Richard Gill "Topics in quantum statistics" March 23, 2006 (1)

Aim : estimate $\vec{\theta} = \theta \in \mathbb{R}^p$

Quantum system : $\rho(\theta) \in \mathbb{C}^{d\times d}$, ≥ 0, trace $= 1$.

Measurement : M with outcomes $x \in (\mathcal{X}, \mathcal{B})$ ← σ-algebra

is a collection of matrices

$$M(B) \in \mathbb{C}^{d\times d},\ \geq 0, \quad M(\cup B_i) = \Sigma M(B_i) \text{ when } B_i \text{ disjoint}$$

$$M(\mathcal{X}) = \mathbb{1}.$$

Then $\Pr_\theta(\text{outcome} \in dx) = \text{trace}\left[\rho(\theta)\, M(dx)\right]$

Estimator of θ : $\hat{\theta}$ is a function of x

$(\hat{\theta}, M)$ is locally unbiased (l.u.) at θ_0
$$\begin{cases} E_{\theta_0} \vec{\hat{\theta}} = \vec{\theta}_0 \\ \dfrac{d}{d\theta_i} E_\theta (\hat{\theta}_j)\Big|_{\theta=\theta_0} = \delta_{ij} \end{cases}$$

Suppose $(\hat{\theta}, M)$ is l.u. at θ_0

Define $X_i = \int_{\mathcal{X}} \left(\hat{\theta}_i(x) - \theta_{0i}\right) M(dx)$ $\quad i = 1..p$ collection of matrices

self adjoint

l.u. $\Rightarrow \begin{cases} \text{trace}(\rho_{\theta_0} X_i) = 0 \\ \text{trace}(\rho'_i(\theta_0) X_j) = \delta_{ij} \end{cases}$ $\qquad \rho'_i = \dfrac{d}{d\theta_i}\rho(\theta)\Big|_{\theta_0}$

Notation $\quad (*) \begin{cases} \text{trace } \rho_{\theta_0} \vec{X} = \vec{0} \\ \text{trace } \vec{\rho'}_{\theta_0} \otimes \vec{X} = Id. \end{cases}$ $\qquad \vec{X}, \rho'$ vector of matrices.

↑ ⚠ tensor product $\vec{u} \otimes \vec{v} = \left(u_i v_j \right)$

$$\Rightarrow \text{Var}_{\theta_0}(\hat{\theta}) \geq \left(\text{trace } \rho_{\theta_0} X_i X_j\right)_{ij=1..p} = Z(\vec{X})$$

(cf last course)

real matrix $\qquad\qquad$ hermitian matrix ≥ 0.

$$\mathcal{V}_{att} = \left\{ \text{Var}_{\theta_0}(\hat{\theta}) \text{ such that } (\hat{\theta}, M) \text{ lu at } \theta_0 \right\}$$

attainable

and we are looking at a best variance matrix in this set.

In fact we are looking at,

$\mathcal{I}_{att} = \{ I_M$: non singular Fisher information matrix for some $M \}$

$\mathcal{I}_{att}^{-1\uparrow}$ = {all the matrices larger than I_M^{-1} for some M}

then $\mathcal{I}_{att}^{-1\uparrow} = \mathcal{V}_{att}$ then we look for the best M (ie the smallest I_M^{-1}).

$\mathcal{V}_{Hol} = \{ V$: real symmetric non neg matrix such that $\exists \vec{X}$ satisfying $\boxed{*}$ and $V \geq Z_{\theta_0}(\vec{X})\}$

Holevo

and by def, $\mathcal{V}_{att} \subset \mathcal{V}_{Hol}$.

~~Last time~~ Thus if $\rho(\theta)$ is a pure state, then $\mathcal{V}_{att} = \mathcal{V}_{Hol}$

and if one has n copies $n\, \mathcal{V}_{att}^{(n)} = \mathcal{V}_{att}^{(1)}$.

MATSUMOTO (2002) For pure states ie $\rho(\theta) = |\psi(\theta)\rangle\langle\psi(\theta)|$

$\mathcal{V}_{att} = \mathcal{V}_{Hol}$ and $\mathcal{V}_{att}^{(N)} = \frac{1}{N}\mathcal{V}_{att}$.

when $\rho^{(N)}(\theta) = \rho(\theta)^{\otimes N}$ and $|\psi^{(N)}(\theta)\rangle = |\psi(\theta)\rangle^{\otimes N}$

Proof (2) We want to prove that $\mathcal{V}_{Hol} \subset \mathcal{V}_{att}$.

Suppose $V \in \mathcal{V}_{Hol}$ ie $\exists X_1 \dots X_p$ satisfying $\boxed{*}$ and $V \geq Z(\vec{X})$.

Define $|x_i\rangle = X_i|\psi\rangle$, write $|\psi_i'\rangle$ for derivative of $|\psi\rangle$ wrt θ_i.

and θ always fixed $= \theta_0$.

$\boxed{*} \Rightarrow \begin{cases} \forall i \quad \langle\psi|x_i\rangle = 0. \\ 2\,\mathrm{Re}\,\langle\psi_i'|x_j\rangle = \delta_{ij} \end{cases}$ (since $\rho = |\psi\rangle\langle\psi| \Rightarrow \rho' = |\psi'\rangle\langle\psi| + |\psi\rangle\langle\psi'|$)

$V \geq Z \Rightarrow \quad V \geq (\langle x_i|x_j\rangle)_{i,j=1,\dots,p} = Z$.

Look at $V - Z$: non neg complex matrix

Consider $\mathbb{C}^d$ as the subspace $\mathbb{C}^d \oplus 0$ of $\mathbb{C}^d \oplus \mathbb{C}^p$

then $|\psi\rangle, |\psi_i'\rangle, |x_i\rangle$ belong to $\mathbb{C}^d \oplus \mathbb{C}^p$.

$\exists \quad |u_i\rangle \in \mathbb{C}^p$ s.t. $|y_i\rangle = |x_i\rangle \oplus |u_i\rangle$

then $\langle \Psi | y_i \rangle = 0 \ \forall i$

$2 \mathrm{Re} \langle \Psi'_i | y_j \rangle = \delta_{ij} \ \forall i j$.

such that $V = (\langle y_i | y_j \rangle)_{i=1 \dots p}$

Last time

$|u_i\rangle$ were $|y_i\rangle$.

$|y_i\rangle$ were $|v_i\rangle$

Choose an onb of span$(|\Psi\rangle, |y_j\rangle \ j = 1 \dots p)$. $|j\rangle \ j = 0, \dots, p$

such that $|y_i\rangle = \sum_j |j\rangle \langle j | y_i \rangle$.

$$= \sum_j \frac{\langle j | y_i \rangle}{\langle j | \Psi \rangle} |j\rangle \langle j | \Psi \rangle .$$

such that $\langle j | \Psi \rangle \neq 0$ and $\langle j | y_i \rangle$ real and $\langle j | \Psi \rangle$ real

(this is possible since $(\langle y_i | y_j \rangle) = V$ which is a real matrix)

Let us define $Y_i = \sum_j \frac{\langle j | y_i \rangle}{\langle j | \Psi \rangle} |j\rangle \langle j|$

so $|y_i\rangle = Y_i |\Psi\rangle$.

the Y_i's are self adjoint and they commute (since they are composed of commuting projectors).

and they satisfy $\underbrace{\mathrm{tr}\, \rho_{\theta_0} \vec{Y}}_{\langle \Psi | \vec{y} \rangle} = \vec{0}$, $\underbrace{\mathrm{tr}\, \rho'_{\theta_0} \otimes \vec{Y}} = \mathrm{Id}_p$. $\boxed{*}$

since

$$(\mathrm{tr}\, \vec{\rho}' \otimes \vec{Y})_{ij} = 2 \mathrm{Re} \langle \Psi'_i | y_j \rangle$$

<u>Proposition</u>: Suppose Y_i commute and satisfy $\boxed{*}$

Then $\exists M, \hat{\theta}$ l.u. at θ_0 such that $\mathrm{Var}_{\theta_0}(\hat{\theta}) = Z(\vec{Y})$.

$$= (\mathrm{tr}\, \rho Y_i Y_j)_{ij} .$$

(since they commute then Z is real).

Recall if Y is self adjoint then $Y = \sum y \, \Pi_y$

eigenvalue — proj on eigenspace.

then one can define $f(Y) = \sum f(y) \, \Pi_y$, for any real function f.

$$E_\rho \left[f(\mathrm{meas}(Y)) \right] = \mathrm{trace} \left[\rho \cdot f(Y) \right]$$

More generally if $Y_1, \dots, Y_p$ commute then

they can be measured simultaneously and defining $g(Y_1, \dots, Y_p) = \sum g(y_1 \dots y_p)\, \Pi_{y_1} \cdots \Pi_{y_p}$.
$g: \mathbb{R}^p \to \mathbb{R}$
and then $E_\rho\Big(g\big(\text{meas}(Y_1), \dots, \text{meas}(Y_p)\big)\Big) = \text{trace}\big[\rho\, g(Y_1, \dots, Y_p)\big]$

Here the Y_i's commute so they can be measured simultaneously.
Let M be a sim meas of $Y_1 \dots Y_p$.
Let $\hat\theta = \theta_0 + \text{meas}(\vec{Y})$
Then $E_{\theta_0}(\hat\theta - \theta_0) = \text{trace}\, \rho \vec{Y} = 0$
and all the other properties follow in the same way.
So $(\hat\theta, M)$ is l.u. by $\boxed{*}$ and $\text{Var}_{\theta_0}(\hat\theta) = E_{\theta_0}(\text{meas}\, \vec{Y}^{\otimes 2})$
$= \text{tr}\, \rho\, \vec{Y} \otimes \vec{Y} = V$.

$\perp$

Recall I started with $V \in \mathcal{V}_{\text{Hol}}$ ie $V \geq Z(\vec{X})$
What I've found is $(\hat\theta, M)$ on $\mathbb{C}^d \oplus \mathbb{C}^p$ l.u. at θ_0 and with $\text{Var}_{\theta_0}(\hat\theta) = V$
Then we have to go back to $\mathbb{C}^d$.

Lemma: Consider $\mathcal{H}$ as the subspace $\mathcal{H} \oplus 0$ of $\mathcal{H} \oplus \mathcal{K}$
Let M be a measurement on $\mathcal{H} \oplus \mathcal{K}$
Suppose ρ is a state on $\mathcal{H}$ then $\rho \oplus 0$ is a state σ on $\mathcal{H} \oplus \mathcal{K}$
Define $M|_{\mathcal{H}}$ by $M|_{\mathcal{H}}(B) = \Pi_{\mathcal{H}}\, M(B)|_{\mathcal{H}}$
$= \Pi_{\mathcal{H}}\, M(B)\, \Pi_{\mathcal{H}}|_{\mathcal{H}}$.
(so they are non neg, check all the properties of the measurement
$M|_{\mathcal{H}}(\mathcal{X}) = \text{Id}_{\mathcal{H}}$)
and $\text{trace}\, \rho\, M|_{\mathcal{H}}(\cdot) = \text{trace}\, \sigma M(\cdot)$.

So $(M, \hat\theta)$ on $\mathbb{C}^d \oplus \mathbb{C}^p$ generate $M|_{\mathbb{C}^d}, \hat\theta$ on $\mathbb{C}^d$ which is still l.u. at θ_0 and with the same variance matrix.
(since one has the same probability distribution for $\hat\theta$)

$$v_{att} = V_{Hol} = \{ V : \exists\ |y_i\rangle \in \mathcal{H} \oplus \overset{\to \dim \geq p}{\mathcal{K}} \quad \begin{matrix} \langle \Psi | y_i \rangle = 0 \\ 2\mathrm{Re} \langle \Psi_i' | y_j \rangle = \delta_{ij} \end{matrix} \qquad V_{ij} = \langle y_i | y_j \rangle \}$$

(for pure states).

If one has N copies $|\Psi\rangle$ is replaced by $|\Psi\rangle^{\otimes N}$

$|\Psi_i'\rangle$ by $|\Psi_i'\rangle \otimes |\Psi\rangle \otimes \ldots \otimes |\Psi\rangle$

N terms $\left(\begin{matrix} + \ldots \\ + |\Psi\rangle \otimes \ldots \otimes |\Psi_i'\rangle \otimes \ldots \otimes |\Psi\rangle \\ \vdots \end{matrix} \right.$

$\rho = |\Psi\rangle\langle\Psi|$

I can with no loss of generality define $|\Psi(\theta)\rangle$ st $\langle \Psi_i' | \Psi \rangle = 0$.

(the new $|y_i\rangle = \cancel{\frac{|y_i\rangle^{\otimes N}}{\sqrt{N}}(?)}$)

we replace $|\tilde{\Psi}(\theta)\rangle = e^{i \,\mathrm{Im} \langle \Psi_0' | \Psi \rangle \theta} |\Psi(\theta)\rangle$.
($\dim \theta = 1$).

$$\frac{1}{N} \left(|y_i\rangle \otimes |\Psi\rangle \otimes \ldots \otimes |\Psi\rangle + (N-1) \text{ terms permutations of } |\Psi\rangle . \right).$$

~~Now observe that~~ Consider $\mathcal{H} \oplus \mathcal{K}$ as a subspace of $\mathcal{H}^{(N)} \oplus \mathcal{K}^{(N)}$ ($\dim \mathcal{K} = \dim \mathcal{K}^{(N)} = p$) by extending the mapping

$$|\Psi\rangle \longrightarrow |\Psi^{(N)}\rangle$$

$$|\Psi_i'\rangle \longrightarrow \frac{1}{\sqrt{N}} |\Psi_i^{(N)\prime}\rangle$$

Because we arranged that $\langle \Psi | \Psi_i' \rangle = 0$, the inner products between all $p+1$ vectors on the left and on the right are the same. So one can construct an isometric embedding in this way.

Now under this mapping, any feasible $|y_1\rangle, \ldots |y_p\rangle$ in $\mathcal{H} \oplus \mathcal{K}$ map to vectors, which when divided by $1/\sqrt{N}$, are feasible in $\mathcal{H}^{(N)} \oplus \mathcal{K}^{(N)}$! Hence any $V \in V_{att}$ corresponds to $\frac{1}{N} V \in V_{att}^{(N)}$.

Conversely, I can consider $\mathcal{H}^{(N)} \otimes \mathcal{K}^{(N)}$ ($\dim \mathcal{K}^{(N)} = p$) as a subspace of $\mathcal{H} \otimes \mathcal{K}$ if $\dim(\mathcal{K})$ is taken large enough. In the same way as before, any feasible $|y_1^{(N)}\rangle, \dots |y_p^{(N)}\rangle$ in $\mathcal{H}^{(N)} \otimes \mathcal{K}^{(N)}$ corresponds to vectors which, multiplied by $\sqrt{N}$, are feasible in $\mathcal{H} \otimes \mathcal{K}$. Hence any $V^{(N)} \in V_{att}^{e(N)}$ corresponds to $N V^{(N)} \in V_{att}^{e}$.

Thus $N V^{e(N)} = V^{e(1)}$