Hermitian symmetric spaces of infinite dimension and maximal representations

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Symmetric spaces of non-compact type

Definition

A symmetric space is a manifold $M$ such that for any $x \in M$, there exists an isometry, $\sigma_x$, fixing $x$ with $T_x \sigma_x = -\text{Id}$.

Exemples : Euclidean space $\mathbb{E}^n$, spheres $\mathbb{S}^n$, hyperbolic spaces $\mathbb{H}^n$.

A symmetric space $X$ is of non-compact type if it has non-positive sectionnal curvature and no Euclidean factor.
A dictionary

$X$ symmetric space of non-compact type

$\uparrow$

- $G = \text{Isom}(X)^{\circ}$ is a connected semi-simple Lie group without compact factor and trivial center.
- $K = \text{Stab}_G(x)$ is a maximal compact subgroup.
- $X \cong G/K$. 

Classical examples:

- \( \text{SL}_n(\mathbb{R})/\text{SO}_n(\mathbb{R}) \leftrightarrow \text{ellipsoids centered at 0 and unit volume.} \)

- \( H^n = \text{O}(1, n)/\text{O}(1) \times \text{O}(n) \leftrightarrow \text{upper half of the hyperboloid.} \)
Let $\mathcal{H}$ a Hilbert space with basis $(e_i)_{i \in \mathbb{N}}$. $(x, y) = x_0 y_0 - \sum_{i > 0} x_i y_i$

$\mathcal{H}^\infty = \{ x \in \mathcal{H}, (x, x) = 1, \text{ et } x_0 > 0 \}$
Let $L^2(\mathcal{H})$ be the set of Hilbert-Schmidt operators of $\mathcal{H}$, those operators $M$ with

$$\sum_{i,j} \langle e_i, Me_j \rangle^2 < \infty.$$ 

Then $GL^2(\mathcal{H})/O^2(\mathcal{H})$ is a Riemannian symmetric space of non-positive curvature.
Let us consider the quadratic form

\[ Q(x) = \sum_{i=1}^{p} x_i^2 - \sum_{j>p} x_j^2. \]

Then \( \mathcal{X}_R(p, \infty) = \{ P \subset \mathcal{H}, \dim(P) = p, \ Q|_P > 0 \} \) is a Riemannian symmetric space of non-positive curvature.

One has the identification

\[ \mathcal{X}_R(p, \infty) = O(p, \infty) / O(p) \times O(\infty). \]
Infinite dimension

Let us consider the following Hermitian form

\[ Q(x) = \sum_{i=1}^{p} |x_i|^2 - \sum_{j>p} |x_i|^2. \]

Then \( \mathcal{K}_C(p, \infty) = \{ P \subset \mathcal{H}, \dim(P) = p, \, Q|_P > 0 \} \) is a Hermitian symmetric space of non-positive curvature. One has the identification

\[ \mathcal{K}_C(p, \infty) = \text{U}(p, \infty)/ \text{U}(p) \times \text{U}(\infty). \]
Let $\mathcal{H} = L^2(S^1)$. For $g \in \text{SL}_2(\mathbb{R})$ and $f \in \mathcal{H}$, one define

$$\pi_s(g)(f) = \text{Jac}(g)^{1/2+s} f \circ g^{-1}.$$ 

For $p \in \mathbb{N}$ and $s \in (p - 1/2, p + 1/2)$, $\pi_s$ preserves a quadratic form of signature $(p, \infty)$. One obtains an action of $\text{SL}_2(\mathbb{R})$ on $\mathcal{X}_R(p, \infty)$. 
Let $C$ be the group of birational transformations $\mathbb{P}^2(\mathbb{C})$.

There exists a rich action of $C$ on $\mathbb{H}^\infty$ by isometries. Thanks to this action, one can show that $C$ satisfies the Tits alternative (Cantat 2012) and has many normal subgroups (Cantat-Lamy 2013).
What are all symmetric spaces of infinite dimension? Can one classify them?

Does the strategy of Élie Cartan still work in infinite dimension?

There is no classification of Banach algebras.
Theorem (D. 2015)

Let \((M, g)\) be a symmetric space with non-positive curvature operator \(M\), then it is isometric to the Hilbertian product

\[ M \simeq \prod_{i}^{2} M_i \]

Where each \(M_i\) is irreducible of finite dimension or isometric to one of the following:

\[
\begin{align*}
GL^2_\infty(\mathbb{R})/O^2_\infty(\infty), & \quad U^*^2(\infty)/Sp^2(\infty), & \quad U^2(p, \infty)/U^2(p) \times U^2(\infty), \\
O^2(p, \infty)/O^2(p) \times O^2(\infty), & \quad O^*^2(\infty)/U^2(\infty), & \quad Sp^2_\infty(\mathbb{R})/U^2(\infty), & \quad Sp^2(p, \infty)/Sp^2(p) \times Sp^2(\infty), \\
GL^2_\infty(\mathbb{C})/U^2(\infty), & \quad O^2_\infty(\mathbb{C})/O^2(\infty), & \quad Sp^2_\infty(\mathbb{C})/Sp^2(\infty).
\end{align*}
\]
Classification of infinite dimensional symmetric spaces

The *rank* is the maximal dimension of a flat subspace.

**Corollary**

Let \((M, g)\) be a symmetric space with non-positive curvature operator, irreducible with rank \(p < \infty\) and infinite dimension then \(M\) is isometric to

\[
O(p, \infty)/O(p) \times O(\infty), \quad U(p, \infty)/U(p) \times U(\infty),
\]

\[
Sp(p, \infty)/Sp(p) \times Sp(\infty).
\]
“This spaces look to me as cute and sexy as their finite dimensional siblings but they have been for years shamefully neglected by geometers and algebraists a like.”

– Gromov, *Asymptotic invariants of infinite groups.*
Lett $G$ be a Lie group, a lattice of $G$ is a discrete subgroup $\Gamma$ of finite covolume.

**Theorem (Margulis 1974)**

Let $G, H$ be two semi-simple Lie groups with finite center and no compact factors. Let $\Gamma < G$ be an irreducible lattice and $\rho: \Gamma \rightarrow H$, a representation with Zariski dense image.

If $\text{Rank}_\mathbb{R}(G) \geq 2$ then there exists a representation $\overline{\rho}: G \rightarrow H$ such that $\rho$ is the restriction of $\overline{\rho}$ to $H$. 
Theorem (D. 2015)

Let $G$ be a semi-simple Lie group with finite center and no compact factor with $\text{Rank}_\mathbb{R}(G) \geq 2$. Let $\Gamma$ be an irreducible lattice of $G$ without torsion. Let $\mathcal{Y}$ be a simply connected Riemannian manifold with non-positive curvature and finite rank.

If $\Gamma$ acts by isometries on $\mathcal{Y}$ without fixed points in $\partial \mathcal{Y}$ then $\Gamma$ stabilizes a totally geodesic subspace of $\mathcal{Y}$ isometric to a product of factors of $\mathcal{X}_G$.

Ideas: Existence of a $\Gamma$-equivariant harmonic map $\mathcal{X}_G \to \mathcal{Y}$ then a Bochner type inequality due to Mok-Siu-Yeung.
Let $g \in \text{Isom}(\mathcal{X})$, the translation length
\[
\ell_{\mathcal{X}}(g) = \inf_{x \in \mathcal{X}} d(gx, x).
\]

**Theorem (Monod-Py)**  
For each $t \in (0, 1]$ there is, up to conjugacy, exactly one irreducible continuous representation $\rho_t : \text{Isom}(\mathcal{H}^n) \to \text{Isom}(\mathcal{H}^\infty)$ such that
\[
\ell_{\mathcal{H}^\infty}(\rho_t(g)) = t \ell_{\mathcal{H}^n}(g).
\]
Moreover, there is an equivariant harmonic map $\mathcal{H}^n \to \mathcal{H}^\infty$ that is totally geodesic if and only if $t = 1$. The group $\text{Isom}(\mathcal{H}^n)$ acts cocompactly on the convex hull of the image if this map.
A symmetric space \((M, g)\) is **Hermitian** if there is a complex structure \(J\) that is invariant under the connected component of the isometry group. The Kähler form is \(\omega(X, Y) = g(X, JY)\).

**Examples**: \(\mathcal{X}_C(p, \infty)\) and \(\mathcal{X}_R(2, \infty)\) (associated to \(\text{PU}(p, \infty)\) and \(\text{PO}^+(2, \infty)\)) are Hermitian.
Bounded cohomology

The *bounded cohomology* $H^n_b(G, \mathbb{R})$ of a group $G$ is the cohomology of the complex $C^n_b(G, \mathbb{R})^G = \{ f : G^{n+1} \to \mathbb{R} | f \text{ is } G\text{-invariant, } \sup_{(g_0, \ldots, g_n) \in G^{n+1}} |f(g_0, \ldots, g_n)| < \infty \}$ whose coboundary operator is defined by the formula

$$df(g_0, \ldots, g_{n+1}) = \sum_{i=0}^{n+1} (-1)^i f(g_0, \ldots, \hat{g_i}, \ldots, g_{n+1}).$$

If $G$ is a locally compact group, one defines the *continuous bounded cohomology* $H^2_{cb}(G, \mathbb{R})$. 
Let $G$ be a simple Lie group of non-compact type and finite center. Then $H^2_{cb}(G, \mathbb{R}) \neq 0$ if and only if $G$ is Hermitian and in that case

$$H^2_{cb}(G, \mathbb{R}) = R\kappa^c_G$$

Where $\kappa^c_G$ is the bounded Kähler class.
The bounded Kähler class of an Hermitian simple Lie group $G$ is the class $\kappa^b_G \in H^2_b(G, \mathbb{R})$ defined by the cocycle

$$C^x_\omega(g_0, g_1, g_2) = \frac{1}{\pi} \int_{\Delta(g_0 x, g_1 x, g_2 x)} \omega$$

where $x$ is any base point in the corresponding symmetric space $\mathcal{X}_G$. 
Let $\Gamma$ be a lattice in $\text{SU}(1, n)$.

The restriction map $i^* : H^2_{cb}(\text{SU}(1, n), \mathbb{R}) \to H^2_b(\Gamma, \mathbb{R})$.

The transfer map $T^*_b : H^2_b(\Gamma, \mathbb{R}) \to H^2_{cb}(\text{SU}(1, n), \mathbb{R})$. 
The bounded Kähler class of the groups $G = \text{PU}(p, \infty)$ and $G = \text{PO}^+(2, \infty)$ is the class $\kappa^b_G \in H^2_b(G, \mathbb{R})$ defined by the cocycle

$$C^x_\omega(g_0, g_1, g_2) = \frac{1}{\pi} \int_{\Delta(g_0^x, g_1^x, g_2^x)} \omega$$

where $x$ is any base point in the corresponding symmetric space $\mathcal{X}$.

$$\|\kappa^b_G\|_\infty = \text{rank}(\mathcal{X})$$
Toledo invariant and maximal representations

Let $G$ be a lattice in $SU(1, n)$ and let $\kappa^b_n$ be the bounded Kähler class of $SU(1, n)$.

Definition

Let $G \in \{PO(2, \infty), PU(p, \infty)\}$ and let $\rho : \Gamma \rightarrow G$ be an homomorphism. The Toledo invariant of the representation $\rho$ is the number $i_\rho$ such that

$$T^*_b \rho^* \kappa^b_G = i_\rho \kappa^b_n$$

Milnor-Wood inequality : $|i_\rho| \leq \text{rank}(\mathcal{X})$. The representation is maximal when there is equality.
A bit of history

Let $\Gamma < \text{SU}(1, n)$ and $\rho: \Gamma \to G$ be a maximal representation where $G$ is Hermitian.

- [Goldman 1988] $\Gamma$ is cocompact, $n = 1$, $G = \text{SU}(1, 1)$. Then maximal representations are Fuchsian.
- [Toledo 1989] $\Gamma$ is cocompact, $n = 1$, $G = \text{SU}(1, n)$. Then there is an invariant complex geodesic line.
- [Burger-Lozzi-Wienhard 2003] $n = 1$. Then the Zariski closure is of tube type (e.g. $\text{SU}(k, k)$).
- [Pozzetti 2015] $G = \text{SU}(k, l)$. Then the image is not Zariski-dense for $k \neq l$.
- [Koziarz-Maubon 2017] $\Gamma$ is cocompact. Necessarily $G = \text{SU}(k, l)$ with $l \geq kn$ and the representation is rigid.
A maximal representation is obtained this way:

\[ \Gamma \to \text{SU}(1, n) \to \text{SU}(p, pn) \]

where \( \text{SU}(1, n) \to \text{SU}(p, pn) \) is the diagonal inclusion.
Theorem (D.-Lécureux-Pozzetti)

Let $\Gamma$ be a lattice of $\text{SU}(1, n)$ with $n \geq 1$ and $\rho: \Gamma \to \text{PU}(p, \infty)$ be a maximal representation. If $p \leq 2$ then there exists a finite dimensional totally geodesic subspace $\mathcal{Y} \subset \mathcal{X}_\mathbb{C}(p, \infty)$ that is $\Gamma$-invariant.

More generally, there is no Zariski-dense maximal representation $\Gamma \to \text{PU}(p, \infty)$. 
Steps of the proof

Steps:

1. Existence of a boundary map $\partial X_c(1, n) \rightarrow \partial X_c(p, \infty)$.
2. This boundary map sends chains to chains.

Difficulties:

1. The space is no more locally compact.
2. There is no Zariski topology.
Let $\Sigma$ be a torus with one puncture and $\Gamma_{\Sigma}$.

**Theorem (BLP)**

*There are geometrically dense maximal representations $\rho : \Gamma_{\Sigma} \to \text{PO}_R(2, \infty)$.*
Boundaries at infinity

Let $\mathcal{X}$ a symmetric space of non-positive curvature. The boundary at infinity $\partial \mathcal{X}$ is the set of classes of geodesic rays that are at bounded distance.

For $H^n_\mathbb{C} = \mathcal{X}_\mathbb{C}(1, n)$, $\partial H^n_\mathbb{C} \simeq \{\text{isotropic lines}\}$.

For $\mathcal{X}_\mathbb{C}(p, \infty)$, $\partial \mathcal{X}_\mathbb{C}(p, \infty)$ has a structure of spherical building. Each cell corresponds to a flag of isotropic subspaces.

Let $I_p = \{\text{maximal isotropic subspaces}\}$.
Theorem (BLP)

Let $\Gamma < \text{SU}(1, n)$ be a countable subgroup, $B = \partial H^n_C$ and $p \in \mathbb{N}$. If $\Gamma$ acts \textit{geometrically densely} on $\mathcal{X}_K(p, \infty)$ with $p \leq 2$, then there is a measurable $\Gamma$-equivariant map $\phi: B \rightarrow \mathcal{I}_p$. Moreover, for almost all pair $(b, b') \in B^2$, $\phi(b)$ and $\phi(b')$ are transverse.

If $\Gamma \rightarrow \text{PO}_K(p, \infty)$ is a representation with a Zariski-dense image, then there is a measurable $\Gamma$-equivariant map $\phi: B \rightarrow \mathcal{I}_p$. Moreover, for almost all pair $(b, b') \in B^2$, $\phi(b)$ and $\phi(b')$ are transverse.
A chain in $\partial H^n_C$ is the boundary of a complex geodesic.

A chain in $\mathcal{I}_p$ corresponds to the boundary of a totally geodesic copy of $\mathcal{X}_C(p, p)$. 
Cartan and Bergmann invariants

The *Cartan invariant* is a map $c : (\partial H^n_C)^3 \to [-1, 1]$ such that $|c(\xi_1, \xi_2, \xi_3)|$ is maximal iff $\xi_1, \xi_2, \xi_3$ lie in a common complex geodesic.

The *Bergmann invariant* is a map $\beta : \mathcal{I}_p^3 \to [-p, p]$ such that $|\beta(\xi_1, \xi_2, \xi_3)|$ is maximal iff $\xi_1, \xi_2, \xi_3$ lie in a common copy of $\partial X_C(p, p)$.

**Lemma**

*For every $V \in \mathcal{I}_p$, the cocycle $C^V_\beta$ defined by*

$$C^V_\beta (g_0, g_1, g_2) = \beta_c (g_0 V, g_1 V, g_2 V)$$

*represents the bounded Kähler class.*
Theorem

Let $\Gamma < SU(1, n)$ be a lattice. Assume that a representation $\rho : \Gamma \to PU(p, \infty)$ is maximal and admits an equivariant boundary map $\phi : \partial X_{\mathbb{C}}(1, n) \to \mathcal{I}_p(p, \infty)$. Then the boundary map $\phi$ almost surely maps chains to chains.
**Theorem**

Let $n \geq 2$ and let $\Gamma < SU(1, n)$ be a complex hyperbolic lattice, and let $\rho : \Gamma \to \text{POC}(p, \infty)$ be a maximal representation. If there is a $\rho$-equivariant measurable map $\phi : \partial H^n_C \to \mathcal{I}_p$ then there is a finite dimensional totally geodesic Hermitian symmetric subspace $\mathcal{Y} \subset \mathcal{X}(p, \infty)$ that is invariant by $\Gamma$. Furthermore, the representation $\Gamma \to \text{Isom}(\mathcal{Y})$ is maximal.

**Idea:** One can reconstruct $\partial H^n_C$ with finitely many chains. So, the same is true for the essential image of $\phi$. 