Richard Gill   Topics in quantum statistics   March 30 2006   1

Pure states (last time) $\mathcal{V}_{att} = \mathcal{V}_{Hol} = N\,\mathcal{V}_{Hol}^{(N)} = N\,\mathcal{V}_{att}^{(N)}$

= Result of Matsumoto

↳ Heuristic argument for designing separate measurements

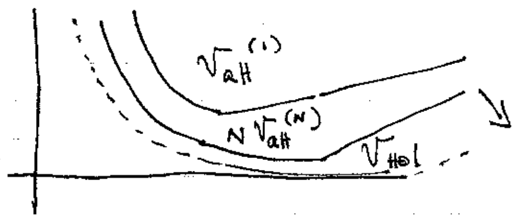
⇒ $\hat{\theta}$ is as good as the best estimator based on the best measurement.

So there exists adaptive sequential measurement schemes and estimation which are asymptotically optimal.

Mixed states $\boxed{\mathcal{V}_{Hol} = N\,\mathcal{V}_{Hol}^{(N)}}$ Hayashi and Matsumoto (2005) (based on Holevo (1982))

$\underbrace{N\,\mathcal{V}_{att}^{(N)}}$ $\subseteq N\,\mathcal{V}_{Hol}^{(N)} = \mathcal{V}_{Hol}^{(1)}$ (and not = typically)

typically these sets are increasing with $\mathcal{V}_{Hol}^{(1)}$ as limit.



Notations: $\rho(\theta)$ one state

$\rho^{(N)}(\theta) = \rho(\theta)^{\otimes N}$ N copies. and we want to measure them simultaneously.

Fix $\theta = \theta_0$ $\mathcal{V}_{Hol} = \{ V : \exists \vec{X} : \mathrm{tr}\,\rho\vec{X} = 0,\ \mathrm{tr}\,\vec{\rho}' \otimes \vec{X} = \mathrm{Id}$

$V \geq \underbrace{\mathrm{tr}\,\rho\,\vec{X}\otimes\vec{X}}_{\text{complex valued}} \}$.

real   selfadjoint

because it's ≠ it is not clear what to do.

Define
$$\langle\langle X, Y \rangle\rangle_\rho = \mathrm{tr}\,\rho X^* Y$$ for X and Y arbitrary d×d complex matrix.
$$\langle X, Y \rangle_\rho = \mathrm{Re}\left(\langle\langle X, Y\rangle\rangle_\rho\right)$$ for X, Y self adjoint

self adjoint = real vector space

so $\langle X, Y\rangle_\rho$ is a real semi-inner product on a real vector space.

(since one do not know if $\langle X, X\rangle_\rho = 0 \Leftrightarrow X = 0$)

Rmk: $\langle X, X\rangle_\rho = \langle\langle X, X\rangle\rangle_\rho$ since $\mathrm{tr}\,\rho X^* X \in \mathbb{R}$.

Introduce <u>symmetric logarithmic derivatives</u>

d_i selfadjoint $i = 1 .. p$. by demanding

$$\rho_i' = \frac{1}{2}(\rho d_i + d_i \rho) . \qquad (*)$$

If ρ is strictly positive (then $\langle\langle\ \rangle\rangle$ and $\langle\ \rangle_\rho$ are real inner products)

write out elementwise in a basis making ρ diagonal $\rho = \begin{pmatrix} p_1 & & 0 \\ & \ddots & \\ 0 & & p_d \end{pmatrix}$ $\Sigma p_i = 1$.

$$(\rho_i')_{jk} = \frac{1}{2}\left(p_j (d_i)_{jk} + (d_i)_{jk} p_k \right)$$

$$\Rightarrow \quad (d_i)_{jk} = \frac{(\rho_i')_{jk}}{\frac{1}{2}(p_j + p_k)} .$$

$\Rightarrow$ there always exists a solution to $(*)$

Note

If $\rho(\theta) = \begin{pmatrix} p_1(\theta) & & 0 \\ & \ddots & \\ 0 & & p_d(\theta) \end{pmatrix}$

then $d_i(\theta) = \frac{d}{d\theta} \log \rho(\theta) = \frac{d}{d\theta} \begin{pmatrix} \log p_1(\theta) & & 0 \\ & \ddots & \\ 0 & & \log p_d(\theta) \end{pmatrix}$

d_i are the generalización of <u>score functions</u> in classical statistics.

then $$\operatorname{tr} \rho_i' X_j = \operatorname{tr}\left[\frac{1}{2}(\rho d_i + d_i \rho) X_j\right]$$
$$= \frac{1}{2}\left[\operatorname{tr}(\rho d_i X_j) + \operatorname{tr}(d_i \rho X_j)\right]$$

But $$\overline{\operatorname{tr}(d_i \rho X_j)} = \operatorname{tr}\left[(d_i \rho X_j)^*\right] = \operatorname{tr}(X_j \rho d_i) = \operatorname{tr}(\rho d_i X_j)$$

So $$\operatorname{tr} \rho_i' X_j = \operatorname{Re} \operatorname{tr} \rho d_i X_j = \langle d_i , X_j \rangle_\rho$$

So $$\mathcal{V}_{Hol} = \left\{ V : \exists \vec{X} , \ \underset{\langle\langle \mathbb{1}, \vec{X} \rangle\rangle_\rho}{\langle \mathbb{1} , \vec{X} \rangle_\rho} = \vec{0} , \ \langle \vec{d} , \vec{X} \rangle_\rho = Id , \ V \geq \langle\langle \vec{X}, \vec{X} \rangle\rangle_\rho \right\}$$

since $\mathbb{1}$ and $\vec{X}$ commute so it's a real number

Notice that $\operatorname{tr} \rho d_i = 0 \quad \forall i$. since $\operatorname{tr} \rho_i' = \operatorname{tr}(\rho d_i)$

$$\operatorname{tr} \rho_i' = \frac{d}{d\theta_i} \operatorname{tr} \rho = \frac{d}{d\theta_i} 1 = 0$$

Suppose $V \geqslant Z$ real symmetric & complex selfadjoint

$$V = V^T \geqslant Z^T = \bar{Z}$$

so: $$V \geqslant \frac{Z + \bar{Z}}{2} = \mathrm{Re}(Z).$$

So we see that $\mathcal{V}_{Hol} \subseteq \{ V : \exists \vec{X} \quad \langle 1, \vec{X} \rangle_\rho = 0 , \langle \vec{d}, \vec{X} \rangle_\rho = \mathrm{Id}, V \geqslant \langle \vec{X}, \vec{X} \rangle_\rho \}$.

write $\vec{X} = \vec{X}_{\mathcal{L}} + \vec{X}_{\mathcal{L}^\perp}$

where $\mathcal{L} = \mathrm{Re\ Span}(d_i)$

and $\langle 1, \vec{X} \rangle = 0 \quad , \quad \langle \vec{d}, \vec{X} \rangle = \mathrm{Id}$

$\Rightarrow \quad \langle 1, \vec{X}_{\mathcal{L}} \rangle = 0 \qquad \langle \vec{d}, \vec{X}_{\mathcal{L}} = \mathrm{Id}$

$\langle \vec{X}, \vec{X} \rangle_\rho = \langle \vec{X}_{\mathcal{L}}, \vec{X}_{\mathcal{L}} \rangle_\rho + \langle \vec{X}_{\mathcal{L}^\perp}, \vec{X}_{\mathcal{L}^\perp} \rangle.$

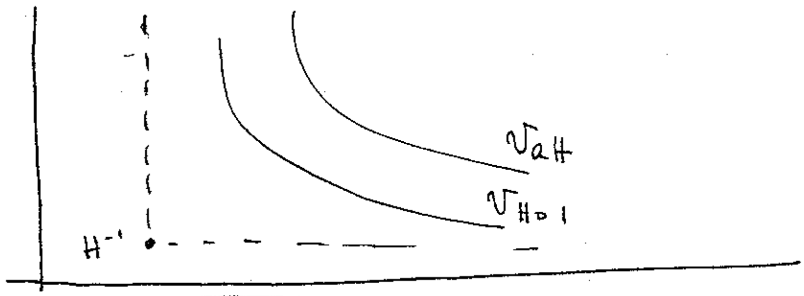
Defining $H = \langle \vec{d}, \vec{d} \rangle$ supposed non singular

$\boxed{\text{HELSTROM} = \text{quantum information matrix}}$

then $\vec{X} = H^{-1} \vec{d}$ is the unique $\vec{X} \in \mathcal{L}$ satisfying the constraint

So $\{ V : \exists \vec{X} \quad \langle 1, \vec{X} \rangle_\rho = 0 , \langle \vec{d}, \vec{X} \rangle_\rho = \mathrm{Id} , V \geqslant \langle \vec{X}, \vec{X} \rangle_\rho \}$

$= \{ V \geqslant \langle H^{-1} \vec{d}, d^{-1} \vec{d} \rangle = H^{-1} \}.$



Conclusion: $\boxed{\forall M, \quad I_M \leq H.}$ Helstrom's quantum Cramer Rao Bound

If the d_i's commute, then the measurement, M, which measures all the d_i's together (simultaneously) attains H. $(I_M = H)$

Exercise express H with $|y_i\rangle$ and $|x_i\rangle$ and $|v_i\rangle$ of the proof of last time when $\rho(0)$ is pure.

Now we go back to $N \tilde{V}^{(N)}_{Hol} = \tilde{V}^{(1)}_{Ho}$?

But $V^{(1)}_{Hol} = \{V : \exists \vec{X} \text{ s.a. } \langle \mathbb{1}, \vec{X} \rangle_\rho = \vec{0}, \langle \vec{A}, \hat{x} \rangle_\rho = Id,$
$V \geq \langle\langle \vec{X}, \vec{X} \rangle\rangle_\rho \}$.

Given a s.a. X, define $\underline{U = \mathcal{D}_\rho(X)}$ by

$\| \ i(X\rho - \rho X) = \rho U + U\rho$ (solve it if $\rho > 0$ in the same way as before)

(if it is not unique, in fact you want to solve
$\forall Y \quad tr[i(X\rho - \rho X) Y] = tr[(\rho U + U\rho) Y]$.
this is linear in Y $= 2 \langle U, Y \rangle_\rho$
Riesz Representation Thm

Remark: Suppose $\mathcal{L}$ is a subspace of s.a. operators closed under $\mathcal{D}$.
then if $Y \perp \mathcal{L}$ wrt $\langle , \rangle_\rho$
then $Y \perp \mathcal{L}$ wrt $\langle\langle , \rangle\rangle_\rho$

Because Γ $\mathrm{Im} \langle\langle X, Y \rangle\rangle_\rho = \mathrm{Re} \langle\langle \mathcal{D}(X), Y \rangle\rangle_\rho = \langle \mathcal{D}(X), Y \rangle_\rho$
$\quad \| $
$\llcorner \frac{1}{2i}\left(tr\, \rho X Y - \overline{tr\, \rho X Y}\right) = \frac{1}{2i}\left(tr(\rho X - X\rho) Y \right)$.

Temporary conclusion:
Suppose $\mathcal{L}$ contains the A_i's, and is closed under $\mathcal{D}$ ↓ is orthogonal in either sense to $\mathbb{1}$.
then $V_{Hol} = \{ V : \exists \vec{X}, X_i \in \mathcal{L}, \dots \}$.
(you project on $\mathcal{L}$ wrt $\langle\langle , \rangle\rangle$)

(Rmk: if $X \perp 1 \Rightarrow \mathcal{D}(X) \perp 1$)