X_\gamma^0 - X^0\|_{\mathcal{C}^{-\nu}} + C \gamma^{\frac{\kappa}{2}}.$$

For the last term on the right-hand side of (7.24), using (8.28) once more, we get for any $\bar{\kappa} > 0$

$$\begin{aligned} & \left\| \int_0^t P_{t-s}^\gamma K_\gamma \star (\text{Err}_m^{(1)}(s, \cdot) + \text{Err}_m^{(2)}(s, \cdot)) ds \right\|_{\mathcal{C}^{\frac{1}{2}}} \\ & \leq C(\bar{\kappa}, T) \int_0^t (t-s)^{-\frac{1}{4}-\bar{\kappa}} \|\text{Err}_m^{(1)}(s, \cdot) + \text{Err}_m^{(2)}(s, \cdot)\|_{L^\infty} ds. \end{aligned}$$

Recall that according to (7.9), for $s > \tau_{\gamma, \mathbf{m}}$ we have $\text{Err}_m^{(1)}(s) = 0$. Else, according to (7.10), we have

$$\begin{aligned} & \|\text{Err}_m^{(1)}(s, \cdot)\|_{L^\infty(\mathbb{T}^2)} \\ & \leq C(T, \mathbf{m}) \left(\gamma^{\frac{1}{2}} s^{-\frac{1}{3}} + \gamma^{-\frac{1}{6}} \|X_\gamma^{\text{high}}(s, \cdot)\|_{L^\infty(\mathbb{T}^2)} + \gamma^{-\frac{1}{6}} \|Q_{\gamma, s}(s, \cdot)\|_{L^\infty(\Lambda_\varepsilon)} \right). \end{aligned}$$

For the second term on the right-hand side of this estimate we write

$$\begin{aligned} & \|X_\gamma^{\text{high}}(s, \cdot)\|_{L^\infty(\mathbb{T}^2)} \\ & \leq \|Z_\gamma^{\text{high}}(s, \cdot)\|_{L^\infty(\mathbb{T}^2)} + \|v_{\gamma, \mathbf{m}}^{\text{high}}(s, \cdot)\|_{L^\infty(\mathbb{T}^2)} + \|Y_\gamma^{\text{high}}(s, \cdot)\|_{L^\infty(\mathbb{T}^2)} \\ & \leq \|Z_\gamma^{\text{high}}(s, \cdot)\|_{L^\infty(\mathbb{T}^2)} + C\gamma^1 \|v_{\gamma, \mathbf{m}}(s, \cdot)\|_{\mathcal{C}^{\frac{1}{2}}} + C(\nu, \bar{\kappa}, T) \gamma^1 t^{-\frac{1+\nu}{2}-\bar{\kappa}} \|X_\gamma^0\|_{\mathcal{C}^{-\nu}}. \end{aligned}$$

Here, $v_{\gamma, \mathbf{m}}^{\text{high}}$ and $Y_\gamma^{\text{high}}(s, \cdot)$ are defined analogously to (7.3). From this definition, it follows immediately that $\|v_{\gamma, \mathbf{m}}^{\text{high}}(s, \cdot)\|_{L^\infty(\mathbb{T}^2)}$ is controlled by $C\gamma^1 \|v_{\gamma, \mathbf{m}}(s, \cdot)\|_{\mathcal{C}^{\frac{1}{2}}}$, and similarly for Y_γ^{high} . We have also used once more the estimates provided in Lemma 8.4 to control the $\mathcal{C}^{\frac{1}{2}}$ norm of $Y_\gamma(s)$.

According to (7.21), we have for any $\bar{\kappa} > 0$

$$\|\text{Err}_m^{(2)}\|_{L^\infty(\mathbb{T}^2)} \leq C(t, \nu, \bar{\kappa}) \left(\|E_{\gamma, s, \mathbf{m}}^{3:}\|_{L^\infty(\mathbb{T}^2)} + s^{-\nu-\bar{\kappa}} \|E_{\gamma, s, \mathbf{m}}^{2:}\|_{L^\infty(\mathbb{T}^2)} \right).$$

Summarising all of these calculations, we get

$$\begin{aligned} & \|v_{\gamma, \mathbf{m}}(t, \cdot) - \bar{v}_{\gamma, \mathbf{m}}(t, \cdot)\|_{\mathcal{C}^1} \leq \bar{C}_1 \int_0^t (t-s)^{-\frac{1}{3}} s^{-\frac{1}{6}} \|v_{\gamma, \mathbf{m}}(s) - \bar{v}_{\gamma, \mathbf{m}}(s)\|_{\mathcal{C}^1} ds \\ & \quad + \bar{C}_1 (\gamma^{\frac{\kappa}{2}} + \|X_\gamma^0 - X^0\|_{\mathcal{C}^{-\nu}}) + \text{Err}^{(3)}(t), \end{aligned}$$

for a constant \bar{C}_1 that depends on all of the quantities listed below (7.29) and (7.30).

The error term takes the form

$$\begin{aligned} \text{Err}^{(3)}(t) & \leq C(\bar{\kappa}, T) \int_0^t (t-s)^{-\frac{1}{4}-\bar{\kappa}} \left(\gamma^{-\frac{1}{6}} \|X_\gamma^{\text{high}}(s, \cdot)\|_{L^\infty(\mathbb{T}^2)} + \gamma^{-\frac{1}{6}} \|Q_{\gamma, s}(s, \cdot)\|_{L^\infty(\Lambda_\varepsilon)} \right. \\ & \quad \left. + \|E_{\gamma, s, \mathbf{m}}^{3:}\|_{L^\infty(\mathbb{T}^2)} + s^{-\nu-\bar{\kappa}} \|E_{\gamma, s, \mathbf{m}}^{2:}\|_{L^\infty(\mathbb{T}^2)} \right) ds, \end{aligned}$$

In particular, we can conclude from Lemmas 4.6, 5.1 and 5.3 that for any $p \geq 2$,

$$\mathbb{E} \sup_{0 \leq t \leq T} |\text{Err}^{(3)}(t)|^p \leq C(p, T, \lambda) \gamma^{\lambda p},$$

for any $\lambda < \frac{5}{6}$. So the desired result follows. \square

We are now ready to conclude the proof of our main result.

Proof of Theorem 2.1. For r and $m \geq 1$, we define the events $\mathcal{A}_r^Z = \mathcal{A}_r^Z(\gamma, m)$, and $\mathcal{A}^E = \mathcal{A}^E(\gamma, m)$ by

$$\begin{aligned} \mathcal{A}_r^Z &:= \left\{ \|Z_{\gamma,m}\|_{\mathcal{C}^{-\nu}} \leq r, \|Z_{\gamma,m}^{:2:}\|_{\mathcal{C}^{-\nu}} \leq r, \|Z_{\gamma,m}^{:3:}\|_{\mathcal{C}^{-\nu}} \leq r \text{ on } [0, T] \right\}, \\ \mathcal{A}^E &:= \left\{ \sup_{0 \leq t \leq T} |\text{Err}^{(3)}(t)| \leq \gamma^{\frac{1}{2}} \right\}. \end{aligned}$$

Recall that

$$\bar{v}_{\gamma,m} = \mathcal{S}_T(Z_{\gamma,m}, Z_{\gamma,m}^{:2:}, Z_{\gamma,m}^{:3:}).$$

By Theorem 3.2, there exists a constant $C(T, r)$ such that on the event \mathcal{A}_r^Z ,

$$\sup_{0 \leq t \leq T} \|\bar{v}_{\gamma,m}(t, \cdot)\|_{\mathcal{C}^{\frac{1}{2}}} \leq C(T, r). \quad (7.31)$$

Let $r_0 = C(T, r) + 2$ and let $\tilde{T} = \tilde{T}(\gamma)$ be the stopping time defined by

$$\tilde{T} = \inf\{t \geq 0 : \|\bar{v}_{\gamma,m}(t, \cdot)\|_{\mathcal{C}^{\frac{1}{2}}} \geq r_0\}.$$

By a suitable version of Gronwall's inequality (see e.g. [34, Lemma 5.7]), we deduce that there exists a constant C depending on T, r and the choice of small constants $\kappa, \nu > 0$ in Theorem 2.1 such that on the event $\mathcal{A}_r^Z \cap \mathcal{A}^E$, we have

$$\sup_{0 \leq t \leq \tilde{T} \wedge T} \|v_{\gamma,m}(t, \cdot) - \bar{v}_{\gamma,m}(t, \cdot)\|_{\mathcal{C}^{\frac{1}{2}}} \leq C \left(\gamma^{\frac{\kappa}{2}} + \|X_\gamma^0 - X^0\|_{\mathcal{C}^{-\nu}} \right) \quad (7.32)$$

(recall that the quantity in the supremum above is a continuous function of t). In particular, for γ sufficiently small,

$$\sup_{0 \leq t \leq \tilde{T} \wedge T} \|v_{\gamma,m}(t, \cdot) - \bar{v}_{\gamma,m}(t, \cdot)\|_{\mathcal{C}^{\frac{1}{2}}} \leq 1.$$

Together with (7.31), this implies that

$$\|\bar{v}_{\gamma,m}(\tilde{T} \wedge T, \cdot)\|_{\mathcal{C}^{\frac{1}{2}}} \leq r_0 - 1.$$

By continuity of $t \mapsto \|\bar{v}_{\gamma,m}(t, \cdot)\|_{\mathcal{C}^{\frac{1}{2}}}$, this implies that $\tilde{T} \geq T$, and thus that (7.32) can be upgraded to

$$\sup_{0 \leq t \leq T} \|v_{\gamma,m}(t, \cdot) - \bar{v}_{\gamma,m}(t, \cdot)\|_{\mathcal{C}^{\frac{1}{2}}} \leq C \left(\gamma^{\frac{\kappa}{2}} + \|X_\gamma^0 - X^0\|_{\mathcal{C}^{-\nu}} \right), \quad (7.33)$$

on the event $\mathcal{A}_r^Z \cap \mathcal{A}^E$ and for $\gamma > 0$ small enough (depending in r). Recalling that

$$X_{\gamma,m} = v_{\gamma,m} + Z_{\gamma,m} + Y_\gamma \quad \text{and} \quad \bar{X}_{\gamma,m} = \bar{v}_{\gamma,m} + Z_{\gamma,m} + Y,$$

we deduce from (7.33) and Lemma 7.3 that for every $m, r \geq 1$ and every bounded uniformly continuous mapping $F : \mathcal{D}([0, T], \mathcal{C}^{-\nu}) \rightarrow \mathbb{R}$,

$$\lim_{\gamma \rightarrow 0} \mathbb{E} \left[|F(\bar{X}_{\gamma,m}) - F(X_{\gamma,m})| \mathbf{1}_{\mathcal{A}_r^Z \cap \mathcal{A}^E} \right] = 0. \quad (7.34)$$

Let us write $\bar{\mathcal{A}}_r^Z$ and $\bar{\mathcal{A}}^E$ for the complementary events of \mathcal{A}_r^Z and \mathcal{A}^E respectively. Note that in (7.34), the indicator function of the event \mathcal{A}^E does no harm, since it follows from (7.23) and Chebyshev's inequality that

$$\lim_{\gamma \rightarrow 0} \mathbb{P} \left[\bar{\mathcal{A}}^E \right] = 0. \quad (7.35)$$

Let $\varepsilon > 0$. We now argue that by choosing r sufficiently large, we can make the probability of the event $\overline{\mathcal{A}}_r^Z$ smaller than ε as $\gamma \rightarrow 0$. In order to do so, let us introduce the stopping times

$$\begin{aligned}\tau_{\gamma,m}^{Z,r} &= \inf \left\{ t \geq 0 : \max \left(\|Z_{\gamma,m}(t)\|_{\mathcal{C}^{-\nu}}, \|Z_{\gamma,m}^{2:}(t)\|_{\mathcal{C}^{-\nu}}, \|Z_{\gamma,m}^{3:}(t)\|_{\mathcal{C}^{-\nu}} \right) \geq r \right\}, \\ \tau^{Z,r} &= \inf \left\{ t \geq 0 : \max \left(\|Z(t)\|_{\mathcal{C}^{-\nu}}, \|Z^{2:}(t)\|_{\mathcal{C}^{-\nu}}, \|Z^{3:}(t)\|_{\mathcal{C}^{-\nu}} \right) \geq r \right\}.\end{aligned}$$

Recall from Theorem 6.2 that $(Z_{\gamma,m}, Z_{\gamma,m}^{2:}, Z_{\gamma,m}^{3:})$ converges in law to $(Z, Z^{2:}, Z^{3:})$ for the topology of $\mathcal{D}(\mathbb{R}_+, \mathcal{C}^{-\nu})^3$. Arguing as in the proof of Theorem 6.1, we see that for every r outside of a countable set Loc (the set of r such that $t \mapsto \max(\|Z_{\gamma,m}(t)\|_{\mathcal{C}^{-\nu}}, \|Z_{\gamma,m}^{2:}(t)\|_{\mathcal{C}^{-\nu}}, \|Z_{\gamma,m}^{3:}(t)\|_{\mathcal{C}^{-\nu}})$ attains the value r as a local maximum with positive probability), the random variable $\tau_{\gamma,m}^{Z,r}$ converges in law to $\tau^{Z,r}$ as $\gamma \rightarrow 0$ (keeping m fixed). By Proposition 3.1, the random variable

$$\sup_{0 \leq t \leq T} \max \left(\|Z(t)\|_{\mathcal{C}^{-\nu}}, \|Z^{2:}(t)\|_{\mathcal{C}^{-\nu}}, \|Z^{3:}(t)\|_{\mathcal{C}^{-\nu}} \right)$$

is finite a.s., so it suffices to choose r sufficiently large to ensure that

$$\mathbb{P}[\tau^{Z,r} \leq T] \leq \varepsilon.$$

Enlarging r if necessary, we can also make sure that $r \notin \text{Loc}$, and thus get

$$\limsup_{\gamma \rightarrow 0} \mathbb{P}[\overline{\mathcal{A}}_r^Z] \leq \limsup_{\gamma \rightarrow 0} \mathbb{P}[\tau_{\gamma,m}^{Z,r} \leq T] \leq \mathbb{P}[\tau^{Z,r} \leq T] \leq \varepsilon. \quad (7.36)$$

We decompose

$$\begin{aligned}|\mathbb{E}[F(X_{\gamma,m})] - \mathbb{E}[F(X)]| &\leq |\mathbb{E}[F(\overline{X}_{\gamma,m})] - \mathbb{E}[F(X)]| \\ &\quad + \mathbb{E}[|F(\overline{X}_{\gamma,m}) - F(X_{\gamma,m})| \mathbf{1}_{\mathcal{A}_r^Z \cap \mathcal{A}^E}] + \|F\|_{L^\infty} \mathbb{P}[\overline{\mathcal{A}}_r^Z \cup \overline{\mathcal{A}}^E].\end{aligned}$$

By Lemma 7.2, (7.34), (7.35) and (7.36), we obtain that

$$\limsup_{\gamma \rightarrow 0} |\mathbb{E}[F(X_{\gamma,m})] - \mathbb{E}[F(X)]| \leq \varepsilon \|F\|_{L^\infty}.$$

Since $\varepsilon > 0$ was arbitrary, this proves that $X_{\gamma,m}$ converges in law to X as γ tends to 0, for any fixed value of m .

We can now remove m by a similar reasoning. Recall the definition of $\tau_{\gamma,m}$ in (6.1) as

$$\tau_{\gamma,m} = \inf \left\{ t \geq 0 : \|X_{\gamma,m}(t)\|_{\mathcal{C}^{-\nu}} \geq m \right\},$$

and set

$$\tau_m = \inf \left\{ t \geq 0 : \|X(t)\|_{\mathcal{C}^{-\nu}} \geq m \right\}.$$

Arguing as above, we obtain that for every m outside of a countable set Loc' , the stopping time $\tau_{\gamma,m}$ converges in law to τ_m . Moreover, we know from Theorem 3.2 that $\sup_{0 \leq t \leq T+1} \|X(t)\|_{\mathcal{C}^{-\nu}}$ is almost surely finite. Hence, for any given $\varepsilon > 0$, we can choose $m = m(T, \varepsilon)$ sufficiently large and outside of Loc' so that

$$\limsup_{\gamma \rightarrow 0} \mathbb{P}[\tau_{\gamma,m} \leq T] \leq \varepsilon.$$

Recalling that $X_{\gamma,m}$ and X_γ coincide up to $\tau_{\gamma,m}$, this implies that

$$\limsup_{\gamma \rightarrow 0} \mathbb{P}[X_{\gamma,m} \neq X_\gamma] \leq \varepsilon.$$

Since $X_{\gamma,m}$ converges in law to X , this concludes the proof of Theorem 2.1. \square

8. SOME BOUNDS ON THE KERNELS K_γ AND P_t^γ

In this section, we collect some facts about the kernels K_γ and the approximate heat semigroups P_t^γ . We start by summarising some properties of the Fourier transform \hat{K}_γ of K_γ . Recall that for $0 < \gamma < \frac{1}{3}$ and $\omega \in \{-N, \dots, N\}^2$, it is given by

$$\hat{K}_\gamma(\omega) = \sum_{x \in \Lambda_\varepsilon} \varepsilon^2 K_\gamma(x) e^{-i\pi\omega \cdot x} = c_{\gamma,1} \sum_{x \in \gamma\mathbb{Z}_*^2} \gamma^2 \mathfrak{K}(x) e^{-i\pi(\varepsilon/\gamma)\omega \cdot x}, \quad (8.1)$$

where \mathfrak{K} is the smooth function introduced at the beginning of Section 2. In the second equality, we have used the fact that \mathfrak{K} has compact support in $B(0, 3)$ to replace the sum over $\frac{\gamma}{\varepsilon}\Lambda_\varepsilon^*$ by a sum over $\gamma\mathbb{Z}_*^2 := \gamma\mathbb{Z}^2 \setminus \{0\}$. Recall our choice of scaling (2.16), (2.17), in particular $\varepsilon = \gamma^2 c_{\gamma,2}$.

For some of the following calculations, it is useful to view \hat{K}_γ as a function of a continuous parameter by evaluating (8.1) for all $\omega \in \mathbb{R}^2$. The function \hat{K}_γ defined in this way is smooth and $(2N+1)$ periodic in both coordinates. We will typically evaluate it only for $\omega \in [-N - \frac{1}{2}, N + \frac{1}{2}]^2$. Furthermore, we have for all ω and for $j = 1, 2$

$$\partial_j \hat{K}_\gamma(\omega) = -i \frac{\varepsilon}{\gamma} \pi c_{\gamma,1} \sum_{x \in \gamma\mathbb{Z}^2} \gamma^2 x_j \mathfrak{K}(x) e^{-i\pi(\varepsilon/\gamma)\omega \cdot x}, \quad (8.2)$$

and

$$\partial_j^2 \hat{K}_\gamma(\omega) = -\frac{\varepsilon^2}{\gamma^2} \pi^2 c_{\gamma,1} \sum_{x \in \gamma\mathbb{Z}^2} \gamma^2 x_j^2 \mathfrak{K}(x) e^{-i\pi(\varepsilon/\gamma)\omega \cdot x}. \quad (8.3)$$

In (8.2) and (8.3), we can sum over $\gamma\mathbb{Z}^2$ instead of $\gamma\mathbb{Z}_*^2$ because the summand at $x = 0$ vanishes. This will be slightly more convenient below.

For small γ , the expression (8.1) approximates $\widehat{\mathfrak{K}}(\gamma\omega)$, where the continuous Fourier transform $\widehat{\mathfrak{K}}(\omega)$ is defined as

$$\widehat{\mathfrak{K}}(\omega) := \int_{\mathbb{R}^2} \mathfrak{K}(x) e^{-i\pi\omega \cdot x} dx \quad (\omega \in \mathbb{R}^2).$$

The following lemmas state that some properties of $\widehat{\mathfrak{K}}(\gamma\omega)$ also hold for \hat{K}_γ , uniformly in γ . We begin with pointwise estimates.

Lemma 8.1. *There exists a constant $C > 0$ such that for all $0 < \gamma < \frac{1}{3}$ and for $|\omega| \leq \gamma^{-1}$ we have for $j = 1, 2$*

$$|\gamma^{-2}(1 - \hat{K}_\gamma(\omega)) - \pi^2|\omega|^2| \leq C\gamma|\omega|^3, \quad (8.4)$$

$$|-\gamma^{-2}\partial_j \hat{K}_\gamma(\omega) - 2\pi^2\omega_j| \leq C\gamma|\omega|^2, \quad (8.5)$$

$$|-\gamma^{-2}\partial_j^2 \hat{K}_\gamma(\omega) - 2\pi^2| \leq C\gamma|\omega|. \quad (8.6)$$

Proof. For any $|\omega| \leq \gamma^{-1}$, a Taylor expansion yields

$$\begin{aligned} 1 - \hat{K}_\gamma(\omega) &= c_{\gamma,1} \sum_{x \in \gamma\mathbb{Z}^2} \gamma^2 \mathfrak{K}(x) (1 - e^{-i\pi(\varepsilon/\gamma)\omega \cdot x}) \\ &= c_{\gamma,1} \sum_{x \in \gamma\mathbb{Z}^2} \gamma^2 \mathfrak{K}(x) \left(i\pi \frac{\varepsilon}{\gamma} \omega \cdot x + \frac{1}{2} \left(\pi \frac{\varepsilon}{\gamma} \omega \cdot x \right)^2 \right) + \text{Err}, \end{aligned}$$

where

$$|\text{Err}| \leq \frac{\varepsilon^3}{\gamma^3} |\omega|^3 \frac{\pi^3}{6} c_{\gamma,1} \sum_{x \in \gamma \mathbb{Z}^2} \gamma^2 |x|^3 \mathfrak{K}(x) \leq C \gamma^3 |\omega|^3 .$$

By the symmetry of the kernel $\mathfrak{K}(x)$, we have

$$\sum_{x \in \gamma \mathbb{Z}^2} \gamma^2 \mathfrak{K}(x) (\omega \cdot x) = 0 \quad \text{and} \quad \sum_{x \in \gamma \mathbb{Z}^2} \gamma^2 x_1 x_2 \mathfrak{K}(x) = 0 .$$

Furthermore, for $j = 1, 2$, the sums

$$c_{\gamma,1} \sum_{x \in \gamma \mathbb{Z}_*^2} \gamma^2 \mathfrak{K}(x) x_j^2$$

converge to $\int_{\mathbb{R}^2} \mathfrak{K}(x) x_j^2 dx = 2$ as $\gamma \rightarrow 0$ and the error is controlled by $C\gamma^2$, so (8.4) follows.

The remaining bounds (8.5) and (8.6) follow in a similar manner: For (8.5), we write

$$\begin{aligned} -\partial_j \hat{K}_\gamma(\omega) &= i \frac{\varepsilon}{\gamma} \pi c_{\gamma,1} \sum_{x \in \gamma \mathbb{Z}^2} \gamma^2 x_j \mathfrak{K}(x) (e^{-i\pi(\varepsilon/\gamma)\omega \cdot x} - 1) \\ &= \gamma^2 \pi^2 \omega_j c_{\gamma,1} c_{\gamma,2}^2 \sum_{x \in \gamma \mathbb{Z}^2} \gamma^2 x_j^2 \mathfrak{K}(x) + \text{Err}' , \end{aligned}$$

for an error term Err' that is bounded by $C|\omega|^2 \gamma^3$ uniformly for $|\omega| \leq \gamma^{-1}$. Here we have used the symmetry of the kernel \mathfrak{K} to add the term -1 in the first equality and to remove the sum over $x_1 x_2$ in the Taylor expansion in the second line. The bound then follows as above.

In the same way, we write

$$\begin{aligned} -\partial_j^2 \hat{K}_\gamma(\omega) &= \frac{\varepsilon^2}{\gamma^2} \pi^2 c_{\gamma,1} \sum_{x \in \gamma \mathbb{Z}^2} \gamma^2 x_j^2 \mathfrak{K}(x) e^{-i\pi(\varepsilon/\gamma)\omega \cdot x} \\ &= \gamma^2 \pi^2 c_{\gamma,1} c_{\gamma,2}^2 \sum_{x \in \gamma \mathbb{Z}^2} \gamma^2 x_j^2 \mathfrak{K}(x) + \text{Err}'' , \end{aligned}$$

where $\text{Err}'' \leq C\gamma|\omega|$ and the bound (8.6) follows. \square

Lemma 8.2. *There exists a constant C such that for all $0 < \gamma < \frac{1}{3}$, $\omega \in [-N - \frac{1}{2}, N + \frac{1}{2}]^2$ and $j = 1, 2$,*

(1) *(Estimates most useful for $|\omega| \leq \gamma^{-1}$)*

$$|\hat{K}_\gamma(\omega)| \leq 1 , \tag{8.7}$$

$$|\partial_j \hat{K}_\gamma(\omega)| \leq C\gamma (|\gamma\omega| \wedge 1) , \tag{8.8}$$

$$|\partial_j^2 \hat{K}_\gamma(\omega)| \leq C\gamma^2 , \tag{8.9}$$

(2) *(Estimates most useful for $|\omega| \geq \gamma^{-1}$)*

$$|\gamma\omega|^2 |\hat{K}_\gamma(\omega)| \leq C , \tag{8.10}$$

$$|\gamma\omega|^2 |\partial_j \hat{K}_\gamma(\omega)| \leq C\gamma , \tag{8.11}$$

$$|\gamma\omega|^2 |\partial_j^2 \hat{K}_\gamma(\omega)| \leq C\gamma^2 . \tag{8.12}$$

Furthermore, there exist constants $C_1 > 0$ and $\gamma_0 > 0$ such that for all $0 < \gamma < \gamma_0$ and $\omega \in [-N - \frac{1}{2}, N + \frac{1}{2}]^2$,

$$(1 - \hat{K}_\gamma(\omega)) \geq \frac{1}{C_1} (|\gamma\omega|^2 \wedge 1). \quad (8.13)$$

Proof. To see (8.7), we write

$$|\hat{K}_\gamma(\omega)| \leq c_{\gamma,1} \sum_{x \in \gamma\mathbb{Z}_*^2} \gamma^2 \mathfrak{K}(x) = 1.$$

For $|\omega| \leq \gamma^{-1}$, equations (8.8) and (8.9) follow directly from Lemma 8.1. For $|\omega| \geq \gamma^{-1}$, the bounds (8.10) – (8.12) are stronger, so it suffices to establish those.

The argument for (8.10) is very similar to the argument used in the proof of Lemma B.1. Indeed, for any function $f: \gamma\mathbb{Z}^2 \rightarrow \mathbb{R}$, we set

$$\Delta_\gamma f(x) = \gamma^{-2} \sum_{\substack{\bar{x} \in \gamma\mathbb{Z}^2 \\ \bar{x} \sim x}} (f(\bar{x}) - f(x)),$$

where $\bar{x} \sim x$ means that \bar{x} and x are adjacent in $\gamma\mathbb{Z}^2$. Similar to (B.1), for fixed $\omega \in \mathbb{R}^2$ we set $e_\omega : x \mapsto e^{-i\pi(\varepsilon/\gamma)\omega \cdot x}$, and we get that

$$-\Delta_\gamma e_\omega = 2\gamma^{-2} \sum_{j=1}^2 (1 - \cos(\varepsilon\pi\omega_j)) e_\omega.$$

After a summation by parts, we get

$$\hat{K}_\gamma(\omega) = \frac{1}{2\gamma^{-2} \sum_{j=1}^2 (1 - \cos(\varepsilon\pi\omega_j))} c_{\gamma,1} \sum_{x \in \gamma\mathbb{Z}^2} \gamma^2 (-\Delta_\gamma \tilde{\mathfrak{K}}(x)) e_\omega(x), \quad (8.14)$$

where $\tilde{\mathfrak{K}}(x) = \mathfrak{K}(x)$ for $x \neq 0$ and $\tilde{\mathfrak{K}}(0) = 0$.

Recalling our scaling ((2.16) and (2.17)), it is easy to see that

$$\frac{1}{\gamma^{-2} \sum_{j=1}^2 (1 - \cos(\varepsilon\pi\omega_j))} \leq C \frac{1}{\gamma^2 |\omega|^2},$$

uniformly over ε and $|\omega_j| \leq N + \frac{1}{2}$. On the other hand the fact that \mathfrak{K} is a \mathcal{C}^2 function with compact support shows that the sum on the right-hand side of (8.14) is bounded when restricted to points x that are not 0 or adjacent to 0. For those five points x which are, we bound $\Delta_\gamma \tilde{\mathfrak{K}}(x) \leq \frac{8}{\gamma^2}$. Hence, the sum over these five points weighted with γ^2 is bounded by 40. This finishes the proof of (8.10).

The arguments for (8.11) and (8.12) are almost identical to the argument for (8.10). Indeed, performing the same summation by parts as in (8.14), we get

$$|\partial_j \hat{K}_\gamma(\omega)| \leq C \frac{\gamma}{|\gamma\omega|^2} \sum_{x \in \gamma\mathbb{Z}^2} \gamma^2 |-\Delta_\gamma(x_j \mathfrak{K}(x))|,$$

and

$$|\partial_j^2 \hat{K}_\gamma(\omega)| \leq C \frac{\gamma^2}{|\gamma\omega|^2} \sum_{x \in \gamma\mathbb{Z}^2} \gamma^2 |-\Delta_\gamma(x_j^2 \mathfrak{K}(x))|.$$

Note that the problem caused by the fact that $\mathfrak{K}(0)$ was defined replaced by 0 has disappeared in these expressions. By assumption, both $x_j \mathfrak{K}(x)$ and $x_j^2 \mathfrak{K}(x)$ are \mathcal{C}^2 functions with compact support, so that the sums appearing here are uniformly bounded. This shows (8.11) and (8.12).

Let us proceed to the proof of (8.13). From (8.10), we get the existence of $\bar{c} > 0$ such that for $\omega \geq \bar{c}\gamma^{-1}$, we have $\hat{K}_\gamma(\omega) \leq \frac{1}{2}$. Hence, (8.13) holds for such ω . Next, we treat ω with $|\omega| < \underline{c}\gamma^{-1}$ for a $\underline{c} > 0$ to be fixed below. For such ω (8.4) implies the existence of \underline{C} such that

$$1 - \hat{K}_\gamma(\omega) \geq \pi^2 |\omega|^2 \gamma^2 - \underline{C} |\omega|^3 \gamma^3 \geq (\pi^2 - \underline{C} \underline{c}) |\omega|^2 \gamma^2,$$

which can be bounded from below by $\frac{\pi^2}{2} |\omega|^2 \gamma^2$ if we choose \underline{c} small enough.

Finally, in order to treat the case $\underline{c}\gamma^{-1} \leq |\omega| \leq \bar{c}\gamma^{-1}$, we observe that the Riemann sums

$$K_\gamma(\gamma^{-1}\omega) = c_{\gamma,1} \sum_{x \in \gamma \mathbb{Z}_*^2} \gamma^2 \hat{\mathfrak{K}}(x) e^{-i\pi c_{\gamma,2} \omega \cdot x}$$

approximate $\widehat{\mathfrak{K}}(\omega)$ uniformly for $|\omega| \in [\underline{c}, \bar{c}]$. On the other hand, $\widehat{\mathfrak{K}}$ is the Fourier transform of a probability measure with a density on \mathbb{R}^2 , and as such, it is continuous and has $|\widehat{\mathfrak{K}}(\omega)| < 1$ if $\omega \neq 0$. In particular, $|\widehat{\mathfrak{K}}(\omega)|$ is bounded away from 1 uniformly for $|\omega| \in [\underline{c}, \bar{c}]$. Combining these facts, we see that for γ small enough, $K_\gamma(\omega)$ is bounded away from 1 uniformly in $\underline{c}\gamma^{-1} \leq |\omega| \leq \bar{c}\gamma^{-1}$. This shows (8.13). \square

The next lemma provides some pointwise estimates on the kernels $P_t^\gamma \star K_\gamma$ that are used in the proof of Lemma 4.5. On Λ_ε , $P_t^\gamma(\cdot)$ is a Markov kernel for every $t \geq 0$:

$$x \in \Lambda_\varepsilon \Rightarrow P_t^\gamma(x) \geq 0 \quad \text{and} \quad \sum_{x \in \Lambda_\varepsilon} \varepsilon^2 P_t^\gamma(x) = 1. \quad (8.15)$$

Recall our convention to define K_γ and P_t^γ for all $x \in \mathbb{T}^2$ by extending them as trigonometric polynomials of degree $\leq N$. We note that for $x \notin \Lambda_\varepsilon$, the properties $K_\gamma(x) = c_{\gamma,1} \frac{\gamma^2}{\varepsilon^2} \hat{\mathfrak{K}}(x)$ and $P_t^\gamma(x) \geq 0$ do not hold in general.

Lemma 8.3. *Let $\gamma_0 > 0$ be the constant introduced in Lemma 8.2. For every $T > 0$, there exists a constant $C = C(T)$ such that for all $0 < \gamma < \gamma_0$, $0 \leq t \leq T$ and $x \in \mathbb{T}^2$, we have*

$$|P_t^\gamma \star K_\gamma(x)| \leq C(t^{-1}(\log(\gamma^{-1}))^2 \wedge \gamma^{-2} \log(\gamma^{-1})), \quad (8.16)$$

and

$$|x|^2 |P_t^\gamma \star K_\gamma(x)| \leq C \log(\gamma^{-1}). \quad (8.17)$$

Proof. We write

$$|P_t^\gamma \star K_\gamma(x)| \leq \frac{1}{4} \sum_{\omega \in \{-N, \dots, N\}^2} \exp(t\gamma^{-2}(\hat{K}_\gamma(\omega) - 1)) |\hat{K}_\gamma(\omega)|.$$

We first bound the sum over ω satisfying $|\omega| \leq \gamma^{-1}$. According to (8.7) and (8.13), there exists $C_1 > 0$ such that for $0 < \gamma < \gamma_0$

$$\sum_{|\omega| \leq \gamma^{-1}} \exp(t\gamma^{-2}(\hat{K}_\gamma(\omega) - 1)) |\hat{K}_\gamma(\omega)| \leq \sum_{|\omega| \leq \gamma^{-1}} \exp\left(-\frac{t}{C_1} |\omega|^2\right) \leq C(t^{-1} \wedge \gamma^{-2}),$$

for a universal constant C . On the other hand, using (8.10) and (8.13), we get

$$\begin{aligned} & \sum_{\substack{\omega \in \{-N, \dots, N\}^2 \\ |\omega| > \gamma^{-1}}} \exp(t\gamma^{-2}(\hat{K}_\gamma(\omega) - 1)) |\hat{K}_\gamma(\omega)| \\ & \leq C \sum_{\substack{\omega \in \{-N, \dots, N\}^2 \\ |\omega| > \gamma^{-1}}} \exp\left(-\frac{t}{C_1 \gamma^2}\right) \frac{1}{|\gamma \omega|^2}. \end{aligned}$$

If $t \geq 4C_1\gamma^2 \log(\gamma^{-1})$ (and recalling that according to (2.16), $N \leq \gamma^{-2}$), then

$$\sum_{\substack{\omega \in \{-N, \dots, N\}^2 \\ |\omega| > \gamma^{-1}}} \exp\left(-\frac{t}{C_1\gamma^2}\right) \frac{1}{|\gamma\omega|^2} \leq 5\gamma^{-4} \exp\left(-\frac{t}{C_1\gamma^2}\right) \leq 5\gamma^{-4}\gamma^4 = 5.$$

Else, if $t < 4C_1\gamma^2 \log(\gamma^{-1})$, we have

$$\sum_{\substack{\omega \in \{-N, \dots, N\}^2 \\ |\omega| > \gamma^{-1}}} \exp\left(-\frac{t}{C_1\gamma^2}\right) \frac{1}{|\gamma\omega|^2} \leq C\gamma^{-2} \log(\gamma^{-1}) \leq Ct^{-1} (\log(\gamma^{-1}))^2,$$

which shows (8.16).

The proof of (8.17) is again similar to the proof of Lemma B.1 and the proof of Lemma 8.2. As in those arguments, we define a discrete Laplace operator, this time acting on functions $f: \mathbb{Z}^2 \rightarrow \mathbb{R}$ and defined as

$$\underline{\Delta}f(\omega) = \sum_{\substack{\bar{\omega} \in \mathbb{Z}^2 \\ \bar{\omega} \sim \omega}} (f(\bar{\omega}) - f(\omega)). \quad (8.18)$$

As above, for fixed $x \in \mathbb{T}^2$ we set $e_x: \omega \mapsto e^{i\pi\omega \cdot x}$, and we get that

$$-\underline{\Delta}e_x = 2 \sum_{j=1}^2 (1 - \cos(\pi x_j)) e_x.$$

In this way, after a summation by parts over \mathbb{Z}^2 , we get for all $x \in \mathbb{T}^2$ and $t \geq 0$ that

$$\begin{aligned} |P_t^\gamma \star K_\gamma(x)| &\leq \frac{1}{2 \sum_{j=1}^2 (1 - \cos(\pi x_j))} \\ &\times \frac{1}{4} \sum_{\omega \in \{-N-1, \dots, N+1\}^2} \left| -\underline{\Delta} [\exp(t\gamma^{-2}(\hat{K}_\gamma(\omega) - 1)) \hat{K}_\gamma(\omega)] \right|. \end{aligned} \quad (8.19)$$

Recall that the \hat{P}_t^γ and \hat{K}_γ are defined as zero outside of $\{-N, \dots, N\}^2$. Hence by (8.18), the discrete Laplacian of $\hat{P}_t^\gamma \hat{K}_\gamma$ is zero outside of $\{-N-1, \dots, N+1\}^2$.

We treat the sum over the boundary points, i.e. over those $\omega = (\omega_1, \omega_2)$ for which at least one ω_j satisfies $|\omega_j| = N$ or $|\omega_j| = N+1$, first. For such an ω we bound brutally using (8.10)

$$\left| -\underline{\Delta} \hat{P}_t^\gamma \hat{K}_\gamma(\omega) \right| \leq 4 \left| \hat{P}_t^\gamma \hat{K}_\gamma(\omega) \right| + \sum_{\substack{\bar{\omega} \in \mathbb{Z}^2 \\ \bar{\omega} \sim \omega}} \left| \hat{P}_t^\gamma \hat{K}_\gamma(\bar{\omega}) \right| \leq \frac{C}{|\gamma\omega|^2} \leq C\gamma^2.$$

There are $16N + 8 \leq C\gamma^{-2}$ such boundary points, so that the sum over these points in (8.19) is bounded uniformly in $0 < \gamma < \gamma_0$ and t .

In order to bound the discretised Laplacian appearing on the right-hand side of (8.19) at interior points ω , it is useful to pass to continuous coordinates. If $f: \mathbb{R}^2 \rightarrow \mathbb{R}$ is C^2 , then for $\omega \in \mathbb{Z}^2$, equation (8.18) can be rewritten as

$$\begin{aligned} \left| \underline{\Delta}f(\omega) \right| &= \left| \sum_{j=1,2} \int_{-1}^1 \partial_j^2 f(\omega + \tau e_j) (1 - |\tau|) d\tau \right| \\ &\leq \sum_{j=1,2} \sup_{\tau \in [-1,1]} \left| \partial_j^2 f(\omega + \tau e_j) \right|, \end{aligned} \quad (8.20)$$

where e_1 and e_2 are the standard unit basis vectors of \mathbb{R}^2 . In order to apply this to bound (8.19), we calculate for $j = 1, 2$

$$\begin{aligned} & \partial_j^2 (e^{\lambda(\hat{K}_\gamma-1)} \hat{K}_\gamma) \\ &= e^{\lambda(\hat{K}_\gamma-1)} \left(\lambda^2 (\partial_j \hat{K}_\gamma)^2 \hat{K}_\gamma + \lambda (\partial_j^2 \hat{K}_\gamma) \hat{K}_\gamma + \partial_j^2 \hat{K}_\gamma + 2\lambda (\partial_j \hat{K}_\gamma)^2 \right), \end{aligned} \quad (8.21)$$

where to improve readability we have dropped the arguments ω and set $\lambda := t\gamma^{-2}$.

We now use the bounds derived in Lemmas 8.1 and 8.2 to bound the terms on the right-hand side of this expression one by one. For the first term, we write for $0 < \gamma < \gamma_0$, $|\omega| \leq \gamma^{-1}$ and $j = 1, 2$

$$\begin{aligned} & \sup_{\tau \in [-1,1]} e^{\lambda(\hat{K}_\gamma(\omega+\tau e_j)-1)} \lambda^2 (\partial_j \hat{K}_\gamma(\omega+\tau e_j))^2 \hat{K}_\gamma(\omega+\tau e_j) \\ & \leq C \sup_{\tau \in [-1,1]} \exp\left(-\frac{t}{C_1} |\omega+\tau e_j|^2\right) t^2 |\omega+\tau e_j|^2 \\ & \leq C \exp\left(-\frac{t}{2C_1} |\omega|^2\right) e^{\frac{t}{C_1}} 2t^2 (|\omega|^2 + 1), \end{aligned}$$

where we have used (8.7), (8.8), and (8.13) in the first inequality. The sum over $\{\omega \in \mathbb{Z}^2: |\omega| \leq \gamma^{-1}\}$ of the terms appearing in the last line is bounded uniformly in $0 \leq t \leq T$ and $0 < \gamma < 1$.

For the second term, we write in a similar way for $|\omega| \leq \gamma^{-1}$ and $j = 1, 2$

$$\begin{aligned} & \sup_{\tau \in [-1,1]} e^{\lambda(\hat{K}_\gamma(\omega+\tau e_j)-1)} \lambda \partial_j^2 \hat{K}_\gamma(\omega+\tau e_j) \hat{K}_\gamma(\omega+\tau e_j) \\ & \leq C \exp\left(-\frac{t}{2C_1} |\omega|^2\right) e^{\frac{t}{C_1}} t, \end{aligned}$$

where we have used (8.7), (8.9), and (8.13). The sum over $|\omega| \leq \gamma^{-1}$ of these terms is again uniformly bounded for $0 \leq t \leq T$ and $0 < \gamma < 1$.

For the third term on the right-hand side of (8.21) and $|\omega| \leq \gamma^{-1}$, we use the ω -independent bound

$$e^{\lambda(\hat{K}_\gamma-1)} (\partial_j^2 \hat{K}_\gamma) \leq C\gamma^2,$$

which follows immediately from (8.9) once we observe that $e^{\lambda(\hat{K}_\gamma-1)} \leq 1$.

Finally, for the fourth term we write

$$2e^{\lambda(\hat{K}_\gamma(\omega)-1)} \lambda (\partial_j \hat{K}_\gamma(\omega))^2 \leq C \exp\left(-\frac{t}{C_1} |\omega|^2\right) \frac{t}{\gamma^2} \gamma^4 |\omega|^2 \leq C\gamma^2,$$

which shows that the sum over $|\omega| \leq \gamma^{-1}$ of these terms is uniformly bounded.

For the sums over $|\omega| \geq \gamma^{-1}$, we note that by (8.13), for $0 < \gamma < \gamma_0$ the terms $e^{\lambda(\hat{K}_\gamma-1)} \lambda^2$ and $e^{\lambda(\hat{K}_\gamma-1)} \lambda$ are uniformly bounded for $|\omega| \geq \gamma^{-1}$, so that we can bound the whole expression on the right-hand side of (8.21) by

$$\begin{aligned} & C \left((\partial_j \hat{K}_\gamma)^2 \hat{K}_\gamma + (\partial_j^2 \hat{K}_\gamma) \hat{K}_\gamma + \partial_j^2 \hat{K}_\gamma + 2(\partial_j \hat{K}_\gamma)^2 \right) \\ & \leq C \left(\frac{\gamma^2}{|\gamma\omega|^6} + \frac{\gamma^2}{|\gamma\omega|^4} + \frac{\gamma^2}{|\gamma\omega|^2} \right). \end{aligned}$$

Taking local maxima over $\omega + \tau e_j$ only changes the constant in this expression. The sum over $\{\omega \in \{-N+1, \dots, N-1\}^2: |\omega| > \gamma^{-1}\}$ of this expression can be bounded by a constant times $\log(\gamma^{-1})$. Hence the proof is complete. \square

Finally, we summarise the regularising properties of these approximate heat operators P_t^γ as well as $P_t^\gamma \star K_\gamma$. We recall that the *usual* heat operator P_t satisfies

$$\|P_t X\|_{\mathcal{C}^{\nu+\beta}} \leq C(\beta) t^{-\frac{\beta}{2}} \|X\|_{\mathcal{C}^\nu} \quad (8.22)$$

for all $\nu \in \mathbb{R}$, $\beta \geq 0$ and $t \geq 0$.

In the following lemma, we discuss analogous bounds for the approximate operators. Recall that the operators P_t^γ and $K_\gamma \star$ are naturally defined on trigonometric polynomials of degree $\leq N$, and are otherwise viewed as Fourier multiplication operators (that is, they evaluate to 0 on trigonometric monomials of degree $> N$). Their behaviour is quite different on frequencies that are small compared to γ^{-1} and on frequencies that are large compared to γ^{-1} , so we treat those two cases separately. Our argument essentially follows [28, Lemma A.5].

Lemma 8.4. *Let γ_0 be the constant introduced in Lemma 8.2 and fix $c_1, c_2 > 0$, $T > 0$, and $\kappa > 0$.*

- (1) *For every $\beta > 0$ and $0 \leq \lambda \leq 1$, there exists a constant $C = C(c_1, T, \kappa, \beta, \lambda)$ such that for all functions $X: \mathbb{T}^2 \rightarrow \mathbb{R}$ with $\hat{X}(\omega) = 0$ for $|\omega| \geq c_1 \gamma^{-1}$, we have for all $0 < \gamma < \gamma_0$, $0 \leq t \leq T$ and $\nu \in \mathbb{R}$*

$$\|P_t^\gamma X\|_{\mathcal{C}^{\nu+\beta-\kappa}} \leq C t^{-\frac{\beta}{2}} \|X\|_{\mathcal{C}^\nu}, \quad (8.23)$$

$$\|(P_t^\gamma - P_t)X\|_{\mathcal{C}^{\nu-\kappa}} \leq \left(C \gamma^\lambda t^{-\frac{\lambda}{2}} \|X\|_{\mathcal{C}^\nu} \right) \wedge \left(C \gamma^\lambda \|X\|_{\mathcal{C}^{\nu+\lambda}} \right), \quad (8.24)$$

$$\|K_\gamma \star X\|_{\mathcal{C}^{\nu-\kappa}} \leq C \|X\|_{\mathcal{C}^\nu}, \quad (8.25)$$

$$\|K_\gamma \star X - X\|_{\mathcal{C}^{\nu-\kappa}} \leq C \gamma^{2\lambda} \|X\|_{\mathcal{C}^{\nu+2\lambda}}. \quad (8.26)$$

- (2) *For every $\beta > 0$, there exists $C = C(c_2, T, \kappa, \beta)$ such that for any distribution X on \mathbb{T}^2 with $\hat{X}(\omega) = 0$ for $|\omega| \leq c_2 \gamma^{-1}$, we have for all $0 < \gamma < \gamma_0$, $0 \leq t \leq T$ and $\nu \in \mathbb{R}$*

$$\|P_t^\gamma X\|_{\mathcal{C}^{\nu+\beta-\kappa}} \leq C t^{-\beta} \|X\|_{\mathcal{C}^\nu}. \quad (8.27)$$

If furthermore $0 \leq \beta \leq 2$, then

$$\|P_t^\gamma (K_\gamma \star X)\|_{\mathcal{C}^{\nu+\beta-\kappa}} \leq C t^{-\frac{\beta}{2}} \|X\|_{\mathcal{C}^\nu}. \quad (8.28)$$

Remark 8.5. The (arbitrarily small) loss of regularity κ in these estimates has two reasons. It is caused on the one hand by the logarithmic divergence that appears in the last line of the estimate (8.29) below. This could probably be removed by performing additional integrations by parts. A second (arbitrarily small) loss of regularity is caused by the fact that in (8.29), we derive a bound on a discrete L^1 norm over Λ_ε rather than on the L^1 norm over \mathbb{T}^2 . This allows us to avoid boundary terms in the integration by parts.

Remark 8.6. The approximate heat operator P_t^γ is not as regularising on high frequencies as P_t is, as can be seen in the estimate (8.27). The usual rescaling is recovered when one also convolves with K_γ as in (8.28), but only for $\beta \leq 2$.

Proof. We start by discussing the regularisation properties of an abstract Fourier multiplication operator. Suppose that T is a Fourier multiplication operator with symbol $\hat{T}: \mathbb{Z}^2 \rightarrow \mathbb{C}$ vanishing for $\omega \notin \{-N, \dots, N\}^2$. We aim to derive a bound on $\|\delta_k(TX)\|_{L^\infty(\mathbb{T}^2)} = \|T\delta_k X\|_{L^\infty(\mathbb{T}^2)}$ for k satisfying $2^k \frac{3}{4} \leq N$. By definition, $\delta_k(TX) = 0$ for all larger k .

In order to avoid boundary terms in (8.29) below, we will actually derive bounds on $\|T\delta_k X\|_{L^\infty(\Lambda_\varepsilon)}$ and then use Lemma B.6 to conclude that

$$\|T\delta_k X\|_{L^\infty(\mathbb{T}^2)} \leq C(\kappa)\varepsilon^{-\frac{\kappa}{4}} \|T\delta_k X\|_{L^\infty(\Lambda_\varepsilon)} \leq C(\kappa)2^{\frac{\kappa}{2}k} \|T\delta_k X\|_{L^\infty(\Lambda_\varepsilon)},$$

i.e. we encounter an arbitrarily small loss of regularity.

We start by defining auxiliary cutoff functions $\bar{\chi}_k$ for $k \geq -1$ as follows. Suppose that $\bar{\chi}: \mathbb{R}^2 \rightarrow \mathbb{R}_+$ is smooth, symmetric, and that it coincides with the constant function 1 on $B(0, 3) \setminus B(0, \frac{1}{2})$ and that $\bar{\chi}$ vanishes outside of the annulus $B(0, 4) \setminus B(0, \frac{1}{3})$. For $k \in \mathbb{N}_0$, set $\bar{\chi}_k(\omega) = \bar{\chi}(2^{-k}\omega)$. Let $\bar{\chi}_{-1}: \mathbb{R}^2 \rightarrow \mathbb{R}_+$ be smooth and suppose that it coincides with 1 on $B(0, 3/2)$ and that it vanishes outside of $B(0, 2)$. Finally, set $\bar{\eta}_k(x) = \frac{1}{4} \sum_{\omega \in \mathbb{Z}^2} \bar{\chi}_k(\omega) e^{i\pi\omega \cdot x}$.

For any $k \geq -1$, we have according to Young's inequality

$$\begin{aligned} \|T\delta_k X\|_{L^\infty(\Lambda_\varepsilon)} &= \|T \star_\varepsilon \bar{\eta}_k \star_\varepsilon \delta_k X\|_{L^\infty(\Lambda_\varepsilon)} \leq \|T \star_\varepsilon \bar{\eta}_k\|_{L^1(\Lambda_\varepsilon)} \|\delta_k X\|_{L^\infty(\Lambda_\varepsilon)} \\ &\leq \|T \star_\varepsilon \bar{\eta}_k\|_{L^1(\Lambda_\varepsilon)} 2^{-k\nu} \|\delta_k X\|_{C^\nu}. \end{aligned}$$

In this calculation, we identified T with its integral kernel. In order to bound the L^1 norm appearing on the right-hand side of this estimate, we write for $k \geq 4$

$$\begin{aligned} &\|T \star \bar{\eta}_k\|_{L^1(\Lambda_\varepsilon)} \\ &= \sum_{x \in \Lambda_\varepsilon} \varepsilon^2 \left| \frac{1}{4} \sum_{\omega \in \{-N, \dots, N\}^2} \hat{T}(\omega) \bar{\chi}(2^{-k}\omega) e^{i\pi\omega \cdot x} \right| \\ &= \sum_{x \in 2^k \Lambda_\varepsilon} 2^{2k} \varepsilon^2 \left| \frac{1}{4} \sum_{\substack{\omega \in 2^{-k} \mathbb{Z}^2 \\ |\omega_j| \leq 2^{-k} N}} 2^{-2k} \hat{T}(2^k \omega) \bar{\chi}(\omega) e^{i\pi\omega \cdot x} \right| \\ &= \sum_{x \in 2^k \Lambda_\varepsilon} 2^{2k} \varepsilon^2 \frac{1}{1 + 2^{2k} \sum_{j=1,2} 2(1 - \cos(\pi 2^{-k} x_j))} \\ &\quad \times \left| \frac{1}{4} \sum_{\substack{\omega \in 2^{-k} \mathbb{Z}^2 \\ |\omega_j| \leq 2^{-k} N}} 2^{-2k} \hat{T}(2^k \omega) \bar{\chi}(\omega) (1 - \underline{\Delta}_k) e^{i\pi\omega \cdot x} \right| \\ &\leq C \sum_{x \in 2^k \Lambda_\varepsilon} 2^{2k} \varepsilon^2 \frac{1}{1 + |x|^2} \left| \frac{1}{4} \sum_{\substack{\omega \in 2^{-k} \mathbb{Z}^2 \\ |\omega_j| \leq 2^{-k} N}} 2^{-2k} (1 - \underline{\Delta}_k) (\hat{T}(2^k \omega) \bar{\chi}(\omega)) \right| \\ &\leq C \log(2^k) \sup_{\frac{1}{5} \leq |\omega| \leq 5} \left| (1 - \underline{\Delta}_k) (\hat{T}(2^k \omega) \bar{\chi}(\omega)) \right|. \end{aligned} \tag{8.29}$$

This time, the discrete Laplacian is defined as

$$\underline{\Delta}_k f(\omega) = 2^{2k} \sum_{\substack{\bar{\omega} \in 2^{-k} \mathbb{Z}^2 \\ |\omega_j| \leq 2^{-k} N \\ \bar{\omega} \sim \omega}} (f(\bar{\omega}) - f(\omega)),$$

and the nearest neighbour relation is to be understood with periodic boundary conditions on $\{\omega \in 2^{-k} \mathbb{Z}^2: |\omega_j| \leq 2^{-k} N\}$. The integration by parts is also understood with periodic boundary conditions on this discrete torus. (This is the reason why we had to restrict the sum to $x \in \Lambda_\varepsilon$ — only for such x is $\omega \mapsto e^{i\pi\omega \cdot x}$ an eigenfunction for $\underline{\Delta}_k$). In the fourth inequality in (8.29), we have used the fact that $\bar{\chi}$ has compact

support contained in $B(0, 4) \setminus B(0, \frac{1}{3})$, which ensures that for $k \geq 4$ the discrete Laplacian is supported in $B(0, 5) \setminus B(0, \frac{1}{5})$.

In all of our applications, \hat{T} is defined and smooth on all of \mathbb{R}^2 , and not only on the grid points in \mathbb{Z}^2 . As in (8.20), we can then replace the local supremum over the discrete Laplacian by a local supremum over the derivatives in the continuum. We get for $k \geq 4$ and $2^{k+1}5 \leq N$

$$\begin{aligned} & \sup_{\frac{1}{5} \leq |\omega| \leq 5} \left| (1 - \underline{\Delta}_k) \left(\hat{T}(2^k \omega) \bar{\chi}(\omega) \right) \right| \\ & \leq C \sup_{\frac{1}{5} \leq |\omega| \leq 5} \left| \hat{T}(2^k \omega) \right| + C \sup_{\substack{\frac{1}{5} \leq |\omega| \leq 5 \\ j=1,2}} \left| \partial_j \hat{T}(2^k \omega) \right| + C \sup_{\substack{\frac{1}{5} \leq |\omega| \leq 5 \\ j=1,2}} \left| \partial_j^2 \hat{T}(2^k \omega) \right|. \end{aligned} \quad (8.30)$$

In the case $k = -1, \dots, 3$, we get easily

$$\|T \star \bar{\eta}_k\|_{L^1(\mathbb{T}^2)} \leq \int_{\mathbb{T}^2} \left| \frac{1}{4} \sum_{\omega \in \{-N, \dots, N\}^2} \hat{T}(\omega) \bar{\chi}_k(\omega) \right| dx \leq C \sup_{|\omega| \leq 32} |T(\omega)|. \quad (8.31)$$

The case where boundary terms appear in (8.29) i.e. those k for which $2^k \frac{3}{4} \leq N < 2^k 5$ require some extra care (recall that we had already excluded k with $2^k \frac{3}{4} \geq N$ from our discussion in the beginning of the proof). For such k the term

$$\left| \frac{1}{4} \sum_{\substack{\omega \in 2^{-k} \mathbb{Z}^2 \\ |\omega_j| \leq 2^{-k} N}} 2^{-2k} (1 - \underline{\Delta}_k) \left(\hat{T}(2^k \omega) \bar{\chi}(\omega) \right) \right|$$

appearing in the sixth line of (8.29) has to be bounded in a different way. The sum over those $\omega = (\omega_1, \omega_2)$ for which $|\omega_1|, |\omega_2| < 2^{-k} N$ can be bounded by $C \sup_{\frac{1}{5} \leq |\omega| \leq 5} \left| (1 - \underline{\Delta}_k) \left(\hat{T}(2^k \omega) \bar{\chi}(\omega) \right) \right|$ as in (8.29) but the boundary terms for which at least one of the $|\omega_j| = 2^{-k} N$ require a different argument: We illustrate the case where $\omega = (\omega_1, N)$ with $|\omega_1| < 2^{-k} N$, the argument for the other cases being the same. In this case the discrete second derivative in the ω_2 direction appearing in $-\underline{\Delta}_k \left(\hat{T}(2^k \omega) \bar{\chi}(\omega) \right)$ gives (dropping the argument ω_1 to improve readability)

$$\begin{aligned} & \bar{\chi}(2^{-k} N) 2^{2k} \left(\hat{T}(N-1) + \hat{T}(-N) - 2\hat{T}(N) \right) \\ & + 2^k \left(\bar{\chi}(-N) - \bar{\chi}(N) \right) 2^k \left(\hat{T}(-N) - \hat{T}(N) \right) \\ & + 2^k \left(\bar{\chi}(2^{-k} N) - \bar{\chi}(2^{-k}(N-1)) \right) 2^k \left(\hat{T}(N) - \hat{T}(N-1) \right) \\ & + \hat{T}(2^{-k} N) 2^{2k} \left(\bar{\chi}(2^{-k}(N-1)) + \bar{\chi}(-2^{-k} N) - 2\bar{\chi}(2^{-k} N) \right). \end{aligned} \quad (8.32)$$

In the application we have in mind in this Lemma \hat{T} is the restriction of a $2N+1$ periodic function to the grid which is well adapted to our choice of the periodic boundary condition for $\underline{\Delta}_k$. Hence, the discrete first and second derivatives of \hat{T} can be bounded as before. However, $\bar{\chi}$ does not have this property. The discrete first derivative $(\bar{\chi}(-N) - \bar{\chi}(N))$ vanishes by the symmetry assumption on $\bar{\chi}$, but the discrete second derivative appearing in the last line of (8.32) can only be bounded

by

$$\begin{aligned} & 2^{2k} \left| \bar{\chi}(2^{-k}(N-1)) + \bar{\chi}(-2^{-k}N) - 2\bar{\chi}(2^{-k}N) \right| \\ &= 2^{2k} \left| \bar{\chi}(2^{-k}(N-1)) - \bar{\chi}(2^{-k}N) \right| \leq 2^k \sup_{\omega} |\nabla \bar{\chi}(\omega)|. \end{aligned}$$

Fortunately, there are only $\leq C2^k$ such boundary terms so that the Riemann factor 2^{-2k} gives the uniform bound (8.30) on these terms as well.

We are now ready to apply these bounds to the operators of interest. The bounds on $\delta_k X$ for $k \leq 3$ follow from (8.31), so from now on we assume that $k \geq 4$. For P_t^γ we have for $0 < \gamma < \gamma_0$, for $|\omega| \leq \gamma^{-1}$, and $j = 1, 2$

$$\begin{aligned} \hat{P}_t^\gamma(\omega) &= \exp\left(-\frac{t}{\gamma^2}(1 - \hat{K}_\gamma(\omega))\right) \leq \exp\left(-\frac{t|\omega|^2}{C_1}\right), \\ \partial_j \hat{P}_t^\gamma(\omega) &= \hat{P}_t^\gamma(\omega) \frac{t}{\gamma^2} \partial_j \hat{K}_\gamma(\omega) \leq C\sqrt{t} \exp\left(-\frac{t|\omega|^2}{C_1}\right) (\sqrt{t}|\omega|), \\ \partial_j^2 \hat{P}_t^\gamma(\omega) &= \hat{P}_t^\gamma(\omega) \left(\frac{t^2}{\gamma^4} (\partial_j \hat{K}_\gamma(\omega))^2 + \frac{t}{\gamma^2} \partial_j^2 \hat{K}_\gamma(\omega) \right) \\ &\leq Ct \exp\left(-\frac{t|\omega|^2}{C_1}\right) (t|\omega|^2 + 1), \end{aligned} \quad (8.33)$$

where we have used the bounds on \hat{K}_γ and its derivatives stated in Lemmas 8.1 and 8.2. The first of these bounds immediately yields, for any $\beta \geq 0$,

$$\sup_{\frac{1}{5} \leq |\omega| \leq 5} |\hat{P}_t^\gamma(2^k \omega)| \leq \exp\left(-\frac{t}{25C_1} 2^{2k}\right) \leq (2^{-k} t^{-\frac{1}{2}})^\beta \sup_{x>0} x^\beta e^{-\frac{x^2}{25C_1}}.$$

Similar bounds on the first and second derivatives of \hat{P}_t^γ follow in the same way from the remaining estimates in (8.33). Indeed, both of these bounds behave slightly better for small t – we gain a factor $t^{\frac{1}{2}}$ in the bound for the first derivative, and a factor t in the bound for the second derivative. The desired bound (8.23) then follows.

To get bounds on $P_t^\gamma - P_t$, we write

$$\begin{aligned} \left| \hat{P}_t^\gamma(\omega) - \hat{P}_t(\omega) \right| &= \left| \exp\left(-\frac{t}{\gamma^2}(1 - \hat{K}_\gamma(\omega))\right) - \exp(-t\pi^2|\omega|^2) \right| \\ &\leq \exp\left(-\frac{t|\omega|^2}{C_1}\right) \left| \frac{t}{\gamma^2}(1 - \hat{K}_\gamma(\omega)) - t\pi^2|\omega|^2 \right| \\ &\leq C \exp\left(-\frac{t|\omega|^2}{C_1}\right) t|\omega|^2 \gamma |\omega| \leq C(\lambda) \gamma^\lambda (t^{-\frac{\lambda}{2}} \wedge |\omega|^\lambda), \end{aligned}$$

the last inequality being valid for any $0 \leq \lambda \leq 1$. In the third inequality we have made use of (8.4). For the derivatives, we write

$$\begin{aligned} & \left| \partial_j \hat{P}_t^\gamma(\omega) - \partial_j \hat{P}_t(\omega) \right| \\ & \leq 2\pi^2 t |\omega| \left| \hat{P}_t^\gamma(\omega) - \hat{P}_t(\omega) \right| + \left| \hat{P}_t(\omega) \right| \left| -\frac{t}{\gamma^2} \partial_j \hat{K}_\gamma(\omega) - 2t\pi^2 \omega_j \right| \\ & \leq C\gamma + C e^{-\frac{t|\omega|^2}{C_1}} t \gamma |\omega|^2 \leq C\gamma. \end{aligned}$$

Here, in the second inequality we have used (8.5). Note that as above in the bound for $\partial_j \hat{P}_t^\gamma$ we gain a factor \sqrt{t} with respect to the bound for \hat{P}_t^γ .

Finally, for the second derivatives we have using (8.5) and (8.6)

$$\begin{aligned} & \left| \partial_j^2 \hat{P}_t^\gamma(\omega) - \partial_j^2 \hat{P}_t(\omega) \right| \\ & \leq |4\pi^4 t^2 \omega_j^2 - 2\pi^2 t| \left| \hat{P}_t^\gamma(\omega) - \hat{P}_t(\omega) \right| \\ & \quad + \hat{P}_t^\gamma(\omega) \left(\left| \frac{t^2}{\gamma^4} (\partial_j \hat{K}_\gamma(\omega))^2 - 4\pi^4 t^2 \omega_j^2 \right| + \left| \frac{t}{\gamma^2} \partial_j^2 \hat{K}_\gamma(\omega) + 2\pi^2 t \right| \right) \\ & \leq C t^{\frac{1}{2}} \gamma. \end{aligned}$$

We can now deduce (8.24) by repeating the argument below (8.33).

The bounds (8.25) and (8.26) follow immediately from (8.7), (8.8), and (8.9) as well as the bound

$$|\hat{K}_\gamma(\omega) - 1| \leq C \gamma^2 |\omega|^2,$$

uniformly in $|\omega| \leq \gamma^{-1}$, which is an immediate consequence of (8.4).

We now pass to the bounds for $|\omega| \geq \gamma^{-1}$. In order to get (8.27), we write for $|\omega| \geq \gamma^{-1}$ with $\omega \in \{-N, \dots, N\}^2$

$$\begin{aligned} \hat{P}_t^\gamma(\omega) &= \exp\left(-\frac{t}{\gamma^2}(1 - \hat{K}_\gamma(\omega))\right) \leq \exp\left(-\frac{t}{C_1 \gamma^2}\right) \leq \exp\left(-\frac{t|\omega|}{2C_1}\right) \\ \partial_j \hat{P}_t^\gamma(\omega) &= \hat{P}_t^\gamma(\omega) \frac{t}{\gamma^2} \partial_j \hat{K}_\gamma(\omega) \leq C \gamma \exp\left(-\frac{t}{C_1 \gamma^2}\right) \frac{t}{\gamma^2}, \\ \partial_j^2 \hat{P}_t^\gamma(\omega) &= \hat{P}_t^\gamma(\omega) \left(\frac{t^2}{\gamma^4} (\partial_j \hat{K}_\gamma(\omega))^2 + \frac{t}{\gamma^2} \partial_j^2 \hat{K}_\gamma(\omega) \right) \\ &\leq C \gamma^2 \exp\left(-\frac{t}{C_1 \gamma^2}\right) \left(\frac{t^2}{\gamma^4} + \frac{t}{\gamma^2} \right). \end{aligned}$$

In this case, we write for $\beta \geq 0$

$$\sup_{\frac{1}{5} \leq |\omega| \leq 5} |\hat{P}_t^\gamma(2^k \omega)| \leq \exp\left(-\frac{t}{25C_1} 2^k\right) \leq (2^{-k} t^{-1})^\beta \sup_{x>0} x^\beta e^{-\frac{x}{25C_1}}.$$

and as before similar bounds on the first and second derivatives of \hat{P}_t^γ follow in the same way, so that (8.27) follows. Observe the exponent $t^{-\beta}$ which yields a weaker bound than the $t^{-\frac{\beta}{2}}$ one gets in the case of the heat operator P_t .

Finally, for $P_t^\gamma K_\gamma$ we write for $|\omega| \geq \gamma^{-1}$

$$|\hat{P}_t^\gamma(\omega) \hat{K}_\gamma(\omega)| \leq \exp\left(-\frac{t}{C_1 \gamma^2}\right) \frac{C}{\gamma^2 |\omega|^2} \leq C(\lambda) \frac{1}{t^\lambda |\omega|^{2\lambda}},$$

for any $0 \leq \lambda \leq 1$. For the derivatives, we get

$$\begin{aligned} & \left| \partial_j \hat{P}_t^\gamma(\omega) \hat{K}_\gamma(\omega) \right| + \left| \hat{P}_t^\gamma(\omega) \partial_j \hat{K}_\gamma(\omega) \right| \\ & \leq C \gamma \exp\left(-\frac{t}{C_1 \gamma^2}\right) \frac{t}{\gamma^2} \frac{1}{\gamma^2 |\omega|^2} + C \exp\left(-\frac{t}{C_1 \gamma^2}\right) \frac{\gamma}{\gamma^2 |\omega|^2} \leq C(\lambda) \frac{\gamma}{t^\lambda |\omega|^{2\lambda}}, \end{aligned}$$

and

$$\begin{aligned}
& \left| \partial_j^2 \left(e^{\frac{t}{\gamma^2} (\hat{K}_\gamma - 1)} \hat{K}_\gamma \right) \right| \\
& \leq e^{\frac{t}{C_1 \gamma^2}} \left(\frac{t^2}{\gamma^4} (\partial_j \hat{K}_\gamma)^2 \hat{K}_\gamma + \frac{t}{\gamma^2} (\partial_j^2 \hat{K}_\gamma) \hat{K}_\gamma + \partial_j^2 \hat{K}_\gamma + 2 \frac{t}{\gamma^2} (\partial_j \hat{K}_\gamma)^2 \right) \\
& \leq C e^{\frac{t}{C_1 \gamma^2}} \left(\frac{t^2}{\gamma^4} \frac{\gamma^2}{\gamma^6 |\omega|^6} + \frac{t}{\gamma^2} \frac{\gamma^2}{\gamma^6 |\omega|^6} + \frac{\gamma^2}{\gamma^2 |\omega|^2} + 2 \frac{t}{\gamma^2} \frac{\gamma^2}{\gamma^4 |\omega|^4} \right) \\
& \leq C \frac{\gamma^2}{t^\lambda |\omega|^{2\lambda}}.
\end{aligned}$$

Hence (8.28) follows as well. \square

Corollary 8.7. *Let γ_0 be the constant appearing in Lemma 8.2. For any $T > 0$, $\kappa > 0$, $\beta > 0$ and $0 \leq \lambda \leq 1$, there exists a constant $C = C(T, \kappa, \beta, \lambda)$ such that for every $0 < \gamma < \gamma_0$, $0 \leq t \leq T$, $\nu \in \mathbb{R}$ and distribution X on \mathbb{T}^2 ,*

$$\|(P_t^\gamma - P_t)X\|_{\mathcal{C}^{\nu-\kappa}} \leq \left(C \gamma^\lambda t^{-\lambda} \|X\|_{\mathcal{C}^\nu} \right) \wedge \left(C \gamma^\lambda \|X\|_{\mathcal{C}^{\nu+\lambda}} \right). \quad (8.34)$$

$$\|(P_t^\gamma K_\gamma - P_t)X\|_{\mathcal{C}^{\nu-\kappa}} \leq \left(C \gamma^\lambda t^{-\frac{\lambda}{2}} \|X\|_{\mathcal{C}^\nu} \right) \wedge \left(C \gamma^\lambda \|X\|_{\mathcal{C}^{\nu+\lambda}} \right). \quad (8.35)$$

Proof. To see (8.34), we split X into its Littlewood-Paley blocks $\delta_k X$, separating those with $2^k \leq \gamma^{-1}$ from those with $2^k > \gamma^{-1}$. For the blocks with $2^k \leq \gamma^{-1}$, the necessary bound is stated in (8.24). Indeed, this bound is even better, because it has the term $t^{-\lambda}$ replaced by $t^{-\frac{\lambda}{2}}$.

For the blocks with $2^k > \gamma^{-1}$, we bound brutally

$$\begin{aligned}
\|\delta_k (P_t^\gamma - P_t)X\|_{L^\infty(\mathbb{T}^2)} & \leq \|P_t^\gamma \delta_k X\|_{L^\infty(\mathbb{T}^2)} + \|P_t \delta_k X\|_{L^\infty(\mathbb{T}^2)} \\
& \leq 2^{-k(\nu+\lambda)} \left(\|P_t^\gamma \delta_k X\|_{\mathcal{C}^{\nu+\lambda}} + \|P_t \delta_k X\|_{\mathcal{C}^{\nu+\lambda}} \right) \\
& \leq 2^{-\nu k} \gamma^\lambda \left(\|P_t^\gamma \delta_k X\|_{\mathcal{C}^{\nu+\lambda}} + \|P_t \delta_k X\|_{\mathcal{C}^{\nu+\lambda}} \right). \quad (8.36)
\end{aligned}$$

Then the bound (8.34) follows from (8.27), and this is where we pick up the term $t^{-\lambda}$.

To get (8.35), we argue in a similar way. For $\delta_k X$ with $2^k \leq \gamma^{-1}$, we bound

$$\begin{aligned}
\|\delta_k (P_t^\gamma K_\gamma - P_t)X\|_{L^\infty(\mathbb{T}^2)} \\
\leq \|\delta_k (P_t^\gamma (K_\gamma - 1)X)\|_{L^\infty(\mathbb{T}^2)} + \|\delta_k (P_t^\gamma - P_t)X\|_{L^\infty(\mathbb{T}^2)}.
\end{aligned}$$

The bound then follows from a combination of (8.23), (8.24), and (8.26). For $2^k > \gamma^{-1}$, we argue as in (8.36) with (8.27) replaced by (8.28). \square

APPENDIX A. EXTENSION BY TRIGONOMETRIC POLYNOMIALS

In this appendix, we recall some properties of the extensions of functions on a grid by trigonometric polynomials. For any function $Y: \Lambda_\varepsilon \rightarrow \mathbb{R}$, we define the Fourier transform \hat{Y} through

$$\hat{Y}(\omega) = \begin{cases} \sum_{x \in \Lambda_\varepsilon} \varepsilon^2 Y(x) e^{-i\pi\omega \cdot x} & \text{if } \omega \in \{-N, \dots, N\}^2, \\ 0 & \text{if } \omega \in \mathbb{Z}^2 \setminus \{-N, \dots, N\}^2. \end{cases} \quad (\text{A.1})$$

In this context, Fourier inversion states

$$Y(x) = \frac{1}{4} \sum_{\omega \in \mathbb{Z}^2} \hat{Y}(\omega) e^{i\pi\omega \cdot x} \quad \text{for all } x \in \Lambda_\varepsilon. \quad (\text{A.2})$$

It is thus natural to extend Y to all of $\mathbb{T}^2 = [-1, 1]^2$ by taking (A.2) as a definition of $Y(x)$ for $x \in \mathbb{T}^2 \setminus \Lambda_\varepsilon$. More explicitly, for $Y: \Lambda_\varepsilon \rightarrow \mathbb{R}$, we define $(\text{Ext } Y): \mathbb{T}^2 \rightarrow \mathbb{R}$ as

$$\begin{aligned} \text{Ext } Y(x) &= \frac{1}{4} \sum_{\omega \in \{-N, \dots, N\}^2} \sum_{y \in \Lambda_\varepsilon} \varepsilon^2 e^{i\pi\omega \cdot (x-y)} Y(y) \\ &= \sum_{y \in \Lambda_\varepsilon} \frac{\varepsilon^2}{4} Y(y) \prod_{j=1,2} \frac{\sin\left(\frac{\pi}{2}(2N+1)(x_j - y_j)\right)}{\sin\left(\frac{\pi}{2}(x_j - y_j)\right)}. \end{aligned}$$

The extension of Y defined in this way is smooth and real valued. The function $\text{Ext } Y$ is the unique trigonometric polynomial of degree $\leq N$ that coincides with Y on Λ_ε . This extension has the nice property that several identities that should hold approximately are in fact exactly true.

First of all, by the continuous Fourier inversion we see that for $\omega \in \mathbb{Z}^2$

$$\hat{Y}(\omega) = \int_{\mathbb{T}^2} e^{-i\pi\omega \cdot x} (\text{Ext } Y)(x) dx,$$

i.e. the discrete Fourier transform of Y on Λ_ε and the continuous Fourier transform of its extension coincide. Furthermore, if $X, Y: \Lambda_\varepsilon \rightarrow \mathbb{R}$ and $Z := X \star_\varepsilon Y$, then the discrete Fourier transform (A.1) satisfies $\hat{Z}(\omega) = \hat{X}(\omega)\hat{Y}(\omega)$. As the Fourier transform of $\tilde{Z}(x) = \int_{\mathbb{T}^2} (\text{Ext } X)(x-y) (\text{Ext } Y)(y) dy$ satisfies the same identity, we can conclude that Z and \tilde{Z} coincide on Λ_ε . Finally, in the current context, Parseval's identity states

$$\begin{aligned} \sum_{x \in \Lambda_\varepsilon} \varepsilon^2 X(x) Y(x) &= \frac{1}{4} \sum_{\omega \in \mathbb{Z}^2} \hat{X}(\omega) \overline{\hat{Y}(\omega)} \\ &= \int_{\mathbb{T}^2} (\text{Ext } X)(x) (\text{Ext } Y)(x) dx. \end{aligned} \quad (\text{A.3})$$

Throughout the article, we use these nice identities to freely jump from discrete to continuous expressions, depending on which one is more convenient. When no confusion is possible, we omit the operator Ext and simply use the same symbol for a function $\Lambda_\varepsilon \rightarrow \mathbb{R}$ and its extension.

APPENDIX B. BESOV SPACES

The aim of this appendix is to recall some facts about Besov spaces on the torus \mathbb{T}^n , which we identify with $[-1, 1]^n$. We begin with a technical lemma.

Lemma B.1. *Let χ be a smooth function with compact support. For every $\varepsilon > 0$, let*

$$\eta_\varepsilon(x) = \sum_{\omega \in \mathbb{Z}^n} e^{i\pi\omega \cdot x} \chi(\varepsilon\omega) \quad (x \in \mathbb{T}^n).$$

For every $p, p' \in [1, +\infty]$ with $1/p + 1/p' = 1$, we have

$$\sup_{0 < \varepsilon \leq 1} \varepsilon^{n/p'} \|\eta_\varepsilon\|_{L^p(\mathbb{T}^n)} < +\infty.$$

Remark B.2. This lemma would follow from very simple scaling arguments if we were considering the continuous Fourier transform instead of the discrete Fourier series.

Proof. Since $\|\eta_\varepsilon\|_{L^p(\mathbb{T}^n)}^p \leq \|\eta_\varepsilon\|_{L^\infty(\mathbb{T}^n)}^{p-1} \|\eta_\varepsilon\|_{L^1(\mathbb{T}^n)}$, it suffices to prove the lemma for $p = 1$ and $p = \infty$. Since χ is bounded and compactly supported, the result is clear for $p = \infty$. In order to cover the case $p = 1$, we define

$$\check{\chi}_\varepsilon(x) = \varepsilon^n \sum_{\omega \in \varepsilon\mathbb{Z}^n} e^{i\pi\omega \cdot x} \chi(\omega),$$

so that $\varepsilon^n \eta_\varepsilon(\varepsilon x) = \check{\chi}_\varepsilon(x)$. It is then clear that $\check{\chi}_\varepsilon$ is bounded uniformly over ε and that $\|\eta_\varepsilon\|_{L^1(\mathbb{T}^n)} = \|\check{\chi}_\varepsilon\|_{L^1(\mathbb{T}^n/\varepsilon)}$. In order to conclude the proof, it thus suffices to check that $\check{\chi}_\varepsilon$ decays uniformly fast enough at infinity. In order to do so, we introduce

$$\underline{\Delta}_\varepsilon f(\omega) = \varepsilon^{-2} \sum_{\substack{\omega' \in \varepsilon\mathbb{Z}^n \\ \omega' \sim \omega}} (f(\omega') - f(\omega)),$$

where $\omega' \sim \omega$ is the usual nearest-neighbour relationship in $\varepsilon\mathbb{Z}^n$. Writing $e_x : \omega \mapsto e^{ix \cdot \omega}$, we observe that

$$\underline{\Delta}_\varepsilon e_x = 2\varepsilon^{-2} \sum_{i=1}^n (\cos(\varepsilon\pi x_i) - 1) e_x. \quad (\text{B.1})$$

For every positive integer k , repeated integration by parts yields

$$\sum_{\omega \in \varepsilon\mathbb{Z}^n} \underline{\Delta}_\varepsilon^k e_x(\omega) \chi(\omega) = \sum_{\omega \in \varepsilon\mathbb{Z}^n} e_x(\omega) \underline{\Delta}_\varepsilon^k \chi(\omega). \quad (\text{B.2})$$

Since χ is smooth and compactly supported, $|\underline{\Delta}_\varepsilon^k \chi|$ is bounded uniformly over ε , and thus

$$\left| \varepsilon^n \sum_{\omega \in \varepsilon\mathbb{Z}^n} e_x(\omega) \underline{\Delta}_\varepsilon^k \chi(\omega) \right| \leq C_k$$

for some constant C_k . Combining this with (B.1) and (B.2), we obtain that

$$|\check{\chi}_\varepsilon(x)| \leq \frac{C_k}{[\varepsilon^{-2} \sum_{i=1}^n (1 - \cos(\varepsilon\pi x_i))]^k}.$$

Hence, there exists C'_k such that uniformly over $\varepsilon \in (0, 1]$ and $x \in \mathbb{T}^n/\varepsilon$,

$$|\check{\chi}_\varepsilon(x)| \leq \frac{C'_k}{|x|^{2k}}, \quad (\text{B.3})$$

and this is sufficient to conclude the proof. \square

We learn from [3, Proposition 2.10] that one can find smooth rotationally invariant functions $\tilde{\chi}, \chi$ taking values in $[0, 1]$ and such that

$$\text{supp } \tilde{\chi} \subseteq B(0, 4/3), \quad (\text{B.4})$$

$$\text{supp } \chi \subseteq B(0, 8/3) \setminus B(0, 3/4), \quad (\text{B.5})$$

$$\forall \omega \in \mathbb{R}^d, \quad \tilde{\chi}(\omega) + \sum_{k=0}^{+\infty} \chi(\omega/2^k) = 1, \quad (\text{B.6})$$

where we write $\text{supp } f$ to denote the support of f , and $B(0, r)$ to denote the Euclidean ball of radius r . Recall that any distribution Z on \mathbb{T}^n can be written (see e.g. [3, Proposition 2.12]) as the Fourier series

$$Z(x) = \frac{1}{2^n} \sum_{\omega \in \mathbb{Z}^d} \hat{Z}(\omega) e^{i\pi\omega \cdot x},$$

where

$$\hat{Z}(\omega) = \int_{\mathbb{T}^d} Z(x) e^{-i\pi\omega \cdot x} dx;$$

(this expression has to be interpreted as testing the distribution Z against the test function $x \mapsto e^{-i\pi\omega \cdot x}$). We use the partition of unity provided by (B.6) to decompose any distribution u over \mathbb{T}^n as a sum of (smooth) functions with localized spectrum. More precisely, we let

$$\chi_{-1} = \tilde{\chi}, \quad \chi_k = \chi(\cdot/2^k) \quad (k \geq 0), \quad (\text{B.7})$$

$$\delta_k Z(x) = \frac{1}{2^n} \sum_{\omega \in \mathbb{Z}^d} \chi_k(\omega) \hat{Z}(\omega) e^{i\pi\omega \cdot x} \quad (k \geq -1),$$

so that

$$Z = \sum_{k=-1}^{+\infty} \delta_k Z.$$

For any $\nu \in \mathbb{R}$, and $1 \leq p, q \leq \infty$ we define

$$\|Z\|_{\mathcal{B}_{p,q}^\nu} := \left(\sum_{k=-1}^{\infty} (2^{k\nu} \|\delta_k Z\|_{L^p(\mathbb{T}^n)})^q \right)^{\frac{1}{q}} = \left\| (2^{k\nu} \|\delta_k Z\|_{L^p(\mathbb{T}^n)})_{k \geq -1} \right\|_{\ell^q}$$

with the usual interpretation (as a supremum) for $q = \infty$. We define the Besov space $\mathcal{B}_{p,q}^\nu$ as the closure of $\mathcal{C}^\infty(\mathbb{T}^n)$ with respect to this norm. Note that if $p, q = \infty$, this deviates slightly from the standard definition to take all distributions for which this norm is finite. Our choice to take a slightly smaller space has the advantage to yield separable spaces.

The operator δ_k is best thought of as a convolution operator. Letting

$$\eta_k(x) = \frac{1}{2^n} \sum_{\omega \in \mathbb{Z}^d} \chi_k(\omega) e^{i\pi\omega \cdot x} \quad (k \geq -1), \quad (\text{B.8})$$

we observe that

$$\delta_k Z = \eta_k \star Z, \quad (\text{B.9})$$

where \star denotes the convolution in \mathbb{T}^n .

Lemma B.3. *For every $\nu > 0$, there exists $C = C(\nu)$ such that for any $R \geq 1$, if $\hat{Z}(\omega) = 0$ for all $\omega \in \mathbb{Z}^n$ such that $|\omega| > R$, then*

$$\|Z\|_{L^\infty(\mathbb{T}^n)} \leq CR^\nu \|Z\|_{\mathcal{C}^{-\nu}}.$$

Proof. By definition of the Besov norm

$$\|\delta_k Z\|_{L^\infty(\mathbb{T}^n)} \leq 2^{k\nu} \|Z\|_{\mathcal{C}^{-\nu}}.$$

Let l be the smallest integer such that $2^l > R$. The assumption on Z ensures that

$$Z = \sum_{k=0}^l \delta_k Z,$$

hence

$$\|Z\|_{L^\infty(\mathbb{T}^n)} \leq \sum_{k=0}^l \|\delta_k Z\|_{L^\infty(\mathbb{T}^n)} \leq \|Z\|_{C^{-\nu}} \sum_{k=0}^l 2^{k\nu} \leq C 2^{l\nu} \|Z\|_{C^{-\nu}}.$$

We get the result by observing that $2^l \leq 2R$. \square

The following *Kolmogorov-like* lemma is used for deriving a priori bounds in Section 4.

Proposition B.4. *Let $r \mapsto Z(r, \cdot)$ be a cadlag process taking values in $C^\infty(\mathbb{T}^n)$. For any $\nu \in \mathbb{R}$, $1 < p < \infty$ and $\nu_* < \nu - \frac{n}{p}$, there exists a constant $C = C(p, n, \nu, \nu_*)$ such that*

$$\mathbb{E} \sup_{0 \leq r < \infty} \|Z(r, \cdot)\|_{C^{\nu_*}}^p \leq C \sup_{k \geq 0} \sup_{x \in \mathbb{T}^n} 2^{k\nu p} \mathbb{E} \sup_{0 \leq r < \infty} |\delta_k Z(r, x)|^p.$$

Proof. Let $\bar{\chi}: \mathbb{R}^n \rightarrow \mathbb{R}$ be a smooth non-negative function which is identical to 1 on the unit ball $B(0, 3)$ and with compact support contained in $B(0, 4)$. For $k \geq -1$, set

$$\bar{\eta}_k(x) = \frac{1}{2^n} \sum_{\omega \in \mathbb{Z}^n} e^{i\pi\omega \cdot x} \bar{\chi}(2^{-k}\omega) \quad (x \in \mathbb{T}^n).$$

For any $k \geq -1$, we have $\delta_k Z \star \bar{\eta}_k = \delta_k Z$, and hence by Hölder's inequality we get that for any $0 \leq r < \infty$,

$$\sup_{x \in \mathbb{T}^n} |\delta_k Z(r, x)| = \sup_{x \in \mathbb{T}^n} \left| \int_{\mathbb{T}^n} \delta_k Z(r, x-y) \bar{\eta}_k(y) dy \right| \leq \|\delta_k Z(r, \cdot)\|_{L^p(\mathbb{T}^n)} \|\bar{\eta}_k\|_{L^{p'}(\mathbb{T}^n)},$$

where $\frac{1}{p} + \frac{1}{p'} = 1$. Lemma B.1 implies that

$$\|\bar{\eta}_k\|_{L^{p'}(\mathbb{T}^n)} \leq C(p, n) 2^{\frac{kn}{p}},$$

so that

$$\|\delta_k Z(r, \cdot)\|_{L^\infty(\mathbb{T}^n)}^p \leq C(p, n) 2^{kn} \|\delta_k Z(r, \cdot)\|_{L^p(\mathbb{T}^n)}^p.$$

Using this estimate, we get

$$\begin{aligned} & \sup_{0 \leq r < \infty} \sup_{k \geq -1} 2^{kp\nu_*} \|\delta_k Z(r, \cdot)\|_{L^\infty(\mathbb{T}^n)}^p \\ & \leq \sup_{0 \leq r < \infty} \sum_{k \geq -1} 2^{kp\nu_*} \|\delta_k Z(r, \cdot)\|_{L^p(\mathbb{T}^n)}^p \\ & \leq C(p, n) \sum_{k \geq -1} 2^{kp(\nu_* + \frac{n}{p})} \sup_{0 \leq r < \infty} \|\delta_k Z(r, \cdot)\|_{L^p(\mathbb{T}^n)}^p. \end{aligned}$$

Therefore,

$$\begin{aligned} \mathbb{E} \sup_{0 \leq r < \infty} \|Z(r, \cdot)\|_{C^{\nu_*}}^p & \leq C(p, n) \sum_{k \geq 0} 2^{kp(\nu_* + \frac{n}{p})} \int_{\mathbb{T}^n} \mathbb{E} \sup_{0 \leq r < \infty} |\delta_k Z(r, x)|^p dx \\ & \leq C(p, n) \sum_{k \geq 0} 2^{kp(\nu_* + \frac{n}{p} - \nu)} \sup_{x \in \mathbb{T}^n} \sup_{k \geq 0} 2^{k\nu p} \mathbb{E} \sup_{0 \leq r < \infty} |\delta_k Z(r, x)|^p \\ & \leq C(p, n, \nu, \nu_*) \sup_{x \in \mathbb{T}^n} \sup_{k \geq 0} 2^{k\nu p} \mathbb{E} \sup_{0 \leq r < \infty} |\delta_k Z(r, x)|^p. \end{aligned}$$

\square

In Section 7, we make use of the following multiplicative inequality.

Lemma B.5. *Let $\beta < 0 < \nu$ with $\nu + \beta > 0$. Then the mapping $(Z_1, Z_2) \mapsto Z_1 Z_2$ defined for $Z_1, Z_2 \in \mathcal{C}^\infty$ extends uniquely to a continuous bilinear mapping from $\mathcal{C}^\nu \times \mathcal{C}^\beta \rightarrow \mathcal{C}^\beta$. That is, there exists a constant C depending only on ν and β such that*

$$\|Z_1 Z_2\|_{\mathcal{C}^\beta} \leq C \|Z_1\|_{\mathcal{C}^\nu} \|Z_2\|_{\mathcal{C}^\beta}.$$

The proof of this statement in the case of the full space can be found in Sections 2.6 and 2.5 of [3] (see in particular Theorem 2.85). The argument on the torus is exactly the same, so we do not replicate it here.

Finally, we provide a bound on the L^∞ norm of a function defined by extension from a grid by trigonometric polynomials.

Lemma B.6. *For $N \in \mathbb{N}$, set $\varepsilon = \frac{2}{2N+1}$ and let $\Lambda_\varepsilon = \{x \in \varepsilon\mathbb{Z}^n: -1 < x_i < 1 \text{ for } i = 1, \dots, n\}$. For any $Z: \Lambda_\varepsilon \rightarrow \mathbb{R}$, we define the extension*

$$\text{Ext } Z(x) = \sum_{z \in \Lambda_\varepsilon} \frac{\varepsilon^n}{2^n} Z(z) \prod_{j=1}^n \frac{\sin\left(\frac{\pi}{2}(2N+1)(x_j - z_j)\right)}{\sin\left(\frac{\pi}{2}(x_j - z_j)\right)},$$

defined for all $x \in \mathbb{T}^n$. Then for any $\kappa > 0$, there exists a constant $C = C(\kappa, n)$ such that

$$\|\text{Ext } Z\|_{L^\infty(\mathbb{T}^n)} \leq C \varepsilon^{-\kappa} \|Z\|_{L^\infty(\Lambda_\varepsilon)}.$$

Proof. For $p > 1$ we get from Young's inequality that

$$\begin{aligned} & \sum_{z \in \Lambda_\varepsilon} \frac{\varepsilon^n}{2^n} \prod_{j=1}^n \left| \frac{\sin\left(\frac{\pi}{2}(2N+1)(x_j - z_j)\right)}{\sin\left(\frac{\pi}{2}(x_j - z_j)\right)} \right| \\ & \leq \left(\sum_{z \in \Lambda_\varepsilon} \frac{\varepsilon^n}{2^n} \prod_{j=1}^n \left| \frac{\sin\left(\frac{\pi}{2}(2N+1)(x_j - z_j)\right)}{\sin\left(\frac{\pi}{2}(x_j - z_j)\right)} \right|^p \right)^{\frac{1}{p}} \\ & = (2N+1)^{\frac{n(p-1)}{p}} \left(\sum_{z \in \Lambda_\varepsilon} \prod_{j=1}^n \left| \frac{\sin\left(\frac{\pi}{2}(2N+1)(x_j - z_j)\right)}{(2N+1)\sin\left(\frac{\pi}{2}(x_j - z_j)\right)} \right|^p \right)^{\frac{1}{p}} \\ & \leq C(n) \varepsilon^{-\frac{n(p-1)}{p}} \sum_{\tilde{z} \in \{-N, \dots, N\}^n} (1 \wedge |\tilde{z}|^{-np}). \end{aligned}$$

Here in the last line we have made use of the index change $\tilde{z}_i = \lfloor 2(N+1)(x_i - z_i) \rfloor$ (interpreted with periodic boundary condition on $\{-N, \dots, N\}$). Now by choosing $p = \frac{n}{n-\kappa} > 1$ we obtain the desired result. \square

APPENDIX C. MARTINGALES AND AN ITÔ FORMULA WITH JUMPS

Let $M = (M(t))_{t \geq 0}$ be a cadlag square-integrable martingale (with $M(0) = 0$). Its *predictable quadratic variation* $\langle M \rangle_t$ is the unique predictable increasing (in the wide sense) process such that $M^2(t) - \langle M \rangle_t$ is a martingale. Its *bracket process* is

$$[M]_t = M^2(t) - 2 \int M(s^-) dM(s),$$

where $M(s^-)$ is the left limit of M at time s . A convenient way to think about these processes consists in noting that the bracket process is the limit in probability of

$$\sum_{i=0}^{n-1} (M(t_{i+1}) - M(t_i))^2$$

as the subdivision $0 = t_0 \leq \dots \leq t_n = t$ gets finer and finer ([36, Theorem 4.47]), while the predictable quadratic variation is the limit of

$$\sum_{i=0}^{n-1} \mathbb{E}[(M(t_{i+1}) - M(t_i))^2 \mid \mathcal{F}_{t_i}],$$

where $(\mathcal{F}_t)_{t \geq 0}$ is the underlying filtration ([36, Proposition 4.50]). In particular, if M is of finite total variation, then the bracket process is simply the sum of the jumps squared:

$$[M]_t = \sum_{0 < s \leq t} (\Delta_s M)^2,$$

where $\Delta_s M = M(s) - M(s^-)$. If M has continuous sample paths, then the predictable quadratic variation and the bracket process coincide ([36, Theorem 4.47]). In every case, $[M]_t - \langle M \rangle_t$ is a martingale ([36, Proposition 4.50]).

Let $(\sigma(t))_{t \geq 0}$ be a Feller process. Denote by \mathcal{L} its infinitesimal generator, with domain $\mathcal{D}(\mathcal{L})$. For $f \in \mathcal{D}(\mathcal{L})$, the process

$$M_f(t) := f(\sigma(t)) - f(\sigma(0)) - \int_0^t \mathcal{L}f(\sigma(s)) ds$$

is a martingale. Moreover, if $f^2 \in \mathcal{D}(\mathcal{L})$, then its predictable quadratic variation is given by

$$\langle M_f \rangle_t = \int_0^t \Gamma_f(\sigma(s)) ds,$$

where $\Gamma_f = \mathcal{L}(f^2) - 2f\mathcal{L}f$ is the so-called ‘‘carré du champ’’ (see for instance [41, Appendix B]). When \mathcal{L} is the operator defined in (2.5), we have

$$\Gamma_f(\sigma) = \sum_{j \in \Lambda_N} c_\gamma(\sigma, j) (f(\sigma^j) - f(\sigma))^2.$$

The Burkholder-Davis-Gundy inequality below plays a key role in our analysis.

Lemma C.1 (Burkholder-Davis-Gundy inequality). *Let M be a cadlag square integrable martingale. For any $p > 0$, there exists a constant $C = C(p)$ such that for all $T > 0$,*

$$\mathbb{E} \left[\sup_{0 \leq t \leq T} |M(t)|^p \right] \leq C \left(\mathbb{E} \left[\langle M \rangle_T^{\frac{p}{2}} \right] + \mathbb{E} \left[\sup_{0 \leq t \leq T} |\Delta_t M|^p \right] \right),$$

where we recall that $\Delta_t M := M(t) - M(t^-)$ denotes the jump of M at t .

Proof. The case of discrete-time martingales is covered by [35, Theorem 2.11]. The continuous-time case can be recovered by approximation. Alternatively, one can consult for instance [39]. \square

The following result, which can be found for instance in [49, Theorem II.7.31], is akin to the fundamental theorem of calculus, but with integrands that may have jumps.

Lemma C.2. *Let $X = (X^{(1)}, \dots, X^{(n)}) : \mathbb{R}_+ \rightarrow \mathbb{R}^n$ be a cadlag process such that for every i , $X^{(i)}$ is of bounded total variation. Let $f : \mathbb{R}^n \rightarrow \mathbb{R}$ be a continuously*

differentiable function. Then $f(X) : \mathbb{R} \rightarrow \mathbb{R}$ is of bounded total variation and

$$\begin{aligned} & f(X(t)) - f(X(0)) \\ &= \sum_{i=1}^n \int_{s=0}^t \frac{\partial f}{\partial x_i}(X(s^-)) dX^{(i)}(s) + \sum_{0 < s \leq t} \left\{ \Delta_s f(X) - \sum_{i=1}^n \frac{\partial f}{\partial x_i}(X(s^-)) \Delta_s X^{(i)} \right\}, \end{aligned} \quad (\text{C.1})$$

where we recall our notation $\Delta_s F = F(s) - F(s^-)$ and $\int_{s=0}^t = \int_{s \in (0,t]}$.

Remark C.3. In the setting of Lemma C.2, let us assume for simplicity that $n = 1$.

Recall that the bracket process of X is then given by

$$[X]_t = \sum_{0 < s \leq t} (\Delta_s X)^2.$$

If f is twice continuously differentiable, we can rewrite (C.1) as

$$\begin{aligned} f(X(t)) - f(X(0)) &= \int_{s=0}^t f'(X(s^-)) dX(s) + \frac{1}{2} \int_{s=0}^t f''(X(s^-)) d[X]_s \\ &\quad + \sum_{0 < s \leq t} \left\{ \Delta_s f(X) - f'(X(s^-)) \Delta_s X - \frac{1}{2} f''(X(s^-)) (\Delta_s X)^2 \right\}, \end{aligned}$$

which is nothing but Itô's formula, see [49, Theorem II.7.32].

APPENDIX D. MARTINGALE CHARACTERIZATION OF THE STOCHASTIC HEAT EQUATION

Recall that we write $(P_t)_{t \geq 0} = (e^{t\Delta})_{t \geq 0}$ for the semigroup associated with the Laplacian on \mathbb{T}^2 . Let W be a cylindrical Wiener process over $L^2(\mathbb{T}^2)$. The solution Z of the stochastic heat equation

$$\begin{cases} dZ = \Delta Z dt + \sqrt{2} dW \\ Z(0, \cdot) = z_0 \end{cases} \quad (\text{D.1})$$

is defined by

$$Z(t) = P_t z_0 + \sqrt{2} \int_0^t P_{t-s} dW(s).$$

If $\phi \in C^\infty(\mathbb{T}^2)$ is a smooth test function, and if we write (z, ϕ) for the evaluation of $z \in \mathcal{S}'(\mathbb{T}^2)$ against ϕ , we have

$$(Z(t), \phi) = (z_0, P_t \phi) + \sqrt{2} \int_0^t P_{t-s} \phi dW(s)$$

for every $t \geq 0$.

The goal of this section is to give a martingale characterization of the law of Z .

Theorem D.1 (Uniqueness for the martingale problem). *Let \bar{Z} be a random process in $\mathcal{C}(\mathbb{R}_+, \mathcal{S}'(\mathbb{T}^2))$. For every $\phi \in C^\infty(\mathbb{T}^2)$, define*

$$\mathcal{M}_\phi(t) = (\bar{Z}(t), \phi) - (\bar{Z}(0), \phi) - \int_0^t (\bar{Z}(s), \Delta \phi) ds$$

and

$$\Gamma_\phi(t) = \mathcal{M}_\phi^2(t) - 2t \|\phi\|_{L^2}^2.$$

If for every $\phi \in C^\infty(\mathbb{T}^2)$, the processes $(\mathcal{M}_\phi(t))_{t \geq 0}$ and $(\Gamma_\phi(t))_{t \geq 0}$ are local martingales, then \bar{Z} has the same law as Z the solution of (D.1) with initial condition $z_0 = \bar{Z}(0)$.

Remark D.2. Since \bar{Z} is assumed to take values in $\mathcal{C}(\mathbb{R}_+, \mathcal{S}'(\mathbb{T}^2))$, the martingales $(\mathcal{M}_\phi(t))_{t \geq 0}$ and $(\Gamma_\phi(t))_{t \geq 0}$ are automatically continuous.

Remark D.3. The fact that $(\Gamma_\phi(t))_{t \geq 0}$ is a local martingale identifies $(2t\|\phi\|_{L^2}^2)_{t \geq 0}$ as the quadratic variation of $(\mathcal{M}_\phi(t))_{t \geq 0}$. In particular, for every $t \geq 0$, $\mathcal{M}_\phi(t)$ is square-integrable,

$$\mathbb{E}[\mathcal{M}_\phi^2(t)] = 2t\|\phi\|_{L^2}^2, \quad (\text{D.2})$$

and $(\mathcal{M}_\phi(t))_{t \geq 0}$ and $(\Gamma_\phi(t))_{t \geq 0}$ are (true) martingales.

Remark D.4. Theorem D.1 remains true if we only know that \mathcal{M}_ϕ and Γ_ϕ are continuous local martingales for every trigonometric polynomial ϕ . Indeed, it suffices to argue by density using (D.2).

Our proof of Theorem D.1 begins by showing that the martingale property can be extended to smooth test functions that also depend on the time variable.

Lemma D.5 (Time-dependent martingale characterization). *Let \bar{Z} be a random process in $\mathcal{C}(\mathbb{R}_+, \mathcal{S}'(\mathbb{T}^2))$. For every $\psi \in \mathcal{C}^\infty(\mathbb{R}_+ \times \mathbb{T}^2)$, we let*

$$\mathcal{M}_\psi(t) = (\bar{Z}(t), \psi(t)) - (\bar{Z}(0), \psi(0)) - \int_0^t (\bar{Z}(s), (\partial_s + \Delta)\psi(s)) ds$$

and

$$\Gamma_\psi(t) = \mathcal{M}_\psi^2(t) - 2 \int_0^t \|\psi(s)\|_{L^2}^2 ds.$$

Under the assumptions of Theorem D.1, the processes $(\mathcal{M}_\psi(t))_{t \geq 0}$ and $(\Gamma_\psi(t))_{t \geq 0}$ are continuous martingales.

Proof. We decompose $(\bar{Z}(t), \psi(t)) - (\bar{Z}(0), \psi(0))$ into

$$(\bar{Z}(t), \psi(t)) - (\bar{Z}(0), \psi(t)) + (\bar{Z}(0), \psi(t)) - (\bar{Z}(0), \psi(0)). \quad (\text{D.3})$$

By the assumptions of Theorem D.1, we can rewrite the first difference as

$$\int_0^t (\bar{Z}(s), \Delta\psi(t)) ds + \mathcal{M}_{\psi(t)}(t),$$

and we decompose the integral further into

$$\begin{aligned} & \int_0^t (\bar{Z}(s), \Delta\psi(s)) ds + \int_0^t (\bar{Z}(s), \Delta(\psi(t) - \psi(s))) ds \\ &= \int_0^t (\bar{Z}(s), \Delta\psi(s)) ds + \int_{s=0}^t \int_{u=s}^t (\bar{Z}(s), \Delta\partial_u\psi(u)) du ds. \end{aligned} \quad (\text{D.4})$$

The second difference appearing in (D.3) can be rewritten as

$$\int_0^t (\bar{Z}(0), \partial_s\psi(s)) ds = \int_0^t (\bar{Z}(s), \partial_s\psi(s)) ds - \int_0^t (\bar{Z}(s) - \bar{Z}(0), \partial_s\psi(s)) ds.$$

By the assumptions of Theorem D.1, we can rewrite the second integral in the right-hand side above as

$$\int_{s=0}^t \left\{ \int_{u=0}^s (\bar{Z}(u), \Delta\partial_s(\psi(s))) du + \mathcal{M}_{\partial_s\psi(s)}(s) \right\} ds.$$

The double integral cancels with that appearing in the right-hand side of (D.4). We thus obtain

$$\begin{aligned} & (\bar{Z}(t), \psi(t)) - (\bar{Z}(0), \psi(0)) \\ &= \int_0^t (\bar{Z}(s), (\partial_s + \Delta)\psi(s)) ds + \mathcal{M}_{\psi(t)}(t) - \int_0^t \mathcal{M}_{\partial_s\psi(s)}(s) ds, \end{aligned}$$

so that

$$\mathcal{M}_\psi(t) = \mathcal{M}_{\psi(t)}(t) - \int_0^t \mathcal{M}_{\partial_s \psi(s)}(s) ds. \quad (\text{D.5})$$

By Remark D.3, $\mathcal{M}_\psi(t)$ is square-integrable for any $t \geq 0$. If we denote by $(\mathcal{F}_t)_{t \geq 0}$ the underlying filtration, we observe that

$$\mathbb{E}[\mathcal{M}_\psi(t+s) - \mathcal{M}_\psi(t) \mid \mathcal{F}_t] = \mathcal{M}_{\psi(t+s)}(t) - \mathcal{M}_{\psi(t)}(t) - \int_t^{t+s} \mathcal{M}_{\partial_r \psi(r)}(t) dr.$$

By linearity, this is equal to 0, and thus $(\mathcal{M}_\psi(t))_{t \geq 0}$ is a martingale. We now compute its quadratic variation. Since

$$t \mapsto \int_0^t \mathcal{M}_{\partial_s \psi(s)}(s) ds$$

and $t \mapsto \mathcal{M}_{\psi(t)}(t_0)$ are of bounded variation, we can write

$$d\langle \mathcal{M}_\psi \rangle_t = d\langle \mathcal{M}_{\psi(t)} \rangle_t = 2\|\psi(t)\|_{L^2}^2 dt,$$

that is,

$$\langle \mathcal{M}_\psi \rangle_t = 2 \int_0^t \|\psi(s)\|_{L^2}^2 ds,$$

and the proof is complete. \square

Proof of Theorem D.1. Let $\phi \in \mathcal{C}^\infty(\mathbb{T}^2)$, and $\bar{\phi} \in \mathcal{C}^\infty([0, t] \times \mathbb{T}^2)$ be the bounded solution of the backward heat equation

$$\begin{cases} (\partial_s + \Delta)\bar{\phi}(s) = 0 & \text{for } s \in (0, t), \\ \bar{\phi}(t) = \phi. \end{cases}$$

We learn from Lemma D.5 that

$$\mathcal{M}_{\bar{\phi}}(t) = (\bar{Z}(t), \bar{\phi}(t)) - (\bar{Z}(0), \bar{\phi}(0))$$

is a martingale with quadratic variation

$$2 \int_0^t \|\bar{\phi}(s)\|_{L^2}^2 ds.$$

Clearly, $\bar{\phi}(s) = P_{t-s}\phi$, so we can rewrite $\mathcal{M}_{\bar{\phi}}(t)$ as

$$\mathcal{M}_{\bar{\phi}}(t) = (\bar{Z}(t), \phi) - (\bar{Z}(0), P_t\phi)$$

and the quadratic variation as

$$2 \int_0^t \|P_{t-s}\phi\|_{L^2}^2 ds.$$

By polarization, it follows that the quadratic covariation at time t between $\mathcal{M}_{\bar{\phi}_1}$ and $\mathcal{M}_{\bar{\phi}_2}$ is

$$2 \int_0^t (P_{t-s}\phi_1, P_{t-s}\phi_2) ds,$$

for every $\phi_1, \phi_2 \in \mathcal{C}^\infty(\mathbb{T}^2)$. Since $\mathcal{M}_{\bar{\phi}}$ is a continuous martingale for every ϕ , the fact that the quadratic covariations are deterministic implies that the law of $(\mathcal{M}_{\bar{\phi}}(t))_{t \geq 0, \phi \in \mathcal{C}^\infty(\mathbb{T}^2)}$ is determined uniquely. Letting Z be the solution of (D.1) with $z_0 = \bar{Z}(0)$, we observe that the quadratic covariations of

$$\mathcal{M}'_\phi(t) := (Z(t), \phi) - (Z(0), P_t\phi)$$

are the same as those of $(\mathcal{M}_{\bar{\phi}}(t))_{t \geq 0, \phi \in \mathcal{C}^\infty(\mathbb{T}^2)}$, so the proof is complete. \square

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