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À ma grand-mère Nicole,

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Contents

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Chapter 0

Introduction

Almost all the results presented in this habilitation address, or were directly inspired by, questions coming from real algebraic geometry. The aim of Section 0.1, which constitutes the first part of this introduction, is to present in a condensed way some of the major topics of interest in this field. For each topic, we put forward significant open questions and, when relevant, examples of our contributions in this direction.

Another common theme unifying most of the works which we discuss in this memoir is the influence of algebraic cycles. In Section 0.2, we explain the reasons for their importance in our work. In addition, we discuss how we were led to study these various questions, and how they relate to each other. This section is also intended to serve as a roadmap for this text.

All the results mentioned in this introduction are discussed in more detail, with more context, and often in greater generality, in the core of the memoir. In an effort to simplify the presentation, we decided to only state and discuss results over the field $\mathbb{R}$ of real numbers, and to disregard generalizations and counterexamples over arbitrary real closed fields.

0.1 Real algebraic geometry

The field of real algebraic geometry is well illustrated by the following two questions extracted from Hilbert's celebrated 1900 list of problems.

Hilbert's 17th problem. Let $f \in \mathbb{R}[x_1, \ldots, x_n]$ be a nonnegative real polynomial. Is f a sum of squares in $\mathbb{R}(x_1, \ldots, x_n)$?

Hilbert's 16th problem (first part). Fix $d \geq 1$. Which configurations of ovals in $\mathbb{P}^2(\mathbb{R})$ arise from real loci of smooth degree d real plane curves?

Both questions were not entirely new in 1900. The first question arises from Hilbert's works on sums of squares [Hil88, Hil93], which include a solution to Hilbert's 17th problem in two variables. The second one is a natural extension of Harnack's determination of the possible number of connected components of the real locus of a smooth degree d real plane curve [Har76].

It can be argued that these two problems contained the seeds of most later directions of research in real algebraic geometry. Indeed, the 17th problem fostered the development of questions pertaining to

(i) real algebra, positivity and sums of squares;

(ii) arithmetic of real function fields;

whereas Hilbert's 16th problem paved the way for the study of

(iii) classification of real algebraic varieties;

(iv) algebraic approximation of topological or differentiable objects.

We now review a few major open questions and present some of our results in these directions.

0.1.1 Positivity and sums of squares

In 1927, a positive answer to Hilbert's 17th problem was obtained by Artin [Art27]. A beautiful quantitative refinement of Artin's theorem, bounding the number of squares required, was discovered by Pfister [Pfi67] in 1967: a nonnegative real polynomial $f \in \mathbb{R}[x_1, \ldots, x_n]$ is a sum of 2^n squares in $\mathbb{R}(x_1, \ldots, x_n)$.

Pfister's 2^n bound is known to be optimal in two variables, thanks to Cassels, Ellison and Pfister [CEP71]. The question whether it is also optimal in more variables was asked explicitly by Pfister in [Pfi71, §4, Problem 1], and has remained open ever since.

Question 0.1.1. For $n \geq 1$, does there exist a nonnegative polynomial $f \in \mathbb{R}[x_1, \ldots, x_n]$ that is not a sum of $2^n - 1$ squares in $\mathbb{R}(x_1, \ldots, x_n)$?

In order to grasp this question, it is natural to try to understand better in which situations Pfister's bound may be improved. The following theorems provide answers for polynomials in two variables, and for low degree polynomials respectively.

Theorem 0.1.2 (Benoist [Ben18])**.** *Fix $d \geq 2$ even. Let $\Pi_d \subset \mathbb{R}[x, y]_d$ be the set of nonnegative real polynomials of degree $\leq d$. The set of those polynomials that are sums of 3 squares in $\mathbb{R}(x, y)$ is analytically dense in Π_d.*

Theorem 0.1.3 (Benoist [Ben17])**.** *Let $f \in \mathbb{R}[x_1, \ldots, x_n]$ be nonnegative of degree d. If $d \leq 2n - 2$, or if $d = 2n$ and n is even or equal to 3 or 5, then f is a sum of $2^n - 1$ squares in $\mathbb{R}(x_1, \ldots, x_n)$.*

Hilbert's 17th problem has many variants, where one considers algebraic functions that may be defined over other fields than the reals, or real functions that are not necessarily algebraic. In particular, the real-analytic analogue of Hilbert's 17th problem is a tantalizing open question, only known to have a positive answer in ≤ 2 variables [BR75].

Question 0.1.4. Let $f : \mathbb{R}^n \to \mathbb{R}$ be a nonnegative real-analytic function. Is f a sum of squares of real-analytic meromorphic functions?

The following two theorems provide quantitative refinements à la Pfister to questions of this kind, in the local case and in the compact case respectively.

Theorem 0.1.5 (Benoist [Ben20b])**.** *Let $f \in \mathbb{R}\{\{x_1, \ldots, x_n\}\}$ be a convergent real power series in $n \geq 1$ variables, which is nonnegative near the origin. Then f is a sum of 2^{n-1} squares in $\mathrm{Frac}(\mathbb{R}\{\{x_1, \ldots, x_n\}\})$.*

Theorem 0.1.6 (Benoist [Ben23])**.** *Let M be a compact real-analytic manifold of dimension n. Any nonnegative real-analytic function $f : M \to \mathbb{R}$ is a sum of 2^n squares of real-analytic meromorphic functions.*

In the local case, Pfister's 2^n bound was known to hold as a consequence of the Milnor conjectures, and the 2^{n-1} bound that we obtain was conjectured by Choi, Dai, Lam and Reznick [CDLR82, §9, Problem 6 and below] (and due to Hu [Hu15] in dimension ≤ 3). In the compact case where we obtain a 2^n bound exactly as in Pfister's theorem, no quantitative result was previously known in dimension ≥ 3 (in dimension 2, see [Jaw82, ABFR05]).

0.1.2 Arithmetic of real function fields

The arithmetic study of the function field $\mathbb{R}(X)$ of a connected smooth real algebraic variety X differs chiefly from the complex case because of the influence of the field orderings of $\mathbb{R}(X)$, which are induced by the real points of X. Surprisingly, even when X has no real points, questions about $\mathbb{R}(X)$ may lead to arithmetic difficulties not present in the complex case.

To illustrate it, recall that a field K is said to be C_n if any homogeneous polynomial of degree d in $> d^n$ variables with coefficients in K has a nontrivial zero. Function fields of connected smooth complex varieties of dimension n (such as $\mathbb{C}(x_1, \ldots, x_n)$) are known to be C_n thanks to the Tsen–Lang theorem [Lan52]. In [Lan53, p. 379], Lang conjectured that the same should hold for real algebraic varieties of dimension n with no real points.

Question 0.1.7. Let X be a connected smooth real algebraic variety of dimension n such that $X(\mathbb{R}) = \varnothing$. Is $\mathbb{R}(X)$ a C_n field?

It is an old theorem of Witt [Wit37] that Question 0.1.7 has a positive answer for degree $d = 2$ equations when $n = 1$, and Lang showed in [Lan53] that it always has a positive answer for odd degree equations, as the proof of the Tsen–Lang theorem may be adapted in this situation. The following theorem, which deals with degree 2 equations when $n = 2$, settles the first new case of Lang's conjecture since its formulation.

Theorem 0.1.8 (Benoist [Ben19])**.** *Let S be a connected smooth real algebraic surface such that $S(\mathbb{R}) = \varnothing$. Then any quadratic form of rank ≥ 5 over $\mathbb{R}(S)$ has a nontrivial zero.*

One can also ask finer questions than existence of rational points. For instance, if B is a connected smooth projective real algebraic curve, when does a variety X over $F = \mathbb{R}(B)$ satisfy the *weak approximation property*, in the sense that the diagonal map

$$X(F) \to \prod_{b \in B} X(F_b)$$

has dense image (where b runs over all closed points of B, and F_b is the completion of F with respect to the discrete valuation associated with b)? Wittenberg and I showed that this is the case for homogeneous spaces under connected linear algebraic groups, thereby confirming a conjecture of Colliot-Thélène [CT96, p. 151], and improving on results of him [CT96] (for trivial stabilizers) and Scheiderer [Sch96] (for connected stabilizers).

Theorem 0.1.9 (Benoist–Wittenberg [BW21])**.** *Let B be a connected smooth real algebraic curve. Homogeneous spaces under connected linear algebraic groups over $\mathbb{R}(B)$ satisfy the weak approximation property.*

Another classical problem concerning the function field $k(X)$ of an algebraic variety X over a field k is to decide when it is purely transcendental over k (i.e., k-isomorphic to $k(x_1, \ldots, x_n)$), or equivalently, when the variety X is k-rational. This question is arithmetically interesting for varieties that are known to be rational over the algebraic closure $\overline{k}$ of k.

Such is the case for linear algebraic groups. Indeed, structure theorems imply that they are always rational over algebraically closed fields, but even tori over p-adic fields are not rational in general [Che54]. Over the reals, the question, asked by Platonov in [Pla81, p. 169], is open.

Question 0.1.10. Let G be a connected linear algebraic group over $\mathbb{R}$. Is G an $\mathbb{R}$-rational real algebraic variety?

In a collaboration with Wittenberg, we adapted to such arithmetic settings the Clemens–Griffiths method exploiting intermediate Jacobians to obstruct the rationality of complex algebraic varieties. Here is the simplest new example that we obtained in this way over the reals.

Theorem 0.1.11 (Benoist–Wittenberg [BW20a])**.** *The real algebraic variety with equation $\{s^2 + t^2 = x^4 + y^4 + 1\}$ is $\mathbb{C}$-rational but not $\mathbb{R}$-rational.*

Our strategy was subsequently refined by Hassett and Tschinkel [HT21b, HT21a]. The next theorem, which is an application of this refinement, was first discovered by them over the reals, and proven by us in general. It answers positively a conjecture of Kuznetsov and Prokhorov.

Theorem 0.1.12 (Benoist–Wittenberg [BW23])**.** *Let k be a field and let $X \subset \mathbb{P}^5_k$ be a smooth complete intersection of two quadrics. Then X is k-rational if and only if it contains a line defined over k.*

0.1.3 Classification of real algebraic varieties

Hilbert's 16th problem is still very much open to this day.

To illustrate its legacy, let us mention, among the works undertaking the topological classification of significant classes of real algebraic varieties, Kharlamov's classification of real K3 surfaces [Kha76] and the classification of real Enriques surfaces by Degtyarev, Itenberg and Kharlamov [DIK00].

I have not contributed to this line of research. To conclude this paragraph, we state the following basic open question which illustrates how little we know, even in the classical setting of smooth real plane curves considered by Hilbert in his 16th problem.

Question 0.1.13. Fix $d \geq 1$. Let Ω_d be the set of smooth degree d real plane curves with connected real locus. Is Ω_d connected?

0.1.4 Algebraic approximation

Another line of investigation which we like to think of as an outgrowth of Hilbert's 16th problem aims at understanding how far apart differential topology and real algebraic geometry really are. For instance, when can topological or differentiable objects (manifolds, maps, vector bundles, submanifolds...) be realized or approximated by real algebraic objects? The archetypal theorem in this direction is the Nash–Tognoli theorem [Nas52, Tog73] according to which any compact $\mathcal{C}^\infty$ manifold is diffeomorphic to the real locus of a smooth projective real algebraic variety.

In view of this theorem, it is natural to look for algebraizations (or algebraic approximations) of $\mathcal{C}^\infty$ maps between compact $\mathcal{C}^\infty$ manifolds.

Question 0.1.14. Let $f : N \to M$ be a $\mathcal{C}^\infty$ map of compact $\mathcal{C}^\infty$ manifolds. When can one approximate f in $\mathcal{C}^\infty(N, M)$ by maps of the form $\phi^{-1} \circ g(\mathbb{R}) \circ \psi$, where $g : Y \to X$ is a morphism of smooth projective varieties over $\mathbb{R}$, and $\psi : N \xrightarrow{\sim} Y(\mathbb{R})$ and $\phi : M \xrightarrow{\sim} X(\mathbb{R})$ are diffeomorphisms?

This approximation property is known to hold if f is an embedding [BT80, AK81b], but to fail in general [BD84]. As far as we know, it is open whether it always holds when f is an immersion, or a submersion.

Such questions have many variants, depending on whether one fixes real algebraic structures on the source and/or on the target manifold. In the case of submanifolds of the real projective space, Kucharz and van Hamel [KvH09, p. 269] ask the following.

Question 0.1.15. Let $M \subset \mathbb{P}^n(\mathbb{R})$ be a compact $\mathcal{C}^\infty$ submanifold, and let $\mathcal{U} \subset \mathcal{C}^\infty(M, \mathbb{P}^n(\mathbb{R}))$ be a neighborhood of the inclusion. Do there exist $j \in \mathcal{U}$ and an algebraic subvariety $X \subset \mathbb{P}^n_{\mathbb{R}}$ smooth along $X(\mathbb{R})$ with $X(\mathbb{R}) = j(M)$?

We answer this question positively for submanifolds of less than half the ambient dimension.

Theorem 0.1.16 (Benoist [Ben20a])**.** *A compact $\mathcal{C}^\infty$ manifold $M \subset \mathbb{P}^n(\mathbb{R})$ with $\dim(M) < n/2$ can be approximated in the sense of Question 0.1.14 by real algebraic subvarieties that are smooth along their real loci.*

We conclude this section with another algebraic approximation question, asked by Wittenberg and myself in [BW21, Question 1.1]. Recall that a smooth projective real algebraic variety X is said to be *rationally connected* if any two complex points of X can be joined by a complex rational curve.

Question 0.1.17. Let X be a smooth projective real rationally connected variety. Are algebraic maps dense in $\mathcal{C}^\infty(\mathbb{P}^1(\mathbb{R}), X(\mathbb{R}))$?

This question is still open for del Pezzo surfaces in general. It was shown to have a positive answer if X is $\mathbb{R}$-rational by Bochnak and Kucharz [BK99]. We obtained the first positive answers beyond this case.

Theorem 0.1.18 (Benoist–Wittenberg [BW21])**.** *Let X be a real cubic hypersurface or a real complete intersection of two quadrics, which is smooth of dimension ≥ 2. Then algebraic maps are dense in $\mathcal{C}^\infty(\mathbb{P}^1(\mathbb{R}), X(\mathbb{R}))$.*

0.2 Influence of algebraic cycles

We now explain why algebraic cycles may be thought of as a thread connecting the seemingly disparate questions and results listed in Section 0.1.

0.2.1 Towards the real integral Hodge conjecture

My interest in real algebraic geometry was first raised by Lang's conjecture (Question 0.1.7). This conjecture in particular predicts that if C is a connected smooth projective curve over $\mathbb{R}$ with no real points, then $\mathbb{R}(C)$ is a C_1 field. According to heuristics due to Kollár and Manin, one expects that many more $\mathbb{R}(C)$-varieties than low degree hypersurfaces should automatically have $\mathbb{R}(C)$-points, namely all rationally connected varieties. When C is the conic with no real points and the rationally connected variety is defined over $\mathbb{R}$, one is left with the following question, first raised by Kollár (see [AK03, Remarks 20] and [Kol13, §42]).

Question 0.2.1. Let C be the real conic with no real points, and let X be a smooth projective rationally connected variety over $\mathbb{R}$. Does there exist a morphism $C \to X$?

If X has a real point, then the above question obviously has a positive answer: there exists a constant morphism $C \to X$. However, if X has no real points, Question 0.2.1 is nontrivial, and predicts that X contains interesting algebraic curves inside X (namely geometrically rational curves). This suggests that one might try to give counterexamples to Question 0.2.1 by utilizing methods from the theory of algebraic cycles. Maybe such an algebraic curve in X does not exist because its cohomology class does not exist? Maybe there are Hodge-theoretic obstructions, or more subtle obstructions, to the algebraicity of the possible cohomology classes?

With this kind of applications in mind, Wittenberg and I undertook a systematic study of algebraic cycles on real algebraic varieties, in a series of two articles [BW20b, BW20c]. One of the main features of this work is the formulation and the study of the *real integral Hodge conjecture* (see §1.2.2), a statement which we expect to play the same role and to hold in the same generality, over the reals, as the usual integral Hodge conjecture over the complex numbers. In particular, we expect the following real analogue of a complex conjecture of Voisin to have a positive answer, which would dash the hope of finding cohomological obstructions to Question 0.2.1.

Question 0.2.2. Do smooth projective real rationally connected varieties satisfy the real integral Hodge conjecture for 1-cycles?

Divisors [Kra91, MvH98, vH00b] and zero-cycles [CTS96, vH00a] on real algebraic varieties were already well understood, and most of our new results concern curves on threefolds or higher-dimensional real algebraic varieties. Let us only state here the following concrete statement as a sample.

Theorem 0.2.3 (Benoist–Wittenberg [BW20c])**.** *Any smooth real quartic threefold contains a smooth curve of even genus.*

This falls short of solving Question 0.2.1 for smooth quartic threefolds, as the curves of even genus that we construct might not be conics. We believe that Theorem 0.2.3 is nontrivial for quartic threefolds with no real points. Our proof makes essential use of the real integral Hodge conjecture point of view, and relies on infinitesimal methods in Hodge theory.

Chapter 1 is devoted to this circle of ideas. After reviewing cycle class maps in complex and real algebraic geometry in Section 1.1, we present the statement of the real integral Hodge conjecture, some of its implications, and the positive and negative results known about it in Section 1.2.

In Section 1.3, we give another application of infinitesimal methods in Hodge theory to real algebraic geometry, more precisely to period-index problems in this context. The next theorem is one of our main results in this direction. It was known to imply Theorem 0.1.8, and thereby constitues a step forward in the direction of Lang's conjecture.

Theorem 0.2.4 (Benoist [Ben19]). *Let S be a connected smooth real algebraic surface with $S(\mathbb{R}) = \varnothing$. The period and the index of any class in the Brauer group of $\mathbb{R}(S)$ coincide.*

0.2.2 Connections with algebraic approximation

We discovered during the preparation of [BW20b, BW20c] that the real integral Hodge conjecture has applications to algebraic approximation problems. For instance, if Question 0.2.2 has a positive answer for a real rationally connected variety X, then $H_1(X(\mathbb{R}), \mathbb{Z}/2)$ is generated by fundamental classes of real loci of algebraic curves in X, and it follows from known approximation theorems (see Theorem 2.3.1) that all loops in $X(\mathbb{R})$ may be approximated by real loci of algebraic curves in X. As X is assumed to be rationally connected, it is natural to wonder whether these algebraic curves may be required to be rational, and this led us to Question 0.1.17.

Conversely, we had to use that kind of algebraic approximation statements when trying to answer Question 0.2.2 in situations where X has a fibration structure, such as a conic bundle structure. This suggested us to put algebraic approximation at the center of the study of Question 0.2.2, and to formulate a geometric statement in this direction of which Question 0.2.2 would be a cohomological counterpart.

In view of the $n = 1$ case of Lang's conjecture (Question 0.1.7), of complex results such as the Graber–Harris–Starr theorem, and of our intended applications to rationally connected varieties with a fibration structure, it became clear that such a statement should be formulated for one-parameter families of varieties, i.e., for varieties over function fields of real algebraic curves. This line of thought led Wittenberg and I to define in [BW21] the *tight approximation property* for such varieties (see §2.4.1), which incorporates both algebraic approximation and weak approximation conditions, and to ask the following question.

Question 0.2.5. Let B be a connected smooth real curve. Do rationally connected varieties over $\mathbb{R}(B)$ satisfy the tight approximation property?

Chapter 2 gathers several results related to algebraic approximation.

We first recall the Nash–Tognoli theorem and some of its extensions in Section 2.1. In Section 2.2, we study the algebraizability of cohomology classes of compact $\mathcal{C}^\infty$ manifolds, a topic at the crossroads of algebraic cycles and differential geometry. We then focus in Section 2.3 on the construction of algebraic approximations of submanifolds of the real locus of a smooth projective real algebraic variety, a problem which is strongly influenced by the abundance (or the scarcity) of algebraic cycles on the ambient variety. The proof of Theorem 0.1.15 is explained there.

Finally, in Section 2.4, we give the definition of the tight approximation property, we describe remarkable features of this property (birational invariance, compatibility with fibrations, descent à la Colliot-Thélène and Sansuc), and we explain how Theorems 0.1.9 and 0.1.18 can be deduced from them.

0.2.3 Connections with sums of squares problems

Each real algebraic variety carries canonical constant cohomology classes: those that come by pull-back from the cohomology of the base field $\mathbb{R}$, i.e., from the group cohomology of $\mathrm{Gal}(\mathbb{C}/\mathbb{R}) \simeq \mathbb{Z}/2$. These constant cohomology classes yield interesting questions in the theory of real algebraic cycles (and in the study of the real integral Hodge conjecture), which are absent from the complex picture. When are they algebraic? What is their coniveau?

When these questions emerged during the writing of [BW20b, BW20c], I realized that one way in which they had appeared in the literature was in connection with Hilbert's 17th problem. To the best of my knowledge, the first work where this connection is made is [CT93], where Colliot-Thélène gives an alternative proof of the Cassels–Ellison–Pfister theorem (that is, a solution to Question 0.1.1 in two variables). Let $f \in \mathbb{R}[x, y]$ be a nonnegative polynomial. Assume that its homogenization $F \in \mathbb{R}[X, Y, Z]$ defines a smooth plane curve, and let S be the smooth projective real algebraic surface with equation

$$W^2 + F(X, Y, Z) = 0.$$

Colliot-Thélène notices that f is a sum of three squares in $\mathbb{R}(x, y)$ if and only if the constant cohomology classes in $H^2_{\text{ét}}(S, \mathbb{Z}/2)$ are algebraic. This allows him to reformulate the question as an algebraic cycles problem, and to solve it by applying the classical Noether–Lefschetz theorem.

It turns out to be a general principle that sums of few squares questions in the spirit of Pfister's quantitative refinement of Hilbert's 17th problem may be translated into cycle-theoretic questions. In dimension ≥ 3, such translations are often nontrivial, as they rely on the Milnor conjectures proven by Voevodsky [Voe03]. I made a systematic use of this point of view in my works on Hilbert's 17th problem, which led me to use varied tools of algebraic cycles theory.

The proof of Theorem 0.1.2 is parallel to Colliot-Thélène's above-mentioned argument, and eventually reduces to density theorems for (real) Noether–Lefschetz loci in the spirit of [CHM88]. The main tool on which the proof of Theorem 0.1.3 relies is Bloch–Ogus theory [BO74]. The main idea of the proof of Theorem 0.1.5 is a technique to show the vanishing of an unramified cohomology class which I borrowed from [KS12]. Theorem 0.1.6 is further away from classical cycle-theoretic methods, for the reason that it is rooted in Stein analytic geometry instead of algebraic geometry.

Chapter 3 collects our contributions to sums of squares problems. We first recall in §3.1.1 and §3.1.2 the results of Hilbert and Artin on sums of squares of polynomials and rational functions. In §3.1.3, we present a few results on *bad points*, which are local obstructions for a nonnegative real polynomial to be a sum of squares of polynomials, and which we studied in [Ben22b] with the hope that they could be used to produce a counterexample to Question 0.1.4. Finally, Section 3.2 contains the results on sums of few squares that were discussed above.

0.2.4 Connections with rationality problems

Algebraic approximation problems or real algebraic cycles questions are much easier on those real algebraic varieties that are $\mathbb{R}$-rational, i.e., birational to projective space. Indeed, these problems are often birational invariants, and accessible in the case of projective spaces. This is the reason why Question 0.1.17 is known to have a positive answer for $\mathbb{R}$-rational varieties [BK99].

In order to give positive answers (for instance, to Question 0.1.17) for more rationally connected varieties, it is tempting to try to relate them to $\mathbb{R}$-rational varieties and to take advantage of this known case. In general, this seems out of reach. However, there is some hope in the important case of del Pezzo surfaces, as one can relate them geometrically to their universal torsors in the sense of Colliot-Thélène and Sansuc [CTS87]. An application of the descent method (see §2.4.5) would allow us to conclude if we knew that these universal torsors are $\mathbb{R}$-rational if they have a real point.

It is very possible that this strategy is doomed to failure. Still, this line of thought led Wittenberg and I to investigate the $\mathbb{R}$-rationality of real algebraic varieties, with an emphasis on the arithmetically interesting case where they are known to be $\mathbb{C}$-rational. The results we obtained on this topic are described in Chapter 4, where we explain in particular the strategy of proof of Theorems 0.1.11 and 0.1.12.

0.2.5 Other topics

Several of our results that do not fit in the scope of real algebraic geometry, but that emerged from our real interests, are touched upon in the text. Among them are new counterexamples to the integral Hodge conjecture constructed in collaboration with Ottem ([BO20], see Theorem 1.2.2), topological obstructions to the algebraicity of cohomology classes over non-closed fields ([Ben22a], mentioned in §1.2.4), and new examples of algebraic cohomology classes of smooth projective complex varieties that are not a linear combinations of classes of smooth subvarieties obtained in a joint work with Debarre ([BD23], see Theorem 2.3.6).

A last work on algebraic cycles that we will not develop later in this memoir, but that we would like to briefly mention here is our construction with Ottem [BO21] of cohomology classes for which coniveau and strong coniveau differ. Let X be a smooth variety over $\mathbb{C}$. A cohomology class $\alpha \in H^k(X, A)$ is said to have *coniveau* $\geq c$ if it vanishes in the complement of a closed subset of codimension $\geq c$ in X. Define it to have *strong coniveau* $\geq c$ if it comes by proper push-forward from a smooth complex variety of dimension $\leq \dim(X) - c$. Clearly, if α has strong coniveau $\geq c$, it also has coniveau $\geq c$. Deligne [Del74] showed that the converse holds if X is proper and $A = \mathbb{Q}$. The question whether this converse holds in general goes back to Grothendieck [Gro69, p. 300], and is answered in the negative by the next theorem.

Theorem 0.2.6 (Benoist–Ottem [BO21]). *Fix $c \geq 1$ and $k \geq 2c+1$. Then there exist a smooth variety X over $\mathbb{C}$ and a class $\alpha \in H^k(X, A)$ which has coniveau $\geq c$ but strong coniveau $< c$, and where either*

(i) X is projective and $A = \mathbb{Z}$, or

(ii) X is quasi-projective and $A = \mathbb{Q}$.

Chapter 1

Real algebraic cycles

1.1 Cycle class maps

1.1.1 The complex cycle class map

Let X be a smooth projective variety over $\mathbb{C}$. For each integer $c \geq 0$, the *complex cycle class map*

$$\mathrm{cl}_{\mathbb{C}} : \mathrm{CH}^c(X) \to H^{2c}(X(\mathbb{C}), \mathbb{Z}(c))$$

associates with the class $[Z]$ of a codimension c closed subvariety Z of X the fundamental class of $Z(\mathbb{C})$ in $X(\mathbb{C})$. More precisely, let $\widetilde{Z} \to Z$ be any resolution of singularities, and let $\pi : \widetilde{Z} \to X$ be the induced morphism. Then $\mathrm{cl}_{\mathbb{C}}([Z]) = \pi_*(1)$, where $\pi_* : H^0(\widetilde{Z}(\mathbb{C}), \mathbb{Z}) \to H^{2c}(X(\mathbb{C}), \mathbb{Z}(c))$ is the Gysin morphism with respect to the canonical orientations of $X(\mathbb{C})$ and $\widetilde{Z}(\mathbb{C})$.

Hodge theory constrains the image of the complex cycle class map: it is included in the subgroup $\mathrm{Hdg}^{2c}(X(\mathbb{C}), \mathbb{Z}(c)) \subset H^{2c}(X(\mathbb{C}), \mathbb{Z}(c))$ of integral Hodge classes, defined to be the inverse image of the piece $H^{c,c}(X)$ of the Hodge decomposition by the natural morphism

$$H^{2c}(X(\mathbb{C}), \mathbb{Z}(c)) \to H^{2c}(X(\mathbb{C}), \mathbb{C}) = \bigoplus_{p+q=2c} H^{p,q}(X).$$

In this complex setting, the Tate twist $\mathbb{Z}(c)$ is mostly decorative and could be omitted. It will become essential in §1.1.3.

1.1.2 The Borel–Haefliger real cycle class map

Let now X be a smooth projective variety over $\mathbb{R}$. Borel and Haefliger have defined in [BH61] an analogous *real cycle class map*

$$\mathrm{cl}_{\mathbb{R}} : \mathrm{CH}^c(X) \to H^c(X(\mathbb{R}), \mathbb{Z}/2),$$

associating with the class $[Z]$ of a codimension c closed subvariety $Z \subset X$ the fundamental class of $Z(\mathbb{R})$ in $X(\mathbb{R})$. More precisely, with the notation of §1.1.1, one has $\mathrm{cl}_{\mathbb{R}}([Z]) = \pi_*(1)$, where π_* is now the Gysin morphism $H^0(\widetilde{Z}(\mathbb{R}), \mathbb{Z}/2) \to H^c(X(\mathbb{R}), \mathbb{Z}/2)$.

Here, one is compelled to use $\mathbb{Z}/2$ coefficients instead of integral coefficients, because the real locus of a smooth projective variety over $\mathbb{R}$ is not canonically oriented, and may indeed not be orientable.

The image of $\mathrm{cl}_{\mathbb{R}}$, which is the group of algebraic cohomology classes of the real locus, will be denoted by $H^c(X(\mathbb{R}), \mathbb{Z}/2)_{\mathrm{alg}}$. Its study is a classical topic in real algebraic geometry (see [Sil89, BCR98, BK98, Man20]).

1.1.3 Krasnov's equivariant cycle class map

In order to bridge the gap between the Borel–Haefliger cycle class map which is adapted to questions of interest in real algebraic geometry, and the complex cycle class map which brings Hodge theory into the picture, Krasnov [Kra91] introduced a third cycle class map: the *equivariant cycle class map*.

To define it, we introduce the group $G := \mathrm{Gal}(\mathbb{C}/\mathbb{R}) \simeq \mathbb{Z}/2$ generated by the complex conjugation. It acts on the set $X(\mathbb{C})$ of complex points of X through an antiholomorphic involution whose fixed locus is precisely the set $X(\mathbb{R})$ of real points of X. Any G-module M therefore gives rise to a constant G-equivariant sheaf on $X(\mathbb{C})$ of which we can consider the G-equivariant cohomology groups $H^k_G(X(\mathbb{C}), M)$.

Let $\mathbb{Z}(c)$ be the G-module which is isomorphic to $\mathbb{Z}$ as an abelian group, and on which G acts through multiplication by $(-1)^c$. Then Krasnov's equivariant cycle class map reads:

$$\mathrm{cl} : \mathrm{CH}^c(X) \to H^{2c}_G(X(\mathbb{C}), \mathbb{Z}(c)).$$

Its definition is entirely similar to the ones given in §§1.1.1–1.1.2. Using the notation introduced there, one has $\mathrm{cl}([Z]) = \pi_*(1)$, where π_* is now the Gysin morphism $H^0_G(\widetilde{Z}(\mathbb{C}), \mathbb{Z}) \to H^{2c}_G(X(\mathbb{C}), \mathbb{Z}(c))$. Here, the role of the Tate twist is crucial: it takes into account whether the action of G on $X(\mathbb{C})$ and $\widetilde{Z}(\mathbb{C})$ preserve or reverse the canonical orientations.

1.1.4 Relations between cycle class maps

The relation between the complex and the equivariant cycle class maps is clear: the composition

$$\mathrm{CH}^c(X) \xrightarrow{\mathrm{cl}} H^{2c}_G(X(\mathbb{C}), \mathbb{Z}(c)) \xrightarrow{\xi} H^{2c}(X(\mathbb{C}), \mathbb{Z}(c))$$

of the equivariant cycle class map and of the natural morphism ξ between equivariant and non-equivariant sheaf cohomology is induced by the complex cycle class map $\mathrm{cl}_{\mathbb{C}}$. We deduce Hodge-theoretic restrictions on the image of cl: it is included in the subgroup

$$\mathrm{Hdg}^{2c}_G(X(\mathbb{C}), \mathbb{Z}(c)) := \xi^{-1}(\mathrm{Hdg}^{2c}(X(\mathbb{C}), \mathbb{Z}(c)))$$

of equivariant integral Hodge classes.

The relation between the Borel–Haefliger and the equivariant cycle class map is more subtle. Consider the composition

$$\begin{aligned}\chi : H^{2c}_G(X(\mathbb{C}),\mathbb{Z}(c)) \to H^{2c}_G(X(\mathbb{R}),\mathbb{Z}/2) &= H^{2c}(X(\mathbb{R})\times BG,\mathbb{Z}/2)\\ &= \bigoplus_{k=0}^{2c} H^k(X(\mathbb{R}),\mathbb{Z}/2)\end{aligned} \tag{1.1}$$

of the morphism restricting to $X(\mathbb{R})$ and reducing coefficients modulo 2, of the computation of the equivariant cohomology of a space with trivial G-action (here, $BG \simeq \mathbb{P}^\infty(\mathbb{R})$ is the classifying space of G), and of the identification given by the Künneth formula. Krasnov has shown that

$$\chi(\mathrm{cl}(\alpha)) = (0,\dots,0,\mathrm{cl}_{\mathbb{R}}(\alpha),\mathrm{Sq}^1(\mathrm{cl}_{\mathbb{R}}(\alpha)),\dots,\mathrm{Sq}^c(\mathrm{cl}_{\mathbb{R}}(\alpha))) \tag{1.2}$$

for all $\alpha \in \mathrm{CH}^c(X)$, where the Sq^i are the Steenrod squares (see [Kra94], or [Kah87] for earlier results of Kahn when α is a Chern class). It follows at once that $\mathrm{cl}(\alpha)$ determines $\mathrm{cl}_{\mathbb{R}}(\alpha)$. In addition, as Sq^c acts on $H^c(X(\mathbb{R}),\mathbb{Z}/2)$ as the squaring map, we see that

$$\text{the image of } \mathrm{cl}_{\mathbb{C}}(\alpha)|_{X(\mathbb{R})} \text{ in } H^{2c}(X(\mathbb{R}),\mathbb{Z}/2) \text{ is } \mathrm{cl}_{\mathbb{R}}(\alpha)^2. \tag{1.3}$$

We also deduce topological restrictions on the image of cl: it is included in the subgroup

$$H^{2c}_G(X(\mathbb{C}),\mathbb{Z}(c))_0 \subset H^{2c}_G(X(\mathbb{C}),\mathbb{Z}(c))$$

of classes whose images by χ have the form $(0,\dots,0,\beta,\mathrm{Sq}^1(\beta),\dots,\mathrm{Sq}^c(\beta))$.

1.2 The real integral Hodge conjecture

1.2.1 The complex case

Let X be a smooth projective variety over $\mathbb{C}$ and fix $c \geq 0$. The *integral Hodge conjecture* for codimension c cycles on X is the statement that

$$\mathrm{cl}_{\mathbb{C}}(\mathrm{CH}^c(X)) = \mathrm{Hdg}^{2c}(X(\mathbb{C}),\mathbb{Z}(c)).$$

Unlike the classical Hodge conjecture which is formulated with rational coefficients, the integral Hodge conjecture is known to fail in general. More precisely, it holds when $c \in \{0,1,\dim(X)\}$ (the $c=1$ case being the Lefschetz $(1,1)$ theorem), but may fail whenever $2 \leq c \leq \dim(X)-1$.

Varied strategies have been used to contradict the integral Hodge conjecture: topological methods going back to Atiyah and Hirzebruch [AH62, Tot97], degeneration techniques pioneered by Kollár [BCC92, Tot13], or its relation with unramified cohomology first exploited by Colliot-Thélène and Voisin [CTV12, Sch19].

It has been advocated by Voisin that one could get interesting positive results in the direction of the integral Hodge conjecture by restricting the geometry of the varieties that one considers. In particular she asked the following question [Voi07, Question 16].

Question 1.2.1. Do rationally connected varieties over $\mathbb{C}$ satisfy the integral Hodge conjecture for 1-cycles?

In [Voi06], she answered positively this question in dimension three. More generally, she proved the integral Hodge conjecture for uniruled threefolds and, with additions by Totaro [Tot21], for threefolds with trivial canonical bundle. The latter result is sharp and does not extend to threefolds with torsion canonical bundle, as we showed in a joint work with Ottem.

Theorem 1.2.2 (Benoist–Ottem [BO20])**.** *For any complex Enriques surface S, there exists a complex elliptic curve E such that the integral Hodge conjecture for 1-cycles on $S \times E$ fails.*

1.2.2 Statement of the real integral Hodge conjecture

Let now X be a smooth projective variety over $\mathbb{R}$, and fix $c \geq 0$. In order to formulate in this context a statement that could possibly hold in the same generality as the integral Hodge conjecture over the complex numbers, it is necessary to take into account not only the Hodge-theoretic restrictions on the image of the equivariant cycle class map, but also the Kahn–Krasnov topological restrictions described in §1.1.4. To this effect, we consider the subgroup

$$\mathrm{Hdg}^{2c}_G(X(\mathbb{C}), \mathbb{Z}(c))_0 := \mathrm{Hdg}^{2c}_G(X(\mathbb{C}), \mathbb{Z}(c)) \cap H^{2c}_G(X(\mathbb{C}), \mathbb{Z}(c))_0$$

of $H^{2c}_G(X(\mathbb{C}), \mathbb{Z}(c))$. The *real integral Hodge conjecture* for codimension c cycles on X, introduced by Wittenberg and myself in [BW20b, BW20c], is the statement that

$$\mathrm{cl}(\mathrm{CH}^c(X)) = \mathrm{Hdg}^{2c}_G(X(\mathbb{C}), \mathbb{Z}(c))_0.$$

1.2.3 General properties

The following basic results, which extend known facts in the complex case, are established in [BW20b] and indicate that our definition of the real integral Hodge conjecture is reasonable.

Theorem 1.2.3 (Krasnov, Benoist–Wittenberg)**.** *The real integral Hodge conjecture holds for $c \in \{0, 1, \dim(X)\}$. Its validity for $c \in \{2, \dim(X) - 1\}$ is a birational invariant.*

The $c = 0$ case is obvious. The $c = 1$ case, a real analogue of the Lefschetz $(1,1)$ theorem, was known to Krasnov [Kra91] (see also [MvH98, vH00b]). The case $c = \dim(X)$ of zero-cycles is more delicate than in the complex situation because the Kahn–Krasnov topological restrictions really come into play. Finally, the birational invariance results follow from the fact that the action of correspondences preserves the subgroups $\mathrm{Hdg}^{2c}_G(X(\mathbb{C}), \mathbb{Z}(c))_0$ of $H^{2c}_G(X(\mathbb{C}), \mathbb{Z}(c))$. That pull-backs and cup products preserve this group is easy; in contrast, their compatibility with push-forwards is more delicate and relies on the Atiyah–Hirzebruch relative Wu formula [AH61].

These good formal properties encourage us to investigate the real integral Hodge conjecture in cases where it is known or expected to hold in the complex setting. For instance, in view of the results and conjectures of Voisin recalled in §1.2.1, we are led to ask:

Question 1.2.4. Let X be a smooth projective variety over $\mathbb{R}$. Does X satisfy the real integral Hodge conjecture for 1-cycles if:

(i) X is rationally connected;

(ii) X is a geometrically uniruled threefold;

(iii) X is a threefold with trivial canonical bundle?

1.2.4 Counterexamples

Exactly as the integral Hodge conjecture, the real integral Hodge conjecture fails in general, and all the methods alluded to in §1.2.1 can be employed to construct counterexamples to it. However, degeneration techniques, which may be the most efficient tool over the complex numbers, can only produce counterexamples that already fail the usual complex integral Hodge conjecture.

To produce arithmetically more interesting examples, one can rely on the topological obstructions discovered by Atiyah and Hirzebruch [AH62]. Over the complex numbers, Kawai recovered and generalized these obstructions by proving that algebraic cohomology classes are preserved by Steenrod operations. In particular, they are killed by odd degree Steenrod operations.

This fact turns out to fail over nonclosed fields such as the real numbers. A substitute is provided by our next theorem. In its statement, we make use of Brosnan's Steenrod operations on mod 2 Chow groups [Bro03] and we denote by cl_2 the reduction mod 2 of the equivariant cycle class map cl. We also let $\omega \in H^1(G, \mathbb{Z}/2)$ be the nontrivial class and still write ω for its image in $H^1_G(X(\mathbb{C}), \mathbb{Z}/2)$ when X is a real algebraic variety.

Theorem 1.2.5 (Benoist [Ben22a]). *Let X be a smooth projective variety over $\mathbb{R}$. For all $c \geq 0$ and $\alpha \in \mathrm{CH}^c(X)/2$, one has*

$$\sum_{i\geq 0} \mathrm{Sq}^i(\mathrm{cl}_2(\alpha)) = \sum_{i\geq 0} (1+\omega)^{c-i}\mathrm{cl}_2(\mathrm{Sq}^{2i}(\alpha)).$$

We actually prove a more general version of Theorem 1.2.5, for mod ℓ Steenrod operations over any field in which ℓ is invertible. This gives rise to an extension of Kawai's obstructions to the algebraicity of cohomology classes over such fields. Over the reals, exploiting the obstructions stemming from Theorem 1.2.5 yields the following counterexample to the real integral Hodge conjecture.

Theorem 1.2.6 (Benoist [Ben22a])**.** *There exists a smooth projective fourfold over $\mathbb{R}$ failing the real integral Hodge conjecture for codimension 2 cycles, but such that $X_{\mathbb{C}}$ satisfies the integral Hodge conjecture.*

We are however unable to answer the following question, first raised in [BW20b, Question 4.9].

Question 1.2.7. Does there exist a smooth projective variety X over $\mathbb{R}$ such that $X_{\mathbb{C}}$ satisfies the integral Hodge conjecture but X fails the integral Hodge conjecture for 1-cycles?

1.2.5 Geometric consequences

Let us now explain the impact of the real integral Hodge conjecture on more classical problems in real algebraic geometry.

Nonalgebraic cohomology classes of the real locus

First, it follows from the results described in §1.1.4 that if the real integral Hodge conjecture holds for codimension c cycles on X, then the group $H^c(X(\mathbb{R}), \mathbb{Z}/2)_{\mathrm{alg}}$ is entirely determined: it is the image of the morphism

$$\mathrm{Hdg}^{2c}_G(X(\mathbb{C}), \mathbb{Z}(c))_0 \to H^c(X(\mathbb{R}), \mathbb{Z}/2) \tag{1.4}$$

induced by (1.1).

The same line of reasoning shows that, regardless of the validity of the real integral Hodge conjecture for X, the inclusion

$$H^c(X(\mathbb{R}), \mathbb{Z}/2)_{\mathrm{alg}} \subset H^c(X(\mathbb{R}), \mathbb{Z}/2)$$

is strict if (1.4) is not surjective. As we noticed in [BW20b, Remarks 2.7], this remark explains all the seemingly disparate examples of varieties with $H^c(X(\mathbb{R}), \mathbb{Z}/2)_{\mathrm{alg}} \neq H^c(X(\mathbb{R}), \mathbb{Z}/2)$ appearing in the literature. We were able to construct an example of a new kind, not explained by a defect of surjectivity of (1.4), and based instead on a counterexample to the real integral Hodge conjecture (obtained by adapting over $\mathbb{R}$ the degeneration techniques of Totaro [Tot13]).

Theorem 1.2.8 (Benoist–Wittenberg [BW20b])**.** *There is a smooth hypersurface of bidegree $(4, 4)$ in $\mathbb{P}^1_{\mathbb{R}} \times \mathbb{P}^3_{\mathbb{R}}$ with $H_1(X(\mathbb{R}), \mathbb{Z}/2)_{\mathrm{alg}} \neq H_1(X(\mathbb{R}), \mathbb{Z}/2)$.*

The case of 1-cycles

A topological analysis of the morphism (1.4) implies the following.

Theorem 1.2.9 (Benoist–Wittenberg [BW20b]). *Let X be a smooth projective variety over $\mathbb{R}$ with $X(\mathbb{R}) \neq \varnothing$ and $H_1(X(\mathbb{C}), \mathbb{Z}/2) = H^2(X, \mathcal{O}_X) = 0$.*

(i) If the real integral Hodge conjecture for 1-cycles on X holds, then $H_1(X(\mathbb{R}), \mathbb{Z}/2)_{\mathrm{alg}} = H_1(X(\mathbb{R}), \mathbb{Z}/2)$.

(ii) The converse holds if $X_{\mathbb{C}}$ satisfies the integral Hodge conjecture for 1-cycles.

In particular, a positive answer to Question 1.2.4 would imply that $H_1(X(\mathbb{R}), \mathbb{Z}/2)_{\mathrm{alg}} = H_1(X(\mathbb{R}), \mathbb{Z}/2)$ for all smooth projective rationally connected varieties over $\mathbb{R}$. In combination with Theorem 2.3.1, this would yield approximation results of loops in the real locus of real rationally connected varieties by real loci of real algebraic curves (a weak version of Question 0.1.17 considered in the introduction).

Theorem 1.2.9 is restricted to real varieties with $X(\mathbb{R}) \neq \varnothing$ because the property that $H_1(X(\mathbb{R}), \mathbb{Z}/2)_{\mathrm{alg}} = H_1(X(\mathbb{R}), \mathbb{Z}/2)$ is vacuous otherwise. When $X(\mathbb{R}) = \varnothing$, we discovered that it should be replaced with a more exotic property: the existence of a curve of even genus.

Theorem 1.2.10 (Benoist–Wittenberg [BW20b]). *Let X be a smooth projective variety over $\mathbb{R}$ with $X(\mathbb{R}) = \varnothing$ and $H_1(X(\mathbb{C}), \mathbb{Z}/2) = H^2(X, \mathcal{O}_X) = 0$.*

(i) If the real integral Hodge conjecture for 1-cycles on X holds, then X admits a morphism from a smooth projective curve of even genus.

(ii) The converse holds if $X_{\mathbb{C}}$ satisfies the integral Hodge conjecture for 1-cycles.

Let us emphasize that while smooth projective varieties over $\mathbb{R}$ with real points always admit morphisms from curves of even genus, some smooth projective varieties over $\mathbb{R}$ with no real points do not. As above, in view of Theorem 1.2.10, Question 1.2.4 predicts the existence of curves of even genus in all smooth projective rationally connected varieties over $\mathbb{R}$, a weakening of cohomological nature of Question 0.2.1.

To conclude this paragraph, we indicate that this point of view leads to new results even in the case of surfaces, where the real integral Hodge conjecture was known to hold. For instance, combining the above analysis with a new duality theorem for equivariant cohomology, we prove:

Theorem 1.2.11 (Benoist–Wittenberg [BW20b]). *Let X be a smooth projective surface over $\mathbb{R}$ with $H^2(X, \mathcal{O}_X) = 0$.*

(i) If $H_1(X(\mathbb{R}), \mathbb{Z}/2) \neq 0$, then $H_1(X(\mathbb{R}), \mathbb{Z}/2)_{\mathrm{alg}} \neq 0$.

(ii) The variety X contains a curve of even geometric genus if and only if the natural morphism $\mathrm{Pic}(X)[2^\infty] \to \mathrm{Pic}(X_{\mathbb{C}})^G[2^\infty]$ is onto.

1.2.6 Positive results

We finally state some partial positive results obtained in [BW20c] in the direction of Question 1.2.4.

Theorem 1.2.12 (Benoist–Wittenberg [BW20c])**.** *The real integral Hodge conjecture holds for smooth projective real Fano threefolds with no real points.*

Theorem 1.2.13 (Benoist–Wittenberg [BW20c])**.** *Let $f : X \to B$ be a morphism of smooth projective varieties over $\mathbb{R}$ with a conic as generic fibre. If B satisfies the real integral Hodge conjecture for* 1*-cycles, then so does X.*

Theorem 1.2.14 (Benoist–Wittenberg [BW20c])**.** *Let $f : X \to B$ be a morphism of smooth projective varieties over $\mathbb{R}$ whose generic fibre is a degree δ del Pezzo surface, with B a curve. If $\delta \in \{9, 8, 7, 6, 5, 3\}$, or if $\delta = 4$ and $B(\mathbb{R}) = \varnothing$, or if $\delta = 1$ and the smooth real fibers of f have connected real loci, then X satisfies the real integral Hodge conjecture for* 1*-cycles.*

When investigating the real integral Hodge conjecture for 1-cycles on a geometrically uniruled threefold X (Question 1.2.4 (ii)), it is natural in view of the birational invariance of this property (see Theorem 1.2.3) to study separately the various possible outcomes of the minimal model program applied to X: possibly singular Fano threefolds, conic bundles over surfaces, del Pezzo fibrations over curves. Theorems 1.2.12, 1.2.13 and 1.2.14 respectively concern these three cases. Note that only the case of conic bundles over surfaces is solved entirely by Theorem 1.2.13 (as the real integral Hodge conjecture is known to hold on surfaces by Theorem 1.2.3).

All the above positive results are about geometrically uniruled varieties. Much less is known about the Calabi–Yau threefolds considered in Question 1.2.4 (iii) (see however the partial results of de Gaay Fortman for real abelian threefolds [dGF22]).

In view of Theorem 1.2.10, we deduce from Theorem 1.2.12 the existence of curves of even geometric genus in all smooth projective real Fano threefolds, of which Theorem 0.2.3 is a particular case (one can ensure that the curve is smooth thanks to the smoothing method of [Hir68]). Similarly, Theorems 1.2.13 and 1.2.14 have applications to the computation of $H_1(X(\mathbb{R}), \mathbb{Z}/2)_{\mathrm{alg}}$ (using Theorem 1.2.9) and hence to algebraic approximation problems (using Theorem 2.3.1).

The techniques of proof of Theorems 1.2.12, 1.2.13 and 1.2.14 are varied.

The proof of Theorem 1.2.12 is an outgrowth of an attempt to adapt over the reals Voisin's proof of the integral Hodge conjecture for complex uniruled threefolds X. Voisin's idea is to fix a very ample smooth surface $S \subset X$. By the weak Lefschetz theorem, any Hodge class $\alpha \in \mathrm{Hdg}^4(X(\mathbb{C}), \mathbb{Z}(2))$ comes by push-forward from a class $\beta \in H^2(S(\mathbb{C}), \mathbb{Z}(1))$. If β were Hodge, it would be algebraic by the Lefschetz $(1, 1)$ theorem, and one would deduce the algebraicity of α. There is however no reason why β should be Hodge. Voisin's idea is to let S vary in its linear system. For some values of the parameters (along Noether–Lefschetz loci), the surface S will carry extra Hodge classes, hence extra algebraic cycles. One can hope that their push-forwards to X, for all possible choices of parameters, will span the group $\mathrm{Hdg}^4(X(\mathbb{C}), \mathbb{Z}(2))$ of integral Hodge classes and hence imply their algebraicity. One therefore needs an abundance result for Noether–Lefschetz loci, which is the heart of [Voi06]. Over the reals, there is a parallel theory of real Noether–Lefschetz loci (see §1.3.5 for generalities and §1.3.6 and §3.2.2 for applications). Unfortunately, we do not know how to show an abundance result for real Noether–Lefschetz loci in the same generality as Voisin over $\mathbb{C}$. However, when X is a smooth Fano threefold, one can often choose S to be a member of the anticanonical linear system of X, in which case S is a K3 surface and $h^{2,0}(S) = 1$. This leads to huge simplifications which allow us to apply this strategy to prove Theorem 1.2.12.

Both Theorem 1.2.13 and 1.2.14 are proved by using geometrically the fibration structure of the real algebraic variety X. Let us now explain some key features of their proofs.

Let $f : X \to B$ be a conic bundle as in Theorem 1.2.13. Imagine that we want to prove directly the algebraicity of some class $\alpha \in H_1(X(\mathbb{R}), \mathbb{Z}/2)$ which is predicted to be algebraic by Theorem 1.2.13. Represent α by a collection of loops $\gamma_i : \mathbb{S}^1 \to X(\mathbb{R})$. By the hypothesis on the base, the collection of loops $f(\mathbb{R}) \circ \gamma_i : \mathbb{S}^1 \to B(\mathbb{R})$ is known to be algebraic: there exists a curve $C \subset B$ such that $\mathrm{cl}_{\mathbb{R}}(C) = \sum_i (f(\mathbb{R}) \circ \gamma_i)_*[\mathbb{S}^1]$. It is then natural to try to algebraize $\alpha = \sum_i (\gamma_i)_*[\mathbb{S}^1]$ by somehow lifting the curve C to X. This is however impossible in general, as X might have no real points over some real points of C. This difficulty would however be resolved if one could choose $C(\mathbb{R})$ to appropriately approximate the $f(\mathbb{R}) \circ \gamma_i$, instead of only having the same homology class in $B(\mathbb{R})$, as X obviously has real points above the images of $f(\mathbb{R}) \circ \gamma_i$. This can be done using Theorem 2.3.1. The proof of Theorem 1.2.13 hinges on this approximation theorem, as well as on others due to Akbulut and King, to Bröcker, and to Ischebeck and Schülting [AK85a, Brö80, IS88].

Finally, let $f : X \to B$ be a del Pezzo fibration as in Theorem 1.2.14. We assume, as we may, that its reduced fibers are strict normal crossings divisors in X. To prove the algebraicity of a class in $H^4_G(X(\mathbb{C}), \mathbb{Z}(2))_0$, our main idea is to write it as a linear combination of the class of an algebraic multisection of f and of classes supported on fibers of f. On the one hand, this requires to be able to produce sufficiently many algebraic multisections of f. This part of the proof is the reason for the restrictions on the degree δ of the del Pezzo fibration. The hardest case, when $\delta = 4$ and $B(\mathbb{R}) = \varnothing$, relies on Corollary 1.3.6 below. On the other hand, this requires to prove that the group $H^4_{G,X_b(\mathbb{C})}(X(\mathbb{C}),\mathbb{Z}(2))$ of classes supported on a fiber $X_b \subset X$ of f is spanned by classes of algebraic curves lying on X_b. Over algebraically closed fields, such results were obtained by Esnault and Wittenberg [EW16] and we adapt their arguments over the reals.

1.3 The period-index problem for real surfaces

In this section, we present an application of Hodge theory to the arithmetic of real function fields.

1.3.1 The period-index problem

Let K be a field, and let $\mathrm{Br}(K)$ be its Brauer group. A Brauer class $\alpha \in \mathrm{Br}(K)$ has two main invariants. The *period* $\mathrm{per}(\alpha)$ is the order of α in the torsion group $\mathrm{Br}(K)$. The *index* $\mathrm{ind}(\alpha)$ is the smallest degree of a finite field extension of K splitting α (equivalently, the gcd of the degrees of such extensions).

These invariants are not unrelated. They share the same prime divisors, and the period always divides the index. Finding further constraints on the period and the index (usually with the intent to compute the index which is the geometrically interesting invariant) is the so-called period-index problem. Here are two significant results in this direction.

Theorem 1.3.1 (de Jong [dJ04])**.** *Let K be the function field of a surface over an algebraically closed field. If $\alpha \in \mathrm{Br}(K)$, then $\mathrm{ind}(\alpha) = \mathrm{per}(\alpha)$.*

Theorem 1.3.2 (Lieblich [Lie15])**.** *Let K be the function field of a surface over a finite field. If $\alpha \in \mathrm{Br}(K)$, then $\mathrm{ind}(\alpha) \mid \mathrm{per}(\alpha)^2$.*

1.3.2 Function fields of real algebraic surfaces

We will be interested in analogues of Theorems 1.3.1 and 1.3.2 for function fields K of real algebraic surfaces. In this setting, if $\alpha \in \mathrm{Br}(K)$, de Jong's theorem and a norm argument imply that either $\mathrm{ind}(\alpha) = \mathrm{per}(\alpha)$ or $\mathrm{ind}(\alpha) = 2\,\mathrm{per}(\alpha)$. The case $\mathrm{ind}(\alpha) = 2\,\mathrm{per}(\alpha)$ really may happen as was first noticed by Albert [Alb32], for instance for $K = \mathbb{R}(x, y)$ and $\alpha = (-1,-1) + (x, y) \in \mathrm{Br}(K)[2]$. Our goal is therefore to determine conditions under which the equality $\mathrm{ind}(\alpha) = \mathrm{per}(\alpha)$ holds.

One can for instance suppose that α vanishes in restriction to the real points of the surface.

Theorem 1.3.3 (Benoist [Ben19])**.** *Let S be a connected smooth surface over $\mathbb{R}$. Let $\alpha \in \mathrm{Br}(S) \subset \mathrm{Br}(\mathbb{R}(S))$ be such that $\alpha|_x = 0$ for all $x \in S(\mathbb{R})$. Then $\mathrm{ind}(\alpha) = \mathrm{per}(\alpha)$.*

As a particular case, we deduce the next corollary.

Corollary 1.3.4 (Benoist [Ben19]). *Let S be a connected smooth surface over $\mathbb{R}$ with $S(\mathbb{R}) = \varnothing$. If $\alpha \in \mathrm{Br}(\mathbb{R}(S))$, then* $\mathrm{ind}(\alpha) = \mathrm{per}(\alpha)$.

We are also able to entirely compute the index of unramified classes: those classes that extend to a Brauer class of a whole smooth projective model of the surface. We refrain from writing down the precise statement here (see [Ben19, Theorem 0.5]).

1.3.3 Applications to the u-invariant

We were led to prove Theorem 1.3.3 because of the following arithmetic applications. The *u-invariant* of a field K has been defined by Elman and Lam [EL73] to be the maximal rank of an anisotropic quadratic form over K with trivial signature with respect to all the field orderings of K. It is measure of the complexity of quadratic forms over K. Of course, if K admits no field ordering, the condition on signatures is vacuous.

As was noted by Pfister [Pfi82], the next theorem is a consequence of Theorem 1.3.3.

Theorem 1.3.5 (Benoist [Ben19]). *The u-invariant of the function field of a real algebraic surface is equal to* 4.

The function field of a connected smooth real algebraic surface S admits no field ordering if and only if S has no real points [Art27]. In this case, Theorem 1.3.5 reduces to Theorem 0.1.8, which we state again as a corollary.

Corollary 1.3.6 (Benoist [Ben19]). *Let S be a connected smooth surface over $\mathbb{R}$ such that $S(\mathbb{R}) = \varnothing$. Then any quadratic form of rank ≥ 5 over $\mathbb{R}(S)$ has a nontrivial zero.*

1.3.4 A Hodge-theoretic approach

We now explain the strategy of the proof of Theorem 1.3.3.

Let S be a connected smooth surface over $\mathbb{R}$, and fix a Brauer class $\alpha \in \mathrm{Br}(S) \subset \mathrm{Br}(\mathbb{R}(S))$ that vanishes in restriction to the real points of S. A norm argument based on de Jong's theorem allows us to assume that $\mathrm{per}(\alpha)$ is a power of 2, and an additional dévissage reduces us to the critical case where $\mathrm{per}(\alpha) = 2$. One then has to show that $\mathrm{ind}(\alpha) = 2$.

We fix a smooth projective compactification $\overline{S}$ of S such that $R := \overline{S} \setminus S$ is a strict normal crossings divisor. Showing that $\mathrm{ind}(\alpha) = 2$ amounts to finding a morphism $\pi : T \to \overline{S}$ of smooth projective surfaces over $\mathbb{R}$ that is generically finite of degree 2, and such that $\pi^*\alpha = 0$ in $\mathrm{Br}(\mathbb{R}(T))$.

In order to ensure that $\pi^*\alpha$ is unramified on T, i.e., belongs to the subgroup $\mathrm{Br}(T) \subset \mathrm{Br}(\mathbb{R}(T))$, we arrange that π ramifies along R (as ramification kills ramification). More precisely, we fix a sufficiently ample line bundle $\mathcal{L}$ on S, we choose a general divisor $D \in |\mathcal{L}^{\otimes 2}(R)|$, and we let π

be (the minimal resolution of singularities of) a double cover of $\overline{S}$ ramified along $R \cup D$.

At this point, we have made sure that $\pi^*\alpha \in \mathrm{Br}(T)[2]$. To analyze this group, we make use of the short exact sequence

$$0 \to H^2_G(T(\mathbb{C}), \mathbb{Z}(1))/\langle \mathrm{cl}(\mathrm{Pic}(T)), 2\rangle \to \mathrm{Br}(T)[2] \to H^3_G(T(\mathbb{C}), \mathbb{Z}(1))[2] \to 0$$

breaking down $\mathrm{Br}(T)[2]$ into a topological component $H^3_G(T(\mathbb{C}), \mathbb{Z}(1))[2]$ and a cycle-theoretic component $H^2_G(T(\mathbb{C}), \mathbb{Z}(1))/\langle \mathrm{cl}(\mathrm{Pic}(T)), 2\rangle$. Ensuring that the image of $\pi^*\alpha$ in $H^3_G(T(\mathbb{C}), \mathbb{Z}(1))[2]$ vanishes is a purely topological matter. It makes use of a real analogue of the weak Lefschetz theorem, and only works after having chosen the divisor D (more precisely, the topology of its real locus $D(\mathbb{R}) \subset \overline{S}(\mathbb{R})$) with great care, thanks to appropriate algebraic approximation theorem such as Theorem 2.3.1.

We have now arranged that $\pi^*\alpha \in H^2_G(T(\mathbb{C}), \mathbb{Z}(1))/\langle \mathrm{cl}(\mathrm{Pic}(T)), 2\rangle$. It remains to show that this class vanishes. There is however no reason why this should happen. Indeed, this property is deeply influenced by the size of the image of $\mathrm{cl} : \mathrm{Pic}(T) \to H^2_G(T(\mathbb{C}), \mathbb{Z}(1))$, hence, in view of the real Lefschetz $(1,1)$ theorem (see Theorem 1.2.3), by the Hodge theory of the surface T. The idea around this difficulty is to let the divisor D vary in a family. The surface T will vary accordingly, and one may hope that for some value of the parameters (when they hit so-called real Noether–Lefschetz loci), the Picard group $\mathrm{Pic}(T)$ of T will jump in a way that kills $\pi^*\alpha$.

In §1.3.5, we explain abundance results for real Noether-Lefschetz loci that will allow us to achieve this goal in §1.3.6.

1.3.5 Density of real Noether–Lefschetz loci

Over the complex numbers, an infinitesimal criterion for the density of Noether–Lefschetz loci has been discovered by Green [CHM88, §5] (see also [Voi02]). In [Ben18], we proved a real analogue of this criterion.

The setting is the following. Let B be a smooth real algebraic variety. Let $\mathbb{H}^2_{\mathbb{Q}}$ be a $\mathbb{Q}$-local system on $B(\mathbb{C})$ endowed with a weight 2 variation of Hodge structures. In particular, each fiber of the holomorphic vector bundle $\mathcal{H}^2 := \mathbb{H}^2_{\mathbb{Q}} \otimes_{\mathbb{Q}} \mathcal{O}_{B(\mathbb{C})}$ is endowed with a weight 2 Hodge structure such that the Hodge filtration $F^\bullet\mathcal{H}^2$ varies holomorphically, and such that the connection $\nabla : \mathcal{H}^2 \to \mathcal{H}^2 \otimes \Omega^1_{B(\mathbb{C})}$ induced by $\mathbb{H}^2_{\mathbb{Q}}$ satisfies Griffiths transversality.

We fix an action of $G := \mathrm{Gal}(\mathbb{C}/\mathbb{R})$ on $\mathbb{H}^2_{\mathbb{Q}}$ which is compatible with its natural action on $B(\mathbb{C})$. We let G act on the total space of $\mathcal{H}^2$ through the involution which is induced, for $b \in B(\mathbb{C})$, by the diagonal action

$$\mathcal{H}^2_b \xrightarrow{\sim} \mathbb{H}^2_{\mathbb{Q},b} \otimes_{\mathbb{Q}} \mathbb{C} \xrightarrow{\sigma\otimes\sigma} \mathbb{H}^2_{\mathbb{Q},\sigma(b)} \otimes_{\mathbb{Q}} \mathbb{C} \xrightarrow{\sim} \mathcal{H}^2_{\sigma(b)}. \tag{1.5}$$

We assume that (1.5) preserves the factors $\mathcal{H}^{p,q}$ of the Hodge decompositions. In this situation, adapting the complex arguments of Voisin in [Voi02], we prove the following density theorem for the *real Noether–Lefschetz locus*

$$\mathrm{NL}_{\mathbb{R}} := \{b \in B(\mathbb{R}) \mid (\mathbb{H}^2_{\mathbb{Q}} \cap \mathcal{H}^{1,1})^G \neq 0\}.$$

Proposition 1.3.7 (Benoist [Ben18])**.** *With the above notation, assume that there exist $b \in B(\mathbb{R})$ and $\lambda \in (\mathcal{H}^{1,1}_b)^G$ such that the map*

$$\overline{\nabla}(\lambda) : T_{B(\mathbb{C}),b} \to \mathcal{H}^{0,2}_b \tag{1.6}$$

obtained by evaluating ∇ on λ is onto. Then $\mathrm{NL}_{\mathbb{R}}$ is analytically dense in the connected component of $B(\mathbb{R})$ containing b.

Over the complex numbers, several strategies have been devised to verify in practice the hypothesis of the Green density criterion: the original degeneration method of Ciliberto, Harris and Miranda [CHM88], computations with Jacobian rings [Kim91], use of explicit Noether–Lefschetz loci [CL91], or the much more general arguments of Voisin [Voi06].

Over the reals, the requirement that λ be G-invariant makes the verification of the hypothesis of Proposition 1.3.7 much harder. In particular, we are unable to adapt Voisin's work [Voi06] in this setting.

1.3.6 End of the proof

Let us now explain how Proposition 1.3.7 is applied to finish the proof of Theorem 1.3.3 sketched in §1.3.4. We denote by $\mathcal{T} \to B$ the family of smooth projective double covers of S obtained by letting the divisor D vary. We consider the G-equivariant variation of Hodge structures $\mathbb{H}^2_{\mathbb{Q}}$ on $B(\mathbb{C})$, as in §1.3.5, given by

$$\mathbb{H}^2_{\mathbb{Q},b} := \mathrm{Ker}\big[\pi_* : H^2(\mathcal{T}_b(\mathbb{C}), \mathbb{Q}(1)) \to H^2(S(\mathbb{C}), \mathbb{Q}(1))\big].$$

To prove the abundance result for real Noether–Lefschetz loci that is required to make the argument work, we need to verify the hypothesis of Proposition 1.3.7 for $\mathbb{H}^2_{\mathbb{Q}}$. We use the idea of Ciliberto and Lopez [CL91]: we choose (b, λ) such that λ is a Hodge class in $\mathbb{H}^2_{\mathbb{Q},b}$ constructed as the class of an algebraic cycle.

To do so, we argue geometrically. We arrange that for a particular value of $b \in B(\mathbb{R})$, there is a curve $\Gamma \subset \overline{S}(\mathbb{R})$ that splits in T, i.e., such that $\pi^{-1}(\Gamma)$ is the union of two curves Γ_1 and Γ_2. We then choose λ to be the projection in $\mathbb{H}^2_{\mathbb{Q},b}$ of $\mathrm{cl}_{\mathbb{C}}(\Gamma_1)$. It turns out to be quite hard to ensure the desired surjectivity of (1.6). This requires a very delicate choice of the curve Γ, which is only possible if one has blown up beforehand, at the very beginning of the proof, the surface S at many general points.

Another application of Proposition 1.3.7 in the same vein will be given in §3.2.2.

Chapter 2

Algebraic approximation

2.1 The Nash–Tognoli theorem

Let us first state the celebrated Nash–Tognoli theorem.

Theorem 2.1.1 (Nash–Tognoli [Nas52, Tog73]). *Any compact $\mathcal{C}^\infty$ manifold is diffeomorphic to the real locus of a smooth projective variety over $\mathbb{R}$.*

It would be desirable to prove a variant of the Nash–Tognoli theorem applicable to $\mathcal{C}^\infty$ maps. However, one cannot expect to algebraize all $\mathcal{C}^\infty$ maps $f : M \to N$ of compact $\mathcal{C}^\infty$ manifolds. Indeed, the fibers of such maps may in general be quite wild (arbitrary closed subsets of M), which precludes them from being algebraizable on the nose. The best one could hope for is to approximate f (in the $\mathcal{C}^\infty$ topology) by algebraizable maps. This hope is still too optimistic (as shown by Benedetti and Dedò [BD84]), and understanding when this is possible is the content of Question 0.1.14, which we now recall.

Question 2.1.2. Let $f : N \to M$ be a $\mathcal{C}^\infty$ map of compact $\mathcal{C}^\infty$ manifolds. When can one approximate f in $\mathcal{C}^\infty(N, M)$ by maps of the form $\phi^{-1} \circ g(\mathbb{R}) \circ \psi$, where $g : Y \to X$ is a morphism of smooth projective varieties over $\mathbb{R}$, and $\psi : N \xrightarrow{\sim} Y(\mathbb{R})$ and $\phi : M \xrightarrow{\sim} X(\mathbb{R})$ are diffeomorphisms?

While this question is widely open at present, the variant of this question where the algebraic structure on the target manifold M is fixed is fully understood, thanks to a relative variant of the Nash–Tognoli theorem due to Benedetti–Tognoli and Akbulut–King. To state it, we define the *unoriented cobordism group* $\mathrm{MO}_d(M)$ of a compact $\mathcal{C}^\infty$ manifold M to be the set of cobordism classes of $\mathcal{C}^\infty$ maps $f : N \to M$ of compact $\mathcal{C}^\infty$ manifolds. When $M = X(\mathbb{R})$ is the real locus of a smooth projective variety X over $\mathbb{R}$, we let

$$\mathrm{MO}_d(X(\mathbb{R}))_{\mathrm{alg}} \subset \mathrm{MO}_d(X(\mathbb{R}))$$

be the subgroup of algebraic cobordism classes, generated by those maps of the form $g(\mathbb{R})$ for some morphism $g : Y \to X$ of smooth projective varieties over $\mathbb{R}$. We finally let w_i denote the Stiefel–Whitney characteristic classes.

Theorem 2.1.3 (Benedetti–Tognoli, Akbulut–King [BT80, AK81a])**.** *Let X be a smooth projective variety over $\mathbb{R}$ and let $f : N \to X(\mathbb{R})$ be a $\mathcal{C}^\infty$ map of compact $\mathcal{C}^\infty$ manifolds. The following are equivalent.*

(i) For all neighborhoods $\mathcal{U} \subset \mathcal{C}^\infty(N, X(\mathbb{R}))$ of f, there exist a morphism $g : Y \to X$ of smooth projective varieties over $\mathbb{R}$ and a diffeomorphism $\psi : N \xrightarrow{\sim} Y(\mathbb{R})$ such that $g(\mathbb{R}) \circ \psi \in \mathcal{U}$.

(ii) One has $[f] \in \mathrm{MO}_*(X(\mathbb{R}))_{\mathrm{alg}}$.

(iii) For all $i_1, \dots, i_r$, one has $f_*[w_{i_1}(N) \dots w_{i_r}(N)] \in H^*(X(\mathbb{R}), \mathbb{Z}/2)_{\mathrm{alg}}$.

What is really proven in [BT80, AK81a] is the equivalence (i)$\Leftrightarrow$(ii). That (ii)$\Leftrightarrow$(iii) may be deduced from the work of Ischebeck and Schülting [IS88]. Note that Theorem 2.1.3 does indeed reduce to the classical Nash–Tognoli theorem when X is a point.

2.2 Algebraizable cohomology classes

Let M be a compact $\mathcal{C}^\infty$ manifold. A cohomology class in $H^*(M, \mathbb{Z}/2)$ is said to be *algebraizable* if it belongs to $\phi^*(H^*(X(\mathbb{R}), \mathbb{Z}/2)_{\mathrm{alg}})$ for some diffeomorphism $\phi : M \xrightarrow{\sim} X(\mathbb{R})$ onto the real locus of a smooth projective variety X over $\mathbb{R}$. In view of condition (iii) in Theorem 2.1.3, solving Question 2.1.2 is essentially equivalent to understanding when a cohomology class is algebraizable (or, more precisely, when a collection of cohomology classes are simultaneously algebraizable). This gives a strong motivation to study the algebraizability of cohomology classes.

On the positive side, Stiefel–Whitney classes of real vector bundles on M, and fundamental classes of submanifolds of M are known to be algebraizable, as are all the elements of the subring $A(M) \subset H^*(M, \mathbb{Z}/2)$ generated by these classes [BT80, AK81a]. Kucharz asked in [Kuc05, Conjecture A] whether, conversely, all algebraizable classes belonged to $A(M)$. We answer this question in the negative.

Theorem 2.2.1 (Benoist [Ben22a])**.** *There exists a compact $\mathcal{C}^\infty$ manifold M carrying an algebraizable cohomology class which does not belong to $A(M)$.*

On the negative side, non-algebraizable classes where previously known to exist in all even degrees ≥ 2 thanks to Kucharz [Kuc08]. We are able to give examples in all degrees ≥ 2, answering a question raised by Benedetti and Dedò [BD84, p. 150] and Kucharz [Kuc05, p. 194].

Theorem 2.2.2 (Benoist [Ben22a])**.** *For any $c \geq 2$, there exists a compact $\mathcal{C}^\infty$ manifold M and a class in $H^c(M, \mathbb{Z}/2)$ that is not algebraizable.*

Our proofs of both Theorems 2.2.1 and 2.2.2 rely on the simple observation that algebraizable classes are preserved by mod 2 Steenrod operations, a fact first noticed by Akbulut and King [AK85b], which is a consequence of the relative Wu theorem of Atiyah and Hirzebruch [AH61].

To prove Theorem 2.2.1, it suffices to exhibit a compact $\mathcal{C}^\infty$ manifold M such that $A(M)$ is not preserved by Steenrod operations.

Squares of algebraizable classes always lift to integral cohomology classes (this is a consequence of (1.3)). Kucharz used this remark in [Kuc08] to obstruct the algebraizability of some cohomology classes of even degree. Unfortunately, the square of a cohomology class of odd degree always has an integral lift, and hence this obstruction cannot be applied directly to odd degree classes. However, Kucharz's obstruction may very well apply to the image of an odd degree cohomology class by a Steenrod operation. This is how we prove Theorem 2.2.2.

2.3 Algebraic approximation of $\mathcal{C}^\infty$ submanifolds

Let X be a smooth projective variety of dimension n over $\mathbb{R}$, and let $i : N \hookrightarrow X(\mathbb{R})$ be a $\mathcal{C}^\infty$ submanifold of dimension d. We let $c := n - d$ denote the codimension of N in $X(\mathbb{R})$.

2.3.1 An approximation problem

Theorem 2.1.3 characterizes when the inclusion $i : N \hookrightarrow X(\mathbb{R})$ may be approximated in the $\mathcal{C}^\infty$ topology by an algebraic map $g(\mathbb{R}) : Y(\mathbb{R}) \to X(\mathbb{R})$. This does not solve on the nose the problem of whether the submanifold N may be approximated by real loci of real algebraic subvarieties of X. Indeed, the real locus of $g(Y) \subset X$ may be bigger than $g(Y(\mathbb{R}))$, if some pairs of complex conjugate points of Y happen to collapse to the same complex point of X. Determining when the next approximation property holds is therefore an interesting problem.

Property (A)**.** For all neighbourhoods $\mathcal{U} \subset \mathcal{C}^\infty(N, X(\mathbb{R}))$ of the inclusion i, there exist $j \in \mathcal{U}$ and a closed subvariety $Z \subset X$ which is smooth along $Z(\mathbb{R})$ such that $j(N) = Z(\mathbb{R})$.

For curves, this problem was resolved in successive works by Akbulut–King, Bochnak–Kucharz and Benoist–Wittenberg [AK88, BK03b, BW20c].

Theorem 2.3.1 (Akbulut–King, Bochnak–Kucharz, Benoist–Wittenberg)**.** *If $d = 1$, then (A) holds if and only if $i_*[N] \in H_1(X(\mathbb{R}), \mathbb{Z}/2)_{\mathrm{alg}}$.*

Can one generalize Theorem 2.3.1 to higher-dimensional cycles? To do so, one has to take into account the following finer necessary condition based on cobordism, which already appeared in the statement of Theorem 2.1.3.

Property (B)**.** One has $[i : N \hookrightarrow X(\mathbb{R})] \in MO_d^{\mathrm{alg}}(X(\mathbb{R}))$.

So, when are (A) and (B) equivalent? We obtain both positive and negative results on this question, explained respectively in §2.3.2 and §2.3.3.

2.3.2 Positive results: linkage

On the positive side, we show the sufficiency of (B) for cycles of low dimension (below the half of the dimension of the ambient variety).

Theorem 2.3.2 (Benoist [Ben20a])**.** *If $d < c$, then Properties (A) and (B) are equivalent.*

In the particular case where $X = \mathbb{P}^n_{\mathbb{R}}$, Property (B) is always satisfied, and Theorem 2.3.2 reduces to Theorem 0.1.16.

To prove Theorem 2.3.2, we first apply Theorem 2.1.3 to approximate the embedding $i : N \hookrightarrow X(\mathbb{R})$ by a map $g(\mathbb{R}) : Y(\mathbb{R}) \to X(\mathbb{R})$ induced by a morphism $g : Y \to X$ of smooth projective varieties over $\mathbb{R}$. As we already explained in §2.3.1, setting $Z := g(Y)$ does not yield a proof of Theorem 2.3.2 because $g(\mathbb{C})$ might not be injective, so that the inclusion $g(Y(\mathbb{R})) \subset Z(\mathbb{R})$ might be strict. However, under the hypothesis that $d < c$, the $\mathcal{C}^\infty$ map $g(\mathbb{C})$ is generically expected to be an embedding, which would allow us to conclude. This heuristic explains the role of the hypothesis that $d < c$ in Theorem 2.3.2 (reminiscent of Whitney's embedding theorem in differential geometry [Whi36]).

Unfortunately, one cannot in general deform g algebraically, let alone render the $\mathcal{C}^\infty$ map $g(\mathbb{C})$ generic in the above sense after an algebraic deformation of g. One therefore needs a different kind of moving technique. To this effect, we use the method of moving by linkage first exploited by Hironaka in [Hir68].

The basic idea is the following. Fix an embedding $e : Y \hookrightarrow \mathbb{P}^N_{\mathbb{R}} \times X$ of Y in a trivial projective bundle over X, in such a way that $g = pr_2 \circ e$. Fix a sufficiently ample line bundle $\mathcal{L}$ on $\mathbb{P}^N_{\mathbb{R}} \times X$, and let $s_1, \ldots, s_{c+N}$ be general sections of $\mathcal{L}$ vanishing on Y. Then

$$\{s_1 = \cdots = s_{N+c} = 0\} = Y \cup Y',$$

where Y' has the same dimension as Y. The variety Y' is said to be *linked* to Y and one writes $Y \sim Y'$. One would like to replace Y with Y' and hope that it serves our purposes better. There are however two difficulties to overcome.

The first difficulty is that the real locus of Y' is not at all close to the real locus of Y. To solve this problem, one performs two links instead of one! More precisely, one chooses a section t of a sufficiently ample line bundle on $\mathbb{P}^N_{\mathbb{R}} \times X$ that has no real zeros, and one considers a link $Y' \sim Y''$ of the form

$$\{s'_1 = \cdots = s'_{N+c} = 0\} = Y' \cup Y'',$$

where s_i' is very close to $s_i \cdot t$ in the analytic topology. This choice ensures that $Y''(\mathbb{R})$ is close to $Y(\mathbb{R})$ (as the second link almost undoes the effect of the first link at the level of real loci).

The second difficulty is that Y'' now inevitably has singularities (away from its real locus). This makes it impossible to reasonably hope that the projection $Y'' \to X$ is an embedding. We however claim that, after repeating this procedure a large number of times (exponential in d), we will eventually reach a subvariety $Y^{(t)} \subset \mathbb{P}^N_{\mathbb{R}} \times X$, whose real locus is close to $Y(\mathbb{R})$ and such that the projection $Y^{(t)} \to X$ is injective at the level of $\mathbb{C}$-points. One can then set $Z := g(Y^{(t)})$ to conclude. The verification of this claim is nontrivial, and requires a study of the linkage construction in families, based on commutative algebra results due to Peskine and Szpiro, and to Huneke and Ulrich [PS74, HU85].

2.3.3 Negative results: the double point formula

To state our counterexamples to the equivalence of (A) and (B), we define $\alpha(m)$ to be the number of ones in the binary expansion of an integer m.

Theorem 2.3.3 (Benoist [Ben20a]). *If $d \geq c$ and $\alpha(c+1) = 2$, there exist X and $i : N \hookrightarrow X(\mathbb{R})$ such that (B) holds but (A) fails.*

We first present the simplest counterexample that we construct, for which $n = 4$ and $c = d = 2$. In this case, one can choose $N = \mathbb{P}^2(\mathbb{R})$ and $X = \mathbb{P}^1_{\mathbb{R}} \times \mathbb{P}^3_{\mathbb{R}}$, and let $i : \mathbb{P}^2(\mathbb{R}) \hookrightarrow \mathbb{R}^4 \subset \mathbb{P}^1(\mathbb{R}) \times \mathbb{P}^3(\mathbb{R})$ be any embedding of $\mathbb{P}^2(\mathbb{R})$ in a standard affine chart of $\mathbb{P}^1(\mathbb{R}) \times \mathbb{P}^3(\mathbb{R})$. Let us explain why (A) cannot hold in this example.

Assume that we managed to approximate the immersion i by an algebraic map $g(\mathbb{R}) : Y(\mathbb{R}) \to X(\mathbb{R})$ induced by a morphism $g : Y \to X$ of smooth projective varieties over $\mathbb{R}$, for instance thanks to an application of Theorem 2.1.3. Our goal is to show that setting $Z := g(Y) \subset X$ cannot solve our approximation problem, i.e., that Z cannot be smooth along its real locus. To do so, we will show that there necessarily exist two complex conjugate points of Y that collapse to the same point of X, which will then be a singular real point of Z.

To prove that Z has such a real double point, the idea is to count them. If δ is the number of double points of Z, we will verify that δ is odd, and hence that at least one of these double points has to be real. Fulton's double point formula [Ful98, §9.3] (which refines earlier works of Todd and Laksov [Tod40, Lak78]) reads:

$$\begin{aligned} 2\delta &= (g_*[Y])^2 - c_2(N_{Y/X}) \\ &= (g_*[Y])^2 + g^*c_2(X) - c_1(Y) \cdot g^*c_1(X) + [c_1^2(Y) - c_2(Y)]. \end{aligned} \tag{2.1}$$

At this point, our knowledge that $X = \mathbb{P}^1_{\mathbb{R}} \times \mathbb{P}^3_{\mathbb{R}}$, that $Y(\mathbb{R}) \simeq \mathbb{P}^2(\mathbb{R})$ and that the embedding $Y(\mathbb{R}) \hookrightarrow X(\mathbb{R})$ is homotopically trivial is sufficient to prove that the first three terms of the right-hand side of (2.1) are divisible by 4. As for the fourth term $c_1^2(Y) - c_2(Y)$, the Noether formula

$$c_1^2(Y) + c_2(Y) = 12\chi(Y, \mathcal{O}_Y) \tag{2.2}$$

implies that it is congruent to $2c_2(Y)$ modulo 4. That $Y(\mathbb{R}) \simeq \mathbb{P}^2(\mathbb{R})$ may then be used to show that it is congruent to 2 modulo 4. It now follows from (2.1) that δ is odd, as required.

Difficulties arising from the fact that Z might have worse singularities than double points are entirely solved by Fulton's refined intersection theory, already taken into account in [Ful98, §9.3].

For higher values of the codimension c, the proof follows a similar path. One must however replace the above application of the Noether formula (2.2) by more general divisibility results for Chern numbers, valid for higher-dimensional compact complex manifolds. To obtain such divisibility results, we exploit our knowledge of the structure of the complex cobordism ring, following a strategy due to Rees and Thomas [RT77] and relying on results of theirs.

Unfortunately, we were not able to use the kind of obstructions sketched in this paragraph to produce a counterexample to Question 0.1.15.

2.3.4 Applications to Chow groups

The techniques described in §2.3.2 and 2.3.3 (use of linkage and of double point formulae) have other applications, to the structure of Chow groups of real algebraic varieties. More specifically, we use them to study the problems of determining the subgroups of Chow groups generated by classes of subvarieties with empty real loci (in Theorem 2.3.4), or of subvarieties that are smooth along their real loci (in Theorem 2.3.5).

Recall that $\alpha(m)$ is the number of ones in the binary expansion of an integer m.

Theorem 2.3.4 (Benoist [Ben20a])**.**

(i) Let X be a smooth projective variety of dimension $c + d$ over $\mathbb{R}$. If $d < c$, then $\mathrm{Ker}[\mathrm{cl}_{\mathbb{R}} : \mathrm{CH}_d(X) \to H_d(X(\mathbb{R}), \mathbb{Z}/2)]$ *is generated by classes of subvarieties of X with empty real loci.*

(ii) If $\alpha(c+1) \geq 2$ and $d \geq c$, there exists a smooth projective variety of dimension $c+d$ over $\mathbb{R}$ such that $\mathrm{Ker}[\mathrm{cl}_{\mathbb{R}} : \mathrm{CH}_d(X) \to H_d(X(\mathbb{R}), \mathbb{Z}/2)]$ *is not generated by classes of subvarieties of X with empty real loci.*

Theorem 2.3.4 (i) was shown to hold when $d = 0$ by Colliot-Thélène and Ischebeck [CTI81], and when $d = 1$ and $c = 2$ by Kucharz [Kuc04]. It is new in all other cases.

Theorem 2.3.4 (ii) was known to hold for all even values of c, again thanks to Kucharz [Kuc04]. We extend his result to all values of c not of the form $2^k - 1$. It is not possible to entirely remove the hypothesis on c, as there are no such counterexamples in the case $c = 1$ of divisors, by a result of Bröcker [Brö80].

Theorem 2.3.5 (Benoist [Ben20a])**.**

(i) Let X be a smooth projective variety of dimension $c + d$ over $\mathbb{R}$. If $d < c$, then $\mathrm{CH}_d(X)$ is generated by classes of subvarieties of X that are smooth along $X(\mathbb{R})$.

(ii) If $\alpha(d + 1) \geq 3$ and $d \geq c$, there exists a smooth projective variety of dimension $c + d$ over $\mathbb{R}$ such that $\mathrm{CH}_d(X)$ is not generated by classes of subvarieties of X that are smooth along $X(\mathbb{R})$.

Theorem 2.3.5 (i) is a consequence of the original smoothing results of Hironaka [Hir68] when $d \leq 3$. Theorem 2.3.5 (ii), on the other hand, is entirely new.

The problem considered in Theorem 2.3.5 is closely related to a classical question going back to Borel and Haefliger [BH61]: is the Chow group of a smooth projective complex variety X generated by classes of smooth subvarieties? After early positive results due to Hironaka and Kleiman [Hir68, Kle69], which in particular settled the question if $\dim(X) \leq 5$, a 9-dimensional counterexample was discovered by Hartshorne, Rees and Thomas [HRT74].

By exploiting in complex algebraic geometry the complex cobordism ring computations alluded to at the end of §2.3.3, Debarre and myself were able to obtain a counterexample of the lowest possible dimension.

Theorem 2.3.6 (Benoist–Debarre [BD23])**.** *There exist a smooth projective variety X of dimension 6 over $\mathbb{C}$ and an algebraic class in $H^4(X(\mathbb{C}), \mathbb{Z})$ that is not a linear combination of classes of smooth algebraic subvarieties of X.*

More precisely, inspired by an earlier work of Debarre [Deb95], we study the question of Borel and Haefliger when X is the Jacobian of a very general curve of genus g polarized by its theta divisor θ, and for the minimal cohomology classes $\frac{\theta^c}{c!} \in H^{2c}(X(\mathbb{C}), \mathbb{Z})$. Theorem 2.3.6 is obtained when $g = 6$ and $c = 2$.

2.4 The tight approximation property

In the algebraic approximation problems considered in Theorem 2.1.3 and in §2.3.1, the target manifold is the real locus of a fixed smooth projective variety over $\mathbb{R}$, but the algebraic structure on the source manifold is allowed to vary. In contrast, algebraic approximation results for $\mathcal{C}^\infty$ maps

$f : Y(\mathbb{R}) \to X(\mathbb{R})$ between the real loci of two fixed smooth projective varieties Y and X over $\mathbb{R}$ cannot possibly hold without strong geometric hypotheses on Y and X. Indeed, there is no reason why there should even exist any nonconstant morphism $Y_{\mathbb{C}} \to X_{\mathbb{C}}$.

Already when $Y = \mathbb{P}^1_{\mathbb{R}}$, one should restrict to varieties X carrying many rational curves. It is therefore natural to impose that X be a rationally connected variety (and it is then reasonable to allow Y to be any smooth projective real algebraic curve).

We cannot rule out the existence of interesting results for other classes of real algebraic varieties X. In the case of K3 surfaces, we do not even know the answer to the following question (see [BW21, Remark 3.8]).

Question 2.4.1. Let X be a real K3 surface with $X(\mathbb{R}) \neq \varnothing$. Does there exist a nonconstant morphism $\mathbb{P}^1_{\mathbb{R}} \to X$?

In contrast, all complex K3 surfaces are known to carry infinitely many rational curves [CGL22].

2.4.1 Definition of tight approximation

It turns out to be useful, both to actually prove approximation results and to extend the scope of their applications, to consider the problem of approximating sections of one-parameter families of rationally connected varieties, instead of morphisms to a fixed rationally connected variety. In [BW21], we define a quite general approximation property in this spirit: the tight approximation property.

Let B be a connected smooth projective curve over $\mathbb{R}$ with function field $F := \mathbb{R}(B)$. Recall that we set $G := \mathrm{Gal}(\mathbb{C}/\mathbb{R})$.

Let $f : \mathfrak{X} \to B$ be a flat projective morphism of smooth varieties over $\mathbb{R}$. One says that f satisfies the *tight approximation property* if for all G-stable open neighborhoods Ω of $B(\mathbb{R})$ in $B(\mathbb{C})$, for all G-equivariant holomorphic sections $u : \Omega \to \mathfrak{X}(\mathbb{C})$ of $f(\mathbb{C})$ over Ω, and for all $b_1, \ldots, b_m \in \Omega$ and $r \geq 0$, there exists a sequence $s_n : B \to \mathfrak{X}$ of sections of f having the same r-jets as u at the b_i and such that the $s_n(\mathbb{C})|_{\Omega}$ converge to u in $\mathcal{C}^{\infty}(\Omega, \mathfrak{X}(\mathbb{C}))$.

We also say that a smooth variety X over F satisfies the tight approximation property if so does some birational model $f : \mathfrak{X} \to B$ of X as above. (It follows from Theorem 2.4.3 below that this property does not depend on the birational model $f : \mathfrak{X} \to B$ of X). Our main motivation to formulate these definitions is to ask the following question (see [BW21, Question 3.6]).

Question 2.4.2. Do smooth projective rationally connected varieties over F satisfy the tight approximation property?

2.4.2 Connections with the literature

In the particular case where $f : \mathfrak{X} \to B$ is a constant fibration with fiber a rationally connected variety X, tight approximation predicts the existence of many algebraic morphisms $B \to X$. An outstanding positive result in this case is Kollár's theorem that there exist real rational curves through any finite collection of real points in a fixed connected component of the real locus of a real rationally connected variety [Kol99]. Very little was known on $\mathcal{C}^\infty$ approximation questions in this context (such as Question 0.1.17), beyond the case of rational varieties (see the works of Bochnak and Kucharz [BK99, BK03a] discussed in §2.4.3).

Let us insist that the curve B may not be geometrically connected. In this case, the curve B can be thought of as a complex curve, and Question 2.4.2 encompasses the Graber–Harris–Starr theorem according to which one-parameter families of complex rationally connected varieties always have a section [GHS03]. It also contains as a particular case the conjecture that such families satisfy the weak approximation property (see [CTG04, HT06] for significant results in this direction).

In the real case, Question 2.4.2 may in particular be thought of as a substitute for the Graber–Harris–Starr theorem. In this direction, it notably implies Question 0.2.1, as well as Lang's conjecture that low degree hypersurfaces over function fields of real curves with no real points have rational points (see Question 0.1.7 for $n = 1$; for singular hypersurfaces, one must also use [HX09]).

There is however no hope that rationally connected varieties over F satisfy weak approximation in the real case in general, because of obstructions induced by the topology of real points: for $f : \mathfrak{X} \to B$ to have an algebraic section, it is necessary that $f(\mathbb{R}) : \mathfrak{X}(\mathbb{R}) \to B(\mathbb{R})$ should have a $\mathcal{C}^\infty$ section. This explains the importance of having imposed a $\mathcal{C}^\infty$ approximation condition in the definition of tight approximation, and of having insisted that $B(\mathbb{R}) \subset \Omega$. These obstructions were first studied as analogues in real algebraic geometry of the Brauer–Manin obstruction over number fields (see [CT96, Sch96, Duc98]). This point of view led in particular to the proof of (a variant of) the tight approximation property for fibrations in Severi–Brauer varieties over $\mathbb{P}^1_F$ in [Duc98].

2.4.3 Birational aspects

The case of constant fibrations was investigated by Bochnak and Kucharz in [BK99] in the real setting (in which case they only consider approximation in $\mathcal{C}^\infty(B(\mathbb{R}), \mathfrak{X}(\mathbb{R}))$), and in [BK03a] in the complex setting. In these articles, they prove two important facts concerning the approximation properties that they consider: birational invariance, and validity for projective spaces. Combining these facts, they deduce their validity for all rational varieties.

We adapted their arguments for the tight approximation property.

Theorem 2.4.3 (Benoist–Wittenberg [BW21])**.** *Assume that $f : \mathfrak{X} \to B$ and $f' : \mathfrak{X}' \to B$ are flat projective morphisms of smooth varieties over $\mathbb{R}$ and that there exists a birational morphism $g : \mathfrak{X} \to \mathfrak{X}'$ such that $f = f' \circ g$. Then f satisfies the tight approximation property if and only if so does f'.*

Theorem 2.4.4 (Benoist–Wittenberg [BW21])**.** *Rational varieties over F satisfy the tight approximation property.*

It is absolutely crucial for the validity of Theorem 2.4.3 that tight approximation incorporates a weak approximation property (controlling the jets of sections at finitely many points of B), as is already apparent in [BK99, BK03a]. Theorem 2.4.4 can be reduced to the case of projective spaces thanks to Theorem 2.4.3. It then follows from a G-equivariant version of the Runge approximation theorem.

2.4.4 Behaviour in fibrations

Our first main result about the tight approximation property is its compatibility with fibrations.

Theorem 2.4.5 (Benoist–Wittenberg [BW21])**.** *Let $g : X \to X'$ be a dominant morphism between smooth varieties over F. If X' and the fibers of g above the F-points of a dense open subset of X' satisfy the tight approximation property, then so does X.*

A fibration theorem for a weaker property than tight approximation, which in addition assumes X' to be F-rational, had been independently proven by Pál and Szabó in [PS20].

Theorem 2.4.5 can be used to show the tight approximation property for some varieties that have a fibration structure. Here is an example.

Theorem 2.4.6 (Benoist–Wittenberg [BW21])**.** *Let X be be a smooth projective variety of dimension ≥ 2 over F that is either*

(i) a cubic hypersurface containing an F-line, or

(ii) a complete intersection of two quadrics with an F-point.

Then X satisfies the tight approximation property.

Indeed, projecting the cubic from an F-line yields a conic bundle over a projective space, and projecting the complete intersection from its tangent space at an F-point yields a quadric bundle over $\mathbb{P}^1_F$.

Theorem 2.4.6 is particularly interesting because it implies the tight approximation property for those cubic hypersurfaces and complete intersections of two quadrics, smooth of dimension ≥ 2, that are defined over $\mathbb{R}$. This is how we prove the next theorem (which appeared as Theorem 0.1.18 in the introduction).

Theorem 2.4.7 (Benoist–Wittenberg [BW21])**.** *Let X be a cubic hypersurface or a complete intersection of two quadrics, which is smooth of dimension ≥ 2 over $\mathbb{R}$. Then algebraic maps are dense in $\mathcal{C}^\infty(\mathbb{P}^1(\mathbb{R}), X(\mathbb{R}))$.*

Let us now sketch the proof of Theorem 2.4.5. Let $g : \mathfrak{X} \to \mathfrak{X}'$ be a model of $g : X \to X'$. Our goal is to construct a section s of $f : \mathfrak{X} \to B$ such that $s(\mathbb{C})$ approximates (in the sense of $\mathcal{C}^\infty$ approximation on compact subsets as well as in the sense of jets) a fixed G-equivariant holomorphic section $u : \Omega \to \mathfrak{X}(\mathbb{C})$ of $f(\mathbb{C})$ over Ω.

By applying tight approximation for f', one can find a section $s' : B \to \mathfrak{X}'$ of f' such that $s'(\mathbb{C})$ approximates $g(\mathbb{C}) \circ u$ in this sense. One then considers the fibration $h : \mathfrak{X} \times_{\mathfrak{X}'} B \to B$ obtained by restricting g over the image of s'. To conclude, it would remain to use the tight approximation property for h in order to construct an appropriate section s of h (which we can also view as a section of f).

For this to work, which holomorphic section of $h(\mathbb{C})$ over Ω would we like $s(\mathbb{C})$ to approximate? It would have to be a small deformation of u whose image in $\mathfrak{X}(\mathbb{C})$ would happen to lie exactly over the image of $s'(\mathbb{C})$. Such a deformation is not hard to construct, by making use of the normal form of submersions, if g is smooth along the image of u (and s' approximates $g(\mathbb{C}) \circ u$ well enough). Unfortunately, possible singularities of g along the image of u create significant difficulties in the construction of such a deformation.

Our solution to this problem is that it is always possible, after possibly replacing $\mathfrak{X}$ and $\mathfrak{X}'$ by different birational models, to assume that the image of u entirely avoids the singular locus of $g(\mathbb{C})$. This surprising fact may be thought of as a substitute, over a higher-dimensional base, of the Néron smoothening process. Our proof of it makes uses toroidal geometry in an essential way.

2.4.5 Descent along torsors

Our second main result concerning tight approximation is a descent theorem for this property.

Theorem 2.4.8 (Benoist–Wittenberg [BW21])**.** *Let X be a smooth variety over F. Let S be a linear algebraic group over F. Let $Q \to X$ be a left S-torsor over X. If the twists of Q by right S-torsors over F all satisfy the tight approximation property, then so does X.*

The statement of Theorem 2.4.8 (and the general mechanism of its proof) is inspired by the descent method of Colliot-Thélène and Sansuc over number fields, as further developed by Harari and Skorobogatov. Crucial inputs are Scheiderer's deep results on homogeneous spaces under linear algebraic groups over functions fields of real curves [Sch96], and Colliot-Thélène and Gille's proof of weak approximation for homogeneous spaces under linear algebraic groups over function fields of complex curves [CTG04].

Combining Theorems 2.4.5 and 2.4.8 with structure results for linear algebraic groups and Scheiderer's Hasse principle [Sch96], we deduce the following concrete application.

Theorem 2.4.9 (Benoist–Wittenberg [BW21])**.** *Homogeneous spaces under connected linear algebraic groups over F satisfy tight approximation.*

Theorem 2.4.9 does not formally imply weak approximation for homogeneous spaces under connected linear algebraic groups over F because, as we already noted in §2.4.2, there are in general topological obstructions to weak approximation in this context. However, relying yet another time on Scheiderer's work [Sch96], one can verify that these obstructions always vanish for such homomogeneous spaces. We may therefore deduce from Theorem 2.4.9 the conjecture of Colliot-Thélène that we already stated in Theorem 0.1.9.

Theorem 2.4.10 (Benoist–Wittenberg [BW21])**.** *Homogeneous spaces under connected linear algebraic groups over F satisfy weak approximation.*

Theorem 2.4.10 had been proven by Colliot-Thélène [CT96] when the stabilizers are trivial and by Scheiderer [Sch96] when they are connected.

Chapter 3

Positivity and sums of squares

3.1 Sums of squares

3.1.1 Sums of squares of polynomials

Let $f \in \mathbb{R}[x_1, \ldots, x_n]$ be a nonnegative polynomial, and let d be its degree. We note that d is even, because a real polynomial of odd degree always changes sign. It is tempting to try to explain the positivity of f by writing it as a sum of squares of polynomials. It was discovered by Hilbert [Hil88] that this is not always possible. Hilbert moreover determined all the possible values of (n, d) for which such a result holds.

Theorem 3.1.1 (Hilbert [Hil88]).

(i) If $n \leq 1$, $d \leq 2$, or $(n, d) = (2, 4)$, all nonnegative $f \in \mathbb{R}[x_1, \ldots, x_n]$ of degree d are sums of squares of polynomials.

(ii) If $n \geq 2$, $d \geq 4$ and $(n, d) \neq (2, 4)$, there exists $f \in \mathbb{R}[x_1, \ldots, x_n]$ nonnegative of degree d which is not a sum of squares of polynomials.

The most famous example of a nonnegative polynomial which is not a sum of squares of polynomials is Motzkin's polynomial

$$1 + x^2y^4 + x^4y^2 - 3x^2y^2. \tag{3.1}$$

3.1.2 Sums of squares of rational functions

As Hilbert conjectured in his 17th problem, and as was eventually established by Artin [Art27], one way to repair the failure of a nonnegative polynomial to be expressible as a sum of squares of polynomials is to allow denominators.

Theorem 3.1.2 (Artin [Art27]). *Any nonnegative $f \in \mathbb{R}[x_1, \ldots, x_n]$ is a sum of squares in $\mathbb{R}(x_1, \ldots, x_n)$.*

Such sums of squares problems pertain to quadratic form theory. That nonnegative polynomials are always sums of squares of rational functions, but are not sums of squares of polynomials in general, can be thought of as a manifestation of the fact that quadratic forms are better behaved over fields (such as $\mathbb{R}(x_1, \ldots, x_n)$) than over general rings (such as $\mathbb{R}[x_1, \ldots, x_n]$).

3.1.3 Sums of squares in local rings: bad points

Let $f \in \mathbb{R}[x_1, \ldots, x_n]$ be a nonnegative polynomial. A point $p \in \mathbb{C}^n$ is said to be a *bad point* for f if, in all representations of f as sums of squares in $\mathbb{R}(x_1, \ldots, x_n)$, some denominator vanishes at the point p. Equivalently, letting $\mathfrak{m}_p \subset \mathbb{R}[x_1, \ldots, x_n]$ denote the maximal ideal of polynomials vanishing at p, the point p is bad for f exactly when f is not a sum of squares in the local ring $\mathbb{R}[x_1, \ldots, x_n]_{\mathfrak{m}_p}$.

Here are two motivations to study bad points. First, they are exactly the local obstructions to writing a nonnegative polynomial as a sum of squares of polynomials. Second, investigating bad points is a problem concerning quadratic forms in local rings, which is of intermediate difficulty between the ideal situation of quadratic forms over fields and the much wilder case of quadratic forms over general rings.

Bad points were first introduced under this name by Delzell in his PhD thesis [Del80], but the first examples appeared in a 1956 letter of Straus to Kreisel: if $f \in \mathbb{R}[x_1, \ldots, x_n]$ is not a sum of squares of polynomials, then its homogenization in $\mathbb{R}[x_1, \ldots, x_{n+1}]$ has a bad point at the origin. A major result, proven in increasing generality by Choi–Lam, Delzell and Scheiderer [CL77, Del80, Sch01] states the set of bad points has codimension ≥ 3 in $\mathbb{C}^n$. In particular, there are no bad points when $n = 2$.

The above-mentioned results solve the problem of determining for which values of (n, d) bad points may exist, except in the case $(n, d) = (3, 4)$. Our first result on bad points sorts this out, yielding an analogue of Theorem 3.1.1 for the existence of bad points.

Theorem 3.1.3 (Scheiderer [Sch01], Benoist [Ben22b])**.**

(i) If $n \leq 2$, $d \leq 2$, or $(n, d) = (3, 4)$, all nonnegative $f \in \mathbb{R}[x_1, \ldots, x_n]$ of degree d have no bad points.

(ii) If $n \geq 3$, $d \geq 4$ and $(n, d) \neq (3, 4)$, there exists $f \in \mathbb{R}[x_1, \ldots, x_n]$ nonnegative of degree d with a bad point.

In three variables, which is the first case of interest, all known examples of bad points shared striking common features. On the one hand, they were all real points. It was asked by Scheiderer in [Sch00, Remark 1.4 2] whether all bad points of a nonnegative $f \in \mathbb{R}[x, y, z]$ are real. On the other hand, if the real bad point is assumed to be the origin, the polynomial f was not even a sum of squares in the ring $\mathbb{R}[[x, y, z]]$ of formal power series.

A question of Brumfiel appearing in [Del80, p. 62] asked whether this is a general phenomenon. We answer both questions in the negative.

Theorem 3.1.4 (Benoist [Ben22b]). *There exists $f \in \mathbb{R}[x, y, z]$ nonnegative with a nonreal bad point.*

Theorem 3.1.5 (Benoist [Ben22b]). *There exists $f \in \mathbb{R}[x, y, z]$ nonnegative which is a sum of squares in $\mathbb{R}[[x, y, z]]$ but has a bad point at the origin.*

To prove Theorems 3.1.4 and 3.1.5, one has to overcome the same obstacle. Let $\mathfrak{m} \subset \mathbb{R}[x, y, z]$ be the ideal of polynomials vanishing on the bad point. In both cases, one has to prove that f is not a sum of squares in $\mathbb{R}[x, y, z]_{\mathfrak{m}}$, although f is necessarily a sum of squares in all the completions of this local ring. We thus need to devise an obstruction to f being a sum of squares which is sufficiently global in nature.

Here is our solution to this difficulty. We arrange things so that f vanishes on an integral real curve $\Gamma \subset \mathbb{A}^3_{\mathbb{R}}$ such that $\Gamma(\mathbb{R})$ is infinite, hence Zariski-dense in Γ. Let $I \subset \mathbb{R}[x, y, z]_{\mathfrak{m}}$ be the ideal of Γ. If f was a sum of squares of elements of $\mathbb{R}[x, y, z]_{\mathfrak{m}}$, all these elements would have to vanish on Γ, and f would belong to I^2. To reach a contradiction, it therefore suffices to ensure that $f \notin I^2$.

This is not easy to achieve, because the positivity of f and the fact that it vanishes on Γ imply that f belongs to I^2 generically along Γ or, in other words, that f belongs to the symbolic square $I^{(2)}$ of I. Starting from a classical example of an ideal $I \subset \mathbb{R}[x, y, z]$ with $I^2 \neq I^{(2)}$, one can construct pairs (f, Γ) for which the above strategy applies, thereby proving Theorems 3.1.4 and 3.1.5.

3.2 Sums of few squares

Let us recall Pfister's quantitative improvement of Artin's theorem, which is the archetype of the sums of few squares problems which we now consider.

Theorem 3.2.1 (Pfister [Pfi67]). *Any nonnegative $f \in \mathbb{R}[x_1, \ldots, x_n]$ is a sum of 2^n squares in $\mathbb{R}(x_1, \ldots, x_n)$.*

3.2.1 Consequences of the Milnor conjectures

The main modern tool to attack sums of few squares problems are the Milnor conjectures proven by Voevodsky [Voe03]. They were first applied in this way in [CTJ91, Jan16], to prove analogues of Pfister's theorem over number fields.

Theorem 3.2.2 (Jannsen [Jan16]). *If $n \geq 2$, any nonnegative polynomial $f \in \mathbb{Q}[x_1, \ldots, x_n]$ is a sum of 2^{n+1} squares in $\mathbb{Q}(x_1, \ldots, x_n)$.*

Let us state concrete consequences of the Milnor conjectures in this spirit, that we will use in §§3.2.2–3.2.5 (see e.g. [Ben20b, Proposition 2.1], [Ben23, Proposition 6.3] and [Lam80, Chapter 11, Theorem 2.7]). In these statements, we denote by $\{f\} \in H^1(K, \mathbb{Z}/2)$ the image of an invertible element f in a field K of characteristic $\neq 2$ by the Kummer isomorphism $K^*/(K^*)^2 \xrightarrow{\sim} H^1(K, \mathbb{Z}/2)$.

Proposition 3.2.3. *Let K be a field of characteristic $\neq 2$. Fix $f \in K^*$ and $r \geq 0$. The following are equivalent:*

(i) the element f is a sum of 2^r squares in K;

(ii) one has $\{f\} \cdot \{-1\}^r = 0$ in $H^{r+1}(K, \mathbb{Z}/2)$.

If f is a sum of squares in K, then (i) and (ii) are implied by:

(iii) the groups $H^k(K[\sqrt{-1}], \mathbb{Z}/2)$ vanish for $k \geq r+1$.

Proposition 3.2.4. *Let K be a field of characteristic $\neq 2$. Fix $f \in K^*$ and $r \geq 0$. The following are equivalent:*

(i) the element f is a sum of $2^r - 1$ squares in K;

(ii) one has $\{-1\}^r = 0$ in $H^r(K[\sqrt{-f}], \mathbb{Z}/2)$.

In real algebraic geometry, these statement are usually used in conjunction with Artin's comparison theorem between étale and Betti cohomology of complex algebraic varieties [SGA43] (or, more precisely, with its extension by Cox to a comparison theorem between étale and equivariant Betti cohomology of real algebraic varieties [Cox79]).

3.2.2 Sums of 3 squares and Noether–Lefschetz loci

The Cassels–Ellison–Pfister theorem [CEP71] states that Pfister's Theorem 3.2.1 is optimal in $n = 2$ variables. More precisely, it is proven in [CEP71] that Motzkin's polynomial (3.1) is not a sum of 3 squares in $\mathbb{R}(x, y)$.

As we explained in §0.2.3, Colliot-Thélène has provided in [CT93] an alternative proof of the Cassels–Ellison–Pfister theorem based on the Noether–Lefschetz theorem. The main idea of [Ben18] is that one can use finer Hodge-theoretic results (namely, density results for Noether–Lefschetz loci) to also obtain abundance results for sums of 3 squares.

Theorem 3.2.5 (Benoist [Ben18])**.** *Fix $d \geq 2$ even. Let $\Pi_d \subset \mathbb{R}[x, y]_d$ be the set of nonnegative real polynomials of degree $\leq d$. The set of those polynomials that are sums of 3 squares in $\mathbb{R}(x, y)$ is analytically dense in Π_d.*

Let $f \in \Pi_d \subset \mathbb{R}[x, y]_d$ be a nonnegative polynomial. Assume that the homogenization $F \in \mathbb{R}[X, Y, Z]$ of F defines a smooth plane curve $C \subset \mathbb{P}^2_{\mathbb{R}}$, and let $S \to \mathbb{P}^2_{\mathbb{R}}$ be the double cover ramified along C with equation

$$S := \{W^2 + F(X, Y, Z) = 0\}.$$

It follows from Proposition 3.2.4 that f is a sum of 3 squares in $\mathbb{R}(x,y)$ if and only if $\{-1\}^2 \in H^2_{\text{ét}}(S, \mathbb{Z}/2)$ vanishes on a Zariski-open subset of S; equivalently, if and only if this class is algebraic. This happens if and only if its image in $H^2_G(S(\mathbb{C}), \mathbb{Z}/2)$ by Cox's comparison theorem between étale cohomology and equivariant Betti cohomology (see [Cox79]) is algebraic. In view of the real Lefschetz $(1,1)$ theorem (see Theorem 1.2.3), such is the case if and only if this image lifts to an integral Hodge class in $\mathrm{Hdg}^2(S(\mathbb{C}), \mathbb{Z}(1))$.

This property is of course influenced by the Hodge theory of the surface S, and the above analysis allows us to completely understand how it depends on the polynomial f. When the coefficients of f vary, the surface S varies in an algebraic family, and one can consider the G-equivariant weight 2 variation of Hodge structures $\mathbb{H}^2_{\mathbb{Q}}$ (in the sense of §1.3.5) modelled on the primitive cohomology groups $H^2(S(\mathbb{C}), \mathbb{Q}(1))_{\text{prim}}$. The polynomial f is then a sum of 3 squares in $\mathbb{R}(x,y)$ exactly along some real Noether–Lefschetz loci of the parameter space of the family.

To prove Theorem 3.2.5, it remains to apply the density criterion for real Noether–Lefschetz loci given by Proposition 1.3.7. To do so, we take inspiration from the complex work of Ciliberto and Lopez [CL77] (as we already did in §1.3.6): we choose the class λ required in the hypotheses of Proposition 1.3.7 to be the class of an algebraic cycle, and more precisely, the class of a determinantal curve $\Gamma \subset S$ (for a particular choice of S).

3.2.3 Low degree equations

In two variables, Hilbert's Theorem 3.1.1 shows that Pfister's 2^n bound may always be improved for equations of degree $d \leq 4$. It is natural to wonder whether this can also be done for polynomials in more variables. In this direction, we obtain the following result.

Theorem 3.2.6 (Benoist [Ben17])**.** *Let $f \in \mathbb{R}[x_1, \dots, x_n]$ be nonnegative of degree d. If $d \leq 2n-2$, or if $d = 2n$ and n is even or equal to 3 or 5, then f is a sum of $2^n - 1$ squares in $\mathbb{R}(x_1, \dots, x_n)$.*

It was pointed out to us by Leep that the $d \leq 2n-2$ case of Theorem 3.2.6 can also be deduced from results of him and Becher [Lee09, BL11].

Let us now sketch our proof of Theorem 3.2.6. Exactly as in §3.2.2, we let $F \in \mathbb{R}[X_0, \dots, X_n]$ denote the homogenization of f, and we consider the double cover $X \to \mathbb{P}^n_{\mathbb{R}}$ defined by the equation

$$X := \{W^2 + F(X_0, \dots, X_n) = 0\}.$$

We assume henceforth that the hypersurface $\{F = 0\}$ is smooth, and hence that so is X. (The case where these varieties are singular can be reduced to the smooth case by a degeneration argument based on the use of nonarchimedean real closed fields.)

At this point, we may explain the role of the hypothesis that $d \leq 2n$ in Theorem 3.2.6: it is equivalent to the rational connectedness of the real algebraic variety X. This shows that it is a very natural hypothesis from the geometric point of view. One might therefore dream that Pfister's bound becomes optimal as soon as $d \geq 2n+2$. It also becomes reasonable to expect that Theorem 3.2.6 could be extended to the case where $d = 2n$ and $n \geq 7$ is odd (note however that it would not hold for $n = 1$ and $d = 2$). We are unable to prove this at present.

Let us resume the proof of Theorem 3.2.6. By the criterion of Proposition 3.2.4, we have to show that, under our hypotheses on d, the class $\{-1\}^n \in H^n(X, \mathbb{Z}/2)$ vanishes in restriction to a Zariski-open subset of X or, in other words, that it vanishes in the unramified cohomology group $H^n_{\mathrm{nr}}(X, \mathbb{Z}/2)$.

A first idea is to work with integral coefficients instead of torsion coefficients (we use 2-adic étale cohomology here, but equivariant Betti cohomology would also work). Let us denote by ω the lift of $\{-1\}$ by the reduction modulo 2 isomorphism $H^1(\mathbb{R}, \mathbb{Z}_2(1)) \xrightarrow{\sim} H^1(\mathbb{R}, \mathbb{Z}/2)$. We actually prove the (seemingly) stronger vanishing of the class $\omega^n \in H^n_{\mathrm{nr}}(X, \mathbb{Z}_2(n))$.

To this effect, the main tool we use is Bloch–Ogus theory [BO74]. Based on the Milnor conjectures, we produce an exact sequence

$$(3.2)\quad H^n_{\mathrm{nr}}(X_{\mathbb{C}}, \mathbb{Z}_2) \to \{\alpha \in H^n_{\mathrm{nr}}(X, \mathbb{Z}_2(n)) \mid \alpha\cdot\omega = 0\} \to H^1(X, \mathcal{H}^n(\mathbb{Z}_2(n+1))).$$

The left group $H^n_{\mathrm{nr}}(X_{\mathbb{C}}, \mathbb{Z}_2)$ of (3.2) vanishes by a combination of the Milnor conjectures and of a decomposition of the diagonal argument due to Bloch–Srinivas and Colliot-Thélène–Voisin [BS83, CTV12] (this is the only place where the rational connectedness of X is used).

We then show by a direct geometric argument that the cohomology class $\omega^{n+1} \in H^{n+1}_{\text{ét}}(X, \mathbb{Z}_2(n+1))$ has coniveau ≥ 2 (in many cases, one even has $\omega^{n+1} = 0$ on the nose). This implies that ω^n belongs to the middle group $\{\alpha \in H^n_{\mathrm{nr}}(X, \mathbb{Z}_2(n)) \mid \alpha \cdot \omega = 0\}$ of (3.2). As the image of ω^n in the right group of (3.2) is precisely the obstruction for ω^{n+1} to have coniveau ≥ 2 (by Bloch–Ogus theory), which we know vanishes, we deduce from the exactness of (3.2) that $\omega^n \in H^n_{\mathrm{nr}}(X, \mathbb{Z}_2(n))$ vanishes, as wanted.

3.2.4 Sums of squares of real power series

In this paragraph, we study a local variant of Hilbert's 17th problem. Let $\mathbb{R}\{\{x_1, \ldots, x_n\}\}$ denote the ring of convergent real power series in n variables. An analogue of Artin's theorem in this context has been proven by Risler [Ris76].

Theorem 3.2.7 (Risler [Ris76])**.** *An element $f \in \mathbb{R}\{\{x_1, \ldots, x_n\}\}$ is a sum of squares in* $\mathrm{Frac}(\mathbb{R}\{\{x_1, \ldots, x_n\}\})$ *if and only if it is nonnegative in some neighborhood of the origin.*

Such elements will be said to be nonnegative. To obtain a quantitative analogue à la Pfister of Risler's theorem, one can apply Proposition 3.2.3 (iii)$\Rightarrow$(i). As the field $\mathrm{Frac}(\mathbb{C}\{\{x_1, \dots, x_n\}\})$ has cohomological dimension n, we see that a nonnegative $f \in \mathbb{R}\{\{x_1, \dots, x_n\}\}$ is a sum of 2^n squares in $\mathrm{Frac}(\mathbb{R}\{\{x_1, \dots, x_n\}\})$.

It is not hard to see that a nonnegative $f \in \mathbb{R}\{\{x\}\}$ is in fact a square, and Choi, Dai, Lam and Reznick [CDLR82] proved that a nonnegative $f \in \mathbb{R}\{\{x_1, x_2\}\}$ is a sum of 2 squares. This led them to conjecture that for all $n \geq 1$, a nonnegative $f \in \mathbb{R}\{\{x_1, \dots, x_n\}\}$ is a sum of 2^{n-1} squares in $\mathrm{Frac}(\mathbb{R}\{\{x_1, \dots, x_n\}\})$. This was confirmed by Hu [Hu15] when $n = 3$, and proved by us in general.

Theorem 3.2.8 (Benoist [Ben20b]). *Fix $n \geq 1$. Any nonnegative element $f \in \mathbb{R}\{\{x_1, \dots, x_n\}\}$ is a sum of 2^{n-1} squares in $\mathrm{Frac}(\mathbb{R}\{\{x_1, \dots, x_n\}\})$.*

We actually prove a more general theorem, where $\mathbb{R}\{\{x_1, \dots, x_n\}\}$ is replaced by an arbitrary n-dimensional regular excellent Henselian local ring whose residue field κ has characteristic 0. The bound 2^{n-1} then has to be replaced by $2^{n+\delta-1}$, where δ is the cohomological dimension of $\kappa[\sqrt{-1}]$. As the proof of Theorem 3.2.8 already illustrates the main difficulties that have to be overcome, we do not consider this greater generality in what follows.

Theorem 3.2.8 follows from a closely related companion theorem. Pfister has defined the *level* $s(K)$ of a field K to be the smallest integer s such that -1 is a sum of s squares in K (or $+\infty$ if -1 is not a sum of squares in K, which happens exactly when K is *formally real*, i.e., when K admits a field ordering). He showed in [Pfi65] that this invariant is always a power of 2 when finite. The next theorem is due to Hu [Hu15] when $n \leq 2$ and to us in general.

Theorem 3.2.9 (Benoist [Ben20b]). *Fix $n \geq 1$. Let L be a finite extension of $\mathrm{Frac}(\mathbb{R}\{\{x_1, \dots, x_n\}\})$ that is not formally real. Then $s(L) \leq 2^{n-1}$.*

We set $A := \mathbb{R}\{\{x_1, \dots, x_n\}\}$ and $K := \mathrm{Frac}(A)$. Define $S := \mathrm{Spec}(A)$, and let $s \in S$ be the closed point of S.

Let us first explain why Theorem 3.2.9 implies Theorem 3.2.8. Let $f \in A$ be nonnegative. By Proposition 3.2.3 (i)$\Leftrightarrow$(ii), one has to show that $\{f\} \cdot \{-1\}^{n-1} = 0$ in $H^n(K, \mathbb{Z}/2)$. Let us first verify that this class is unramified over S. Its only possible nonzero residues are along divisors $D \subset S$ at which f vanishes with odd multiplicity, and these residues are equal to $\{-1\}^{n-1}$. Moreover, the function fields of the divisors that occur cannot be formally real by nonnegativity of f (otherwise, the function f would change sign across the real points of D). It therefore follows from Theorem 3.2.9 that all these residues vanish, and hence that $\{f\}\cdot\{-1\}^{n-1}$ belongs to $H^n_{\mathrm{nr}}(S, \mathbb{Z}/2)$. In view of the isomorphisms $H^n_{\text{ét}}(S, \mathbb{Z}/2) \xrightarrow{\sim} H^n_{\mathrm{nr}}(S, \mathbb{Z}/2)$ and $H^n_{\text{ét}}(S, \mathbb{Z}/2) \xrightarrow{\sim} H^n(\mathbb{R}, \mathbb{Z}/2)$ induced by Panin's solution of the Gersten conjecture in this context [Pan03], and by the proper base change theorem, the class $\{f\}\cdot\{-1\}^{n-1}\in H^n_{\mathrm{nr}}(S,\mathbb{Z}/2)$ is induced by a class in $H^n(\mathbb{R},\mathbb{Z}/2)$. The nonnegativity of f prevents it from being induced by the nonzero class $\{-1\}^n\in H^n(\mathbb{R},\mathbb{Z}/2)$, and the only possibility is that $\{f\}\cdot\{-1\}^{n-1}=0$.

Let us now present the idea of the proof of Theorem 3.2.9. By resolution of singularities, one can find a projective morphism $\pi: X\to S$ such that X is connected, regular, with strict normal crossings reduced special fiber $(X_s)^{\mathrm{red}}$, and with function field equal to L. Applying Proposition 3.2.3 (i)$\Leftrightarrow$(ii) again, one sees that one has to show that $\{-1\}^n\in H^n(L,\mathbb{Z}/2)$ vanishes. In other words, one must show that the class $\{-1\}^n\in H^n_{\text{ét}}(X,\mathbb{Z}/2)$ vanishes in restriction to some Zariski-open subset $U\subset X$. It turns out to be very easy to concretely construct such a Zariski-open subset: define $U:=X\setminus D$, where $D\subset X$ is any π-ample divisor such that $D\cup(X_s)^{\mathrm{red}}$ is a strict normal crossings divisor in X. To verify that U works, one computes

$$H^n_{\text{ét}}(U,\mathbb{Z}/2)\xrightarrow{\sim}H^n_{\text{ét}}(U_s,\mathbb{Z}/2)=0. \tag{3.3}$$

The isomorphism in (3.3) is an adaptation in our setting of a (non-proper) base change theorem due to Saito and Sato in a p-adic context [SS10]. The vanishing in (3.3) is a manifestation of the weak Lefschetz theorem; it holds because U_s is a real affine variety of dimension $n-1$ with no real points (as L is not formally real by hypothesis).

3.2.5 Sums of squares of real-analytic functions

The real-analytic analogue of Hilbert's 17th problem (Question 0.1.4) is open in general. However, an analogue of Artin's theorem in this context has been proven by Jaworski, under a compactness hypothesis (we state a slightly more general result than the one appearing in [Jaw86], possibly authorizing singularities, which can be proven in the exact same way).

Theorem 3.2.10 (Jaworski [Jaw86])**.** *Let M be a normal real-analytic space, let $K\subset M$ be a compact subset, and let $f: M\to\mathbb{R}$ be a nonnegative real-analytic function. Then f is a sum of squares of real-analytic meromorphic functions in some neighborhood of K in M.*

No quantitative result was known in this direction (unless M is a surface, in which case f was known to be a sum of 5 squares [ABFR05], and even of 3 squares if M is a manifold [Jaw82]). We show that, in the generality of Theorem 3.2.10, Pfister's 2^n bound holds.

Theorem 3.2.11 (Benoist [Ben23])**.** *Let M be a normal real-analytic space of dimension n, let $K\subset M$ be a compact subset, and let $f: M\to\mathbb{R}$ be a nonnegative real-analytic function. Then f is a sum of 2^n squares of real-analytic meromorphic functions in some neighborhood of K in M.*

In keeping with the motto that real geometry is complex geometry done equivariantly with respect to the action of the group $G := \mathrm{Gal}(\mathbb{C}/\mathbb{R})$ generated by complex conjugation, it is crucial for the proof of Theorem 3.2.11 to work in the complex-analytic context. The natural setting is that of *Stein spaces*: the complex-analytic analogues of affine varieties. They are those complex-analytic spaces S on which the higher cohomology groups of all coherent sheaves vanish. A compact subset $K \subset S$ is said to be *Stein* if it admits a basis of Stein open neighborhoods in S. We denote by $\mathcal{O}(K)$ and $\mathcal{M}(K)$ the rings of germs of holomorphic and meromorphic functions in a neighborhood of K. Our complex-analytic enhancement of Theorem 3.2.11 is the following theorem.

Theorem 3.2.12 (Benoist [Ben23]). *Let S be a normal Stein space of dimension n on which G acts through an antiholomorphic involution. Fix a G-invariant Stein compact subset $K \subset X$. Any $f \in \mathcal{O}(K)^G$ which is nonnegative on a neighborhood of K^G in S^G is a sum of 2^n squares in $\mathcal{M}(K)^G$.*

Here is a typical example of application of Theorem 3.2.12. Choose $S = \mathbb{C}^n$ endowed with the involution $z \mapsto \bar{z}$, and let $K \subset S$ be the closed unit ball. Then $\mathcal{O}(K)$ is the ring of power series $\sum_I a_I \underline{z}^I$ in $z_1, \ldots, z_n$ with complex coefficients that have radius of convergence > 1 (i.e. that converge in some neighborhood of K), and $\mathcal{O}(K)^G$ is the subring of those that have real coefficients. Theorem 3.2.12 then asserts that any $f \in \mathcal{O}(K)^G$ that takes nonnegative values in a neighborhood of the closed unit ball in $\mathbb{R}^n$ is a sum of 2^n squares in the fraction field $\mathcal{M}(K)^G$ of $\mathcal{O}(K)^G$.

Theorem 3.2.12 follows from Theorem 3.2.11 thanks to the works of Cartan, Grauert and Tognoli establishing the existence of Stein complexifications of normal real-analytic spaces (see [Car57, Gra58, Tog67]).

Theorem 3.2.12 reduces to the following cohomological dimension computation, thanks to the criterion of Proposition 3.2.3 (iii)$\Rightarrow$(i).

Theorem 3.2.13 (Benoist [Ben23]). *Let S be a normal Stein space of dimension n and let $K \subset S$ be a connected Stein compact subset. Then the field $\mathcal{M}(K)$ has cohomological dimension n.*

The conclusion of Theorem 3.2.13 means that the cohomology of the absolute Galois group of $\mathcal{M}(K)$ with value in any finite Galois-module vanishes in degree $> n$. The strategy of its proof is to exploit the fact, due to Hamm [Ham83] and based on Morse theory, that a Stein space of dimension n has the homotopy type of a finite CW-complex of dimension n, and hence that its singular cohomology vanishes in degree $> n$. To conclude, it remains to prove a theorem comparing étale cohomology (which generalizes Galois cohomology) and singular cohomology.

In algebraic geometry, such a comparison theorem is due to Mike Artin (see [SGA43]). We prove the following analytic counterpart in Stein geometry (more precisely, in relative algebraic geometry over a Stein compactum K).

Theorem 3.2.14 (Benoist [Ben23])**.** *Let S be a Stein space. Let X be an $\mathcal{O}(S)$-scheme of finite type and let $\mathbb{L}$ be a constructible torsion étale abelian sheaf on X. If one lets U run over all Stein open neighborhoods of a Stein compact subset K of S, the change of topology morphisms*

$$\operatorname*{colim}_{K\subset U} H^k_{\text{ét}}(X_{\mathcal{O}(U)}, \mathbb{L}_{\mathcal{O}(U)}) \to \operatorname*{colim}_{K\subset U} H^k((X_{\mathcal{O}(U)})^{\text{an}}, \mathbb{L}^{\text{an}}) \tag{3.4}$$

are isomorphisms for $k \geq 0$.

The known proofs of Artin's comparison theorem in algebraic geometry are based on fibration arguments (to somehow reduce to the case of curves that can be dealt with by a direct computation). Unfortunately, we do not know how to implement such fibrations arguments in Stein geometry, and one has to devise a new strategy of proof.

Appropriate dévissage arguments allow us to reduce to the case where $X = \operatorname{Spec}(\mathcal{O}(S))$ and $\mathbb{L} = \mathbb{Z}/m$. One then has to show that the morphisms

$$\operatorname*{colim}_{K\subset U} H^k_{\text{ét}}(\operatorname{Spec}(\mathcal{O}(U)), \mathbb{Z}/m) \to \operatorname*{colim}_{K\subset U} H^k(U, \mathbb{Z}/m)$$

are isomorphisms for $k \geq 0$. To compare the étale topology on $\operatorname{Spec}(\mathcal{O}(U))$ and the classical topology on U, we make use of the Leray spectral sequences associated to the morphisms of sites $\varepsilon_U : (X_{\mathcal{O}(U)})^{\text{an}} \to (X_{\mathcal{O}(U)})_{\text{ét}}$. To conclude, we need to show that the morphism $\mathbb{Z}/m \to \varepsilon_{U,*}\mathbb{Z}/m$ is an isomorphism and that the higher direct images $\mathrm{R}^q\varepsilon_{U,*}\mathbb{Z}/m$ vanish for $q \geq 1$, at least after taking the colimit over all possible U.

The first assertion follows from Bingener's relative GAGA theorem over a Stein compactum [Bin76]. The second assertion requires to show that singular cohomology classes on U (or more generally on analytifications of étale $\mathcal{O}(U)$-schemes) are étale-locally trivial. This is the heart of the proof. Our strategy to attack it is to use Grauert's bump method of exhausting U by Stein compacta with controlled topological and analytical properties, as developed by Henkin and Leiterer, and Forstnerič [HL98, For17].

In order to remove the compactness hypotheses in Theorems 3.2.11 and 3.2.12, one would like to answer positively the following question.

Question 3.2.15. Let S be a connected normal Stein space of dimension n. Does the field $\mathcal{M}(S)$ have cohomological dimension n?

We were unable to use our strategy of proof of Theorem 3.2.13 to answer Question 3.2.15. One reason is that Theorem 3.2.14 does not hold without a compactness hypothesis: the morphisms appearing in (3.4) are not always isomorphisms if one does not take the colimit over all Stein open neighborhoods U of K.

Chapter 4

Rationality and intermediate Jacobians

A variety over a field k is said to be *k-rational* if it is birational to a projective space; equivalently if its function field is k-isomorphic to $k(x_1, \ldots, x_n)$.

Over the complex numbers (and more generally over algebraically closed fields), several techniques have been developed to detect that a variety is not $\mathbb{C}$-rational. The most prominent ones, discovered almost simultaneously, make use of the Brauer group (often in the guise of torsion in H^3) or more generally of the unramified cohomology (Artin–Mumford [AM72]), of the group of birational automorphisms (Iskovskikh–Manin [IM71]) and of the intermediate Jacobian (Clemens–Griffiths [CG72]) of the variety.

Over a nonclosed field k with algebraic closure $\overline{k}$, deciding which $\overline{k}$-rational varieties are k-rational is a problem of arithmetic interest. Obstructions induced by the group of birational automorphisms or by the Brauer group have been used to obstruct the k-rationality of $\overline{k}$-rational varieties, already in dimension 2 (see [Seg51, Man66]).

In this chapter, we explain that it is also possible to use intermediate Jacobians to this effect, and we derive new examples of non-k-rational varieties as a consequence. As the Clemens–Griffiths method only applies to varieties of dimension 3, we henceforth consider a smooth projective $\overline{k}$-rational threefold X over a field k.

4.1 The intermediate Jacobian

4.1.1 Construction methods

Over the complex numbers, the intermediate Jacobian of X is classically constructed by the transcendental Hodge-theoretic method of Griffiths. A construction over an arbitrary algebraically closed field, based on the remark that the intermediate Jacobian of X parametrizes codimension 2 algebraic cycles on X, has been provided by Murre [Mur85]. These constructions have then been showed by Achter, Casalaina–Martin and Vial [ACMV17] to descend over the base field k, if k is perfect.

Unfortunately, we do not know how to implement these constructions if the field k is imperfect, a generality which may be important in applications (see Theorem 4.2.6 and the discussion below).

To covercome this difficulty, Wittenberg and I give in [BW23] an entirely new construction of the intermediate Jacobian of X, which is applicable over an arbitrary field k. As Murre, we take the point of view that the intermediate Jacobian of X is a parameter space for codimension 2 algebraic cycles on X. Our idea, inspired by Grothendieck's construction of the Picard scheme in the case of codimension 1 cycles, is then to define the intermediate Jacobian through its functor of points.

4.1.2 The functor $\mathrm{CH}^2_{X/k}$

Let (Sch) be the category of quasi-compact and quasi-separated k-schemes and let (Ab) be the category of abelian groups. Defining a functor

$$\mathrm{CH}^2_{X/k} : (\mathrm{Sch})^{\mathrm{op}} \to (\mathrm{Ab})$$

which intuitively parametrizes codimension 2 cycles on X is not an easy task, as Chow groups are not contravariantly functorial with respect to arbitrary morphisms. This is the reason why Grothendieck's Picard scheme does not really parametrize codimension 1 cycles, but rather line bundles, which can be thought of as a cohomological counterpart of codimension 1 cycles.

We are therefore led to use a cohomological variant of codimension 2 cycles. Among several possibilities, we chose to base our definition on K-theory. The idea is that on a smooth projective threefold X, the Grothendieck–Riemann–Roch theorem implies that $K_0(X_{\overline{k}})$ is (at least rationally) built from cycles of codimension 0, 1, 2 and 3. Substracting the contributions of cycles of codimension 0, 1 and 3, there only remains the contribution of codimension 2 cycles. Moreover, this may even be made to work integrally, thanks to Jouanolou's Riemann–Roch theorem without denominators [Jou70].

Let us now proceed to give the definition of the functor $\mathrm{CH}^2_{X/k}$ associated with a smooth projective $\overline{k}$-rational threefold X over k. We first define three functors $(\mathrm{Sch})^{\mathrm{op}} \to (\mathrm{Ab})$ by setting

$$\begin{aligned} \mathbb{Z}_{X/k}(T) &= \mathbb{Z}(X \times_k T) \\ \mathrm{Pic}_{X/k}(T) &= \mathrm{Pic}(X \times_k T) \\ \mathrm{K}_{0,X/k}(T) &= \mathrm{K}_0(X \times_k T), \end{aligned}$$

and we let $\mathbb{Z}_{X/k,\mathrm{fppf}}, \mathrm{Pic}_{X/k,\mathrm{fppf}}$ and $\mathrm{K}_{0,X/k,\mathrm{fppf}}$ denote their fppf sheafifications. To remove from $\mathrm{K}_{0,X/k,\mathrm{fppf}}$ the contribution of cycles of codimension ≤ 1, we further define

$$\mathrm{SK}_{0,X/k,\mathrm{fppf}} := \mathrm{Ker}\big[\mathrm{K}_{0,X/k,\mathrm{fppf}} \xrightarrow{(\mathrm{rk},\det)} \mathbb{Z}_{X/k,\mathrm{fppf}} \times \mathrm{Pic}_{X/k,\mathrm{fppf}}\big]$$

to be the kernel of the morphism given by the rank and the determinant of vector bundles. To remove the contribution of codimension 3 cycles, we prove that there exists a unique morphism $\nu_X : \mathbb{Z}_{X/k,\mathrm{fppf}} \to \mathrm{SK}_{0,X/k,\mathrm{fppf}}$ such that the image of $\nu_X(1)$ in $K_0(X_{\overline{k}})$ is the class $[\mathcal{O}_{\{x\}}] \in K_0(X_{\overline{k}})$ of the structure sheaf of any $\overline{k}$-point $x \in X(\overline{k})$. We then set

$$\mathrm{CH}^2_{X/k} := \mathrm{Coker}\big[\mathbb{Z}_{X/k,\mathrm{fppf}} \xrightarrow{\nu_X} \mathrm{SK}_{0,X/k,\mathrm{fppf}}\big].$$

This definition has been fine-tuned so there exists a natural isomorphism

$$\mathrm{CH}^2(X_{\overline{k}}) \xrightarrow{\sim} \mathrm{CH}^2_{X/k}(\overline{k}). \tag{4.1}$$

4.1.3 Representability

Here is our main representability theorem concerning $\mathrm{CH}^2_{X/k}$. We let $\mathrm{CH}^2(X_{\overline{k}})_{\mathrm{alg}} \subset \mathrm{CH}^2(X_{\overline{k}})$ denote the subgroup of algebraically trivial cycles, and we define the Néron–Severi group $\mathrm{NS}^2(X_{\overline{k}}) := \mathrm{CH}^2(X_{\overline{k}})/\mathrm{CH}^2(X_{\overline{k}})_{\mathrm{alg}}$ to be the quotient.

Theorem 4.1.1 (Benoist–Wittenberg [BW23]). *Let X be a smooth $\overline{k}$-rational threefold over a field k.*

(i) There exists a smooth k-group scheme $\mathbf{CH}^2_{X/k}$ representing $\mathrm{CH}^2_{X/k}$.

(ii) The identity component $(\mathbf{CH}^2_{X/k})^0$ of $\mathbf{CH}^2_{X/k}$ is an abelian variety.

(iii) The map (4.1) restricts to a bijection $\mathrm{CH}^2(X_{\overline{k}})_{\mathrm{alg}} \xrightarrow{\sim} (\mathbf{CH}^2_{X/k})^0(\overline{k})$.

To prove Theorem 4.1.1, one may use fppf descent to replace k by any finite extension of it (a clear benefit of the functorial point of view). We may thus assume that X is k-rational. This also allows us to use a resolution of indeterminacies result of Abhyankar [Abh98] to find a diagram

$$X \xleftarrow{q} X_N \to \cdots \to X_j \xrightarrow{p_j} X_{j-1} \to \cdots \to X_0 = \mathbb{P}^3_k \tag{4.2}$$

of smooth projective varieties over k, where q is birational and the p_j are blow-ups along smooth centers. Starting from the basic case of $X_0 = \mathbb{P}^3_k$, we show by induction on j that Theorem 4.1.1 holds for X_j, and then deduce Theorem 4.1.1 for X from its validity for X_N.

We now define the *intermediate Jacobian* of X to be the abelian variety $(\mathbf{CH}^2_{X/k})^0$. In view of Theorem 4.1.1 (iii), the group of geometric connected components of $\mathbf{CH}^2_{X/k}$ may be identified with $\mathrm{NS}^2(X_{\overline{k}})$. If $\gamma \in \mathrm{NS}^2(X_{\overline{k}})^{\mathrm{Aut}(\overline{k}/k)}$, we let $(\mathbf{CH}^2_{X/k})^\gamma$ denote the corresponding connected component of $\mathbf{CH}^2_{X/k}$. These $(\mathbf{CH}^2_{X/k})^0$-torsors also play a role in rationality problems.

The Clemens–Griffiths method requires the construction of an intermediate Jacobian not only as an abelian variety, but as a principally polarized abelian variety. The above strategy of following step by step diagram (4.2) allows us to also construct the desired principal polarization on $(\mathbf{CH}^2_{X/k})^0$.

4.2 Applications to rationality problems

4.2.1 Obstructions to rationality

Here is the most general obstruction to rationality that we derive from our construction.

Theorem 4.2.1 (Benoist–Wittenberg [BW23]). *Let X be a smooth k-rational threefold over a field k. There exists a smooth projective curve C over k such that the k-group scheme $\mathbf{CH}^2_{X/k}$ is a direct factor of $\mathbf{Pic}_{C/k}$, in a way compatible with the canonical principal polarizations.*

The proof of Theorem 4.2.1 again makes use of diagram (4.2), which indeed exists over the base field k if k is perfect. In this case, one can (almost) choose C to be the disjoint union of the smooth projective curves that are blown-up in this diagram.

If the base field k is imperfect, diagram (4.2) still exists by work of Cossart and Piltant [CP08], but the projective varieties that appear in it may only be regular (and not smooth), and the centers of the blow-ups p_j may also only be regular (and not smooth). Note however that the curve C in the statement of Theorem 4.2.1 is required to be smooth. Consequently, in order to prove Theorem 4.2.1 as stated, one has to show that the nonsmooth regular curves that are blown up in (4.2) do not contribute to $\mathbf{CH}^2_{X/k}$, and hence that they can be discarded. This highly nontrivial fact relies on a detailed study of which Jacobians of reduced projective curves over k split as a direct product of an affine k-group scheme and of an abelian variety over k.

One can deduce from Theorem 4.2.1 more concrete obstructions to rationality. Here are the two that we will use later.

Theorem 4.2.2 (Benoist–Wittenberg [BW23]). *Let X be a smooth k-rational threefold over a field k.*

(i) There exists a smooth projective curve C over k and an isomorphism $(\mathbf{CH}^2_{X/k})^0 \simeq \mathbf{Pic}^0_{C/k}$ of principally polarized abelian varieties over k.

(ii) Assume that there exists a curve C as in (i), which is geometrically connected of genus ≥ 2. Then, for all $\gamma \in \mathrm{NS}^2(X_{\overline{k}})^{\mathrm{Aut}(\overline{k}/k)}$, there exists $d \in \mathbb{Z}$ such that $(\mathbf{CH}^2_{X/k})^\gamma \simeq \mathbf{Pic}^d_{C/k}$ as $(\mathbf{CH}^2_{X/k})^0 \simeq \mathbf{Pic}^0_{C/k}$-torsors.

4.2.2 Conic bundles

It was first noticed in [BW20a] that Theorem 4.2.2 (i) may lead to examples of $\overline{k}$-rational varieties that are not k-rational. Indeed, it may happen that the intermediate Jacobian $(\mathbf{CH}^2_{X/k})^0$ of a smooth projective $\overline{k}$-rational threefold is not isomorphic to the Jacobian of a smooth projective curve over k, despite necessarily being so over $\overline{k}$.

Here are simple examples that are obtained in this way, as an application of Theorem 4.2.2 (i).

Theorem 4.2.3 (Benoist–Wittenberg [BW20a])**.** *Let k be a field.*

(i) If k has characteristic $\neq 2$ and $\alpha \in k^* \setminus (k^*)^2$, the variety defined by the affine equation $\{s^2 - \alpha t^2 = x^4 + y^4 + 1\}$ is k-unirational and $k(\sqrt{\alpha})$-rational but not k-rational.

(ii) If k has characteristic 2 and $\beta \in \overline{k} \setminus k$ is such that $\alpha := \beta^2 + \beta \in k$, the variety defined by the affine equation $\{s^2 + st + \alpha t^2 = x^3 y + y^3 + x\}$ is k-unirational and $k(\beta)$-rational but not k-rational.

When $k = \mathbb{R}$ and $\alpha = -1$, Theorem 4.2.3 (i) yields Theorem 0.1.11.

Let us briefly sketch how the nonrationality assertion in Theorem 4.2.3 (i) is proven (the proof of Theorem 4.2.3 (ii) is entirely similar).

Let X be a well-chosen smooth projective model of the considered variety. The intermediate Jacobian of $X_{\overline{k}}$ is computed to be the Jacobian $\mathbf{Pic}^0_{\Gamma_{\overline{k}}/\overline{k}}$ of the smooth plane quartic curve Γ with equation $\{X^4 + Y^4 + Z^4 = 0\}$. However, the actions of $\mathrm{Aut}(\overline{k}/k)$ on the $\overline{k}$-points of these two abelian varieties over $\overline{k}$ are not compatible, and it turns out that the intermediate Jacobian of X is isomorphic to a quadratic twist of $\mathbf{Pic}^0_{\Gamma/k}$ (associated with the involution $-\mathrm{Id}$ and with the field extension $k(\sqrt{\alpha})/k$). If the principally polarized abelian variety $(\mathbf{CH}^2_{X/k})^0$ were the Jacobian of a smooth projective curve C over k, a version of the Torelli theorem due to Serre (see [Lau01, Appendix]) would imply that C is also a quadratic twist of Γ. This is however impossible because no automorphism of Γ acts as $-\mathrm{Id}$ on $\mathbf{Pic}^0_{\Gamma/k}$ since Γ is not hyperelliptic. One may apply Theorem 4.2.2 (i) to conclude.

4.2.3 Complete intersections of two quadrics

That torsors under the intermediate Jacobian may lead to finer obstructions to rationality, as in Theorem 4.2.2 (ii), was first noticed by Hasset and Tschinkel in [HT21b]. There, they used these obstructions to find rationality criteria for smooth three-dimensional complete intersections of two quadrics over $\mathbb{R}$ (then over fields of characteristic 0 in [HT21a]). Thanks to the constructions of §4.1, we extend this result to arbitrary base fields.

Theorem 4.2.4 (Benoist–Wittenberg [BW23])**.** *Let k be a field and let $X \subset \mathbb{P}^5_k$ be a smooth complete intersection of two quadrics. Then X is k-rational if and only if it contains a line defined over k.*

We complement Theorem 4.2.4 with a general (separable) unirationality criterion for smooth complete intersections of two quadrics, extending earlier results appearing in [Man86, CTSSD87, Kne15].

Theorem 4.2.5 (Benoist–Wittenberg [BW23])**.** *Let X be a smooth complete intersection of two quadrics of dimension ≥ 2 over a field k. Then X is separably k-unirational if and only if it contains a k-point.*

Based on Theorems 4.2.4 and 4.2.5, we obtain a counterexample of a new kind to the Lüroth problem.

Theorem 4.2.6 (Benoist–Wittenberg [BW23])**.** *Let κ be an algebraically closed field. There exists a smooth complete intersection of two quadrics $X \subset \mathbb{P}^5_{\kappa((t))}$ which is separably $\kappa((t))$-unirational, $\kappa((t^{\frac{1}{2}}))$-rational, but not $\kappa((t))$-rational.*

Theorem 4.2.6 is particularly interesting when κ has characteristic 2. Dolgachev and Duncan [DD18] have showed that a three-dimensional smooth complete intersection of two quadrics X over a perfect field k of characteristic 2 is always k-rational. Theorem 4.2.6 shows that their result cannot be extended to imperfect base fields, even if X is assumed to have a k-point.

In addition, when κ has characteristic 2, the variety X over the field $k := \kappa((t))$ considered in Theorem 4.2.6 has a k-point, is rational over the perfect closure of k, but not over k itself. No such example was known before. The results of Segre, Iskovskikh and Manin on the rationality of surfaces over nonclosed fields (see [Isk96]) imply that no example of this kind can exist in dimension ≤ 2. As the construction of such an example really requires to distinguish between X and its base change to the perfect closure of k, the theory of intermediate Jacobians over possibly imperfect fields that we developed in Section 4.1 turns out to be absolutely essential.

Let us finally give a hint at how Theorem 4.2.4 is deduced from Theorem 4.2.2 (ii). Let $X \subset \mathbb{P}^5_k$ be a smooth complete intersection of two quadrics. Let F be the variety of lines in X and let Γ be the Albanese image of the variety of conics in X. A geometric study of the variety X and of algebraic curves lying on it shows that the degree map induces an isomorphism $\deg : \mathrm{NS}^2(X_{\overline{k}}) \xrightarrow{\sim} \mathbb{Z}$, that Γ is a geometrically connected smooth projective curve of genus 2, and that there are natural isomorphisms

$$F \xrightarrow{\sim} (\mathbf{CH}^2_{X/k})^1 \quad \text{and} \quad \mathbf{Pic}^1_{\Gamma/k} \xrightarrow{\sim} (\mathbf{CH}^2_{X/k})^2. \tag{4.3}$$

In addition, the second isomorphism in (4.3) is compatible with the canonical principal polarizations. In particular, all the varieties appearing in (4.3) may be considered as $\mathbf{Pic}^0_{\Gamma/k} \simeq (\mathbf{CH}^2_{X/k})^0$-torsors.

Assume that X is k-rational. Then, by Theorem 4.2.2 (ii), there exists $d \in \mathbb{Z}$ such that $\mathbf{Pic}^d_{\Gamma/k} \simeq (\mathbf{CH}^2_{X/k})^1$. As $\mathbf{Pic}^1_{\Gamma/k} \xrightarrow{\sim} (\mathbf{CH}^2_{X/k})^2$, it follows that $\mathbf{Pic}^{1-d}_{\Gamma/k} \simeq (\mathbf{CH}^2_{X/k})^1$. Now either d or $1-d$ is even. Since $\mathbf{Pic}^2_{\Gamma/k}$ has a k-point (the class of the canonical bundle K_Γ of Γ), it follows that the torsor $(\mathbf{CH}^2_{X/k})^1$ also has a k-point. In view of the isomorphism $F \xrightarrow{\sim} (\mathbf{CH}^2_{X/k})^1$, we deduce that $F(k) \neq \varnothing$. This exactly means that X contains a line defined over k.

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RÉSUMÉ

Le thème central de ce mémoire est l'influence des cycles algébriques sur des questions de géométrie algébrique réelle. Nous étudions une variante de la conjecture de Hodge entière pour les variétés algébriques réelles qui prend en compte la topologie de leur lieu réel. Nous considérons ensuite divers problèmes de géométrie algébrique réelle : conjecture de Lang pour les corps de fonctions réels, approximation algébrique d'objets différentiables, représentation de fonctions positives comme sommes de carrés dans l'esprit du 17ème problème de Hilbert, questions de rationalité. Les contributions que nous apportons à chacune de ces directions de recherche requièrent l'utilisation de techniques variées de la théorie des cycles algébriques.

ABSTRACT

The central theme of this memoir is the influence of algebraic cycles on questions of real algebraic geometry. We study a variant of the integral Hodge conjecture for real algebraic varieties that takes into account the topology of their real loci. We then consider several problems pertaining to real algebraic geometry: Lang's conjecture for real function fields, algebraic approximation of differentiable objects, sums of squares representations of positive functions in the spirit of Hilbert's 17th problem, rationality questions. Our contributions to each of these directions of research require the use of varied techniques of the theory of algebraic cycles.