QUANTITATIVE VERSION OF THE KIPNIS-VARADHAN THEOREM AND MONTE-CARLO APPROXIMATION OF HOMOGENIZED COEFFICIENTS

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Abstract. This article is devoted to the analysis of a Monte-Carlo method to approximate effective coefficients in stochastic homogenization of discrete elliptic equations. We consider the case of independent and identically distributed coefficients, and adopt the point of view of the random walk in a random environment. Given some final time $t > 0$, a natural approximation of the homogenized coefficients is given by the empirical average of the final squared positions rescaled by t of n independent random walks in n independent environments. Relying on a quantitative version of the Kipnis-Varadhan theorem combined with estimates of spectral exponents obtained by an original combination of pde arguments and spectral theory, we first give a sharp estimate of the error between the homogenized coefficients and the expectation of the rescaled final position of the random walk in terms of t. We then complete the error analysis by quantifying the fluctuations of the empirical average in terms of n and t, and prove a large-deviation estimate, as well as a central limit theorem. Our estimates are optimal, up to a logarithmic correction in dimension 2.

Keywords: random walk, random environment, stochastic homogenization, effective coefficients, Monte-Carlo method, quantitative estimates.

2010 Mathematics Subject Classification: 35B27, 60K37, 60H25, 65C05, 60H35, 60G50.

1. Main result and structure of the proof

1.1. Main result.

We consider the discrete elliptic operator $-\nabla^* \cdot A\nabla$, where $\nabla^*\cdot$ and ∇ are the discrete backward divergence and forward gradient, respectively. For all $x \in \mathbb{Z}^d$, $A(x)$ is the diagonal matrix whose entries are the conductances $\omega_{x,x+\mathbf{e}_i}$ of the edges $(x, x+\mathbf{e}_i)$ starting at x, where $(\mathbf{e}_i)_{i\in\{1,\ldots,d\}}$ denotes the canonical basis of $\mathbb{R}^d$. Let $\mathbb{B}$ denote the set of *unoriented* edges of $\mathbb{Z}^d$. We call the family of conductances $\omega = (\omega_e)_{e\in\mathbb{B}}$ the *environment*. This environment is *symmetric* in the sense that for all $x, y \in \mathbb{Z}^d$ with $|x-y| = 1$, we have $e = (x,y) = (y,x)$, so that $\omega_{x,y} = \omega_{y,x} = \omega_e$. The environment ω is random, and we write $\mathbb{P}$ for its distribution (with corresponding expectation $\mathbb{E}$). We make the following assumptions :

- (H1) the measure $\mathbb{P}$ is invariant under translations,
- (H2) the conductances are i. i. d.[1],
- (H3) there exists $0 < \alpha < \beta$ such that $\alpha \leqslant \omega_e \leqslant \beta$ almost surely.

Under these conditions, standard homogenization results ensure that there exists some *deterministic* symmetric matrix A_{hom} such that the solution operator of the deterministic continuous differential operator $-\nabla \cdot A_{\mathrm{hom}}\nabla$ describes the large scale behavior of the solution operator of the random discrete differential operator $-\nabla^* \cdot A\nabla$ almost surely (for this statement, (H2) can in fact be replaced by the weaker

[1](H2) obviously implies (H1) in the present form. Yet for most qualitative (and some quantitative) results (H2) can be weakened and may not imply (H1) any longer.

assumption that the measure $\mathbb{P}$ is ergodic with respect to the group of translations, see [Kü83]).

The operator $-\nabla^* \cdot A\nabla$ is the infinitesimal generator of a stochastic process $(X(t))_{t\in\mathbb{R}_+}$ which can be defined as follows. Given an environment ω, it is the Markov process whose jump rate from a site $x \in \mathbb{Z}^d$ to a neighbouring site y is given by $\omega_{x,y}$. We write $\mathbf{P}_x^\omega$ for the law of this process starting from $x \in \mathbb{Z}^d$.

It is proved in [KV86] that under the averaged measure $\mathbb{P}\mathbf{P}_0^\omega$, the rescaled process $\sqrt{\varepsilon}X(\varepsilon^{-1}t)$ converges in law, as ε tends to 0, to a Brownian motion whose infinitesimal generator is $-\nabla \cdot A_{\mathrm{hom}}\nabla$, or in other words, a Brownian motion with covariance matrix $2A_{\mathrm{hom}}$ (see also [AKS82, Kü83, Ko85] for prior results). We will use this fact to construct computable approximations of A_{hom}. As proved in [DFGW89], this invariance principle holds as soon as (H1) is true, (H2) is replaced by the ergodicity of the measure $\mathbb{P}$, and (H3) by the integrability of the conductances. Under the assumptions (H1-H3), [SS04] strengthens this result in another direction, showing that for almost every environment, $\sqrt{\varepsilon}X(\varepsilon^{-1}t)$ converges in law under $\mathbf{P}_0^\omega$ to a Brownian motion with covariance matrix $2A_{\mathrm{hom}}$. This has been itself extended to environments which do not satisfy the uniform ellipticity condition (H3), see [BB07, MP07, BP07, Ma08, BD10].

Let $(Y(t))_{t\in\mathbb{N}}$ denote the sequence of consecutive sites visited by the random walk $(X(t))_{t\in\mathbb{R}_+}$ (note that the "times" are different in nature for $X(t)$ and $Y(t)$). This sequence is itself a Markov chain that satisfies for any two neighbours $x, y \in \mathbb{Z}^d$:

$$\mathbf{P}_x^\omega[Y(1) = y] = \frac{\omega_{x,y}}{p_\omega(x)},$$

where $p_\omega(x) = \sum_{|z|=1} \omega_{x,x+z}$. We simply write $p(\omega)$ for $p_\omega(0)$. Let us introduce a "tilted" version of the law $\mathbb{P}$ on the environments, that we write $\tilde{\mathbb{P}}$ and define by

$$\mathrm{d}\tilde{\mathbb{P}}(\omega) = \frac{p(\omega)}{\mathbb{E}[p]}\,\mathrm{d}\mathbb{P}(\omega). \tag{1.1}$$

The reason why this measure is natural to consider is that it makes the environment seen from the position of the random walk Y a stationary process (see (3.2) for a definition of this process).

Interpolating between two integers by a straight line, we can think of Y as a continuous function on $\mathbb{R}_+$. With this in mind, it is also true that there exists a matrix $A_{\mathrm{hom}}^{\mathrm{disc}}$ such that, as ε tends to 0, the rescaled process $\sqrt{\varepsilon}Y(\varepsilon^{-1}t)$ converges in law under $\tilde{\mathbb{P}}\mathbf{P}_0^\omega$ to a Brownian motion with covariance matrix $2A_{\mathrm{hom}}^{\mathrm{disc}}$. Moreover, $A_{\mathrm{hom}}^{\mathrm{disc}}$ and A_{hom} are related by (see [DFGW89, Theorem 4.5 (ii)]) :

$$A_{\mathrm{hom}} = \mathbb{E}[p]\ A_{\mathrm{hom}}^{\mathrm{disc}} = 2d\mathbb{E}[\omega_e]\ A_{\mathrm{hom}}^{\mathrm{disc}}. \tag{1.2}$$

Given that the numerical simulation of Y saves some operations compared to the simulation of X (there is no waiting time to compute, and the running time is equal to the number of steps), we will focus on approximating $A_{\mathrm{hom}}^{\mathrm{disc}}$. More precisely, we fix once and for all some $\xi \in \mathbb{R}^d$ with $|\xi| = 1$, and define

$$\sigma_t^2 = t^{-1}\ \tilde{\mathbb{E}}\mathbf{E}_0^\omega[(\xi \cdot Y(t))^2], \tag{1.3}$$

$$\sigma^2 = 2\xi \cdot A_{\mathrm{hom}}^{\mathrm{disc}}\xi = \frac{2\xi \cdot A_{\mathrm{hom}}\xi}{\mathbb{E}[p]}. \tag{1.4}$$

It follows from results of [KV86] (or [DFGW89, Theorem 2.1]) that σ_t^2 tends to σ^2 as t tends to infinity. We now describe a Monte-Carlo method to approximate σ_t^2. Using the definition of the tilted measure (1.1), one can see that

$$\sigma_t^2 = \frac{\tilde{\mathbb{E}}\mathbf{E}_0^\omega[(\xi \cdot Y(t))^2]}{t} = \frac{\mathbb{E}\mathbf{E}_0^\omega[p(\omega)(\xi \cdot Y(t))^2]}{t\mathbb{E}[p]}. \tag{1.5}$$

Assuming that we have easier access to the measure $\mathbb{P}$ than to the tilted $\tilde{\mathbb{P}}$, we prefer to base our Monte-Carlo procedure on the r. h. s. of the second identity in (1.5). Let $Y^{(1)}, Y^{(2)}, \ldots$ be independent random walks evolving in the environments $\omega^{(1)}, \omega^{(2)}, \ldots$ respectively. We write $\mathbf{P}_0^{\overline{\omega}}$ for their joint distribution, all random walks starting from 0, where $\overline{\omega}$ stands for $(\omega^{(1)}, \omega^{(2)}, \ldots)$. The family of environments $\overline{\omega}$ is itself random, and we let $\mathbb{P}^{\otimes}$ be the product distribution with marginal $\mathbb{P}$. In other words, under $\mathbb{P}^{\otimes}$, the environments $\omega^{(1)}, \omega^{(2)}, \ldots$ are independent and distributed according to $\mathbb{P}$. Our computable approximation of σ_t^2 is defined by

$$\hat{A}_n(t) = \frac{p(\omega^{(1)})(\xi \cdot Y^{(1)}(t))^2 + \cdots + p(\omega^{(n)})(\xi \cdot Y^{(n)}(t))^2}{nt\ \mathbb{E}[p]}. \tag{1.6}$$

In $\hat{A}_n(t)$, the expectation $\mathbb{E}[p] = 2d\mathbb{E}[\omega_e]$ comes into play. This expectation can be easily computed, so we assumed that we did so beforehand.

The main result of this paper is the following optimal bounds on the distribution of the error $|\hat{A}_n(t) - \sigma^2|$.

Theorem 1.1. *Under the assumptions (H1-H3), there exist $C, c > 0$ such that, for any $n \in \mathbb{N}^*$, any $\varepsilon > 0$ and any t large enough,*

$$\mathbb{P}^{\otimes}\mathbf{P}_0^{\overline{\omega}}\left[\left|\hat{A}_n(t) - \sigma^2\right| \geqslant (C\mu_d(t) + \varepsilon)/t\right] \leqslant \exp\left(-\frac{n\varepsilon^2}{ct^2}\right), \tag{1.7}$$

where σ^2 and $\hat{A}_n(t)$ are defined respectively in (1.4) *and* (1.6)*, and*

$$\mu_d(t) = \left| \begin{array}{ll} \ln^q t & (d=2) \\ 1 & (d>2) \end{array} \right.$$

for some $q > 0$ depending only on α and β.

This result precisely quantifies the convergence rate of a method proposed by Papanicolaou in [Pa83] in the beginning of the eighties to approximate the homogenized coefficients A_{hom} numerically.

For completeness of the analysis we also prove a central limit theorem (and identify the limiting variance) for the quantity $\sqrt{n(t)}(\hat{A}_{n(t)}(t) - \sigma_t^2)$ for all $n : \mathbb{N} \to \mathbb{N}$ such that $n(t)$ tends to infinity with t.

Let us quickly discuss the sharpness of these results. If A was a periodic matrix (or even a constant matrix) we would get the same estimate as in Theorem 1.1, except in dimension 2 for which no logarithmic correction would be needed (in the setting of Theorem 1.1, we conjecture that $q = 1$ is the optimal exponent in (1.7)). Numerical tests illustrating (1.7) for $d = 2$ are reported and commented on in the last section of this article.

1.2. **Structure of the proof.** Although the result of Theorem 1.1 is purely probabilistic (we estimate a distribution) its proof involves both nontrivial probabilistic arguments (martingale decomposition and Kipnis-Varadhan theory, large deviation estimates) and nontrivial arguments of elliptic theory (Harnack inequality, De Giorgi-Nash-Moser theory, and L^p-theory). What allows to combine these arguments is spectral theory. This makes the overall structure of the proof interesting and rather unusual.

The starting point of the proof is the observation that

$$\left|\hat{A}_n(t) - \sigma^2\right| \leqslant \left|\hat{A}_n(t) - \sigma_t^2\right| + \left|\sigma_t^2 - \sigma^2\right|.$$

The result then follows from the following two estimates:

$$\left|\sigma_t^2-\sigma^2\right| \leqslant C\frac{\mu_d(t)}{t}, \tag{1.8}$$

$$\mathbb{P}^{\otimes}\mathbf{P}_0^{\overline{\omega}}\left[\left|\hat{A}_n(t)-\sigma_t^2\right| \geqslant \varepsilon/t\right] \leqslant \exp\left(-\frac{n\varepsilon^2}{ct^2}\right), \tag{1.9}$$

see Theorems 3.1 and 4.1. The second estimate is a large deviation estimate. Its proof is standard once we are given sharp upper bounds on the transition probabilities of the random walk in the random environment — which are also by now standard under assumption (H3). The proof is given in Section 4 for completeness. The central limit theorem for the quantity $\sqrt{n(t)}(\hat{A}_{n(t)}(t)-\sigma_t^2)$ is given in Proposition 5.1 and proved in Section 5.

The core of this article is the estimate (1.8). We call its l. h. s. the systematic error. As proved in the celebrated paper [KV86] by Kipnis and Varadhan (see also [DFGW89]), the systematic error vanishes as t goes to infinity as soon as the measure $\mathbb{P}$ is ergodic under translations. The strategy to prove this result is to find a decomposition of $Y(t)\cdot\xi$ into a martingale plus a remainder, in such a way that the remainder term becomes negligible in the limit, and conclude using the orthogonality of the increments of the martingale and ergodicity. The approach taken up by [KV86] is based on the spectral analysis of the (self-adjoint) operator of the environment viewed by the particle. More precisely, it is shown that in order for this decomposition with negligible remainder to exist, it suffices that the spectral measure of this operator, once projected on the "local drift" $\mathfrak{d}$ (see (3.4)), satisfies some integrability condition (IC) at the edge of the spectrum. Condition (IC) is then seen to be equivalent to asking $\mathfrak{d}$ to belong to the function space H^{-1}, a fact which is automatically true due to certain symmetry considerations that were systematized in [DFGW89].

Our proof of (1.8) consists in two steps. We first make the argument of Kipnis and Varadhan quantitative in Section 2. That is, we show that stronger integrability conditions than (IC) on the spectral measure can be turned into quantitative estimates on the systematic error — this is a general result of independent interest.

In the second step, addressed in Section 3, we prove that indeed condition (IC) can be strengthened to higher integrability properties, provided ergodicity is replaced by the stronger assumption that the conductances are i. i. d., the hypothesis (H2). This result is the main achievement of this article. In [GM10], we had taken advantage of spectral theory to turn results of [GO10b] into bounds on spectral exponents. In the present paper we go the other way around, and make systematic use of the interplay between estimates on the spectral measure and iterates of the elliptic operator. There is a twist in the analysis at this point. In [GM10] spectral theory is somehow only used at the end of the argument to rephrase in terms of spectral exponents the results on systematic errors obtained by pde arguments in [GO10b]. Here spectral theory enters the proof itself and is used in combination with pde arguments. This approach has the advantage to reveal the very nice structure of the problem under consideration.

Let us point out that although the results of this paper are proved under assumptions (H1-H3), the assumption (H2) on the statistics of ω is only used to obtain the variance estimate of [GO10a, Lemma 2.3]. In particular, (H2) can be weakened as follows:

- the distribution of $\omega_{z,z+\mathbf{e}_i}$ may in addition depend on $\mathbf{e}_i$,
- independence can be replaced by finite correlation length $C_L>0$, that is for all $e,e'\in\mathbb{B}$, ω_e and $\omega_{e'}$ are independent if $|e-e'|\geqslant C_L$.

Notation. So far we have already introduced the probability measures $\mathbf{P}_0^\omega$ (distribution of Y), $\mathbf{P}_0^{\overline{\omega}}$ (distribution of $Y^{(1)}, Y^{(2)}, \ldots$), $\mathbb{P}$ (i.i.d. distribution for $\omega = (\omega_e)_{e\in\mathbb{B}}$), $\tilde{\mathbb{P}}$ (tilted measure defined in (1.1)) and $\mathbb{P}^\otimes$ (product distribution of $\overline{\omega}$ with marginal $\mathbb{P}$). It will be convenient to define $\tilde{\mathbb{P}}^\otimes$ the product distribution of $\overline{\omega}$ with marginal $\tilde{\mathbb{P}}$. For convenience, we write $\mathbb{P}_0$ as a short-hand notation for $\mathbb{P}\mathbf{P}_0^\omega$, $\tilde{\mathbb{P}}_0$ for $\tilde{\mathbb{P}}\mathbf{P}_0^\omega$, $\mathbb{P}_0^\otimes$ for $\mathbb{P}^\otimes\mathbf{P}_0^{\overline{\omega}}$, and $\tilde{\mathbb{P}}_0^\otimes$ for $\tilde{\mathbb{P}}^\otimes\mathbf{P}_0^{\overline{\omega}}$. The corresponding expectations are written accordingly, replacing "P" by "E" with the appropriate typography. We write $|\cdot|$ for the Euclidian norm of $\mathbb{R}^d$.

Finally, $\lesssim$ and $\gtrsim$ stand respectively for $\leqslant$ and $\geqslant$ up to multiplicative constants (which depend only on the bounds α and β on the conductances and the dimension d, if not otherwise stated).

2. Quantitative version of the Kipnis-Varadhan theorem

The Kipnis-Varadhan theorem [KV86] concerns additive functionals of reversible Markov processes. It gives conditions for such additive functionals to satisfy an invariance principle. The proof of the result relies on a decomposition of the additive functional as the sum of a martingale term plus a remainder term, the latter being shown to be negligible. In this section, which can be read independently of the rest of the paper, we give conditions that enable to obtain some quantitative bounds on this remainder term.

We consider discrete and continuous times simultaneously. Let $(\eta_t)_{t\geqslant 0}$ be a Markov process defined on some measurable state space $\aleph$ (here, $t \geqslant 0$ stands either for $t \in \mathbb{N}$ or for $t \in \mathbb{R}_+$). We denote by P_x the distribution of the process started from $x \in \aleph$, and by E_x the associated expectation. We assume that this Markov process is reversible and ergodic with respect to some probability measure ν. We write P_ν for the law of the process started from the distribution ν, and E_ν for the associated expectation.

To the Markov process is naturally associated a semi-group $(P_t)_{t\geqslant 0}$ defined, for any $f \in \mathbb{L}_2(\nu)$, by

$$P_t f(x) = E_x[f(\eta_t)].$$

Each P_t is a self-adjoint contraction of $\mathbb{L}^2(\nu)$. In the continuous-time case, we assume further that the semi-group is strongly continuous, that is to say, that $P_t f$ converges to f in $\mathbb{L}^2(\nu)$ as t tends to 0, for any $f \in \mathbb{L}^2(\nu)$. We let L be the $\mathbb{L}^2(\nu)$-infinitesimal generator of the semi-group. It is self-adjoint in $\mathbb{L}^2(\nu)$, and we fix the sign convention so that it is a positive operator (i.e., $P_t = e^{-tL}$).

Note that in general, one can see using spectral analysis that there exists a projection $\overline{P}$ such that $P_t f$ converges to $\overline{P} f$ as t tends to 0, $t > 0$. Changing $\mathbb{L}^2(\nu)$ to the image of the projection $\overline{P}$, and P_0 for $\overline{P}$, one recovers a strongly continuous semigroup of contractions, and one can still carry the analysis below replacing $\mathbb{L}^2(\nu)$ by the image of $\overline{P}$ when necessary.

In discrete time, we set $L = \mathrm{Id} - P_1$. Again, L is a positive self-adjoint operator on $\mathbb{L}^2(\nu)$. Note that we slightly depart from the custom of defining the generator as P_1 in order to match more closely the continuous-time situation.

We denote by $\langle \cdot, \cdot \rangle$ the scalar product in $\mathbb{L}^2(\nu)$. For any function $f \in \mathbb{L}^2(\nu)$ we define the *spectral measure* of L projected on the function f as the measure e_f on $\mathbb{R}_+$ that satisfies, for any bounded continuous $\Psi : \mathbb{R}_+ \to \mathbb{R}$, the relation

$$\langle f, \Psi(L) f \rangle = \int \Psi(\lambda) \, \mathrm{d}e_f(\lambda). \tag{2.1}$$

The Dirichlet form associated to L is given by

$$\|f\|_1^2 = \int \lambda \, \mathrm{d}e_f(\lambda). \tag{2.2}$$

We denote by H^1 the completion of the space $\{f \in \mathbb{L}^2(\nu) : \|f\|_1 < +\infty\}$ with respect to this $\|\cdot\|_1$ norm, taken modulo functions of zero $\|\cdot\|_1$ norm. This turns $(H^1, \|\cdot\|_1)$ into a Hilbert space, and we let H^{-1} denote its dual. One can identify H^{-1} with the completion of the space $\{f \in \mathbb{L}^2(\nu) : \|f\|_{-1} < +\infty\}$ with respect to the norm $\|\cdot\|_{-1}$ defined by

$$\|f\|_{-1}^2 = \int \lambda^{-1} \, \mathrm{d}e_f(\lambda).$$

Indeed, for all $f \in \mathbb{L}^2(\nu)$, the linear form

$$\left\{ \begin{array}{ccc} (\mathbb{L}^2(\nu) \cap H^1, \|\cdot\|_1) & \to & \mathbb{R} \\ \phi & \mapsto & \langle f, \phi \rangle \end{array} \right.$$

has norm $\|f\|_{-1}$, and thus defines an element of H^{-1} (with norm $\|f\|_{-1}$) iff $\|f\|_{-1}$ is finite. The notion of spectral measure introduced in (2.1) for functions of $\mathbb{L}^2(\nu)$ can be extended to elements of H^{-1}. Indeed, let $\Psi : \mathbb{R}_+ \to \mathbb{R}$ be a continuous function such that $\Psi(\lambda) = O(\lambda^{-1})$ as $\lambda \to +\infty$. One can check that the map

$$\left\{ \begin{array}{ccc} (\mathbb{L}^2(\nu) \cap H^{-1}, \|\cdot\|_{-1}) & \to & H^1 \\ f & \mapsto & \Psi(L)f \end{array} \right.$$

extends to a bounded linear map on H^{-1}. One can then define the spectral measure of L projected on the function f as the measure e_f such that for any continuous Ψ with $\Psi(\lambda) = O(\lambda^{-1})$, (2.1) holds. With a slight abuse of notation, for all $f \in H^{-1}$ and $g \in H^1$, we write $\langle f, g \rangle$ for the $H^{-1} - H^1$ duality product between f and g.

For any $f \in H^{-1}$, we define $(Z_f(t))_{t \geqslant 0}$ as

$$Z_f(t) = \int_0^t f(\eta_s) \, \mathrm{d}s \qquad \text{or} \qquad Z_f(t) = \sum_{s=0}^{t-1} f(\eta_s), \tag{2.3}$$

according to whether we consider the continuous or the discrete time cases. In the continuous case, the meaning of (2.3) is unclear a priori. Yet it is proved in [DFGW89, Lemma 2.4] that for any $t \geqslant 0$ the map

$$\left\{ \begin{array}{ccc} \mathbb{L}^2(\nu) \cap H^{-1} & \to & \mathbb{L}^2(P_\nu) \\ f & \mapsto & Z_f(t) \end{array} \right.$$

can be extended by continuity to a bounded linear map on H^{-1}, and moreover, that (2.3) coincides with the usual integral as soon as $f \in \mathbb{L}^1(\nu)$. The following theorem is due to [DFGW89], building on previous work of [KV86].

Theorem 2.1. *(i) For all $f \in H^{-1}$, there exists $(M_t)_{t \geqslant 0}$, $(\xi_t)_{t \geqslant 0}$ such that $Z_f(t)$ defined in* (2.3) *satisfies the identity $Z_f(t) = M_t + \xi_t$, where (M_t) is a square-integrable martingale with stationary increments under P_ν (and the natural filtration), and (ξ_t) is such that :*

$$t^{-1} E_\nu[(\xi_t)^2] \xrightarrow[t \to +\infty]{} 0. \tag{2.4}$$

As a consequence, $t^{-1/2} Z_f(t)$ converges in law under P_ν to a Gaussian random variable of variance $\sigma^2(f) < +\infty$ as t goes to infinity, and

$$t^{-1} E_\nu[(Z_f(t))^2] \xrightarrow[t \to +\infty]{} \sigma^2(f). \tag{2.5}$$

(ii) If, moreover, $f \in \mathbb{L}^1(\nu)$ and, for some $\overline{t} > 0$, $\sup_{0 \leqslant t \leqslant \overline{t}} |Z_f(t)|$ is in $\mathbb{L}^2(\nu)$, then the process $t \mapsto \sqrt{\varepsilon} Z_f(\varepsilon^{-1} t)$ converges in law under P_ν to a Brownian motion of variance $\sigma^2(f)$ as ε goes to 0.

Remarks. The additional conditions appearing in statement (ii) are automatically satisfied in discrete time, due to the fact that $H^{-1} \subseteq \mathbb{L}^2(\nu)$ in this case. In the continuous-time setting and when $f \in \mathbb{L}^1(\nu)$, the process $t \mapsto Z_f(t)$ is almost surely continuous, and $\sup_{0 \leqslant t \leqslant \overline{t}} |Z_f(t)|$ is indeed a well-defined random variable.

Under some additional information on the spectral measure of f, we can estimate the rates of convergence in the limits (2.4) and (2.5). For any $\gamma > 1$ and $q \geqslant 0$, we say that the spectral exponents of a function $f \in H^{-1}$ are at least $(\gamma, -q)$ if

$$\int_0^\mu \mathrm{d}e_f(\lambda) = O\big(\mu^\gamma \ln^q(\mu^{-1})\big) \qquad (\mu \to 0). \tag{2.6}$$

Note that the phrasing is consistent, since if $(\gamma', -q') \leqslant (\gamma, -q)$ for the lexicographical order, and if the spectral exponents of f are at least $(\gamma, -q)$, then they are at least $(\gamma', -q')$. In [Mo11], it was found more convenient to consider, instead of (2.6), a condition of the following form :

$$\int_0^\mu \lambda^{-1} \, \mathrm{d}e_f(\lambda) = O\big(\mu^{\gamma-1} \ln^q(\mu^{-1})\big) \qquad (\mu \to 0). \tag{2.7}$$

One can easily check that conditions (2.6) and (2.7) are equivalent. Indeed, on the one hand, one has the obvious inequality

$$\int_0^\mu \mathrm{d}e_f(\lambda) \leqslant \mu \int_0^\mu \lambda^{-1} \, \mathrm{d}e_f(\lambda),$$

which shows that (2.7) implies (2.6). On the other hand, one may perform a kind of integration by parts, use Fubini's theorem :

$$\begin{aligned} \int_0^\mu \lambda^{-1} \, \mathrm{d}e_f(\lambda) &= \int_0^\mu \int_{\delta=\lambda}^{+\infty} \delta^{-2} \, \mathrm{d}\delta \, \mathrm{d}e_f(\lambda) \\ &= \int_{\delta=0}^{+\infty} \delta^{-2} \int_{\lambda=0}^{\delta \wedge \mu} \mathrm{d}e_f(\lambda) \, \mathrm{d}\delta, \end{aligned}$$

and obtain the converse implication by examining separately the integration over δ in $[0, \mu)$ and in $[\mu, +\infty)$.

For all $\gamma > 1$ and $q \geqslant 0$, we set

$$\psi_{\gamma,q}(t) = \left| \begin{array}{ll} t^{1-\gamma} \ln^q(t) & \text{if } \gamma < 2, \\ t^{-1} \ln^{q+1}(t) & \text{if } \gamma = 2, \\ t^{-1} & \text{if } \gamma > 2. \end{array} \right. \tag{2.8}$$

The quantitative version of Theorem 2.1 is as follows.

Theorem 2.2. *If the spectral exponents of $f \in H^{-1}$ are at least $(\gamma, -q)$, then the decomposition $Z_f(t) = M_t + \xi_t$ of Theorem 2.1 holds with the additional property that*

$$t^{-1} E_\nu[(\xi_t)^2] = O(\psi_{\gamma,q}(t)) \qquad (t \to +\infty).$$

Moreover,

$$\sigma^2(f) - \frac{E_\nu[Z_f(t)^2]}{t} = O(\psi_{\gamma,q}(t)) \qquad (t \to +\infty).$$

Proof. In the continuous-time setting, the argument for the first estimate is very similar to the one of [Mo11, Proposition 8.2], and we do not repeat the details here. It is based on the observation that

$$\frac{1}{t} E_\nu[(\xi_t)^2] = 2 \int \frac{1 - e^{-\lambda t}}{\lambda^2 t} \, \mathrm{d}e_f(\lambda). \tag{2.9}$$

One needs to take into account the possible logarithmic terms that appear in (2.7) and which are not considered in [Mo11]. Some care is also needed because we do not assume that $f \in \mathbb{L}^2(\nu)$. Yet one can easily replace the bound involving the

$\mathbb{L}^2(\nu)$ norm of f by its H^{-1} norm. The second part of the statement is given by [Mo11, Proposition 8.3].

We now turn to the discrete time setting. In this context, identity (2.9) should be replaced by

$$\frac{1}{t} E_\nu[(\xi_t)^2] = 2 \int \frac{1-(1-\lambda)^t}{\lambda^2 t} \, \mathrm{d}e_f(\lambda).$$

By definition, $L = \mathrm{Id} - P_1$, where P_1 is the semi-group at time 1. Hence the spectrum of L is contained in $[0,2]$. One can then follow the same computations as before to prove the first part of Theorem 2.2.

Somewhat surprisingly, the second part of the statement requires additional attention in the discrete time setting. Indeed, in the continuous case, the argument of [Mo11, Proposition 8.3] (which already appears in [DFGW89]) is that $Z_f(t)$ and $\xi(t)$ are orthogonal in $\mathbb{L}^2(P_\nu)$, a fact obtained using the invariance under time symmetry. This orthogonality is only approximately valid in the discrete-time setting. Indeed, let us recall that $Z_f(t)$ is given by (2.3), while ξ_t is obtained as the limit in $\mathbb{L}^2(P_\nu)$ of

$$-u_\varepsilon(\eta_t) + u_\varepsilon(\eta_0),$$

where $u_\varepsilon = (\varepsilon + L)^{-1} f$. Using time symmetry, what we obtain is that ξ_t is orthogonal to $(Z_f(t) + f(\eta_t))$. As a consequence, the cross-product $E_\nu[Z_f(t)\xi_t]$, which is equal to 0 in the proof of [Mo11, Proposition 8.3], is in the present case equal to $-E_\nu[f(\eta_t)\xi_t]$. Yet spectral analysis ensures that this term is equal to

$$\int \frac{1-(1-\lambda)^t}{\lambda} \, \mathrm{d}e_f(\lambda) = O(1) \qquad (t \to +\infty),$$

which is what we need to obtain the second claim of the theorem. □

3. The systematic error

We now come back to the analysis of the Monte-Carlo approximation of the homogenized coefficients within assumptions (H1)-(H3). The aim of this section is to estimate the difference between σ_t^2 and the quantity σ^2 we wish to approximate (both being defined in (1.3)). This difference, that we refer to as the *systematic error* after [GO10a], is shown to be of order $1/t$ as t tends to infinity, up to a logarithmic correction in dimension 2.

Theorem 3.1. *Under assumptions (H1)-(H3), there exists $q \geqslant 0$ such that, as t tends to infinity,*

$$\sigma_t^2 - \sigma^2 = \left| \begin{array}{ll} O\big(t^{-1}\ln^q(t)\big) & \text{if } d=2, \\ O\big(t^{-1}\big) & \text{if } d>2. \end{array} \right. \tag{3.1}$$

Theorem 3.1 is a discrete-time version of [Mo11, Corollary 2.6]. Its proof makes use of an auxiliary process that we now introduce.

Let $(\theta_x)_{x\in\mathbb{Z}^d}$ be the translation group that acts on the set of environments as follows: for any pair of neigbhours $y, z \in \mathbb{Z}^d$, $(\theta_x\ \omega)_{y,z} = \omega_{x+y,x+z}$. The *environment viewed by the particle* is the process defined by

$$\omega(t) = \theta_{Y(t)}\ \omega. \tag{3.2}$$

One can check that $(\omega(t))_{t\in\mathbb{N}}$ is a Markov chain, whose generator is given by

$$-\mathcal{L}f(\omega) = \frac{1}{p(\omega)} \sum_{|z|=1} \omega_{0,z}(f(\theta_z\ \omega) - f(\omega)), \tag{3.3}$$

so that $\mathbf{E}_0^\omega[f(\omega(1))] = (I - \mathcal{L})f(\omega)$. Moreover, the measure $\tilde{\mathbb{P}}$ defined in (1.1) is reversible and ergodic for this process [DFGW89, Lemma 4.3 (i)]. As a consequence, the operator $\mathcal{L}$ is (positive and) self-adjoint in $\mathbb{L}^2(\tilde{\mathbb{P}})$.

The proof of Theorem 3.1 relies on spectral analysis. For any function $f \in L^2(\tilde{\mathbb{P}})$, let e_f be the spectral measure of $\mathcal{L}$ projected on the function f. This measure is such that, for any positive continuous function $\Psi : [0, +\infty) \to \mathbb{R}_+$, one has

$$\tilde{\mathbb{E}}[f\ \Psi(\mathcal{L}) f] = \int \Psi(\lambda)\ \mathrm{d}e_f(\lambda).$$

For any $\gamma > 1$ and $q \geqslant 0$, we recall that we say that the spectral exponents of a function f are at least $(\gamma, -q)$ if (2.6) holds.

Let us define the local drift $\mathfrak{d}$ in direction ξ as

$$\mathfrak{d}(\omega) = \mathbf{E}_0^\omega[\xi \cdot Y(1)] = \frac{1}{p(\omega)} \sum_{|z|=1} \omega_{0,z}\ \xi \cdot z. \tag{3.4}$$

As we shall prove at the end of this section, we have the following bounds on the spectral exponents of $\mathfrak{d}$.

Proposition 3.2. *Under assumptions (H1)-(H3), there exists $q \geqslant 0$ such that the spectral exponents of the function $\mathfrak{d}$ are at least*

$$\left| \begin{array}{ll} (2, -q) & \text{if } d = 2, \\ (d/2+1, 0) & \text{if } 3 \leqslant d \leqslant 5, \\ (4, -1) & \text{if } d = 6, \\ (4, 0) & \text{if } d \geqslant 7. \end{array} \right. \tag{3.5}$$

Let us see how this result implies Theorem 3.1. In order to do so, we also need the following information, that is a consequence of Proposition 3.2.

Corollary 3.3. *Let*

$$\mathfrak{d}_t(\omega) = \mathbf{E}_0^\omega[\mathfrak{d}(\omega(t))] \tag{3.6}$$

be the image of $\mathfrak{d}$ by the semi-group at time t associated with the Markov chain $(\omega(t))_{t \in \mathbb{N}}$. There exists $q \geqslant 0$ such that

$$\tilde{\mathbb{E}}[(\mathfrak{d}_t)^2] = \left| \begin{array}{ll} O\big(t^{-2}\ \ln^q(t)\big) & \text{if } d = 2, \\ O\big(t^{-(d/2+1)}\big) & \text{if } 3 \leqslant d \leqslant 5, \\ O\big(t^{-4}\ \ln(t)\big) & \text{if } d = 6, \\ O\big(t^{-4}\big) & \text{if } d \geqslant 7. \end{array} \right.$$

Proof. This result is the discrete-time analog of [GM10, Corollary 1]. It is obtained the same way, noting that

$$\tilde{\mathbb{E}}[(\mathfrak{d}_t)^2] = \int (1-\lambda)^{2t}\ \mathrm{d}e_{\mathfrak{d}}(\lambda),$$

and that the support of the measure $e_{\mathfrak{d}}$ is contained in $[0, 2]$. □

We are now in position to prove Theorem 3.1.

Proof of Theorem 3.1. The proof has the same structure as for the continuous-time case of [Mo11, Proposition 8.4]. Note that [DFGW89, Theorem 2.1] ensures that

$$\lim_{t \to \infty} \sigma_t^2 \overset{(\mathrm{def})}{=} \lim_{t \to \infty} t^{-1} \mathbb{E}_0[(\xi \cdot Y(t))^2] = \sigma^2. \tag{3.7}$$

The starting point is the observation that, under $\tilde{\mathbb{P}}_0$, the process defined by

$$N_t = \xi \cdot Y(t) - \sum_{s=0}^{t-1} \mathfrak{d}(\omega(s)) \tag{3.8}$$

is a square-integrable martingale with stationary increments. On the one hand, following (2.3), we denote by $Z_{\mathfrak{d}}(t)$ the sum appearing in the r. h. s. of (3.8). From Proposition 3.2 and Theorem 2.2, we learn that there exist $\overline{\sigma}$ and $q \geqslant 0$ such that

$$t\overline{\sigma}^2 - \tilde{\mathbb{E}}_0[(Z_{\mathfrak{d}}(t))^2] = \left| \begin{array}{ll} O\big(\ln^q(t)\big) & \text{if } d = 2, \\ O\big(1\big) & \text{if } d > 2. \end{array} \right. \tag{3.9}$$

On the other hand, since N_t is a martingale with stationary increments,

$$\tilde{\mathbb{E}}_0[(N_t)^2] = t\tilde{\mathbb{E}}_0[(N_1)^2]. \tag{3.10}$$

As in the proof of Theorem 2.2 in the discrete time case, we then use that $\xi \cdot Y(t)$ is orthogonal to $(Z_{\mathfrak{d}}(t) + \mathfrak{d}(\omega(t)))$ to turn (3.8) into

$$t^{-1}\tilde{\mathbb{E}}_0[(N_t)^2] = t^{-1}\tilde{\mathbb{E}}_0[(\xi \cdot Y(t))^2] + t^{-1}\tilde{\mathbb{E}}_0[(Z_{\mathfrak{d}}(t))^2] + 2t^{-1}\tilde{\mathbb{E}}_0[\mathfrak{d}(\omega(t))(\xi \cdot Y(t))]. \tag{3.11}$$

We already control the l. h. s. and the second term of the r. h. s. of (3.11). In order to quantify the convergence of $t^{-1}\tilde{\mathbb{E}}_0[(\xi \cdot Y(t))^2]$ it remains to control the last term. In particular, provided we show that

$$\tilde{\mathbb{E}}_0[\mathfrak{d}(\omega(t))(\xi \cdot Y(t))] = \left| \begin{array}{ll} O\big(\ln^q(t)\big) & \text{if } d = 2, \\ O\big(1\big) & \text{if } d > 2, \end{array} \right. \tag{3.12}$$

(3.11), (3.9), (3.10), and (3.7) imply first that $\sigma^2 = \tilde{\mathbb{E}}_0[(N_1)^2] - \overline{\sigma}^2$, and then the desired quantitative estimate (3.1). We now turn to (3.12) and write

$$\begin{aligned} \tilde{\mathbb{E}}_0[\mathfrak{d}(\omega(t))(\xi \cdot Y(t))] &= \sum_{s=0}^{t-1} \tilde{\mathbb{E}}_0[\mathfrak{d}(\omega(t))(\xi \cdot (Y(s+1) - Y(s))] \\ &= \sum_{s=0}^{t-1} \tilde{\mathbb{E}}_0[\mathfrak{d}_{t-s-1}(\omega(s+1))(\xi \cdot (Y(s+1) - Y(s))], \end{aligned}$$

where we have used the Markov property at time $s+1$, together with the definition (3.6) of $\mathfrak{d}_{t-s-1}$. Using Cauchy-Schwarz inequality and the stationarity of the process $(\omega(t))_{t\in\mathbb{N}}$ under $\tilde{\mathbb{E}}_0$, this sum is bounded by

$$|\xi|^2 \sum_{s=0}^{t-1} \tilde{\mathbb{E}}[(\mathfrak{d}_{t-s-1})^2]^{1/2}.$$

Esimate (3.12) then follows from Corollary 3.3. This concludes the proof of the theorem. ☐

Proposition 3.2 is a discrete-time counterpart of [GM10, Theorem 5]. In [GM10, Theorem 5] however, we had proved in addition that the spectral exponents are at least $(d/2 - 2, 0)$, which is sharper than the exponents of Proposition 3.2 for $d > 10$. In particular for $d > 10$ the bounds of [GM10, Theorem 5] follow from results of [Mo11], whose adaptation to the discrete time setting is not straightforward. As shown above, the present statement is sufficient to prove the optimal scaling of the systematic error, and we do not investigate further this issue (see however Remark 3.10). The proof of Proposition 3.2 is rather involved and one may wonder whether this is worth the effort in terms of the application we have in mind — namely Theorem 3.1. In order to obtain the optimal convergence rate in Theorem 3.1 we need the spectral exponents to be larger than $(2, 0)$. Proving that the exponents are at least $(2, 0)$ is rather direct using results of [GO10a] (see the first three steps of the proof of Proposition 3.2). Yet proving that they are larger than $(2, 0)$ for $d > 2$ is as involved as proving Proposition 3.2 itself. This is the reason why we display the complete proof of Proposition 3.2 — although the precise values of the spectral exponents are not that important in the context of this paper.

There are two new features in the proof of Proposition 3.2 with respect to our previous works:

- First the discrete elliptic operator we consider here is slightly different than the operator considered in [GO10a] since the zero-order term is now random as well — the adaptation of the results of [GO10a] is only technical though;
- The string of arguments is different than in the proof of [GM10, Theorem 5]. In particular, the starting point of [GM10] was an estimate obtained in [GO10b] based on the crucial use of a covariance estimate. In [GO10b] the main quantity of interest was a systematic error. In the present proof the main quantity of interest is the spectral exponents at the first place. This twist of points of view allows to reduce the proof to a suitable use of the variance estimate only, and reveals the general structure of the problem.

This proof does not only complete the proof of Theorem 3.1 but allows us to shed some new light on our conjecture in [GM10] on the optimal values of the spectral exponents — see Remark 3.10.

As already mentioned, this proof makes extensive use of tools developed by the authors, and by Otto. For the reader's convenience, we recall five useful auxiliary results from [GO10a, GO10b, Gl10]: a spectral gap estimate, and bounds on Green's functions.

Lemma 3.4 (Lemma 2.3 of [GO10a])**.** *Let $a = \{a_i\}_{i\in\mathbb{N}}$ be a sequence of i. i. d. random variables with range $[\alpha,\beta]$. Let X be a Borel measurable function of $a \in \mathbb{R}^{\mathbb{N}}$ (i. e. measurable w. r. t. the smallest σ-algebra on $\mathbb{R}^{\mathbb{N}}$ for which all coordinate functions $\mathbb{R}^{\mathbb{N}} \ni a \mapsto a_i \in \mathbb{R}$ are Borel measurable, cf.* [Kl08, Definition 14.4]*). Then we have*

$$\operatorname{var}[X] \leqslant \left\langle \sum_{i=1}^{\infty} \sup_{a_i} \left| \frac{\partial X}{\partial a_i} \right|^2 \right\rangle \operatorname{var}[a_1], \tag{3.13}$$

where $\sup_{a_i}\left|\frac{\partial X}{\partial a_i}\right|$ denotes the supremum of the modulus of the i-th partial derivative

$$\frac{\partial X}{\partial a_i}(a_1, \cdots, a_{i-1}, a_i, a_{i+1}, \cdots)$$

of X with respect to the variable $a_i \in [\alpha,\beta]$.

Let $h : \mathbb{Z}^d \to \mathbb{R}$ be some function. We define its forward and backward discrete gradients ∇ and ∇^* as

$$\nabla h(x) := \left[\begin{array}{l} h(x+\mathbf{e}_1) - h(x) \\ \vdots \\ h(x+\mathbf{e}_d) - h(x) \end{array}\right], \ \nabla^* h(x) := \left[\begin{array}{l} h(x) - h(x-\mathbf{e}_1) \\ \vdots \\ h(x) - h(x-\mathbf{e}_d) \end{array}\right],$$

the discrete backward divergence of some vector field $V : \mathbb{Z}^d \to \mathbb{R}^d$ is given by the "formal" scalar product between ∇^* and V, that is

$$\nabla^* \cdot V(x) = \sum_{i=1}^{d} \big(V_i(x+\mathbf{e}_i) - V_i(x)\big).$$

To avoid confusion, when a function $h : \mathbb{Z}^d \times \mathbb{Z}^d \to \mathbb{R}, (x,z) \mapsto h(x,z)$ depends on two variables, we denote by $\nabla_1 h$ (resp. $\nabla_1^* h$) the forward (resp. backward) discrete gradient with respect to the first variable (x here) and by $\nabla_2 h$ (resp. $\nabla_2^* h$) the forward (resp. backward) discrete gradient with respect to the second variable (z here). We further use the notation $\nabla_{k,i} h := \nabla_k h \cdot \mathbf{e}_i$ for the forward discrete gradients in direction $\mathbf{e}_i$ (and likewise for the backward gradients), $i \in \{1,\ldots,d\}$.

We define discrete Green's functions as follows:

Definition 3.5 (discrete Green's function)**.** *Let $d \geqslant 2$. Let ω be an environment, $p_\omega : \mathbb{Z}^d \to \mathbb{R}, x \mapsto \sum_{|z-x|=1} \omega_{x,z}$, and A be the associated diagonal matrix on $\mathbb{Z}^d$ defined by $A(x) = \mathrm{diag}(\omega_{x,x+\mathbf{e}_1}, \dots, \omega_{x,x+\mathbf{e}_d})$. For all $T > 0$, the Green function $G_T(\cdot,\cdot;\omega) : \mathbb{Z}^d \times \mathbb{Z}^d \to \mathbb{Z}^d, (x,y) \mapsto G_T(x,y;\omega)$ associated with the environment ω is defined for all $y \in \mathbb{Z}^d$ as the unique space square-integrable solution to*

$$\int_{\mathbb{Z}^d} T^{-1} p_\omega(x) G_T(x,y;\omega) v(x)\,dx + \int_{\mathbb{Z}^d} \nabla v(x) \cdot A(x) \nabla_1 G_T(x,y;\omega)\,dx = v(y), \tag{3.14}$$

for all square-integrable functions $v : \mathbb{Z}^d \to \mathbb{R}$, where $\int_{\mathbb{Z}^d} \mathrm{d}y$ denotes the sum over all $y \in \mathbb{Z}^d$.

The existence and uniqueness of discrete Green's functions is a consequence of Riesz' representation theorem. In the rest of this article we use the short-hand notation $G_T(x,y)$ for $G_T(x,y;\omega)$. Note that G_T is stationary in the sense that $(x,y) \mapsto G_T(x+z,y+z)$ has the same statistics as $(x,y) \mapsto G_T(x,y)$. This will be used for the gradient of the Green function as follows: for all $q > 0$,

$$\langle |\nabla_2 G_T(x,y)|^q \rangle \;=\; \langle |\nabla_1 G_T(x-y,0)|^q \rangle\,. \tag{3.15}$$

The next two lemmas give estimates on the Green function and its derivatives.

Lemma 3.6 (Lemma 3.2 of [Gl10])**.** *There exists $c > 0$ depending only on α, β, and d, such that for every environment ω and for all $T > 0$, the Green function G_T satisfies the pointwise estimates: For all $x, y \in \mathbb{Z}^d$,*

$$\text{for } d > 2: \quad G_T(x,y) \;\lesssim\; (1+|x-y|)^{2-d} \exp\left(-c\frac{|x-y|}{\sqrt{T}}\right), \tag{3.16}$$

$$\text{for } d = 2: \quad G_T(x,y) \;\lesssim\; \ln\Big(\frac{\sqrt{T}}{1+|x-y|}\Big) \exp\left(-c\frac{|x-y|}{\sqrt{T}}\right). \tag{3.17}$$

Lemma 3.7 (Lemma 2.9 of [GO10a])**.** *Let ω be an environment, $T > 0$, and let G_T be the associated Green function. Then, for $d \geqslant 2$, there exists $p > 2$ depending only on α, β, and d such that for all $T > 0$, $p \geqslant r \geqslant 2$, $k > 0$ and $R \lesssim 1$,*

$$\int_{R \leqslant |z| \leqslant 2R} |\nabla_1 G_T(z,0)|^r dz \;\;\lesssim\;\; R^d (R^{1-d})^r \min\{1, \sqrt{T} R^{-1}\}^k. \tag{3.18}$$

Note that this lemma shows that $\nabla_1 G_T(z,0)$ has the optimal decay $(1+|z|)^{1-d}$ (that is, the decay of the Green function of the Laplace operator) when integrated on dyadic annuli (plus the exponential, or superalgebraic decay).

Corollary 3.8 (Corollary 2.3 of [GO10a])**.** *For every environment ω and for all $T > 0$ and $x, y \in \mathbb{Z}^d$,*

$$|\nabla_1 G_T(x,y;\omega)|, |\nabla_2 G_T(x,y;\omega)| \;\lesssim\; 1$$

(the multiplicative constant depending only on α, β, and d).

Note that the versions of these lemmas proved in [GO10a] and [Gl10] cover the case when the zero-order term is constant (namely T^{-1} in place of $T^{-1}p_\omega(x)$). The proofs adapt mutadis mutandis using the uniform bounds $0 < 2^d\alpha \leqslant p_\omega \leqslant 2^d\beta$.

The last lemma we shall need is the following double convolution estimate.

Lemma 3.9 (Lemma 6 of [GM10])**.** *Let $d > 2$, $T \gg 1$, and let $g_T : \mathbb{Z}^d \to \mathbb{R}^+$ be given by*

$$g_T(x) \;=\; (1+|x|)^{2-d} \exp\big(-c\frac{|x|}{\sqrt{T}}\big)$$

for some $c > 0$. Let $h_T : \mathbb{Z}^d \to \mathbb{R}^+$ be such that

$$\int_{|x| \leqslant R} h_T(x)^2 \;\lesssim\; 1,$$

and for all $R \gg 1$ *and all* $j \in \mathbb{N}$*,*

$$\int_{2^j R \leqslant |x| < 2^{j+1} R} h_T(x)^2 \mathrm{d}x \lesssim (2^j R)^{d-2(d-1)}.$$

Then,

$$\int_{\mathbb{Z}^d}\int_{\mathbb{Z}^d}\int_{\mathbb{Z}^d} g_T(w) g_T(w') h_T(z-w) h_T(z-w') \mathrm{d}z\mathrm{d}w\mathrm{d}w' \tag{3.19}$$
$$\lesssim 1 + \left| \begin{array}{rl} T^{3-d/2} & \textit{if} \quad 5 \geqslant d > 2, \\ \ln T & \textit{if} \quad d = 6, \\ 1 & \textit{if} \quad d > 6. \end{array} \right.$$

We are in position to prove Proposition 3.2.

Proof of Proposition 3.2. Our starting point is the following inequality which holds for every non-negative measure κ:

$$\int_0^{T^{-1}} \mathrm{d}\kappa(\lambda) \lesssim T^{-4} \int_0^\infty \frac{1}{(T^{-1}+\lambda)^4} \mathrm{d}\kappa(\lambda), \tag{3.20}$$

which follows from the fact that for $\lambda \leqslant T^{-1}$, $\frac{T^{-4}}{(T^{-1}+\lambda)^4} \gtrsim 1$. The variable T^{-1} for T large plays the role of μ in (2.6). In what follows we make the standard identification between stationary functions $(z,\omega) \mapsto f(z,\omega)$ of both the space variable $z \in \mathbb{Z}^d$ and the environment ω and their translated versions at 0 $\omega \mapsto f(0,\theta_z\omega)$ depending on the environment only. We define ϕ_T as the unique stationary solution to

$$T^{-1}\phi_T(x) - \frac{1}{p_\omega(x)} \nabla^* \cdot A(x) \nabla \phi_T = \frac{1}{p_\omega(x)} \nabla^* \cdot A(x)\xi, \tag{3.21}$$

whose existence and uniqueness follow from the Riesz representation theorem in $\mathbb{L}^2(\tilde{\mathbb{P}})$ using the identification between the stationary function ϕ_T and its version defined on the environment only (see a similar argument of [Kü83]). In particular, with the notation $\mathfrak{d} = \frac{1}{p_\omega(x)} \nabla^* \cdot A(x)\xi$,

$$\phi_T = (T^{-1} + \mathcal{L})^{-1}\mathfrak{d},$$

where $\mathcal{L}$ is the operator defined in (3.3), and the spectral theorem ensures that

$$\tilde{\mathbb{E}}(\phi_T^2) = \tilde{\mathbb{E}}(\mathfrak{d}(T^{-1}+\mathcal{L})^{-2}\mathfrak{d}) = \int_0^\infty \frac{1}{(T^{-1}+\lambda)^2} de_{\mathfrak{d}}(\lambda)$$

where $e_{\mathfrak{d}}$ is the spectral measure of $\mathcal{L}$ projected on the drift $\mathfrak{d}$. We also let ψ_T be the unique stationary solution to

$$T^{-1}\psi_T(x) - \frac{1}{p_\omega(x)} \nabla^* \cdot A(x) \nabla \psi_T(x) = \phi_T(x), \tag{3.22}$$

whose existence and uniqueness also follows from the Riesz representation theorem in the probability space as well. This time,

$$\psi_T = (T^{-1} + \mathcal{L})^{-2}\mathfrak{d},$$

and the spectral theorem yields

$$\tilde{\mathbb{E}}(\psi_T^2) = \tilde{\mathbb{E}}(\mathfrak{d}(T^{-1}+\mathcal{L})^{-4}\mathfrak{d}) = \int_0^\infty \frac{1}{(T^{-1}+\lambda)^4} de_{\mathfrak{d}}(\lambda)$$

From now on, we shall use the shorthand notation $\langle u \rangle := \tilde{\mathbb{E}}(u)$ and $\mathrm{var}[u] = \big\langle (u - \langle u \rangle)^2 \big\rangle$ for all $u \in \mathbb{L}^2(\tilde{\mathbb{P}})$. In particular the identity above turns into

$$\int_0^\infty \frac{1}{(T^{-1}+\lambda)^4} \mathrm{d}e_{\mathfrak{d}}(\lambda) = \mathrm{var}[\psi_T], \tag{3.23}$$

since $\langle \psi_T \rangle = \frac{1}{\mathbb{E}[p]} \int \psi_T p \, \mathrm{d}\mathbb{P} = \frac{T}{\mathbb{E}[p]} \int \phi_T p \, \mathrm{d}\mathbb{P} = 0$ using equations (3.22) and (3.21).

The streamline of the proof is to obtain bounds on the spectral exponents via (3.20) and (3.23) by proving bounds on the variance of ψ_T.

The rest of the proof, which is dedicated to the estimate of var $[\psi_T]$, is divided into five steps. As a starting point we appeal to the variance estimate of Lemma 3.4 that we apply to ψ_T. This requires to estimate the susceptibility of ψ_T with respect to the random coefficients. In view of (3.22) it is not surprising that we will have to estimate not only the susceptibility of ψ_T but also of ϕ_T and of some Green function with respect to the random coefficients. In the first step, we establish the susceptibility estimate for the Green function. In Step 2 we turn to the susceptibility estimate for the approximate corrector ϕ_T. We then show in Step 3 that, relying on [GO10a], this implies that the spectral exponents are at least

$$\left| \begin{array}{rll} (2,-q) & \text{for} & d=2, \\ (2,0) & \text{for} & d>2. \end{array} \right. \tag{3.24}$$

In Step 4 we estimate the susceptibility of ψ_T. We conclude the proof of the proposition in Step 5.

Step 1. Susceptility of the Green function.
We shall prove for all $e=(z,z')\in\mathbb{B}$, $z\in\mathbb{Z}^d$, $z'=z+\mathbf{e}_i$,

$$\begin{aligned} \frac{\partial G_T}{\partial \omega_e}(x,y) \;=\; -T^{-1}\Big(G_T(z,y)G_T(x,z)+G_T(z',y)G_T(x,z')\Big) \\ -\nabla_{2,i}G_T(x,z)\nabla_{1,i}G_T(z,y), \end{aligned} \tag{3.25}$$

and

$$\begin{aligned} \sup_{\omega_e}|\nabla_{1,i}G_T(z,y)| \;&\lesssim\; |\nabla_{1,i}G_T(z,y)|+T^{-1}g_T(y-z), \\ \sup_{\omega_e}|\nabla_{2,i}G_T(y,z)| \;&\lesssim\; |\nabla_{2,i}G_T(y,z)|+T^{-1}g_T(y-z), \end{aligned} \tag{3.26}$$

where $g_T:\mathbb{Z}^d\to\mathbb{R}^+$ satisfies for some constant $c>0$ (depending on α,β,d)

$$g_T(x) \;=\; (1+|x|)^{2-d}\exp\big(-c\frac{|x|}{\sqrt{T}}\big) \tag{3.27}$$

for $d>2$, and

$$g_T(x) \;=\; \left|\ln\Big(\frac{\sqrt{T}}{1+|x|}\Big)\right|\exp\big(-c\frac{|x|}{\sqrt{T}}\big) \tag{3.28}$$

for $d=2$.

We define the elliptic operator L_T as

$$(L_T u)(x) \;=\; \sum_{x',|x-x'|=1}\omega_{x,x}T^{-1}u(x)+\sum_{x',|x-x'|=1}\omega_{x,x'}\big(u(x)-u(x')\big),$$

so that for all $y\in\mathbb{Z}^d$, (3.14) takes the form

$$(L_T G_T(\cdot,y))(x) \;=\; \delta(x-y). \tag{3.29}$$

Recalling that the edges are not oriented, a formal differentiation of this equation with respect to $\omega_e=\omega_{z,z'}=\omega_{z',z}$ yields

$$\begin{aligned} &L_T\Big(\frac{\partial G_T}{\partial\omega_e}(\cdot,y)\Big)(x)+T^{-1}G_T(x,y)(\delta(x-z)+\delta(x-z')) \\ &\quad+\big(G_T(z,y)-G_T(z',y)\big)\delta(x-z)+\big(G_T(z',y)-G_T(z',y)\big)\delta(x-z') \;=\; 0. \end{aligned}$$

Using (3.29) this identity turns into

$$L_T\bigg(\frac{\partial G_T}{\partial \omega_e}(\cdot,y)+T^{-1}\big(G_T(z,y)G_T(\cdot,z)+G_T(z',y)G_T(\cdot,z')\big)\\ +\nabla_{2,i}G_T(\cdot,z)\nabla_{1,i}G_T(z,y)\bigg)(x)\;=\;0.$$

Provided that the argument of L_T is well-defined (that is, G_T is differentialble w. r. t. ω_e) and that it is square-integrable on $\mathbb{Z}^d$, it vanishes identically by the Riesz representation theorem — which is the desired identity (3.25).

To turn this into a rigorous argument, one may first consider finite differences of parameter $h>0$ instead of a derivative w. r. t. ω_e, use that L_T is bijective on the set of square-integrable functions on $\mathbb{Z}^d$, and then pass to the limit $h\to 0$. We refer the reader to [GO10a, Proof of Lemma 2.5] for details, and directly turn to (3.26).

From (3.25) with $x=z$ and $x=z'$, we infer that

$$\frac{\partial \nabla_{1,i}G_T(z,y)}{\partial \omega_e}\;=\;-\nabla_{1,i}\nabla_{2,i}G_T(z,z)\nabla_{1,i}G_T(z,y)\\ -T^{-1}\big(G_T(z,y)\nabla_{1,i}G_T(z,z)+G_T(z',y)\nabla_{1,i}G_T(z,z')\big). \tag{3.30}$$

Using the uniform pointwise estimate of Corollary 3.8, the uniform pointwise estimate on the Green function of Lemma 3.6, we obtain (3.26) by considering (3.30) as an ODE for $\nabla_{1,i}G_T(z,y)$ in function of ω_e.

Step 2. Susceptibility of ϕ_T.
In this step we shall prove that for $e=(z,z')\in\mathbb{B}$, $z\in\mathbb{Z}^d$ and $z'=z+\mathbf{e}_i$,

$$\frac{\partial \phi_T}{\partial\omega_e}(x)\;=\;-(\nabla_i\phi_T(z)+\xi_i)\nabla_{2,i}G_T(x,z)\\ -T^{-1}\phi_T(z)\big(G_T(x,z)+G_T(x,z')\big), \tag{3.31}$$

$$\sup_{\omega_e}|\phi_T(x)|\;\lesssim\;|\phi_T(x)|\\ +(|\nabla_i\phi_T(z)|+1)\Big(|\nabla_{2,i}G_T(x,z)|+T^{-1/2}g_T(x-z)\Big), \tag{3.32}$$

$$\sup_{\omega_e}\left|\frac{\partial\phi_T}{\partial\omega_e}(x)\right|\;\lesssim\;(|\nabla_i\phi_T(z)|+1)\Big(|\nabla_{2,i}G_T(x,z)|+T^{-1/2}g_T(x-z)\Big), \tag{3.33}$$

and for all $n\in\mathbb{N}$,

$$\sup_{\omega_e}\left|\frac{\partial(\phi_T(x)^{n+1})}{\partial\omega_e}\right|\;\lesssim\;(|\nabla_i\phi_T(z)|+1)\Big(|\nabla_{2,i}G_T(x,z)|+T^{-1/2}g_T(x-z)\Big)\\ \times\Big(|\phi_T(x)|+(|\nabla_i\phi_T(z)|+1)(|\nabla_{1,i}G_T(z,x)|+T^{-1/2}g_T(x-z)\Big)^n. \tag{3.34}$$

As for the Green function, we rewrite the defining equation for ϕ_T as

$$(L_T\phi_T)(x)-\nabla^*\cdot A(x)\xi\;=\;0. \tag{3.35}$$

Formally differentiating (3.35) w. r. t. ω_e yields

$$L_T\frac{\partial\phi_T}{\partial\omega_e}(x)-(\nabla_i\phi_T(x)+\xi_i)\big(\delta(x-z)-\delta(x-z')\big)\\ +T^{-1}\phi_T(x)\big(\delta(x-z)+\delta(x-z')\big)\;=\;0,$$

which using (3.29) turns into

$$L_T\bigg(\frac{\partial \phi_T}{\partial \omega_e}-(\nabla_i\phi_T+\xi_i)\big(G_T(\cdot,z)-G_T(\cdot,z')\big) \\ +T^{-1}\phi_T\big(G_T(\cdot,z)+G_T(\cdot,z')\big)\bigg)(x)\;=\;0.$$

This (formally) shows (3.31).

To turn this into a rigorous argument, we may combine (3.25) with the Green representation formula

$$\phi_T(x)\;=\;\int_{\mathbb{Z}^d}G_T(x,y)\nabla^*\cdot A(y)\xi\;\mathrm{d}y,$$

which holds since $G_T(x,\cdot)$ is integrable on $\mathbb{Z}^d$ by Lemma 3.6, and use standard results of commutation of integration and differentiation.

We now turn to (3.33). This estimate follows from (3.31), (3.26), and the following two facts:

$$|\phi_T|\;\lesssim\;\sqrt{T} \tag{3.36}$$

and

$$\sup_{\omega_e}|\nabla_i\phi_T(z)|\;\lesssim\;|\nabla_i\phi_T(z)|+1. \tag{3.37}$$

The starting point to prove (3.36) is the Green representation formula in the form of

$$\begin{aligned}|\phi_T(x)|\;&=\;\left|\int_{\mathbb{Z}^d}G_T(x,y)\nabla^*\cdot A(y)\xi\;\mathrm{d}y\right|\\ &=\;\left|\int_{\mathbb{Z}^d}\nabla_2 G_T(x,y)\cdot A(y)\xi\;\mathrm{d}y\right|\\ &\lesssim\;\int_{\mathbb{Z}^d}|\nabla_2 G_T(0,y)|\;\mathrm{d}y. \qquad (3.38)\end{aligned}$$

The claim would easily follow if we had the estimate

$$|\nabla_2 G_T(0,y)|\;\lesssim\;(1+|y|)^{1-d}\exp\bigg(-c\frac{|y|}{\sqrt{T}}\bigg).$$

Although this estimate does not hold *pointwise*, it holds when *square-integrated on dyadic annuli*, as shows Lemma 3.7 with "$p=2$ and k large". The claim (3.36) thus follows from a dyadic decomposition of space in (3.38) combined with Cauchy-Schwarz' inequality and Lemma 3.7 (a similar calculation is displayed for instance in [Gl10, Proof of Lemma 4]).

For (3.37), we first note that (3.31) implies

$$\frac{\partial\nabla_i\phi_T(z)}{\partial\omega_e}\;=\;-(\nabla_i\phi_T(z)+\xi_i)\big(\nabla_{2,i}G_T(z',z)-\nabla_{2,i}G_T(z,z)\big) \\ +T^{-1}\phi_T(z)\big(\nabla_{1,i}G_T(z,z)+\nabla_{1,i}G_T(z,z')\big),$$

which — seen as an ODE w. r. t. ω_e — yields the claim using the uniform bound $|\nabla_1 G_T|,|\nabla_2 G_T|\lesssim 1$ of Corollary 3.8 and (3.36).

Estimate (3.32) is a direct consequence of (3.33), whereas (3.34) follows from the Leibniz' rule combined with (3.31), (3.32), and (3.33).

Step 3. Proof of (3.24).

The estimates (3.24) of the spectral exponents follow from the more general estimates: for all $q>0$ there exists $\gamma(q)>0$ such that

$$\langle|\phi_T|^q\rangle \lesssim \left|\begin{array}{rcl} \ln^{\gamma(q)} T & \text{for} & d=2, \\ 1 & \text{for} & d>2, \end{array}\right. \tag{3.39}$$

combined with the fact that

$$\int_0^{T^{-1}} \mathrm{d}e_{\mathfrak{d}}(\lambda) \lesssim T^{-2}\int_0^\infty \frac{1}{(T^{-1}+\lambda)^2}\,\mathrm{d}e_{\mathfrak{d}}(\lambda) \;=\; T^{-2}\left\langle \phi_T^2\right\rangle.$$

The proof of (3.39) is an easy adaptation of [GO10a, Proof of Proposition 2.1] which already covers the case of a constant coefficient in the zero order term of L_T, that is for $T^{-1}\phi_T$ instead of $T^{-1}p_\omega\phi_T$ (no randomness in the zero order term). We only point out what needs to be changed in [GO10a, Proof of Proposition 2.1]

The first step to apply the variance estimate of Lemma 3.4 is to show that ϕ_T is measurable with respect to the cylindrical topology associated with the random variables. This is proved exactly as in [GO10a, Lemma 2.6].

The auxiliary [GO10a, Lemmas 2.4 & 2.5] are replaced by the susceptibility estimates (3.26), (3.32), (3.33), and (3.34) of Steps 1 and 2, which have however the additional term $T^{-1/2}g_T(x-z)$ next to $|\nabla_{2,i}G_T(x,z)|$.

In the proof of [GO10a, Proposition 2.1], the terms $|\nabla_{2,i}G_T(x,z)|$ are either estimated by the Green function $G_T(x,z)$ itself (in which case the additional term $T^{-1/2}g_T(x-z)$ is of higher order), or they are controlled on dyadic annuli by Lemma 3.7. By the definition (3.27) for $d>2$ and (3.28) for $d=2$ of the function g_T, it is easy to see that for all $r\geqslant 2$, $k>0$ and $R\gg 1$: for $d>2$

$$\int_{R\leqslant|x-z|<2R}\big(T^{-1/2}g_T(x-z)\big)^r dz \lesssim R^d(R^{1-d})^r\min\{1,\sqrt{T}R^{-1}\}^k,$$

whereas for $d=2$

$$\int_{R\leqslant|x-z|<2R}\big(T^{-1/2}g_T(x-z)\big)^r dz \lesssim R^2(R^{-1})^r\ln^q T\min\{1,\sqrt{T}R^{-1}\}^k.$$

These scalings coincide with those of Lemma 3.7 (with a possible additional logarithmic correction for $d=2$).

Hence the proof of [GO10a, Proposition 2.1] adapts mutadis mutandis to the present case, and we have (3.39).

Step 4. Susceptibility of ψ_T.
In this step we shall prove that for all $e=(z,z')$, $z\in\mathbb{Z}^d$ and $z'=z+\mathbf{e}_i$, and for all $x\in\mathbb{Z}^d$

$$\begin{aligned}\frac{\partial\psi_T}{\partial\omega_e}(x) \;=\;& -\nabla_{2,i}G_T(x,z)\nabla_i\psi_T(z)-T^{-1}G_T(x,z)\psi_T(z)\\ &-T^{-1}G_T(x,z')\psi_T(z')\\ &-(\nabla_i\phi_T(z)+\xi_i)\int_{\mathbb{Z}^d}G_T(x,y)p_\omega(y)\nabla_{2,i}G_T(y,z)\,\mathrm{d}y\\ &-T^{-1}\phi_T(z)\int_{\mathbb{Z}^d}G_T(x,y)p_\omega(y)\big(G_T(y,z)+G_T(y,z')\big)\,\mathrm{d}y\\ &+G_T(x,z)\phi_T(z)+G_T(x,z')\phi_T(z'),\end{aligned} \tag{3.40}$$

and

$$\begin{aligned}(3.41)\quad &\sup_{\omega_e}\left|\frac{\partial\psi_T}{\partial\omega_e}(x)\right| \\ &\lesssim\; g_T(z-x)\big(|\nabla_i\psi_T(z)|+T^{-1}|\psi_T(z)|+\nu_d(T)(1+|\phi_T(z)|+|\phi_T(z')|)\big) \\ &\quad +(1+|\phi_T(z)|+|\phi_T(z')|)\int_{\mathbb{Z}^d} g_T(y-x)\big(|\nabla_{2,i}G_T(y,z)|+T^{-1}g_T(y-z)\big)\,\mathrm{d}y,\end{aligned}$$

where

$$\nu_d(T)\;=\;\left|\begin{array}{rll} T & \text{for} & d=2,\\ \sqrt{T} & \text{for} & d=3,\\ \ln T & \text{for} & d=4,\\ 1 & \text{for} & d>4.\end{array}\right. \tag{3.42}$$

The starting point is again the Green representation formula

$$\psi_T(x)\;=\;\int_{\mathbb{Z}^d} G_T(x,y)p_\omega(y)\phi_T(y)\,\mathrm{d}y,$$

associated with (3.22) in the form

$$T^{-1}p_\omega\psi_T-\nabla^*\cdot A\nabla\psi_T\;=\;p_\omega\phi_T.$$

Differentiated w. r. t. ω_e it turns into

$$\begin{aligned}\frac{\partial\psi_T(x)}{\partial\omega_e}\;&=\;\int_{\mathbb{Z}^d}\frac{\partial G_T(x,y)}{\partial\omega_e}p_\omega(y)\phi_T(y)\,\mathrm{d}y \\ &\quad+\int_{\mathbb{Z}^d} G_T(x,y)\frac{\partial p_\omega(y)}{\partial\omega_e}\phi_T(y)\,\mathrm{d}y+\int_{\mathbb{Z}^d} G_T(x,y)p_\omega(y)\frac{\partial\phi_T(y)}{\partial\omega_e}\,\mathrm{d}y.\end{aligned}$$

Combined with (3.25), (3.31), and the Green representation formula itself, this shows (3.40).

We now turn to (3.41) and treat each term of the r. h. s. of (3.40) separately. We begin with the supremum of the third line of (3.40), and claim that

$$\begin{aligned}(3.43)\quad &\sup_{\omega_e}\left|(\nabla_i\phi_T(z)+\xi_i)\int_{\mathbb{Z}^d} G_T(x,y)p_\omega(y)\nabla_{2,i}G_T(y,z)\,\mathrm{d}y\right| \\ &\lesssim\;(1+|\phi_T(z)|+|\phi_T(z')|)\int_{\mathbb{Z}^d} g_T(y-x)\big(|\nabla_{2,i}G_T(y,z)|+T^{-1}g_T(y-z)\big)\,\mathrm{d}y,\end{aligned}$$

which is proved

- using (3.37) to bound the supremum in ω_e of $|\nabla_i\phi_T(z)|$ by $|\nabla_i\phi_T(z)|$ itself,
- bounding $|\nabla_i\phi_T(z)|$ by the triangle inequality $|\phi_T(z)|+|\phi_T(z')|$,
- replacing the Green function G_T by g_T using Lemma 3.6,
- and appealing to (3.26) to estimate the supremum in ω_e of $|\nabla_{2,i}G_T(y,x)|$.

This shows that this term is controlled by the second term of the r. h. s. of (3.41).

The supremum of the term in the fourth line of (3.40) is also estimated by the second term of the r. h. s. of (3.41), namely

$$\begin{aligned}(3.44)\quad &\sup_{\omega_e}\left|T^{-1}\phi_T(z)\int_{\mathbb{Z}^d} G_T(x,y)p_\omega(y)\big(G_T(y,z)+G_T(y,z')\big)\,\mathrm{d}y\right| \\ &\qquad\lesssim\;(1+|\phi_T(z)|+|\phi_T(z')|)T^{-1}\int_{\mathbb{Z}^d} g_T(y-x)g_T(y-z)\,\mathrm{d}y.\end{aligned}$$

It is enough to bound the Green function by g_T using Lemma 3.6, and to apply (3.32) for $x=z$ to control $\sup_{\omega_e}|\phi_T(z)|$, and use that $|\nabla_1 G_T|,|\nabla_2 G_T|,T^{-1/2}G_T\lesssim 1$ by Corollary 3.8 and Lemma 3.6.

The suprema of the last two tems of (3.40) is bounded by

$$\sup_{\omega_e} |G_T(x,z)\phi_T(z) + G_T(x,z')\phi_T(z')| \lesssim (1 + |\phi_T(z)| + |\phi_T(z')|) g_T(z-x), \tag{3.45}$$

and therefore controlled by the first term of the r. h. s. of (3.41). The argument is similar to the proof of (3.44).

The subtle terms are the first three ones, for which we have to estimate the suprema of $|\nabla_i \psi_T(z)|$, $|\psi_T(z)|$, and $|\psi_T(z')|$ w. r. t. ω_e.

We begin with the following two estimates

$$\begin{aligned} \sup_{\omega_e} |\psi_T(z)| &\lesssim |\psi_T(z)| + \big(|\phi_T(z)| + |\phi_T(z')| + 1\big)\nu_d(T) \\ &\quad + \sup_{\omega_e} |\nabla_i \psi_T(z)|, \end{aligned} \tag{3.46}$$

$$\begin{aligned} \sup_{\omega_e} |\nabla_i \psi_T(z)| &\lesssim |\nabla_i \psi_T(z)| + \big(|\phi_T(z)| + |\phi_T(z')| + 1\big)\nu_d(T) \\ &\quad + T^{-1} \sup_{\omega_e} |\psi_T(z)|, \end{aligned} \tag{3.47}$$

which — seen as a linear system — show that there exists some $T_* > 0$ such that for all $T \geqslant T^*$,

$$\begin{aligned} \sup_{\omega_e} |\psi_T(z)| &\lesssim |\psi_T(z)| + \big(|\phi_T(z)| + |\phi_T(z')| + 1\big)\nu_d(T) \\ &\quad + |\nabla_i \psi_T(z)|, \end{aligned} \tag{3.48}$$

$$\begin{aligned} \sup_{\omega_e} |\nabla_i \psi_T(z)| &\lesssim |\nabla_i \psi_T(z)| + \big(|\phi_T(z)| + |\phi_T(z')| + 1\big)\nu_d(T) \\ &\quad + T^{-1} |\psi_T(z)|. \end{aligned} \tag{3.49}$$

To prove (3.46) we consider (3.40) as an ODE on $\psi_T(z)$, bound $\psi_T(z')$ by $\psi_T(z) + |\nabla_i \psi_T(z)|$, and use (3.43), (3.44), and (3.45) (for $x = z$), so that (3.40) turns into

$$\begin{aligned} \left| \frac{\partial \psi_T}{\partial \omega_e}(z) \right| &\lesssim \sup_{\omega_e} \{ |\nabla_{2,i} G_T(z,z)| |\nabla_i \psi_T(z)| \} + T^{-1} G_T(z,z) |\psi_T(z)| \\ &\quad + T^{-1} G_T(z,z') \big(|\psi_T(z)| + \sup_{\omega_e} |\nabla_i \psi_T(z)| \big) \\ &\quad + (1 + |\phi_T(z)| + |\phi_T(z')|) \\ &\qquad \times \left(1 + \int_{\mathbb{Z}^d} g_T(y-z) \big(|\nabla_{2,i} G_T(y,z)| + T^{-1} g_T(y-z) \big) \, \mathrm{d}y \right). \end{aligned}$$

Using Corollary 3.8 and Lemma 3.6 in the form of $|\nabla_1 G_T|, |\nabla_2 G_T|, T^{-1} G_T \lesssim 1$, and bounding the gradient of the Green function by g_T in the integral, we obtain

$$\left| \frac{\partial \psi_T}{\partial \omega_e}(z) \right| \lesssim \sup_{\omega_e} \{ |\nabla_i \psi_T(z)\} + |\psi_T(z)| + (1 + |\phi_T(z)| + |\phi_T(z')|) \left(1 + \int_{\mathbb{Z}^d} g_T(y)^2 \, \mathrm{d}y \right).$$

Noting that by definition (3.27)&(3.28) of g_T we have $\int_{\mathbb{Z}^d} g_T(y)^2 \mathrm{d}y \lesssim \nu_d(T)$, this inequality turns into

$$\left| \frac{\partial \psi_T}{\partial \omega_e}(z) \right| \lesssim \sup_{\omega_e} \{ |\nabla_i \psi_T(z)\} + |\psi_T(z)| + \nu_d(T)(1 + |\phi_T(z)| + |\phi_T(z')|).$$

Seen as an ODE for ψ_T, this implies (3.46).

We now turn to (3.47) and infer from (3.40) that

$$\begin{aligned}\frac{\partial \nabla_i \psi_T(z)}{\partial \omega_e} &= -\nabla_{1,i}\nabla_{2,i}G_T(z,z)\nabla_i\psi_T(z) - T^{-1}\nabla_{1,i}G_T(z,z)\psi_T(z) \\ &\quad -T^{-1}\nabla_{1,i}G_T(z,z')\psi_T(z') \\ &\quad -(\nabla_i\phi_T(z)+\xi_i)\int_{\mathbb{Z}^d}\nabla_{2,i}G_T(z,y)p_\omega(y)\nabla_{2,i}G_T(y,z)\,\mathrm{d}y \\ &\quad -T^{-1}\phi_T(z)\int_{\mathbb{Z}^d}\nabla_{2,i}G_T(z,y)p_\omega(y)\big(G_T(y,z)+G_T(y,z')\big)\,\mathrm{d}y \\ &\quad +\nabla_{1,i}G_T(z,z)\phi_T(z)+\nabla_{1,i}G_T(z,z')\phi_T(z').\end{aligned}$$

Repeating the string of arguments leading from (3.40) to (3.46), we deduce (3.47), and therefore (3.48) and (3.49). Combining the inequality $|\psi_T(z')| \leqslant |\psi_T(z)| + |\nabla_i\psi_T(z)|$ with (3.48) and (3.49) yields the last estimate we need:

$$\sup_{\omega_e}|\psi_T(z')| \lesssim |\psi_T(z)| + \big(|\phi_T(z)|+|\phi_T(z')|+1\big)\nu_d(T) + |\nabla_i\psi_T(z)|. \tag{3.50}$$

We are finally in position to conclude the proof of (3.41). The four last terms are controlled by (3.43), (3.44), and (3.45). Using (3.49), (3.48), and (3.50), and Corollary 3.8 and Lemma 3.6, the first three terms of the r. h. s. of (3.40) are controlled by the first temr of the r. h. s. of (3.41). Estimate (3.41) is proved.

Step 5. Estimate of $\mathrm{var}[\psi_T]$ for $d>2$ and conclusion.

We apply the variance estimate of Lemma 3.4 to ψ_T

$$\mathrm{var}[\psi_T] \lesssim \sum_{e\in\mathbb{B}}\left\langle \sup_{\omega_e}\left|\frac{\partial\psi_T(0)}{\partial\omega_e}\right|^2\right\rangle, \tag{3.51}$$

and appeal to (3.41). We distinguish two contributions in this sum and define

$$\begin{aligned} A_e &:= g_T(z)\big(|\nabla_i\psi_T(z)| + T^{-1}|\psi_T(z)| + \nu_d(T)(1+|\phi_T(z)|+|\phi_T(z')|)\big), \\ B_e &:= (1+|\phi_T(z)|+|\phi_T(z')|)\int_{\mathbb{Z}^d} g_T(y)\big(|\nabla_{2,i}G_T(y,z)| + T^{-1}g_T(y-z)\big)\,\mathrm{d}y.\end{aligned}$$

The contribution associated with A_e is estimated as follows:

$$\begin{aligned}&\sum_{e\in\mathbb{B}}\langle A_e^2\rangle \\ &\lesssim \sum_{e\in\mathbb{B}}\left\langle g_T(z)^2\big(|\nabla_i\psi_T(z)|^2 + T^{-2}|\psi_T(z)|^2 + \nu_d(T)^2(1+|\phi_T(z)|^2+|\phi_T(z')|^2)\big)\right\rangle \\ &\lesssim \left(\sum_{z\in\mathbb{Z}^d} g_T(z)^2\right)\big(\langle|\nabla\psi_T|^2\rangle + T^{-2}\langle\psi_T^2\rangle + \nu_d(T)^2(1+\langle\phi_T^2\rangle)\big) \\ &\lesssim \nu_d(T)\big(\langle|\nabla\psi_T|^2\rangle + T^{-2}\langle\psi_T^2\rangle + \nu_d(T)^2(1+\langle\phi_T^2\rangle)\big),\end{aligned}$$

by stationarity of ϕ_T, ψ_T, and $\nabla\psi_T$. This is a nonlinear estimate since $\langle\psi_T^2\rangle$ and $\langle|\nabla\psi_T|^2\rangle$ appear in the r. h. s. whereas we want to estimate $\langle\psi_T^2\rangle$. We then appeal to the elementary a priori estimate

$$\langle|\nabla\psi_T|^2\rangle \lesssim \langle\phi_T^2\rangle^{1/2}\langle\psi_T^2\rangle^{1/2},$$

which we obtain by testing (3.22) with test the solution ψ_T, integrating by parts, using the bounds on A, and Cauchy-Schwarz' inequality. Using in addition Young's inequality, the estimate turns into

$$\sum_{e\in\mathbb{B}}\langle A_e^2\rangle - \frac{1}{C}\langle\psi_T^2\rangle \lesssim C\nu_d(T)^2\langle\phi_T^2\rangle + \nu_d(T)^3(1+\langle\phi_T^2\rangle),$$

for all $C > 0$ and T large enough.

Combined with (3.39) for $q = 2$ and the definition of $\nu_d(T)$, this turns into: for all $C > 0$,

$$\sum_{e\in\mathbb{B}} \langle A_e^2\rangle - \frac{1}{C}\langle\psi_T^2\rangle \;\lesssim\; C \left| \begin{array}{rll} T^{3/2} & \text{for} & d=3, \\ \ln^3 T & \text{for} & d=4, \\ 1 & \text{for} & d>4. \end{array}\right. \tag{3.52}$$

We now turn to the term associated with B_e, which we split into two terms $B_e = B_{e,1} + B_{e,2}$, where

$$\begin{aligned} B_{e,1} &= (1+|\phi_T(z)|+|\phi_T(z')|)T^{-1}\int_{\mathbb{Z}^d} g_T(y)g_T(y-z)\,\mathrm{d}y \\ B_{e,2} &= (1+|\phi_T(z)|+|\phi_T(z')|)\int_{\mathbb{Z}^d} g_T(y)|\nabla_{2,i}G_T(y,z)|\,\mathrm{d}y. \end{aligned}$$

In particular, we shall prove that

$$\sum_{e\in\mathbb{B}} \langle B_e\rangle^2 \lesssim \sum_{e\in\mathbb{B}} \langle B_{e,1}^2\rangle + \sum_{e\in\mathbb{B}} \langle B_{2,e}^2\rangle \;\lesssim\; \left| \begin{array}{rll} T^{3/2} & \text{for} & d=3, \\ T & \text{for} & d=4, \\ \sqrt{T} & \text{for} & d=5, \\ \ln T & \text{for} & d=6, \\ 1 & \text{for} & d>6. \end{array}\right. \tag{3.53}$$

We start with the sum of $B_{e,1}^2$ on $\mathbb{B}$. Since g_T is deterministic and ϕ_T is stationary,

$$\sum_{e\in\mathbb{B}} \langle B_{e,1}^2\rangle \lesssim (1+\langle\phi_T^2\rangle^{1/2})\int_{\mathbb{Z}^d}\int_{\mathbb{Z}^d}\int_{\mathbb{Z}^d} T^{-1}g_T(y)g_T(y')g_T(y-z)g_T(y'-z)\mathrm{d}y\mathrm{d}y'\mathrm{d}z.$$

Using (3.39) with $q = 2$ and the definition (3.28)&(3.27) of g_T to estimate the integral, we conclude that the first term of the l. h. s. of (3.53) is controlled by the r. h. s. of (3.53). A formal argument to estimate the triple integral is as follows. By the exponential decay of g_T, it is enough to integrate on the set $|y|, |y'| \lesssim \sqrt{T}$ and $|y-z| + |y'-z| \lesssim \sqrt{T}$, and the integral essentially behaves as the integral on the ball of radius $\sqrt{T}$ in $\mathbb{Z}^{3d}$ of $T^{-1}(1+|x|)^{4(2-d)}$, whence the bounds

$$\sum_{e\in\mathbb{B}} \langle B_{e,1}^2\rangle \;\lesssim\; \left| \begin{array}{rll} T^{3/2} & \text{for} & d=3, \\ T & \text{for} & d=4, \\ \sqrt{T} & \text{for} & d=5, \\ 1 & \text{for} & d=6, \\ T^{-1/2} & \text{for} & d=7, \\ T^{-1}\ln T & \text{for} & d=8, \\ T^{-1} & \text{for} & d>8. \end{array}\right.$$

To rigorously prove that the r. h. s. of (3.53) is an upper bound for $\sum_{e\in\mathbb{B}}\langle B_{e,1}^2\rangle$, we may simply note that for $d > 2$ if we define $h_T(z) := \sqrt{T}^{-1}g_T(z)$, then for all $z \in \mathbb{Z}^d$,

$$h_T(z) \lesssim (1+|z|)^{1-d}\exp\left(-c\frac{|z|}{\sqrt{T}}\right),$$

and g_T and h_T satisfy the assumptions of Lemma 3.9, which yields the desired upper bound.

We turn to the sum of $B_{e,2}^2$ on $\mathbb{B}$.

$$\begin{aligned} \sum_{e\in\mathbb{B}} \langle B_{e,2}^2\rangle \;\lesssim\; & \Big\langle \sum_{i=1}^d \int_{\mathbb{Z}^d}(1+|\phi_T(z)|+|\phi_T(z+\mathbf{e}_i)|) \\ & \times\int_{\mathbb{Z}^d}\int_{\mathbb{Z}^d} g_T(y)g_T(y')|\nabla_{2,i}G_T(y,z)||\nabla_{2,i}G_T(y',z)|\mathrm{d}y\mathrm{d}y'\mathrm{d}z\Big\rangle. \end{aligned}$$

Since g_T is deterministic, one can take it out of the expectation. We then choose $p > 2$ such that that the higher integrability result of Lemma 3.7 applies, use Hölder's inequality in probability with exponents $(p/(p-2), p, p)$ so that by stationarity of ϕ_T and of G_T in the form of (3.15) we have

$$\sum_{e\in\mathbb{B}} \left\langle B_{e,2}^2 \right\rangle \;\lesssim\; \left\langle |\phi_T|^{p/(p-2)} \right\rangle^{(p-2)/p} \int_{\mathbb{Z}^d}\int_{\mathbb{Z}^d}\int_{\mathbb{Z}^d} g_T(y) g_T(y') \\ \times \left\langle |\nabla_1 G_T(y-z,0)|^p \right\rangle^{1/p} \left\langle |\nabla_1 G_T(y'-z,0)|^p \right\rangle^{1/p} \mathrm{d}y\mathrm{d}y'\mathrm{d}z.$$

We then introduce the notation $h_T(x) := \langle |\nabla_1 G_T(x,0)|^p \rangle^{1/p}$. By Lemma 3.7 and Corollary 3.8, we have for all $R \lesssim 1$

$$\int_{|z|\leqslant R} h_T(x)^2 \mathrm{d}x \;\lesssim\; 1,$$

and for all $R \lesssim 1$ and $j \in \mathbb{N}$

$$\int_{2^jR\leqslant|z|<2^{j+1}R} h_T(x)^2 \mathrm{d}x \;\leqslant\; \left(\int_{2^jR\leqslant|z|<2^{i+1}R} h_T(x)^p \mathrm{d}x\right)^{2/p} \;\lesssim\; (2^jR)^{2-d}.$$

Hence, g_T and h_T satisfy the assumptions of Lemma 3.9, and $\sum_{e\in\mathbb{B}} \left\langle B_{e,2}^2 \right\rangle$ is bounded by the r. h. s. of (3.53).

We are in position to estimate $\mathrm{var}\left[\psi_T^2\right]$. Choosing C large enough in (3.52) to absorb the term $\frac{1}{C}\mathrm{var}\left[\psi_T^2\right]$ in the l. h. s. of (3.51), and using (3.53) we obtain the estimate

$$\mathrm{var}\left[\psi_T^2\right] \;\lesssim\; \left| \begin{array}{rll} T^{3/2} & \text{for} & d=3, \\ T & \text{for} & d=4, \\ \sqrt{T} & \text{for} & d=5, \\ \ln T & \text{for} & d=6, \\ 1 & \text{for} & d>6. \end{array} \right. \tag{3.54}$$

We may conclude the proof. Estimate (3.24) proved in Step 3 yields the desired spectral exponent for $d = 2$, whereas the combination of (3.54) with (3.20) and (3.23) yields the desired spectral exponents for $d > 2$.

□

Remark 3.10. *The structure of the proof can be summarized as follows:*

(a) *The starting point is the optimal estimates of $\left\langle \phi_T^2 \right\rangle$ (up to logarithmic correction for $d = 2$).*

(b) *The variance estimate applied to ψ_T and combined with elliptic theory shows there exists a map F such that*

$$\left\langle \psi_T^2 \right\rangle \;\leqslant\; F(T, \left\langle |\nabla\psi_T|^2 \right\rangle, \left\langle \psi_T^2 \right\rangle, \left\langle \phi_T^2 \right\rangle).$$

(c) *By an a priori estimate, $\left\langle |\nabla\psi_T|^2 \right\rangle \lesssim \left\langle \psi_T^2 \right\rangle^{1/2} \left\langle \phi_T^2 \right\rangle^{1/2}$.*

(d) *Combined with Young's inequality and (c), (b) turns into*

$$\left\langle \psi_T^2 \right\rangle \;\leqslant\; \tilde{F}(T, \left\langle \phi_T^2 \right\rangle)$$

for some map $\tilde{F}$, and yields the claim.

In view of this, a possible strategy to prove optimal scalings of the spectral exponents in any dimension would be to proceed by induction. Set $\phi_{1,T} \equiv \phi_T$, and for all $k \geqslant 1$ define $\phi_{k,T}$ as the unique weak solution to

$$T^{-1}\phi_{k+1,T}(x) - \frac{1}{p_\omega(x)}\nabla^* \cdot A(x)\nabla\psi_{k+1,T}(x) \;=\; \psi_{k,T}(x),$$

and apply the strategy described above to obtain optimal bounds on $\left\langle \phi_{k+1,T}^2 \right\rangle$ *assuming optimal bounds on* $\left\langle \phi_{k,T}^2 \right\rangle$ *(which would yield optimal spectral exponents up to dimension* $4k-2$ *— with a logarithmic correction in dimension* $4k-2$*). The main difficulty is to work out a suitable map* F_k *in step (b).*

4. The random fluctuations

In this section, we show that the computable quantity $\hat{A}_n(t)$ defined in (1.6) is a good approximation of σ_t^2, in the sense that its random fluctuations are small as soon as n/t^2 is large. We write $\mathbb{N}^*$ for $\mathbb{N} \setminus \{0\}$.

Theorem 4.1. *There exists* $c > 0$ *such that, for any* $n \in \mathbb{N}^*$, $\varepsilon > 0$ *and* t *large enough,*

$$\mathbb{P}_0^{\otimes}\left[\left|\hat{A}_n(t) - \sigma_t^2\right| \geqslant \varepsilon/t\right] \leqslant \exp\left(-\frac{n\varepsilon^2}{ct^2}\right).$$

Note that σ_t^2 is the mean value of $\hat{A}_n(t)$, and moreover, $\hat{A}_n(t)$ consists of a sum of i.i.d. random variables. We will thus obtain Theorem 4.1 by using classical techniques from large deviation theory. The important point is that the i.i.d. random variables under consideration are uniformly exponentially integrable. To see this, we use a sharp upper bound on the transition probabilities of the random walk recalled in the following theorem. We refer the reader to [HS93] or [Wo, Theorem 14.12] for a proof.

Theorem 4.2. *There exists a constant* $c_1 > 0$ *such that, for any environement* ω *with conductances in* $[\alpha, \beta]$, *any* $t \in \mathbb{N}^*$ *and* $x \in \mathbb{Z}^d$,

$$\mathbf{P}_0^{\omega}[Y(t) = x] \leqslant \frac{c_1}{t^{d/2}} \exp\left(-\frac{|x|^2}{c_1 t}\right).$$

From Theorem 4.2 we deduce the following result.

Corollary 4.3. *Let* c_1 *be given by Theorem 4.2. For all* $\lambda < 1/c_1$, *one has*

$$\sup_{t \in \mathbb{N}^*} \tilde{\mathbb{E}}_0\left[\exp\left(\lambda \frac{|Y(t)|^2}{t}\right)\right] < +\infty.$$

Proof. Let $\delta = 1/c_1 - \lambda$. By Theorem 4.2,

$$\mathbf{E}_0^{\omega}\left[e^{\lambda |Y(t)|^2/t}\right] \leqslant c_1 t^{-d/2} \sum_{x \in \mathbb{Z}^d} e^{-\delta |x|^2/t}.$$

If the sum ranges over all $x \in (\mathbb{N}^*)^d$, it is easy to bound it by a convergent integral:

$$t^{-d/2} \sum_{x \in (\mathbb{N}^*)^d} e^{-\delta |x|^2/t} \leqslant t^{-d/2} \int_{\mathbb{R}_+^d} e^{-\delta |x|^2/t} \, \mathrm{d}x = \int_{\mathbb{R}_+^d} e^{-\delta |x|^2} \, \mathrm{d}x.$$

By symmetry, the estimate carries over to the sum over all $x \in (\mathbb{Z}^*)^d$. The same argument applies for the sum over all $x = (x_1, \ldots, x_d)$ having exactly one component equal to 0, and so on. □

The following lemma shows that the log-Laplace transform of $\frac{(\xi \cdot Y(t))^2}{t} - \sigma_t^2$ is bounded by a parabola in a neighbourhood of 0, uniformly over t.

Lemma 4.4. *There exist* $\lambda_1 > 0$ *and* c_2 *such that, for any* $\lambda < \lambda_1$ *and any* $t \in \mathbb{N}^*$,

$$\ln \tilde{\mathbb{E}}_0\left[\exp\left(\lambda\left(\frac{(\xi \cdot Y(t))^2}{t} - \sigma_t^2\right)\right)\right] \leqslant c_2 \lambda^2.$$

Proof. It is sufficient to prove that there exists c_3 such that, for any λ small enough and any t,

$$\tilde{\mathbb{E}}_0 \left[\exp\left(\lambda \left(\frac{(\xi \cdot Y(t))^2}{t} - \sigma_t^2 \right) \right) \right] \leqslant 1 + c_3 \lambda^2.$$

We use the series expansion of the exponential to rewrite this expectation as

$$\sum_{k=0}^{+\infty} \frac{\lambda^k}{k!} \, \tilde{\mathbb{E}}_0 \left[\left(\frac{(\xi \cdot Y(t))^2}{t} - \sigma_t^2 \right)^k \right].$$

The term corresponding to $k = 0$ is equal to 1, whereas the term for $k = 1$ vanishes. The remaining sum, for k ranging from 2 to infinity, can be controlled using Corollary 4.3 combined with the bound

$$\tilde{\mathbb{E}}_0 \left| \frac{(\xi \cdot Y(t))^2}{t} - \sigma_t^2 \right|^k \leqslant 2 \tilde{\mathbb{E}}_0 \left[\frac{(\xi \cdot Y(t))^{2k}}{t^k} \right],$$

which follows from the definition of σ_t^2 and Jensen's inequality. □

We are now in position to prove Theorem 4.1.

Proof of Theorem 4.1. From the definition of $\tilde{\mathbb{P}}$ given in (1.1), we can write

$$\mathbb{P}_0^{\otimes} \left[\hat{A}_n(t) - \sigma_t^2 \geqslant \varepsilon/t \right] = \tilde{\mathbb{P}}_0^{\otimes} \left[\frac{(\xi \cdot Y^{(1)}(t))^2 + \cdots + (\xi \cdot Y^{(n)}(t))^2}{nt} - \sigma_t^2 \geqslant \varepsilon/t \right].$$

Let $\lambda > 0$. We bound the latter probability using Chebyshev's inequality:

(4.1)
$$\begin{aligned}
&\mathbb{P}_0^{\otimes} \left[\hat{A}_n(t) - \sigma_t^2 \geqslant \varepsilon/t \right] \\
&\qquad \leqslant \tilde{\mathbb{E}}_0^{\otimes} \left[\exp\left(\lambda \left(\frac{(\xi \cdot Y^{(1)}(t))^2 + \cdots + (\xi \cdot Y^{(n)}(t))^2}{t} - n\sigma_t^2 \right) \right) \right] \exp\left(-\frac{n\lambda\varepsilon}{t} \right) \\
&\qquad \leqslant \tilde{\mathbb{E}}_0 \left[\exp\left(\lambda \left(\frac{(\xi \cdot Y(t))^2}{t} - \sigma_t^2 \right) \right) \right]^n \exp\left(-\frac{n\lambda\varepsilon}{t} \right).
\end{aligned}$$

By Lemma 4.4, the r. h. s. of (4.1) is bounded by

$$\exp\left(n \left(c_2 \lambda^2 - \frac{\lambda\varepsilon}{t} \right) \right),$$

for all λ small enough. Choosing $\lambda = \varepsilon/2c_2 t$ (which is small enough for t large enough), we obtain

$$\mathbb{P}_0^{\otimes} \left[\hat{A}_n(t) - \sigma_t^2 \geqslant \varepsilon/t \right] \leqslant \exp\left(-\frac{n\varepsilon^2}{4c_2 t^2} \right). \tag{4.2}$$

The probability of the symmetric event

$$\mathbb{P}_0^{\otimes} \left[\sigma_t^2 - \hat{A}_n(t) \geqslant 2\varepsilon/t \right]$$

can be handled the same way, so the proof is complete. □

5. Central limit theorem

In this short section, we complete the analysis by showing that the quantity $\sqrt{n(t)} \left(\hat{A}_{n(t)}(t) - \sigma_t^2 \right)$ satisfies a central limit theorem:

Proposition 5.1. *Let $(n(t))_{t\in\mathbb{N}^*}$ be any sequence tending to infinity with t. Under the measure $\mathbb{P}_0^{\otimes}$ and as t tends to infinity, the random variable*

$$\sqrt{n(t)}\left(\hat{A}_{n(t)}(t)-\sigma_t^2\right)$$

converges in distribution to a Gaussian random variable of variance

$$\mathsf{v}=\left(3\frac{\mathbb{E}[p^2]}{\mathbb{E}[p]^2}-1\right)\sigma^4.$$

Proof. Let us define

$$\mathsf{V}(t)=\frac{p(\omega)(\xi\cdot Y_t)^2}{t\mathbb{E}[p]}-\sigma_t^2$$

and

$$\mathsf{V}^{(k)}(t)=\frac{p(\omega^{(k)})(\xi\cdot Y_t^{(k)})^2}{t\mathbb{E}[p]}-\sigma_t^2,$$

so that

$$\hat{A}_{n(t)}(t)-\sigma_t^2=\frac{1}{n(t)}\sum_{k=1}^{n(t)}\mathsf{V}^{(k)}(t).$$

Let also $\mathsf{v}_t=\mathbb{E}_0[\mathsf{V}(t)^2]$. Note that for any t, $(\mathsf{V}^{(k)}(t))_{k\in\mathbb{N}}$ are i.i.d. centred random variables under $\mathbb{P}_0^{\otimes}$. From the Lindeberg-Feller theorem (see for instance [Du, Theorem 2.4.5]), we know that in order to show

$$\frac{\sum_{k=1}^{n(t)}\mathsf{V}^{(k)}(t)}{\sqrt{n(t)}}\xrightarrow[t\to+\infty]{\text{(distr.)}}\text{Gaussian}(0,\mathsf{v}),$$

it suffices to check that

$$\mathsf{v}_t\xrightarrow[t\to+\infty]{}\mathsf{v}\tag{5.1}$$

and that for any $\varepsilon>0$,

$$\mathbb{E}_0\left[\mathsf{V}(t)^2\ \mathbf{1}_{\{\mathsf{V}(t)\geqslant\varepsilon\sqrt{n(t)}\}}\right]\xrightarrow[t\to+\infty]{}0.\tag{5.2}$$

We learn from [SS04] that for almost every environment and as t tends to infinity, $\xi\cdot Y_t/\sqrt{t}$ converges in distribution under $\mathbf{P}_0^{\omega}$ to a Gaussian random variable of variance σ^2, that we write $\sigma\mathsf{G}$, where G is a standard Gaussian random variable. In order to justify that for almost every environment, $\xi\cdot Y_t/\sqrt{t}$ converges in distribution to $(\sigma\mathsf{G})^2$, we need some uniform integrability property, since the square function is unbounded. But this uniform integrability is a direct consequence of Theorem 4.2. Hence, under $\mathbb{P}_0$ and as t tends to infinity, the random variable

$$\frac{p(\omega)(\xi\cdot Y_t)^2}{t\mathbb{E}[p]}\tag{5.3}$$

converges in distribution to

$$\frac{p(\omega)}{\mathbb{E}[p]}(\sigma\mathsf{G})^2,$$

where ω follows the distribution $\mathbb{P}$, and is independent of G. For the foregoing reason, the squares of the random variables in (5.3) are uniformly integrable as t varies. Since we know moreover that $\lim_{t\to+\infty}\sigma_t^2=\sigma^2$, we thus obtain

$$\lim_{t\to+\infty}\mathsf{v}_t=\mathbb{E}\left[\left(\frac{p(\omega)}{\mathbb{E}[p]}(\sigma\mathsf{G})^2-\sigma^2\right)^2\right].$$

We obtain (5.1) by expanding this expectation, recalling that the fourth moment of G is equal to 3.

t	10	20	40	80	160	320	640	1280
$K(t)$	10^4	10^4	10^4	10^4	10^4	10^4	$4.0\,10^3$	10^3
Systematic error	9.27E-02	5.31E-02	3.09E-02	1.71E-02	9.58E-03	5.45E-03	2.93E-03	1.66E-03

Table 1. Systematic error $|A_{\mathrm{hom}} - \frac{\mathbb{E}[p]}{2}\hat{A}_{n(t)}(t)|$ in function of the final time t for $K(t)t^2$ realizations.

Similarly, Theorem 4.2 gives us sufficient control to guarantee that (5.2) holds, so the proof is complete. □

6. Numerical validation and comments

In this section, we illustrate on a simple two-dimensional example the sharpness of the estimates of the systematic error and of the random fluctuations obtained in Theorems 3.1 and 4.1.

In the numerical tests, each conductivity of $\mathbb{B}$ takes the value $\alpha = 1$ or $\beta = 4$ with probability 1/2. In this simple case, the homogenized matrix is given by Dykhne's formula, namely $A_{\mathrm{hom}} = \sqrt{\alpha\beta}\mathrm{Id} = 2\mathrm{Id}$ (see for instance [Gl10, Appendix A]). For the simulation of the random walk, we generate — and store — the environment along the trajectory of the walk. In particular, this requires to store up to a constant times t data. In terms of computational cost, the expensive part of the computations is the generation of the randomness. In particular, to compute one realization of $\hat{A}_{t^2}(t)$ costs approximately the generation of $t^2 \times 4t = 4t^3$ random variables. A natural advantage of the method is its full scalability: the t^2 random walks used to calculate a realization of $\hat{A}_{t^2}(t)$ are completely independent.

We first test the estimate of the systematic error: up to a logarithmic correction, the convergence is proved to be linear in time. In view of Theorem 4.1, typical fluctuations of $t(\hat{A}_{n(t)}(t) - \sigma_t^2)$ are of order no greater than $t/\sqrt{n(t)}$, and thus become negligible when compared with the systematic error as soon as the number $n(t)$ of realizations satisfies $n(t) \gg t^2$. We display in Table 1 an estimate of the systematic error $t \mapsto |A_{\mathrm{hom}} - \frac{\mathbb{E}[p]}{2}\hat{A}_{n(t)}(t)|$ obtained with $n(t) = K(t)t^2$ realizations. The systematic error is plotted on Figure 1 in function of the time in logarithmic scale (crosses). It matches quite well the function $f : t \mapsto Ct^{-1}\ln t$ (for $C > 0$ chosen so that $f(1280) = |A_{\mathrm{hom}} - \frac{\mathbb{E}[p]}{2}\hat{A}_{n(640)}(1280)|$) which is plotted as a solid line. This is consistent with Theorem 3.1 and supports the fact that the spectral exponents are $(2, 0)$ for $d = 2$ (and not $(2, -q)$ for some $q > 0$).

We now turn to the random fluctuations of $\hat{A}_{n(t)}(t)$. Theorem 4.1 gives us a Gaussian upper bound on the tail of the fluctuations of $t(\hat{A}_n(t) - \sigma_t^2)$, measured in units of $t/\sqrt{n}$, whereas Proposition 5.1 proves the corresponding central limit theorem, that is convergence in distribution of $t(\hat{A}_{t^2}(t) - \sigma_t^2)$ to a Gaussian random variable. The Figures 2-7 display the histograms of $t\frac{\mathbb{E}[p]}{2}(\hat{A}_{t^2}(t) - \sigma_t^2)$ for $t = 10, 20, 40$ and 80 (with 10000 realizations of $\hat{A}_{t^2}(t)$ in each case, and σ_t^2 approximated by the empirical mean of $\hat{A}_{t^2}(t)$ over the 10000 realizations). As expected, they look Gaussian. In addition, Proposition 5.1 also gives the limiting variance. Table 2 displays the limiting variance $(\mathbb{E}[p]/2)^2\,\mathsf{v} = 9.08$ and the empirical variances for $t = 10, 20, 40, 80, 160$ and 320, which are in good agreement.

To conclude this article, let us quickly compare the Monte-Carlo approach under consideration here to other approaches to approximate homogenized coefficients. Another possibility to approximate effective coefficients is to directly solve the so-called corrector equation. In this approach, a first step towards the derivation of

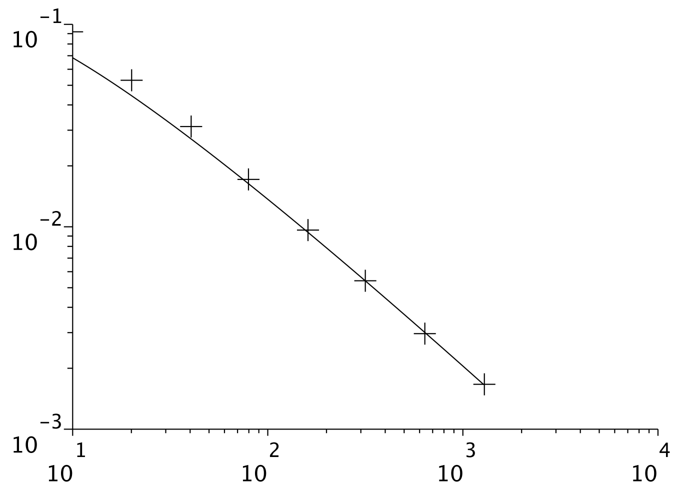

FIGURE 1. Systematic error $|A_{\text{hom}} - \frac{\mathbb{E}[p]}{2}\hat{A}_{n(t)}(t)|$ in function of the final time t for $n(t) = K(t)t^2$ realizations (logarithmic scale)

t	10	20	40	80	160	320	∞
variance	9.86	9.46	9.49	9.46	9.36	9.06	9.08

TABLE 2. Empirical variance of $\frac{\mathbb{E}[p]}{2}t(\hat{A}_{t^2}(t) - \sigma_t^2)$ and limiting variance from Proposition 5.1.

error estimates is a quantification of the qualitative results proved by Künnemann [Kü83] (and inspired by Papanicolaou and Varadhan's treatment of the continuous case [PV79]) and Kozlov [Ko87]. In the stochastic case, such an equation is posed on the whole $\mathbb{Z}^d$, and we need to localize it on a bounded domain, say the hypercube Q_R of side $R > 0$. As shown in a series of papers by Otto and the first author [GO10a, GO10b], and the first author [Gl10], there are three contributions to the $\mathbb{L}^2$-error in probability between the true homogenized coefficients and its approximation. The dominant error in small dimensions takes the form of a variance: it measures the fact that the approximation of the homogenized coefficients by the average of the energy density of the corrector on a box Q_R fluctuates. This error decays at the rate of the central limit theorem R^{-d} in any dimension (with a logarithmic correction for $d = 2$). The second error is a systematic error: it is due to the fact that we have modified the corrector equation by adding a zero-order term of strength $T^{-1} > 0$ (as is standard in the analysis of the well-posedness of the corrector equation). The scaling of this error depends on the dimension and saturates at dimension 4. It is of higher order than the random error up to dimension 8. The last error is due to the use of boundary conditions on the bounded domain Q_R. Provided there is a buffer region, this error is exponentially small in the distance to the buffer zone measured in units of $\sqrt{T}$.

This approach has two main drawbacks. First the numerical method only converges at the central limit theorem scaling in terms of R up to dimension 8, which is somehow disappointing from a conceptual point of view (although this is already fine in practice). Second, although the size of the buffer zone is roughly independent of the dimension, its cost with respect to the central limit theorem scaling

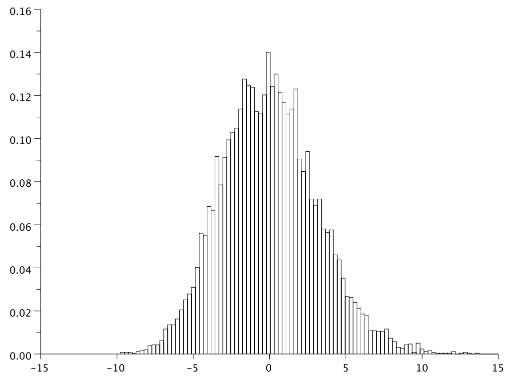

FIGURE 2. Histogram of the rescaled fluctuations for $t = 10$

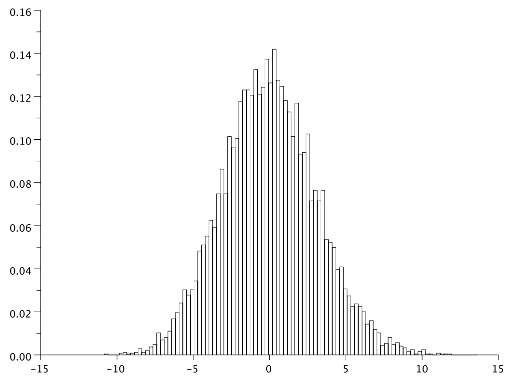

FIGURE 3. Histogram of the rescaled fluctuations for $t = 20$

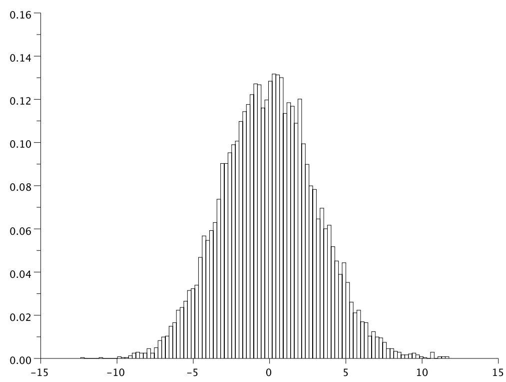

FIGURE 4. Histogram of the rescaled fluctuations for $t = 40$

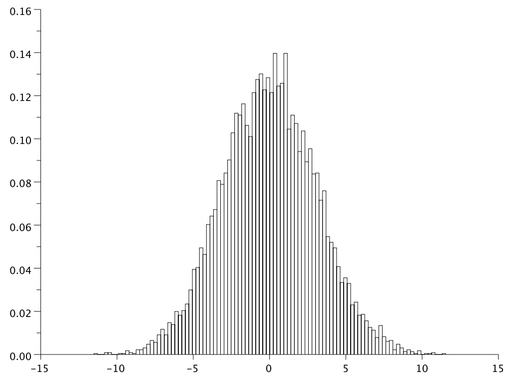

FIGURE 5. Histogram of the rescaled fluctuations for $t = 80$

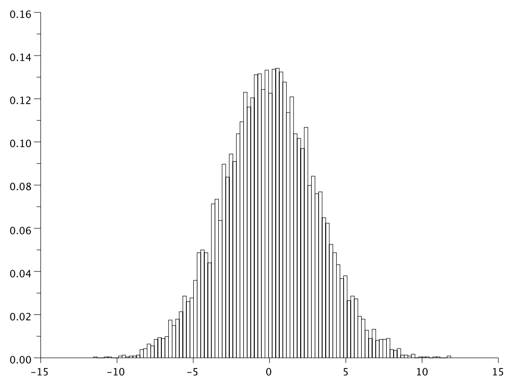

FIGURE 6. Histogram of the rescaled fluctuations for $t = 160$

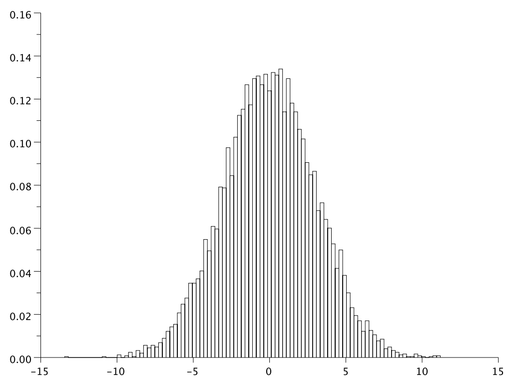

FIGURE 7. Histogram of the rescaled fluctuations for $t = 320$

dramatically increases with the dimension (recall that in dimension d, the CLT scaling is R^{-d}, so that in high dimension, we may consider smaller R for a given precision, whereas the use of boundary conditions requires $R \gg \sqrt{T}$ in any dimension). Based on ideas of the second author in [Mo11], we have taken advantage of the spectral representation of the homogenized coefficients (originally introduced by Papanicolaou and Varadhan to prove their qualitative homogenization result) in order to devise and analyze new approximation formulas for the homogenized coefficients in [GM10]. In particular, this has allowed us to get rid of the restriction on dimension, and exhibit refinements of the numerical method of [Gl10] which converge at the central limit theorem scaling in any dimension (thus avoiding the first mentioned drawback). Unfortunately, the second drawback is inherent to the type of method used: if the corrector equation has to be solved on a bounded domain Q_R, boundary conditions need to be imposed on the boundary ∂Q_R. Since their values are actually also part of the problem, a buffer zone seems mandatory — with the notable exception of the periodization method, whose analysis is yet still unclear to us, especially when spatial correlations are introduced in the coefficients.

In this paper we have analyzed a method which does not suffer from the drawbacks mentioned above: the random walk in random environment approach. In particular, following [Pa83] we have obtained an approximation of the homogenized coefficients by the numerical simulation of a random walk up to some large time. Compared to the deterministic approach based on the approximate corrector equation, the advantage of the present approach is that its convergence rate and computational costs are dimension-independent. In addition, the environment only needs to be generated along the trajectory of the random walker, so that much less information has to be stored during the calculation. This may be quite an important feature of the Monte Carlo method in view of the discussion of [Gl10, Section 4.3].

A more thorough comparison of these numerical approaches in two and three dimensions, for correlated and uncorrelated examples, will be the object of a forthcoming work [EGMN].

Acknowledgements

The authors acknowledge the support of INRIA through the "Action de recherche collaborative" DISCO. This work was also supported by Ministry of Higher Education and Research, Nord-Pas de Calais Regional Council and FEDER through the "Contrat de Projets Etat Region (CPER) 2007-2013".

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