Exercise sheet 4:
Discrete subgroups of Lie groups

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**Exercise 1** (Tits representation of triangle groups). 1. Let $x$, $y$ and $z$ be real numbers. Verify the following identity:

$$\cos^2(x) + \cos^2(y) + \cos^2(z) + 2 \cos(x) \cos(y) \cos(z) - 1 = 4 \prod_{\varepsilon_y,\varepsilon_z \in \{-1,1\}} \cos \left( \frac{x + \varepsilon_y y + \varepsilon_z z}{2} \right).$$

2. Let $p, q, r \in \{2, \ldots, \infty\}$ and let $M$ be the Coxeter diagram

$$
\begin{pmatrix}
1 & p & q \\
p & 1 & r \\
q & r & 1
\end{pmatrix}.
$$

Show that the Tits form of $M$ has signature $(3, 0)$ when

$$\frac{1}{p} + \frac{1}{q} + \frac{1}{r} > 1$$

and signature $(2, 1)$ when

$$\frac{1}{p} + \frac{1}{q} + \frac{1}{r} < 1.$$

**Exercise 2** (Reversed exercise 1). Let $x, y, z$ be real numbers between $0$ and $\frac{\pi}{2}$. We endow $\mathbb{R}^3$ with the bilinear form $q$ whose matrix in the standard basis $e_1, e_2, e_3$ is

$$
\begin{pmatrix}
1 & -\cos(x) & -\cos(y) \\
-\cos(x) & 1 & -\cos(z) \\
-\cos(y) & -\cos(z) & 1
\end{pmatrix}.
$$

1. Assume that $q$ is positive definite. Let $v_1$, $v_2$, $v_3$ be respectively unit vectors in $e_2^+ \cap e_3^+$, $e_3^+ \cap e_1^+$ and $e_1^+ \cap e_2^+$. Show that $v_1$, $v_2$ and $v_3$ form a triangle in the unit sphere $\{q = 1\}$ with angles $x$, $y$ and $z$. 

2. Assume now that $q$ has signature $(2,1)$. Show that $q$ is negative on $e_2^+ \cap e_3^+$, $e_3^+ \cap e_1^+$ and $e_1^+ \cap e_2^+$. 

3. Let $v_1$, $v_2$, $v_3$ be respectively unit vectors in $e_2^+ \cap e_3^+$, $e_3^+ \cap e_1^+$ and $e_1^+ \cap e_2^+$. Show that $v_1$, $v_2$ and $v_3$ form a triangle in the hyperbolic plane $q = -1$ with angles $x$, $y$ and $z$. 

4. Assume now that $q$ is degenerate. Show that $\ker q$ has dimension $1$ and that $q$ is positive definite on $V = \mathbb{R}^3 / \ker q$. 

5. Let $e_1^\perp$, $e_2^\perp$ and $e_3^\perp$ denote the projections of $e_1$, $e_2$ and $e_3$ onto $V$. Show that $\langle e_1, e_2 \rangle = \pi - x$, $\langle e_2, e_3 \rangle = \pi - z$ and $\langle e_3, e_1 \rangle = \pi - y$. 

Exercises 3 (Uniform hyperbolic lattices in all dimensions). Let $q$ be the quadratic form on $\mathbb{R}^n$ given by 

$$q(x) = x_1^2 + \ldots + x_{n-1}^2 - \sqrt{2}x_n^2.$$ 

Define $\Gamma = O(q) \cap GL(n, \mathbb{Z}[\sqrt{2}])$. The goal of this exercise is to prove that $\Gamma$ is a uniform lattice in $O(q)$. 

Let $\overline{q}$ denote the quadratic form 

$$\overline{q}(x) = x_1^2 + \ldots + x_{n-1}^2 + \sqrt{2}x_n^2.$$ 

Image of $q$ by the Galois automorphism of $\mathbb{Q}[\sqrt{2}]$. Let $Q$ and $Q'$ be the quadratic forms on $\mathbb{R}^n \times \mathbb{R}^n$ given respectively by 

$$Q(u, v) = q(u + \sqrt{2}v) + \overline{q}(u - \sqrt{2}v)$$ 

and 

$$Q'(u, v) = \frac{1}{\sqrt{2}} \left( q(u + \sqrt{2}v) - \overline{q}(u - \sqrt{2}v) \right).$$ 

1. Show that $Q$ and $Q'$ take integral values on $\mathbb{Z}^n \times \mathbb{Z}^n$. 

Let $G$ be the subgroup of $GL(\mathbb{R}^n \times \mathbb{R}^n)$ preserving $Q$ and $Q'$. 

2. Assume that there exists a sequence $(u_n, v_n) \in \mathbb{Z}^n \times \mathbb{Z}^n$ and a sequence $g_n \in G$ such that $g_n(u_n, v_n) \xrightarrow{n \to +\infty} 0$. Show that for $n$ large enough, $Q(v_n) = Q'(v_n) = 0$. Deduce that $v_n = 0$ for $n$ large enough. 

3. Show that the $G$-orbit of the lattice $\mathbb{Z}^n \times \mathbb{Z}^n$ is compact in the space of lattices in $\mathbb{R}^n \times \mathbb{R}^n$. 

$$2$$
Define
\[ \varphi : \Gamma \to \text{GL}(\mathbb{Z}^n \times \mathbb{Z}^n) \]
\[ A + \sqrt{2}B \mapsto \begin{pmatrix} A & 2B \\ B & A \end{pmatrix}. \]

4. Show that \( \varphi \) is an injective group morphism and that
\[ \varphi(\Gamma) = G \cap \text{GL}(\mathbb{Z}^n \times \mathbb{Z}^n). \]

5. Show that there exists an isomorphism \( \psi : G \to O(q) \times O(\widetilde{q}) \) such that
\[ \pi \circ \psi \circ \varphi = i, \]
where \( \pi : O(q) \times O(\widetilde{q}) \to O(q) \) denotes the projection on the first factor and \( i : \Gamma \to O(q) \) denotes the inclusion.

6. Conclude.