Deforming discrete groups into Lie groups

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General motivation

The framework of the course is the following: we start with a semisimple Lie group $G$. We will come back to the precise definition. For the moment, one can think of the following examples:

- The group $\text{SL}(n, \mathbb{R})$ $n \times n$ matrices of determinant 1,
- The group $\text{Isom}(\mathbb{H}^n)$ of isometries of the $n$-dimensional hyperbolic space.

Such a group can be seen as the transformation group of certain homogeneous spaces, meaning that $G$ acts transitively on some manifold $X$. A pair $(G, X)$ is what Klein defines as “a geometry” in its famous Erlangen program [Kle72]. For instance:

- $\text{SL}(n, \mathbb{R})$ acts transitively on the space $\mathbb{P}(\mathbb{R}^n)$ of lines in $\mathbb{R}^n$,
- The group $\text{Isom}(\mathbb{H}^n)$ obviously acts on $\mathbb{H}^n$ by isometries, but also on $\partial_{\infty}\mathbb{H}^n \simeq S^{n-1}$ by conformal transformations.

On the other side, we consider a group $\Gamma$, preferably of finite type (i.e. admitting a finite generating set). $\Gamma$ may for instance be the fundamental group of a compact manifold (possibly with boundary). We will be interested in representations (i.e. homomorphisms) from $\Gamma$ to $G$, and in particular those representations for which the intrinsic geometry of $\Gamma$ and that of $G$ interact well.

Such representations may not exist. Indeed, there are groups of finite type for which every linear representation is trivial! This groups won’t be of much interest for us here. We will often start with a group of which we know at least one “geometric” representation. To make things more precise, we can assume the existence of a representation $i : \Gamma \to G$ which is discrete and faithful. In other worlds, we assume that $\Gamma$ can be realized as a discrete subgroup of $G$.

We will for instance consider the following groups $\Gamma$:

- the free group $F_n$ in $n$ generators,
- the fundamental group $\Gamma_g$ of a closed surface of genus $g \geq 2$, 

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• the fundamental group of a closed hyperbolic manifold,
• the group \( \text{SL}(n, \mathbb{Z}) \subset \text{SL}(n, \mathbb{R}) \),
• the Coxeter groups.

One the framework is settled, our course will be motivated by the following questions:

**Question 0.0.1.** Let \( X \) be a \( G \)-homogeneous space. What can be said of the action of \( i(\Gamma) \) on \( X \)?

Since \( i(\Gamma) \) is discrete, it acts properly discontinuously on \( G \) by left multiplication, and more generally on every homogeneous space \( G/K \) where \( K \) is a compact subgroup of \( G \). On the other hand, an infinite subgroup \( \Gamma \) of \( \text{Isom}(\mathbb{H}^n) \) certainly does not act properly on \( \partial_\infty \mathbb{H}^n \), which is compact. One can however hope to find a nice “limit set” \( \Lambda_\Gamma \subset \partial_\infty \mathbb{H}^n \), on which the dynamics is well-understood, and outside of which the action is properly discontinuous.

**Question 0.0.2.** Does the representation \( i : \Gamma \to G \) admit non-trivial deformations?

One can always deform \( i \) by conjugating it with elements of \( G \). Such deformations are “trivial” in the sense that the induced actions of \( \Gamma \) on some \( G \)-homogeneous space are conjugate via a transformation of \( G \).

This question will lead us to introduce the **character variety**, which is the quotient of the space of representations of \( \Gamma \) into \( G \) under the conjugation action of \( G \). If \( i \) is isolated in the character variety, we call \( i \) rigid.

During the second half of the 20th century, many groundbreaking works have contributed to show how many discrete subgroups of Lie groups are rigid. A striking example is that every representation of \( \text{SL}(3, \mathbb{Z}) \) is rigid (as a consequence of Margulis’s superrigidity theorem). We will state those theorems, mostly to exclude those groups of our study. We will then focus on the discrete groups that can actually be deformed. In particular:

• representations of the free group \( F_n \) easily deform, since a representation of \( F_n \) into \( G \) is given by any \( n \)-tuple of elements in \( G \),

• surface groups also have rich character varieties, which carry a symplectic structure invariant under the action of the **mapping class group** of the surface,

• in some cases, we can deform the fundamental group of a closed hyperbolic manifold of dimension \( n \geq 3 \) into a Lie group that contains strictly \( \text{Isom}(\mathbb{H}^n) \).

Finally, when we know how to deform a discrete and faithful representation \( i : \Gamma \to G \) we can ask the following:
**Question 0.0.3.** Which geometric and dynamical properties of \( i(\Gamma) \) are preserved under deformation? In particular, are deformations of \( i \) still discrete and faithful? Is their action on \( G \)-homogeneous spaces still topologically conjugate to that of \( i \)?

We will introduce the notion of **Anosov representation** of a discrete group \( \Gamma \), for which the answer to this question is affirmative in some way. This notion was introduced 15 years ago as a unified setting in which to study:

- convex cocompact subgroups of \( \text{Isom}(\mathbb{H}^n) \),
- divisible convex sets,
- Globally hyperbolic Cauchy compact anti-de Sitter manifolds.

**A beautiful example: Fuchsian and quasi-Fuchsian representations**
Let \( S \) be a closed oriented surface of genus \( g \geq 2 \) and \( \Gamma \) its fundamental group. Let us consider first representations of \( \Gamma \) into the group \( \text{Isom}_+(\mathbb{H}^2) \) of orientation preserving isometries of \( \mathbb{H}^2 \). This group is isomorphic to \( \text{PSL}(2, \mathbb{R}) = \text{SL}(2, \mathbb{R})/\{\pm \text{Id}_2\} \). Among these representations, the ones that are discrete and faithful are called **Fuchsian**.

Let \( j \) be a Fuchsian representation. Then \( j(\Gamma) \) acts freely and properly discontinuously on \( \mathbb{H}^2 \), so that \( \Sigma = j(\Gamma) \backslash \mathbb{H}^2 \) is a smooth hyperbolic surface. Besides, \( j \) induces an isomorphism between \( \Gamma \) and \( \pi_1(\Sigma) \). It follows that \( \Sigma \) diffeomorphic to \( S \). In other words, \( j \) defines a homotopy class of hyperbolic metrics on \( S \).

How many such metrics are there? Poincaré’s uniformization theorem provides an answer: as many as isotopy classes of complex structures on \( S \).

**Theorem 0.0.4** (Poincaré). Let \( \Sigma \) be a Riemann surface of genus at least 2. Then there exists a unique conformal metric on \( \Sigma \) of constant curvature \(-1\). (A conformal metric is a metric that writes \( \alpha dzd\overline{z} \) in local coordinates.)

Putting all this together, we obtain a homeomorphism between the **Teichmüller space** \( \mathcal{T}(\Sigma) \) of isotopy classes of complex structures and the space \( \text{Rep}_{Fuchs}(\Gamma, \text{PSL}(2, \mathbb{R})) \) of Fuchsian representations modulo conjugation.

To complete the picture, one would want to characterize which representations of \( \Gamma \) into \( \text{PSL}(2, \mathbb{R}) \) are Fuchsian. For this, one introduces the **Euler class** \( \text{eu}(\rho) \) of a representation \( \rho \) (which describes the topology of the \( \text{PSL}(2, \mathbb{R}) \)-principal bundle associated to \( \rho \)). The Euler class \( \text{eu} : \text{Rep}(\Gamma, \text{PSL}(2, \mathbb{R})) \to \mathbb{Z} \) is continuous, hence constant on connected components. The following theorem gathers the properties of the Euler class and its relation with the topology of the character variety \( \text{Rep}(\Gamma, \text{PSL}(2, \mathbb{R})) \):

**Theorem 0.0.5.**
- The Euler class takes values between \( 2 - 2g \) and \( 2g - 2 \) (Milnor–Wood, [Mil58]).
• For every $k \in \{2-2g, \ldots, 2g-2\}$, $\text{eu}^{-1}(k)$ is non-empty and connected (Goldman, [Gol88]).

• For every $k \in \{1, \ldots, 2g-2\}$, $\text{eu}^{-1}(k)$ is homeomorphic to a vector bundle of rank $4g - 4 + 2k$ over the symmetric power $\text{Sym}^{2g-2-k}(\Sigma)$ (Hitchin, [Hit87]).

• A representation $\rho$ is Fuchsian if and only if $|\text{eu}(\rho)|2g - 2$ (Goldman, [Gol80]).

In particular, discrete and faithful representations form a connected component in $\text{Rep}(\Gamma, \text{PSL}(2, \mathbb{R}))$.

What happens now if we deform a Fuchsian representation into $\text{PSL}(2, \mathbb{C}) \simeq \text{Isom}_+ (\mathbb{H}^3)$? For small deformations, we obtain quasi-Fuchsian representations. These act properly discontinuously and cocompactly on the complement of a Jordan curve in $\partial_\infty \mathbb{H}^3$. The quotient of this domain of discontinuity gives two Riemann surfaces homeomorphic to $S$. This defines a map from the space of quasi-Fuchsian representations (modulo conjugation) to $T(S) \times T(S)$. A remarkable theorem of Ahlfors–Bers shows that this map is bijective! We thus obtain a complete description of the set of “nice” representations of $\Gamma$ into $\text{PSL}(2, \mathbb{C})$. However, quasi-Fuchsian representations do not form a connected component in $\text{Rep}(\Gamma, \text{PSL}(2, \mathbb{C}))$, and there is a whole research field devoted to understanding limits of quasi-Fuchsian.

We will come back to this beautiful example, which the theory of Anosov representations intend to generalize to Lie groups of higher rank.
Chapter 1

Semisimple Lie groups and their symmetric spaces

This first chapter briefly introduces the first steps of the structure theory of semisimple Lie groups.

1.1 A Reminder of Lie theory

Let us first recall without proofs the basics of Lie theory. We encourage the reader who is not familiar with this material to consider the proofs as exercises.

1.1.1 Lie algebras

Definition 1.1.1. A Lie algebra \( g \) over a field \( K \) is a vector space over \( K \) endowed with a bilinear operation

\[
\left[ \cdot, \cdot \right] : g \times g \to g,
\]

called the Lie bracket, which satisfies the following properties:

- **antisymmetry:** \( [u, v] = -[v, u] \),
- **Jacobi identity:** \( [u, [v, w]] + [v, [w, u]] = [[w, u], v] \).

Example 1.1.2. Let \( A \) be an algebra over \( K \). Then the Lie bracket

\[
[u, v] = uv - vu
\]

endows \( A \) with a Lie algebra structure. For instance.

Example 1.1.3. Let \( M \) be a smooth manifold. Then the space of smooth vector fields on \( M \), with the Lie bracket of vector fields, is a Lie algebra.
Remark 1.1.4. This is the only example we will see of an infinite dimensional Lie algebra. From now on, all our Lie algebras will be assumed finite dimensional.

Definition 1.1.5. Let \( \mathfrak{g} \) and \( \mathfrak{h} \) be two Lie algebras. A linear map \( \varphi : \mathfrak{g} \to \mathfrak{h} \) is a morphism of Lie algebras if

\[
\left[ \varphi(u), \varphi(v) \right] = \varphi([u, v])
\]

pour tous \( u, v \in \mathfrak{g} \).

Example 1.1.6. Let \( \mathfrak{g} \) be a Lie algebra. The Lie bracket defines a linear map

\[
\text{ad} : \mathfrak{g} \to \text{End}(\mathfrak{g})
\]

\[
u \mapsto (v \mapsto [u, v])
\]

Using antisymmetry of the Lie bracket, the Jacobi identity can be rewritten

\[
\text{ad}_u \text{ad}_v w - \text{ad}_v \text{ad}_u w = \text{ad}_{[u, v]} w,
\]

meaning that \( \text{ad} : \mathfrak{g} \to \text{End}(\mathfrak{g}) \) is a Lie algebra morphism (where \( \text{End}(\mathfrak{g}) \) is endowed with the Lie bracket associated to its algebra structure).

Definition 1.1.7. Let \( \mathfrak{g} \) be a Lie algebra. A subspace \( \mathfrak{h} \) of \( \mathfrak{g} \) is a Lie subalgebra if \( [u, v] \in \mathfrak{h} \) for all \( u, v \in \mathfrak{h} \). A Lie subalgebra \( \mathfrak{h} \) is an ideal if \( [u, v] \in \mathfrak{h} \) for all \( u \in \mathfrak{h} \) and all \( v \in \mathfrak{g} \).

Proposition 1.1.8. If \( \varphi : \mathfrak{g} \to \mathfrak{g}' \) is a morphism of Lie algebras, then \( \ker \varphi \) is an ideal of \( \mathfrak{g} \). Conversely, if \( \mathfrak{h} \) is an ideal of \( \mathfrak{g} \), then there exists a unique Lie bracket on \( \mathfrak{g}/\mathfrak{h} \) such that the projection \( \mathfrak{g} \to \mathfrak{g}/\mathfrak{h} \) is a Lie algebra morphism.

1.1.2 Lie groups

Definition 1.1.9. A real (resp. complex) Lie group \( G \) is a smooth (resp. complex) manifold endowed with a group structure such that the operations

\[
G \times G \to G, \quad (g, h) \mapsto gh
\]

and

\[
G \to G, \quad g \mapsto g^{-1}
\]

are smooth (resp. holomorphic).

Remark 1.1.10. One could weaken the regularity assumption in the definition of a real Lie group. Indeed, Gleason, Montgomery and Zippin proved in the 50s that a topological manifold with a continuous group structure has a unique has a canonical smooth (and even real analytic) structure for which it is a smooth Lie group. This answered positively to Hilbert's 5th problem.
If $g$ is an element in $G$, we will denote respectively by $L_g$ and $R_g$ the multiplication by $g$ to the left and to the right, seen as diffeomorphisms of the manifold $G$. We say that a vector field $X$ on $G$ is right invariant if $R_gX = X$ for all $g \in G$. Since the Lie bracket between two right invariant vector fields is again right invariant, the space of right invariant vector fields is a Lie subalgebra of the algebra of smooth vector fields.

**Definition 1.1.11.** The Lie algebra of $G$, denoted Lie($G$), is the space of right invariant vector fields, with the Lie bracket of vector fields.

Since left and right multiplication commute, the image of right-invariant vector field by $L_g$ is still right invariant. The group $G$ thus acts by automorphisms on its Lie algebra.

**Definition 1.1.12.** The adjoint action of $G$ on $\mathfrak{g}$ is given by

$$\text{Ad}_g(X) = L_{g^{-1}}R_gX.$$ 

Let $1_G$ denote the identity element of $G$. Then the linear map

$$: \text{Lie}(G) \rightarrow T_{1_G}G$$

$$X \mapsto X_{1_G}$$

is an isomorphism, identifying Lie($G$) to the tangent space $T_{1_G}$ at $1_G$. From now on, we will alternatively consider Lie($G$) as the space of left invariant vector fields on $G$ or as the tangent space to $1_G$.

Given $g \in G$, denote $C_g = L_g \circ R_{g^{-1}}$ the conjugation by $g$. Then $C_g(1_G) = 1_G$ and for every right-invariant vector field $X$,

$$C_{g^{-1}}X = \text{Ad}_g(X).$$

In other words, the adjoint action of $G$ on Lie($G$) identifies with the differential at $1_G$ of the action of $G$ on itself by conjugation.

**Example 1.1.13.** Let $V$ be a finite dimensional vector space over $\mathbb{R}$ of $\mathbb{C}$. Then the group $\text{GL}(V)$ of invertible endomorphisms of $V$ is a Lie group. Its Lie algebra is $\text{End}(V)$ endowed with the Lie bracket given by its algebra structure. In particular, the group $\text{GL}(n, \mathbb{K})$ ($\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$) of invertible $n \times n$ matrices is a Lie group of Lie algebra $\text{M}(n, \mathbb{K})$, the algebra of square matrices of size $n$.

Every Lie group gives rise to a Lie algebra, conversely, every Lie algebra comes from a Lie group:

**Theorem 1.1.14 (Lie).** If $\mathfrak{g}$ is a real (resp. complex) Lie algebra, then there exists a real (resp. complex) Lie group $G$ such that Lie($G$) is isomorphic to $\mathfrak{g}$. 

1.1.3 Flows and exponential map

Recall that a (smooth) flow on a manifold $M$ is a family $(\Phi_t)_{t \in \mathbb{R}}$ of diffeomorphisms of $M$ such that $\Phi_{t+s} = \Phi_t \circ \Phi_s$ and such that

$$
: M \times \mathbb{R} \to M
(t, x) \mapsto \Phi_t(x)
$$

is smooth.

A flow $\Phi$ on $M$ defines a vector field $X^{\Phi}$ by

$$
X^\Phi_x = \frac{d}{dt}_{|t=0} \Phi_t(x).
$$

Conversely, given a smooth vector field $X$, one constructs the flow $\Phi$ of $X$ by solving the differential equation

$$
\frac{d}{dt} \Phi_t(x) = X_{\Phi_t(x)}.
$$

It may happen that this flow is only defined for a small interval of time depending on $x$. However this is not an issue for right invariant vector fields on a Lie group.

**Definition 1.1.15.** A one parameter subgroup of $G$ is a group morphism $m : \mathbb{R} \to G$ which is smooth.

**Proposition 1.1.16.** Let $m : \mathbb{R} \to G$ be a one parameter subgroup. Then the flow $\Phi_t = L_{m(t)}$ is the flow a right-invariant vector field. Conversely, if $X$ is a right-invariant vector field, then the flow $\Phi_t$ of $X$ is defined for all $t \in \mathbb{R}$ and we have $\Phi_t = L_{\exp(tX)}$, where

$$
t \mapsto \exp(tX)
$$

is a one parameter subgroup of $G$.

**Remark 1.1.17.** Beware that right invariant vector fields generate a flow of left multiplication!

**Definition 1.1.18.** The map $\exp : \text{Lie}(G) \to G$ is called the exponential map of $G$.

Since the flow of $X$ at time $t$ coincides with the flow of $tX$ at time 1, the exponential map is well-defined.

**Proposition 1.1.19.** The exponential map is smooth and is a local diffeomorphism in a neighbourhood of 0. Its differential at 0 is the canonical identification of $\text{Lie}(G)$ with $T_{1_G} G$.

**Example 1.1.20.** For $G = \text{GL}(n, \mathbb{K})$, the exponential map is simply the exponential of matrices:

$$
\exp : M(n, \mathbb{K}) \to \text{GL}(n, \mathbb{K})
A \mapsto \sum_{n \geq 0} \frac{A^n}{n!}.
$$
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1.1.4 Lie group morphisms

**Definition 1.1.21.** A Lie group morphism is a group homomorphism that is smooth.

**Proposition 1.1.22.** If \( \varphi : G \to H \) is a group morphism, then

\[ d_1G \varphi : \text{Lie}(G) \to \text{Lie}(H) \]

is a Lie algebra morphism. Moreover the following diagram commutes

\[
\begin{array}{ccc}
\text{Lie}(G) & \xrightarrow{d_1G \varphi} & \text{Lie}(H) \\
\exp & \downarrow & \exp \\
G & \xrightarrow{\exp} & H
\end{array}
\]

Conversely, one could hope to extend any Lie algebra homomorphism from \( \text{Lie}(G) \) to \( \text{Lie}(H) \) into a Lie group homomorphism from \( G \) to \( H \). The exponential map provides a way to do so in a neighbourhood of the identity. However, to extend this map to the whole group \( G \), two issues appear:

- \( G \) may not be connected, and the exponential map only reaches the connected component of the identity,
- \( G \) may not be simply connected, in which case the global extension of our morphism could raise a monodromy issue.

This issues can be solved, thanks to the following propositions:

**Proposition 1.1.23.** Let \( G \) be a Lie group. Then the connected component \( G_0 \) of \( 1_G \) in \( G \) is a normal subgroup of \( G \).

**Proposition 1.1.24.** The universal cover \( \tilde{G} \) of a Lie group \( G \) has a unique Lie group structure such that the covering map \( \pi : \tilde{G} \to G \) is a morphism. Moreover, \( \ker \pi \cong \pi_1(G) \) is central in \( \tilde{G} \).

**Example 1.1.25.** The group \( \text{GL}(n, \mathbb{R}) \) has two connected components \( \text{GL}_+(n, \mathbb{R}) \) given by the sign of the determinant. One can show that \( \pi_1(\text{GL}_+(n, \mathbb{R})) \simeq \mathbb{Z}/2\mathbb{Z} \).

The group \( \text{GL}(n, \mathbb{C}) \) is connected and the morphism

\[
: \mathbb{C}^* \to \text{GL}(n, \mathbb{C}) \\
\lambda \mapsto \lambda \text{Id}_n
\]

induces an isomorphism

\[ \pi_1(\text{GL}(n, \mathbb{C})) \simeq \pi_1(\mathbb{C}^*) \simeq \mathbb{Z} . \]
The Lie algebra of a Lie group $G$ is isomorphic to that of $\tilde{G}_0$, for which the following proposition applies.

**Proposition 1.1.26.** Let $G$ and $H$ be two Lie groups and $\psi : \text{Lie}(G) \to \text{Lie}(H)$ a Lie algebra homomorphism. Assume that $G$ is connected and simply connected. Then there exists a unique morphism $\varphi : G \to H$ such that

$$d_{\text{Lie}} \varphi = \psi .$$

As a corollary we get that two Lie groups with the same Lie algebra are “almost isomorphic”. Given a group $G$, we denote by $Z(G)$ the center of $G$, i.e.

$$Z(G) = \{ g \in G \mid gh = hg \text{ for all } h \in G \} .$$

**Corollary 1.1.27.** Let $G$ and $H$ be two connected Lie groups. If $\text{Lie}(G)$ and $\text{Lie}(H)$ are isomorphic then

- $\tilde{G}$ and $\tilde{H}$ are isomorphic,
- $G/Z(G)$ and $H/Z(H)$ are isomorphic.

### 1.1.5 Subgroups of Lie groups

**Definition 1.1.28.** A Lie subgroup of a Lie group is a subgroup that is also a submanifold.

If $H$ is a Lie subgroup of $G$, then in particular $H$ is a Lie group. The Lie algebra of $H$ embeds tautologically as a Lie subalgebra of $G$. More precisely, the Lie algebra morphism $\text{Lie}(H) \to \text{Lie}(G)$ induced by the inclusion $H \hookrightarrow G$ is nothing but the inclusion $T_1G \hookrightarrow T_1G$.

**Example 1.1.29.** Let $V$ be a vector space over $\mathbb{R}$ or $\mathbb{C}$. Then the group $\text{SL}(V)$ of endomorphisms of determinant $1$ is a Lie subgroup of $\text{GL}(V)$. Its Lie algebra is the space of endomorphisms of $V$ that are skew-symmetric with respect to $q$.

The subgroup $\text{SO}(q) = \text{O}(q) \cap \text{SL}(V)$ has index $2$ in $\text{O}(q)$.

**Example 1.1.30.** Let $V$ be a vector space over $\mathbb{R}$ or $\mathbb{C}$ and let $q$ be a non-degenerate bilinear form on $V$. Then the group $O(q)$ of linear transformations of $V$ preserving $q$ is a Lie subgroup of $\text{GL}(V)$ its Lie algebra is the space of endomorphisms of $V$ that are skew-symmetric with respect to $q$. The subgroup $\text{SO}(q) = O(q) \cap \text{SL}(V)$ has index $2$ in $O(q)$.

In particular:

- The groups $O(n, \mathbb{K}) = \{ G \in \text{GL}(n, \mathbb{K}) \mid G^T G = \text{Id}_n \}$ and $SO(n, \mathbb{K}) = \{ G \in O(n, \mathbb{K}) \mid \det G = 1 \}$ are Lie subgroups of $\text{GL}(n, \mathbb{K})$, with the same Lie algebra

$$\mathfrak{so}(n, \mathbb{K}) = \{ A \in M(n, \mathbb{K}) \mid A^T = -A \} .$$
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- Given two integers $p$ and $q$, define
  \[
  \text{Id}_{p,q} = \begin{pmatrix}
  \text{Id}_p & 0 \\
  0 & -\text{Id}_q
  \end{pmatrix} \in \text{GL}(p+q, \mathbb{R}) .
  \]

  Then the group $O(p,q) = \{ G \in \text{GL}(p+q, \mathbb{R}) \mid G^T \text{Id}_{p,q} G = \text{Id}_{p,q} \}$ and the group $SO(p,q) = \{ G \in O(p,q) \mid \det G = 1 \}$ are Lie subgroups of $\text{GL}(p+q, \mathbb{R})$, with the same Lie algebra
  \[
  \mathfrak{so}(p,q) = \left\{ \begin{pmatrix} A & B \\ B^T & -C \end{pmatrix} \right\}, A \in \mathfrak{so}(p, \mathbb{R}), D \in \mathfrak{so}(q, \mathbb{R}), B \text{ of size } p \times q \right\} .
  \]

  \textbf{Remark 1.1.31.} The Lie group $O(p,q)$ has four connected components and $SO(p,q)$ has two as soon as $p,q \geq 1$. We needed, we will denote by $SO_0(p,q)$ their identity component.

  \textbf{Example 1.1.32.} The group
  \[
  \text{Par}_k(n, \mathbb{K}) = \left\{ \begin{pmatrix} A & B \\ 0 & C \end{pmatrix} \right\}, A \in \text{GL}(k, \mathbb{K}), C \in \text{GL}(n-k, \mathbb{K}), \det A \det C = 1
  \]
  is a Lie subgroup of $\text{SL}(n, \mathbb{K})$, with Lie algebra
  \[
  \mathfrak{par}_k(n, \mathbb{K}) = \left\{ \begin{pmatrix} A & B \\ 0 & C \end{pmatrix} \right\}, A \in \text{M}(k, \mathbb{K}), C \in \text{M}(n-k, \mathbb{K}), \text{Tr} A + \text{Tr} C = 0 \right\} .
  \]

  It is a good exercise to prove by hand that the examples above are indeed Lie subgroups of linear groups. One could however invoke Cartan–Von Neumann’s theorem:

  \textbf{Theorem 1.1.33 (Von Neumann [vN29]).} Every closed subgroup of a Lie group is a Lie subgroup.

  One could hope that every Lie subalgebra of $\text{Lie}(G)$ is the Lie algebra of a Lie subgroup. This is unfortunately not true. Let $\mathfrak{h}$ be a Lie subalgebra of $\text{Lie}(G)$ and let $H$ be the subgroup of $G$ generated by $\exp(\mathfrak{h})$. Then $H$ can be given a Lie group structure such that $\text{Lie}(H) = \mathfrak{h}$. However, the inclusion $H \hookrightarrow G$ need not be an embedding. In other words, $H$ has no reason to be closed in $G$.

  The proposition 1.1.35 gives a sufficient criterion for the group generated by $\exp(\mathfrak{h})$ to be closed in $G$. Let us first introduce the normalizer of a Lie subalgebra.

  \textbf{Definition 1.1.34.} Let $\mathfrak{h}$ be a Lie subalgebra of $\text{Lie}(G)$. The 	extit{normalizer of $\mathfrak{h}$ in $\mathfrak{g}$} is the Lie subalgebra
  \[
  N_\mathfrak{g}(\mathfrak{h}) = \{ u \in \mathfrak{g} \mid \text{ad}_u(\mathfrak{h}) \subset \mathfrak{h} \} .
  \]

  The normalizer of $\mathfrak{h}$ in $G$ is the subgroup
  \[
  N_G(\mathfrak{h}) = \{ g \in G \mid \text{Ad}_g(\mathfrak{h}) \subset \mathfrak{h} \} .
  \]
The group $N_G(h)$ is obviously closed. It is thus a Lie subgroup of $G$, with Lie algebra $N_g(h)$. We thus have the following:

**Proposition 1.1.35.** Let $h$ be a Lie subalgebra of $\text{Lie}(G)$. If $N_g(h) = h$, then the subgroup $H$ generated by $\exp(h)$ is a Lie subgroup of $G$, with Lie algebra $h$.

### 1.2 Semisimple Lie groups and Lie algebras

In this section we introduce the dicothomy between solvable and semisimple Lie groups.

#### 1.2.1 Simple, semisimple, quasisimple

Let us first recall a few group theoretic notions and their analogs for Lie algebras.

**Definition 1.2.1.** A subgroup $H$ of a group $G$ is normal (in french: distingue) if $ghg^{-1} \in H$ for all $h \in H$ and all $g \in G$. The group $G$ is simple if its only normal subgroups are $G$ itself and $\{1_G\}$.

The corresponding notion for Lie algebras is that of ideal.

**Definition 1.2.2.** A Lie subalgebra $h$ of a Lie algebra $g$ is an ideal if $[u,v] \in h$ for all $u \in g$ and $v \in h$. The Lie algebra $g$ is simple if has dimension at least 2 and its only ideals are $\{0\}$ and $g$.

Recall that a normal subgroup $H$ of $G$ is a subgroup such that $G/H$ admits a group structure for which the projection $G \to G/H$ is a homomorphism. Similarly, we have the following proposition:

**Proposition 1.2.3.** Let $h$ be an ideal of a Lie algebra $g$. Then there exists a unique Lie bracket on $g/h$ such that the projection $g \to g/h$ is a Lie algebra morphism.

**Proposition 1.2.4.** If $G$ is a Lie group and $H$ a normal Lie subgroup, then $\text{Lie}(H)$ is an ideal of $\text{Lie}(G)$. Conversely, if $h$ is an ideal in $\text{Lie}(G)$, then the subgroup spanned by $\exp(h)$ is a normal subgroup of $G$.

**Corollary 1.2.5.** If $G$ is simple, then $\text{Lie}(G)$ is simple. Conversely, if $\text{Lie}(G)$ is simple and $G$ is connected, then the center of $G$ is discrete and $G/Z(G)$ est simple.

**Remark 1.2.6.** The connected component of the identity of a Lie group is a normal subgroup. In particular, a simple Lie group must be connected.

Here, we take as a definition of semisimplicity what is usually presented as a consequences of Cartan’s criteria, which will come in the next subsection.
Definition 1.2.7. A Lie algebra $\mathfrak{g}$ is *semisimple* if it is isomorphic to a product of simple Lie algebras. A Lie group $G$ is semisimple if its Lie algebra is semisimple. We call it *quasi-simple* if its Lie algebra is simple.

**Example 1.2.8.**
- $\mathfrak{sl}(n, \mathbb{K})$ is simple for $n \geq 2$. Hence the group $\text{SL}(n, \mathbb{C})$ is quasi-simple and the group $\text{PSL}(n, \mathbb{C}) = \text{SL}(n, \mathbb{C})/\{\lambda I_n, \lambda^n = 1\}$ is simple. Similarly, $\text{SL}(2n+1, \mathbb{R})$ is simple and $\text{PSL}(2n, \mathbb{R}) = \text{SL}(2n, \mathbb{R})/\pm I_{2n}$ are simple.
- $\mathfrak{so}(n, \mathbb{K})$ is simple for $n \geq 3$, except for $\mathfrak{so}(4, \mathbb{K}) \simeq \mathfrak{so}(3, \mathbb{K}) \times \mathfrak{so}(3, \mathbb{K})$.
- $\mathfrak{so}(2k+1, \mathbb{K})$ is simple, and $\mathfrak{SO}(2k, \mathbb{K})$ is only quasi-simple (except for $k = 2$), while $\mathfrak{PSO}(2k, \mathbb{K}) = \mathfrak{SO}(2k, \mathbb{K})/\pm \text{Id}_{2k}$ is simple.
- $\mathfrak{so}(p, q)$ is simple for $p + q \geq 3$, except for $\mathfrak{so}(2, 2) \simeq \mathfrak{so}(2, 1) \times \mathfrak{so}(2, 1)$.
- $\mathfrak{SO}_0(p, q)$ is simple when $p$ or $q$ is odd. And $\mathfrak{PSO}_0(2k, 2l) = \mathfrak{SO}_0(2k, 2l)/\pm \text{Id}_{2k+2l}$ is simple.

### 1.2.2 Solvable Lie groups and Lie algebras

If $a$ and $b$ are two elements in $G$, the *commutator* of $a$ and $b$ is the element

$$[a, b] = aba^{-1}b^{-1}.$$ 

If $H$ and $H'$ are two subgroups of $G$, we denote by $[H, H']$ the subgroup generated by all the $[a, b]$ with $a \in H$ and $b \in H'$. In particular, the group $D(G) = [G, G]$ is the *derived subgroup* of $G$.

Analoguously, if $\mathfrak{h}$ and $\mathfrak{h}'$ are subspaces of a Lie algebra $\mathfrak{g}$, we denote by $[\mathfrak{h}, \mathfrak{h}']$ the subspace of $\mathfrak{g}$ spanned by all the $[u, v], u \in \mathfrak{h}, v \in \mathfrak{h}'$ (where $[\cdot, \cdot]$ is the Lie bracket.) In particular, the *derived Lie algebra* of $\mathfrak{g}$ is the ideal $D(\mathfrak{g}) = [\mathfrak{g}, \mathfrak{g}]$.

**Definition 1.2.9.** The *derived series* of a group $G$ is the sequence of subgroups $D^i(G)$ defined recursively by

- $D^0(G) = G$,
- $D^{i+1}(G) = D(D^i(G))$.

The group $G$ is called *solvable* if there exists $i$ such that $D^i(G) = \{1_G\}$.

**Definition 1.2.10.** The *derived series* of a Lie algebra $\mathfrak{g}$ is the sequence of Lie subalgebras $D^i(\mathfrak{g})$ defined recursively by

$$D^0(\mathfrak{g}) = \mathfrak{g},$$

$$D^{i+1}(\mathfrak{g}) = D(D^i(\mathfrak{g})).$$
• $D^0(g) = g$,
• $D^{i+1}(g) = D(D^i(g))$.

The Lie algebra $g$ is solvable if there exists $i$ such that $D^i(g) = \{0\}$.

**Proposition 1.2.11.** Let $G$ be a connected Lie group. Then $D^i(G)$ is the subgroup spanned by $\exp(D^i(\text{Lie}(G)))$.

**Corollary 1.2.12.** Let $G$ be a connected Lie group. Then $G$ is solvable if and only if $\text{Lie}(G)$ is solvable.

**Example 1.2.13.** The main example of a solvable Lie algebra is the algebra $t(n, \mathbb{K})$ of upper triangular matrices. We have

$$D^i(t(n, \mathbb{K})) = \{ M \in M(n, \mathbb{K}) \mid M_{k,l} = 0 \text{ when } l < k + i \}.$$

A theorem of Lie states that, conversely, every solvable Lie algebra is “upper triangular”.

**Theorem 1.2.14** (Lie). Let $g$ be a Lie subalgebra of $\text{End}(V)$. Then there exists a basis of $V$ in which every element of $g$ is upper triangular.

### 1.2.3 Killing form and Cartan’s criteria

**Definition 1.2.15.** The Killing form of a Lie algebra $g$ is the bilinear form $\kappa_g$ defined by

$$\kappa_g(u,v) = \text{Tr}(\text{ad}_u \text{ad}_v).$$

The Killing form is invariant under the adjoint action. More precisely, we have:

**Proposition 1.2.16.** Let $G$ be a Lie group with Lie algebra $g$. Then

$$\kappa_g(\text{Ad}_g u, \text{Ad}_g v) = \kappa_g(u,v)$$

for all $g \in G$ and all $u, v \in g$, and

$$\kappa_g([u,v], w) + \kappa_g(v, [u,w]) = 0$$

for all $u, v, w \in g$.

**Corollary 1.2.17.** Let $i$ be an ideal of $g$. Then $i^\perp \overset{\text{def}}{=} \{ u \in g \mid \kappa_g(u, v) = 0 \text{ for all } v \in i \}$ is an ideal of $g$.

**Proposition 1.2.18.** If $g$ is simple, then $\kappa_g$ is the only ad-invariant bilinear form on $g$ (up to scalar multiplication).
Example 1.2.19. The Killing forms on $\mathfrak{sl}(n, \mathbb{K})$, $\mathfrak{so}(n, \mathbb{K})$ and $\mathfrak{so}(p, q)$ are all proportional to the bilinear form

$$(A, B) \mapsto \text{Tr}(AB)$$

when we see those Lie algebras as Lie subalgebras of $\mathbb{M}(n, \mathbb{K})$.

**Theorem 1.2.20** (Cartan’s criterion for solvability). A Lie algebra $\mathfrak{g}$ is solvable if and only if

$$D(\mathfrak{g})^\perp = \mathfrak{g}.$$

**Theorem 1.2.21** (Cartan’s criterion for semisimplicity). Let $\mathfrak{g}$ be a Lie algebra. The following are equivalent:

* $\mathfrak{g}$ is semisimple,
* the only solvable ideal of $\mathfrak{g}$ is $\{0\}$,
* the Killing form $\kappa_{\mathfrak{g}}$ is non-degenerate.

**Corollary 1.2.22** (Solvable/Semisimple decomposition). Let $\mathfrak{g}$ be a Lie algebra. Then there exists a unique solvable ideal $\text{rad}(\mathfrak{g})$ such that $\mathfrak{g}/\text{rad}(\mathfrak{g})$ is semisimple.

Let $G$ be a Lie group. Then there exists a solvable normal subgroup $S$ such that $G/S$ is semisimple.

**Example 1.2.23.** Define

$$\mathfrak{s}_k(n, \mathbb{K}) = \{ \begin{pmatrix} \lambda \text{Id}_k & B \\ 0 & \mu \text{Id}_{n-k} \end{pmatrix}, \ k \lambda + (n-k)\mu = 0 \} .$$

Then $\mathfrak{s}_k(n, \mathbb{K})$ is a solvable ideal of $\mathfrak{par}_k(n, \mathbb{K})$ and we have

$$\mathfrak{par}_k(n, \mathbb{K})/\mathfrak{s}_k(n, \mathbb{K}) \cong \mathfrak{sl}(k, \mathbb{K}) \oplus \mathfrak{sl}(n-k, \mathbb{K}) .$$

### 1.2.4 Semisimple Lie groups as linear algebraic groups

It is often useful to represent a Lie group as a subgroup of a linear group satisfying some equations. This leads to the following definition:

**Definition 1.2.24.** An algebraic subgroup of $\text{GL}(n, \mathbb{K})$ is a subgroup that is the zero locus of a family of polynomial equations on $\mathbb{M}(n, \mathbb{K})$. A linear algebraic group over $\mathbb{K}$ is an algebraic subgroup of $\text{GL}(n, \mathbb{K})$ for some $n$.

**Example 1.2.25.** The groups $\text{SL}(n, \mathbb{K})$, $\text{O}(n, \mathbb{K})$, $\text{SO}(n, \mathbb{K})$, $\text{Par}_k(n, \mathbb{K})$ are linear algebraic groups over $\mathbb{K}$. The groups $\text{O}(p, q)$ and $\text{SO}(p, q)$ are linear algebraic groups over $\mathbb{R}$.

**Remark 1.2.26.** By Cartan–Von Neumann’s theorem, every algebraic subgroup of $\text{GL}(n, \mathbb{K})$ is in particular a Lie subgroup.
A good candidate to embed a Lie group $G$ in a linear group is the adjoint representation. Let us discuss when this representation identifies $G$ to a linear algebraic group.

**Proposition 1.2.27.** Let $\mathfrak{g}$ be a Lie algebra over $\mathbb{K}$. Then the group $\text{Aut}(\mathfrak{g})$ of Lie algebra automorphisms of $\mathfrak{g}$ is an algebraic subgroup of $\text{GL}(\mathfrak{g})$. Its Lie algebra is the Lie subalgebra of $\text{Der}(\mathfrak{g}) \subset \text{End}(\mathfrak{g})$ consisting of derivations $d : \mathfrak{g} \to \mathfrak{g}$, i.e. endomorphisms satisfying

$$d[u, v] = [du, v] + [u, dv].$$

Let $G$ be a Lie group and $\mathfrak{g}$ its Lie algebra. Then the adjoint representation $\text{Ad} : G \to \text{Aut}(\mathfrak{g})$ has kernel

$$C(G_0) = \{g \in G \mid gh = hg \text{ for all } h \in G_0\},$$

where $G_0$ is the connected component of the identity in $G$. The adjoint representation $\text{ad} : \mathfrak{g} \to \text{Der}(\mathfrak{g})$ has kernel

$$Z(\mathfrak{g}) = \{u \in \mathfrak{g} \mid [u, v] = 0 \text{ for all } v \in \mathfrak{g}\}.$$

In particular, if $G$ is connected and has trivial center, then the adjoint representation of $G$ is faithful.

Its image, however, may not be an algebraic subgroup of $\text{GL}(\mathfrak{g})$, and may not even be closed. Luckily for us, this does not happen for semisimple Lie groups.

**Proposition 1.2.28.** Let $\mathfrak{g}$ be a semisimple Lie algebra. Then $\text{ad}(\mathfrak{g}) = \text{Der}(\mathfrak{g})$.

**Corollary 1.2.29.** Every semisimple Lie algebra $\mathfrak{g}$ is the Lie algebra of the linear algebraic group $\text{Aut}(\mathfrak{g})$. If $G$ is a semisimple Lie group with Lie algebra $\mathfrak{g}$, then $\text{Ad}(G)$ is a finite index subgroup of $\text{Aut}(\mathfrak{g})$.

When working with semisimple Lie groups, there is thus not much loss in assuming that our groups are linear algebraic. There is still a small cost to pay for that though, and it is not true that every semisimple Lie group is linear algebraic.

**Example 1.2.30.**

- The group $\text{PSL}(2, \mathbb{R})$ has fundamental group $\mathbb{Z}$. Its universal cover $\tilde{\text{PSL}}(2, \mathbb{R})$ does not admit a faithful linear representation.

- While $\text{SO}(p, q)$ is algebraic, I am pretty sure that $\text{SO}_0(p, q)$ is not algebraic.
1.3. SYMMETRIC SPACES

• Is $\text{PSL}(n, \mathbb{C})$ linear algebraic in general? It is isomorphic to the connected component of the identity in $\text{Aut}(\mathfrak{sl}(n, \mathbb{C}))$. However, $\text{Aut}(\mathfrak{sl}(n, \mathbb{C}))$ has two connected components, one of which contains $
abla : A \mapsto -A^T$.

When $n \equiv 1$ or 2 modulo 4, $\det(\nabla) = 1$, and I don’t see how to discriminate algebraically between those two components...

1.2.5 Complexification, real forms

An interesting feature of algebraic group is that one can consider the same group in different fields. To formulate this in correct algebraic terms, one would need to define a linear algebraic group $G$ over a field $K$ as an affine variety defined over $K$ with a group structure given by regular maps over $K$. One could then consider the group $G(L)$ consisting of $L$-points of $G$, for any field extension $L$ of $K$. To remain more concrete here, we will stick with $K = \mathbb{R}$ and give the following more down to earth definition:

**Definition 1.2.31.** Let $G$ be an algebraic subgroup of $\text{GL}(n, \mathbb{R})$. Then the complexification of $G$ is the subgroup $G_{\mathbb{C}} \subset \text{GL}(n, \mathbb{C})$ defined by the same polynomial equations as $G$, i.e.

$$G_{\mathbb{C}} = \{ g \in \text{GL}(n, \mathbb{C}) \mid P(g) = 0 \text{ for all } P \in \mathbb{R}[\mathbb{M}(n, \mathbb{R})] \text{ such that } P|_G = 0 \}.$$

**Proposition 1.2.32.** Let $G$ be an algebraic subgroup of $\text{GL}(n, \mathbb{R})$ and $\mathfrak{g}$ its Lie algebra. Then the Lie algebra of $G_{\mathbb{C}}$ is the complexification $\mathfrak{g}_{\mathbb{C}}$ of $\mathfrak{g}$:

$$\mathfrak{g}_{\mathbb{C}} = \mathfrak{g} \otimes \mathbb{C} = \mathfrak{g} \oplus i\mathfrak{g}.$$

**Definition 1.2.33.** Let $H$ be a linear algebraic group over $\mathbb{C}$. A real form of $H$ is a real algebraic subgroup $G$ such that $G_{\mathbb{C}} = H$. Let $\mathfrak{h}$ be a complex Lie algebra. A real form of $\mathfrak{h}$ is a real Lie subalgebra $\mathfrak{g}$ such that $\mathfrak{h} = \mathfrak{g} \oplus i\mathfrak{g}$.

**Example 1.2.34.** $\text{SL}(n, \mathbb{R})$ is a real form of $\text{SL}(n, \mathbb{C})$. So is

$$\text{SU}(n) = \{ M \in \mathbb{M}(n, \mathbb{C}) \mid \overline{M}^T M = \text{Id}_n \}.$$

$\text{O}(n, \mathbb{R})$ and $\text{O}(p, q)$ (for $p + q = n$) are real forms of $\text{O}(n, \mathbb{C})$.

1.3 Symmetric spaces

In this section, we turn to a more geometric point of view, namely viewing them as isometry groups of symmetric spaces.
1.3.1 Homogeneous spaces

Definition 1.3.1. A group $G$ acts transitively on a set $X$ if for any $x, y \in X$, there exists $g \in G$ such that $g \cdot x = y$.

Let $G$ be a Lie group. A $G$-homogeneous space is a (smooth) manifold endowed with a transitive action of $G$ by (smooth) diffeomorphisms.

Let $G$ be a Lie group and $H$ a Lie subgroup of $G$. Then the quotient $G/H$ of $G$ by the action of $H$ by right multiplication is a manifold of dimension $\dim G - \dim H$, on which $G$ acts transitively by left multiplication. The space $G/H$ is thus a $G$-homogeneous space. It has a distinguished point – $[1_G] = H$ seen as an $H$-orbit in $G$ – which is fixed by the action of $H$.

Conversely, let $X$ be a $G$-homogeneous space and $o$ a point in $X$. Then the group $H = \text{Stab}(o)$ is a Lie subgroup of $G$ of dimension $\dim G - \dim X$, and the map

$$: G \to X \quad g \mapsto g \cdot o$$

induces a $G$-equivariant diffeomorphism from $G/H$ to $X$ sending $[1_G]$ to $o$. This draws a bijective correspondence between pointed $G$-homogeneous spaces (up to $G$-equivariant diffeomorphisms) and Lie subgroups of $G$.

Example 1.3.2. The group $\text{SL}(n, \mathbb{K})$ acts transitively on the Grassmanian $\text{Gr}_k(\mathbb{K}^n) = \{k\text{-dimensional subspaces of } \mathbb{K}^n\}$.

The stabilizer of $o = \{(x_1, \ldots, x_k, 0, \ldots, 0), (x_1, \ldots, x_k) \in \mathbb{K}^k\} \in \text{Gr}_k(\mathbb{K}^n)$ is the subgroup $\text{Par}_k(n, \mathbb{K})$. Thus

$$\text{Gr}_k(\mathbb{K}^n) \simeq \text{SL}(n, \mathbb{K})/\text{Par}_k(n, \mathbb{K}) .$$

Example 1.3.3. The group $\text{SL}(n, \mathbb{R})$ acts transitively on the space $\mathbf{P}(Q^+(\mathbb{R}^n))$ of positive definite quadratic forms on $\mathbb{R}^n$ modulo scaling. The stabilizer of the standard quadratic form on $\mathbb{R}^n$ is $\text{SO}(n)$. Thus

$$\mathbf{P}(Q^+(\mathbb{R}^n)) \simeq \text{SL}(n, \mathbb{R})/\text{SO}(n) .$$

Example 1.3.4. Let $\mathbb{R}^{p,q}$ denote the space $\mathbb{R}^{p+q}$ endowed with the standard quadratic form of signature $(p, q)$:

$$q(x) = x_1^2 + \ldots + x_p^2 - x_{p+1}^2 - \ldots - x_{p+q}^2 .$$

Then the group $\text{O}(p, q)$ acts transitively on the space

$$\text{Gr}_{k,l}(\mathbb{R}^{p,q}) = \{P \in \text{Gr}_{k+l}(\mathbb{R}^{p+q}) \mid q_P \text{ of signature } (k, l)\} .$$

The stabilizer of a point $P$ also stabilizes $P^\perp$ and is thus conjugate to $\text{O}(k, l) \times \text{O}(p - k, q - l)$. Thus

$$\text{Gr}_{k,l}(\mathbb{R}^{p,q}) \simeq \text{O}(p, q)/\text{O}(k, l) \times \text{O}(p - k, q - l) .$$
Let \((X,o)\) be a pointed \(G\)-homogeneous space and \(H\) the stabilizer of \(o\). Let \(u\) be a vector in \(\text{Lie}(G)\). Then the action of \(\exp(tu)\) on \(X\) is the flow generated by a vector field \(p(u)\), which is the projection of the right invariant vector field \(u\) on \(G\) to \(G/H\). Those vector fields are sometimes called the *Killing fields* of the \(G\)-homogeneous space \(X\). They form a Lie algebra.

Note that \(H\) acts linearly on \(T_oX\) by differentiating its action on \(X\). Besides, the adjoint action of \(H\) on \(\text{Lie}(G)\) preserves \(\text{Lie}(H)\), and thus induces a linear action on the quotient space \(\text{Lie}(G)/\text{Lie}(H)\).

**Proposition 1.3.5.** The Lie subalgebra of Killing fields vanishing at \(o\) is the image of \(\text{Lie}(H)\) by \(p\). This induces a linear isomorphism

\[
\text{Lie}(G)/\text{Lie}(H) \xrightarrow{\sim} T_oH ,
\]

which is equivariant with respect to the action of \(H\) on those spaces.

**Corollary 1.3.6.** \(G\)-invariant tensors on \(X\) are in bijective correspondance with \(H\)-invariant tensors on \(\text{Lie}(G)/\text{Lie}(H)\).

Assume now that \(G\) is semisimple. Recall that The Lie algebra \(g = \text{Lie}(G)\) then carries the *Killing form* \(\kappa_g\), which is a non-degenerate symmetric bilinear form invariant under the adjoint action.

**Definition 1.3.7.** A Lie sublagebra \(h\) of \(g\) is called *reductive* if \(\kappa_g|_h\) is non degenerate. A Lie subgroup \(H\) of \(G\) is called reductive if its Lie algebra is.

Let \(H\) be a reductive Lie subgroup of \(G\). Denote respectively by \(h\) and \(g\) their Lie algebras. Then the adjoint action of \(H\) on \(g\) preserves the decomposition

\[
g = h \oplus h^\perp .
\]

The quotient \(g/h\) is thus isomorphic to \(h^\perp\) (as a \(H\)-module), on which \(H\) preserves the bilinear form \(\kappa_g|_{h^\perp}\). Applying Corollary 1.3.6, we obtain a pseudo-Riemannian metric on \(G/H\). Let us recall its definition.

**Definition 1.3.8.** A *pseudo-Riemannian metric* on a manifold \(M\) is smooth section of \(\text{Sym}^2T^*M\) (the bundle of symmetric bilinear form on the tangent bundle) which is non degenerate at every point. It is a *Riemannian metric* if it is positive definite at every point.

**Proposition 1.3.9.** Let \(G\) be a semisimple Lie group and \(H\) a reductive subgroup. Then the homogeneous space \(G/H\) carries a \(G\)-invariant pseudo-Riemannian metric.

Such homogeneous spaces are called *reductive homogeneous spaces.*
Remark 1.3.10. If $\mathfrak{h}$ is a reductive subalgebra of $\mathfrak{g}$, then in particular $\mathfrak{h}$ carries a non degenerate bilinear form which is invariant under the adjoint action. This is not far from implying that $\mathfrak{h}$ is semisimple. However, Cartan’s criterion does not apply here since $\kappa_{\mathfrak{g}\mid\mathfrak{h}}$ is not equal to $\kappa_{\mathfrak{h}}$.

Actually we have the following implications: if $\mathfrak{h}$ is a semisimple subalgebra of $\mathfrak{g}$, then $\mathfrak{h}$ is reductive. Conversely, if $\mathfrak{h}$ is reductive, then $\mathfrak{h}$ is the direct sum of a semisimple subalgebra and an abelian subalgebra.

1.3.2 (Pseudo-Riemannian) Symmetric spaces

Let $(X, h)$ be a pseudo-Riemannian manifold. An isometry of $X$ is a diffeomorphism $\varphi : X \to X$ such that $\varphi^* h = h$. When $h$ is a Riemannian metric, this is equivalent to being an isometry for the induced distance.

Definition 1.3.11. A pseudo-Riemannian manifold $(X, h)$ is a symmetric space if for every $x \in X$, there exists a (necessarily unique) isometry $s_x$ of $X$ fixing $x$ and such that $d_x s_x = -\text{Id}_{T_x X}$.

Remark 1.3.12. Isometries of a connected pseudo-Riemannian manifold are characterized by their value and differential at a point. In particular, the above condition on $s_x$ implies that $s_x$ is an involution. We call it the central symmetry at $x$.

Using the fact that the central symmetry $s_x$ reflect the geodesics passing through $x$, one can move any point of $x$ to any other point in its connected component by composing finitely many central symmetries, and deduce the following:

Proposition 1.3.13. Let $(X, h)$ be a connected pseudo-Riemannian symmetric space. Then the group $\text{Isom}(X)$ of isometries of $X$ is a Lie group acting transitively on $X$.

Let $G$ be a semisimple Lie group, $\mathfrak{g}$ and $\theta$ an involutive automorphism of $G$ (i.e. a Lie group automorphism such that $\theta \circ \theta = \text{Id}$). We also denote by $\theta$ the induced automorphism of $\mathfrak{g} = \text{Lie}(G)$. Denote by $G^\theta$ the subgroup fixed by $\theta$:

$$G^\theta = \{ g \in G \mid \theta(g) = g \} .$$

Proposition 1.3.14. The Lie subgroup $G^\theta$ is reductive and the reductive homogeneous space $G/G^\theta$ is a pseudo-Riemannian symmetric space.

Proof. Since $\theta^2 = \text{Id}$, the Lie algebra $\mathfrak{g}$ decomposes as

$$\mathfrak{g} = E_1(\theta) \oplus E_{-1}(\theta),$$

where $E_1(\theta)$ and $E_{-1}(\theta)$ denote respectively the eigenspaces of $\theta$ for the eigenvalues 1 and $-1$. Since $\theta$ preserves the Killing form, this splitting is
orthogonal. In particular, $\kappa_{\mathfrak{g}|E_1(\theta)}$ is non degenerate. But we have $E_1(\theta)$ is exactly the Lie algebra $\mathfrak{g}^\theta$ of $G^\theta$. Thus $G^\theta$ is reductive.

Let $(X,o)$ denote the $G$-homogeneous space $G/G_\theta$ with the base point $[1_G]$, and let $h$ be the $G$-invariant pseudo-Riemannian metric on $X$ constructed from $\kappa_{\mathfrak{g}|E_1(\theta)}$. One easily verifies that $\theta$ induces an involution $s_\theta$ of $X$ fixing $o$ and normalizing the action of $G$. In particular, $s_\theta^*h$ is a $G$-invariant metric on $X$. But at the point $o$, we have $d_o s_\theta \simeq \theta_{E_1(\theta)} = -\text{Id}$. In particular, $s_\theta^*h_o = h_o$. We deduce that $s_\theta^*h = h$ and that $s_\theta$ is a central symmetry at $o$. Finally, conjugating by elements of $G$ and using the transitivity of the action of $G$ on $X$, we can construct central symmetries at every point.

Example 1.3.15. Let $G$ be $\text{SL}(n,K)$ and $\theta : g \mapsto g^{T-1}$. Then $\theta$ is an involutive automorphism and $G^\theta = \text{SO}(n,K)$. Thus $\text{SL}(n,K)/\text{SO}(n,K)$ is a symmetric space.

Example 1.3.16. Let $G$ be $O(p,q)$. Let $s$ be the orthogonal symmetry in $\mathbb{R}^{p,q}$ with respect to a $k+l$-dimensional plane $P$ of signature $(k,l)$. Take $\theta : g \mapsto sgs^{-1}$. Then $G^\theta = \text{Stab}(P)$ is isomorphic to $O(k,l) \times O(p-k,q-l)$ and $G/G^\theta = \text{Gr}_{k,l}(\mathbb{R}^{p,q})$.

Two particular cases will be of interest for us:

- $G = O(p,q)$ and $\theta : g \mapsto I_{p,q} g I_{p,q}$. Then $G/G^\theta = O(p,q)/O(p) \times O(q)$.

- $G = O(p,q+1)$ and $\theta : g \mapsto I_{p+q,1} g I_{p+q,1}$. Then $G/G^\theta = O(p,q+1)/O(p,q) \times O(1) \simeq \{\text{negative definite lines in } \mathbb{R}^{p,q}\}$ is the pseudo-Riemannian hyperbolic space of signature $(p,q)$, that we will denote $\mathbb{H}_{p,q}$. For $p = n$ and $q = 0$ we recover the hyperbolic space $\mathbb{H}^n$.

Example 1.3.17. Let $H$ be an algebraic subgroup of $\text{GL}(n,\mathbb{R})$. Take $G = H_\mathbb{C}$, the complexification of $H$, and $\theta : g \mapsto \overline{g}$ the complex conjugation. Then $G^\theta = H$. Thus $H_\mathbb{C}/H$ is a pseudo-Riemannian symmetric space.

Example 1.3.18. Let $G$ be a semisimple Lie group. Consider $G$ as a homogeneous space under the action of $G \times G$ given by $(g,h) \cdot x = gxh^{-1}$.

The stabilizer of $1_G$ is the subgroup $\Delta(G) = \{(g,g), g \in G\} \subset G \times G$, i.e. the fixed point set of the involution $\theta : (g,h) \mapsto (h,g)$.
The group $\Delta(G)$ acts on $T_{1_G}G \simeq \text{Lie}(G)$ via the adjoint action. Since the Killing form of $\text{Lie}(G)$ is invariant under the adjoint action, it extends to a pseudo-Riemannian metric $\kappa_G$ on $G$ that is invariant under the action of $G \times G$. The symmetric space $G \times G/\Delta(G)$ is the space $(G, \kappa_G)$. The central symmetry at $1_G$ is the map $g \mapsto g^{-1}$.

1.3.3 Cartan involutions and Riemannian symmetric spaces

So far, haven’t discussed the signature of the Killing form nor of the induced pseudo-Riemannian metrics on reductive homogeneous spaces. In this section, we describe in more details the symmetric spaces that are Riemannian.

Let us start with the following lemma:

**Lemma 1.3.19.** Let $K$ be a compact subgroup of $\text{GL}(n, \mathbb{R})$ and $\mathfrak{k}$ its Lie algebra. Then the bilinear form

$$(A, B) \mapsto \text{Tr}(AB)$$

is negative definite in restriction to $\mathfrak{k}$.

**Proof.** It is well-known that $K$ preserves a scalar product on $\mathbb{R}^n$. Up to conjugation, we can thus assume that $\mathfrak{k}$ consists of antisymmetric matrices. For all $A \in \mathfrak{k}\setminus\{0\}$, we thus have

$$\text{Tr}(A^2) = -\text{Tr}(AA^T) = -\sum_{1 \leq i,j \leq n} A_{i,j}^2 < 0.$$ 

\qed

**Corollary 1.3.20.** Let $\mathfrak{g}$ be a semisimple Lie algebra. Then $\text{Aut}(\mathfrak{g})$ is a compact group if and only if $\kappa_\mathfrak{g}$ is negative definite.

**Proof.** Assume $\text{Aut}(\mathfrak{g})$ is a compact subgroup of $\text{GL}(\mathfrak{g})$. Since $\mathfrak{g}$ is semisimple, $\text{ad} : \mathfrak{g} \to \text{Der}(\mathfrak{g}) = \text{Lie}(\text{Aut}(\mathfrak{g}))$ is an isomorphism. Thus, for any $u \in \mathfrak{k}\setminus\{0\}$ we have $\text{ad}_u \neq 0$ and $\kappa_\mathfrak{g}(u, u) = \text{Tr}(\text{ad}_u \text{ad}_u) < 0$ by Lemma 1.3.19.

Conversely, if $\kappa_\mathfrak{g}$ is negative definite, then $\text{Aut}(\mathfrak{g}) \subset O(\kappa_\mathfrak{g})$ which is compact. \qed

Now, let $\mathfrak{h}$ be a complex Lie algebra. Note that one can define a complex Killing form $\kappa^C_\mathfrak{h}$ by taking complex traces:

$$\kappa^C_\mathfrak{h}(u, v) = \text{Tr}(\text{ad}_u \text{ad}_v)$$

or a real Killing form $\kappa^R_\mathfrak{h}$ by viewing $\mathfrak{h}$ as a real Lie algebra (and thus taking real traces). The two are related by

$$\kappa^R_\mathfrak{h} = 2\Re(\kappa^C_\mathfrak{h}).$$ 

If $\mathfrak{g}$ is a real form of $\mathfrak{h}$, then $\kappa^C_\mathfrak{h}|_{\mathfrak{g}}$ is real and equals $\kappa_\mathfrak{g}$. Thus $\kappa^C_\mathfrak{h}$ is the $\mathbb{C}$-bilinear extension of $\kappa_\mathfrak{g}$. 
1.3. SYMMETRIC SPACES

**Definition 1.3.21.** Let \( h \) be a complex semisimple Lie algebra. A real form \( g \) of \( h \) is *compact* if \( \kappa_g \) is negative definite.

The term “compact” is motivated by the following definition:

**Definition 1.3.22.** Let \( H \) be a complex semisimple linear algebraic group and \( G \) a real form of \( H \). Then \( \text{Lie}(G) \) is a a compact real form of \( \text{Lie}(H) \) if and only if \( G \) is compact.

**Example 1.3.23.** The group \( \text{SO}(n, \mathbb{R}) \) is a compact real form of \( \text{SO}(n, \mathbb{C}) \).

The group \( \text{SU}(n) = \{ M \in \text{SL}(n, \mathbb{C}) \mid M^T M = \text{Id}_n \} \) is a compact real form of \( \text{SL}(n, \mathbb{C}) \).

**Definition 1.3.24.** Let \( g \) be a real semisimple Lie algebra. An involutive automorphism \( \theta \) of \( g \) is called a *Cartan involution* if the bilinear form

\[
B_\theta(\cdot, \cdot) = \kappa_g(\cdot, \theta \cdot)
\]

is negative definite.

In other words, \( \theta \) is a Cartan involution if \( \kappa_g \) is negative definite on the Lie subalgebra \( g^\theta = \{ u \mid \theta(u) = u \} \) and positive definite on \( g^{\theta \perp} \).

An involutive automorphism \( \theta \) of a semisimple Lie group \( G \) is a Cartan involution if the induced involution on \( \text{Lie}(G) \) is a Cartan involution.

**Example 1.3.25.** Let \( h \) be a complex semisimple Lie algebra and \( g \) a compact real form of \( h \) then the involution

\[
(w \mapsto \overline{w}) : \ h = g \oplus ig \to g \oplus ig
u + iv \mapsto u - iv
\]

is a Cartan involution of \( h \) (seen as a real Lie algebra).

**Example 1.3.26.** Let \( G \) be \( \text{O}(p, q) \). Then \( g \mapsto I_{p,q}gI_{p,q} \) is a Cartan involution of \( G \).

**Example 1.3.27.** Let \( G \) be \( \text{SL}(n, \mathbb{R}) \). Then \( g \mapsto g^{T^{-1}} \) is a Cartan involution of \( G \).

If \( \theta \) is a Cartan involution of \( G \), then the symmetric space \( G/G_\theta \) is Riemannian since \( \kappa_{g^{\theta \perp}} \) is positive definite. The converse is almost true.

**Proposition 1.3.28.** Let \( \theta \) be an involution of \( g \) such that \( \kappa_{g^{\theta \perp}} \) is positive definite. Then \( g^{\theta} \) contains an ideal \( h \) of \( g \) such that the involution induced by \( \theta \) on \( g/h \) is a Cartan involution.

Symmetric spaces of the form \( G/G_\theta \) where \( G \) is semisimple and \( \theta \) is a Cartan involution are called of *non-compact type*.

Note that, if \( K \) is a compact semisimple Lie group with Lie algebra \( \mathfrak{k} \) and \( \theta \) any involution of \( K \), then \( \kappa_{\mathfrak{k}^{\theta \perp}} \) is negative definite by Lemma 1.3.19. Thus
\( \theta \) is not a Cartan involution (unless \( \theta = \text{Id}_K \), in which case \( K/K^\theta \) is reduced to a point). However, the opposite of the Killing form endows \( K/K^\theta \) with the structure of Riemannian symmetric space. Such a symmetric space is called of compact type.

**Example 1.3.29.** Let \( \theta \) be the involution of \( K = \text{O}(n + 1) \) given by
\[
g \mapsto I_n g I_{n,1}.
\]
Then \( G^\theta = \text{O}(n) \times \text{O}(1) \) and \( G/G^\theta \) is the sphere of dimension \( n \) with its round metric.

**Theorem 1.3.30.** Let \( X \) be a connected and simply connected Riemannian symmetric space. Then \( X \) decomposes as the product of a symmetric space of compact type, a symmetric space of non-compact type, and a Euclidean space.

This theorem provides a further motivation for the study of semisimple Lie groups: they are essentially the isometry groups of Riemannian symmetric spaces.

**Theorem 1.3.31 (Cartan).** Every complex semisimple Lie algebra admits a compact real form.

**Corollary 1.3.32.** Every connected real semisimple Lie group \( G \) admits a Cartan involution. All the Cartan involutions of \( G \) are conjugated by an element of \( G \). In particular, if \( G \) is a complex linear algebraic group, then all the Cartan involutions of \( G \) are given by a compact real form and all the compact real forms are conjugate. Finally, if a subgroup \( K \) of \( G \) is compact modulo \( Z(G) \), then \( K \) is fixed by a Cartan involution.

Theorem 1.3.31 was initially obtained by Cartan as a by-product of his classification of complex semisimple Lie algebras. Yet, Cartan was aware that proving the theorem a priori would simplify some of his argument and provide a better geometric insight. He conjectured the existence of an intrinsic proof. Weyl proposed an intrinsic proof which still relied on a deep understanding of the structure of semisimple Lie algebras, in particular their roots systems. Later, Richardson found a more elementary proof, which is the subject of Exercise 2 of the second exercise sheet. Donaldson recently reinterpreted Richardson’s proof in differential geometric terms.

Corollary 1.3.32 follows from Theorem 1.3.31 after working out the geometry of the Riemannian symmetric space \( G/G^\theta \) associated to a Cartan involution. More precisely, we will use the following:

**Lemma 1.3.33.** Let \( G \) be a semisimple Lie group and \( \theta \) a Cartan involution of \( G \). Then the Riemannian symmetric space \( G/G^\theta \) has non-positive sectional curvature.
In the next section, we will recall the theory of principal bundles, connections, and curvature, to describe the curvature form of $G/G^\theta$ and eventually prove this lemma. The main property of negatively curved spaces needed to prove Corollary 1.3.32 is the following:

**Lemma 1.3.34.** Let $K$ be a compact group of isometries of a complete connected simply connected Riemannian manifold of non-positive sectional curvature. Then $K$ has a fixed point.

**Sketch of the proof.** Let $\mu$ be a left-invariant probability measure on $K$. (Such a measure exists and is unique; it is called the Haar measure of $K$.) Let $x$ be some point in $X$. Define the function

$$F : X \rightarrow \mathbb{R}_+$$

$$y \mapsto \int_K d^2(k \cdot x, y) d\mu(k),$$

where $d$ denotes the distance associated to the Riemannian metric on $X$. The negative curvature property implies that $y \mapsto d^2(x, y)$ is strictly convex. Hence $F$ is strictly convex. It is also proper, and thus admits a unique minimum, which is fixed by $K$.

**Proof of Corollary 1.3.32 from Theorem 1.3.31.** Up to quotienting $G$ by its center (which is invariant by any automorphism), we can assume that $G$ is the identity component of $\text{Aut}(g)$.

Let $\theta$ be a Cartan involution. There is a a technical issue in proving that $G/G^\theta$ is simply connected. We assume it here.

Let us first prove the third point: If $K$ is a compact subgroup of $G$, then the action of $K$ on $G/G^\theta$ fixes a point by Lemma 1.3.34. Thus $K$ is contained in a subgroup conjugate to $G^\theta$.

Let now $\theta'$ be another Cartan involution. Let us show that $\theta'$ is conjugate to $\theta$. Since $G^\theta$ is a compact group, it is conjugate to a subgroup of $G^\theta$. Conversely $G^\theta$ is conjugate to a subgroup of $G^\theta$ it follows that $G^\theta$ and $G^\theta$ and $G^\theta$ have the same dimension and are conjugate, from which we deduce that $\theta$ and $\theta'$ are conjugate.

Finally lets prove that $G$ admits a Cartan involution. Let $G^C$ denote the identity component in $\text{Aut}(g_C)$. And $\sigma = G^C \rightarrow G^C$ the involution associated to the real form $G \subset G^C$. By Theorem 1.3.31, we know that $g_C^\theta$ admits a Cartan involution, which induces a Cartan involution of $G^{\sigma C}$. Moreover, from the arguments above, we know that the space of Cartan involutions of $G^C$ is isomorphic to $G^C/K$ where $K$ is the subgroup fixed by a given Cartan involution.

Now, $\sigma$ acts by conjugation on the space of Cartan involutions. Since $\sigma^2 = \text{Id}$, it follows from Lemma 1.3.34 that $\sigma$ has a fixed point. We hence obtain a Cartan involution $\theta$ of $G^C$ that commutes with $\sigma$. Thus $\theta$ preserves $G$, and one verifies that it induces a Cartan involution of $G$.  

$\square$
1.4 Bundles, connections, curvature

1.4.1 \( G \)-bundles

Let \( G \) be a Lie group acting on a manifold \( X \). One can define a notion of \((G, X)\)-bundle over a manifold \( M \), which is morally a fibration over \( X \) whose fibers are identified with \( X \) “up to transformations of \( G \)”. A precise definition is as follows:

**Definition 1.4.1.** A \((G, X)\)-bundle over \( M \) is the data of a manifold \( E \) with a submersion \( \pi : E \to M \), a covering \((U_i)_{i \in I}\) of \( M \) and a family of diffeomorphisms \( \varphi_i : \pi^{-1}(U_i) \to X \times U_i \) such that:

- The following diagram commutes:

\[
\begin{array}{ccc}
\pi^{-1}(U_i) & \xrightarrow{\varphi_i} & X \times U_i \\
\downarrow{\pi} & & \downarrow{p^2} \\
U_i & \xrightarrow{p_2} & X \times U_i \\
\end{array}
\]

where \( p_2 \) denotes the projection on the second factor,

- For every \( i, j \in I \), there exists a function \( g_{ij} : U_i \cap U_j \to G \) such that \( \varphi_i \varphi_j^{-1} : X \times U_i \cap U_j \to X \times U_i \cap U_j \) maps \((x, y)\) to \((g(y) \cdot x, y)\).

When \( X \) is the group \( G \) itself acted on by left multiplication, we obtain what is called a principal bundle. Since that the action of \( G \) on itself by right multiplication commutes with the left multiplications, every principal \( G \)-bundle carries a right action of \( G \). Conversely, a diffeomorphism of \( G \) that commutes with right multiplication is a left multiplication. This eventually leads to the following alternative description of a principal bundle:

**Definition 1.4.2.** A Principal \( G \)-bundle over \( M \) is a manifold \( P \) with a submersion \( \pi : P \to M \) and a right action of \( G \) which preserves the fibers of \( \pi \) and acts simply transitively on each fiber.

From a principal \( G \)-bundle one can construct a \((G, X)\)-bundle for any \( X \) acted on by \( G \).

**Definition 1.4.3.** Let \( P \) be a principal \( G \)-bundle over \( M \) and \( X \) a manifold with a left \( G \) action. The \( X \)-bundle associated to \( P \) is the \((G, X)\)-bundle \( P \times_G X \), quotient of \( P \times X \) by the action of \( G \) given by

\[
g \cdot (y, x) = (y \cdot g^{-1}g \cdot x) .
\]

Conversely, from a \((G, X)\)-bundle, one can recover a principal \( G \)-bundle, as soon as the action of \( G \) on \( X \) is faithful (meaning that only \( 1_G \) acts as the identity of \( X \)).
**Proposition 1.4.4.** Let $X$ be a manifold with a faithful action of $G$ and $E$ a $(G,X)$-bundle over $X$. Then there is a unique principal $G$-bundle $P$ over $M$ (up to isomorphism) such that $E$ is isomorphic to the associated bundle $P \times_G X$.

**Example 1.4.5.** Let $X = G/H$ be a $G$-homogeneous space. Then the $X$-bundle associated a principal $G$-bundle $P$ is the quotient $P/H$ of $P$ by the right action of $H$.

**bundle of automorphisms and adjoint bundle**  Let $P$ be a principal $G$-bundle. Let $X$ be the group $G$ with the action of $G$ on itself by conjugation. Then we call the associated bundle $P \times_G X$ the *automorphism bundle* of $P$ and denote it by $\mathrm{Aut}(P)$. Since the conjugation action of $G$ on itself is by group automorphisms, the bundle $\mathrm{Aut}(P)$ inherits a fiberwise group structure. This bundle has the following interpretation:

**Proposition 1.4.6.** The fiber of $\mathrm{Aut}(P)$ at $x$ is the group of diffeomorphisms of $P_x$ that commute with the right action of $G$. It is a Lie group isomorphic (though not canonically) to $G$.

Let now $\mathfrak{g}$ be the Lie algebra of $G$, with the adjoint action of $G$. Then the associated bundle $P \times_G \mathfrak{g}$ is called the *adjoint bundle* of $P$ and denoted by $\mathrm{Ad}(P)$. Each of its fibers carries a Lie bracket, and $\mathrm{Ad}(P)_x$ is canonically identified with the Lie algebra of $\mathrm{Aut}(P)_x$.

**Principal bundles an vector bundles**  Let $\mathbb{K}$ denote the field $\mathbb{R}$ or $\mathbb{C}$. Let $G$ be the group $\text{GL}(n, \mathbb{K})$ acting on the vector space $X = \mathbb{K}^n$. Then a $(G, X)$-bundle over $M$ is the same thing as a (real or complex) vector bundle of rank $n$.

To be more precise, let $E$ be a (real or complex) vector bundle of rank $n$ over $M$. Define the *frame bundle* $R(E)$ as the set of pairs $(x, \varphi)$ where $x$ is a point in $M$ and $\varphi : \mathbb{K}^n \to E_x$ is a linear isomorphism. Then $\text{GL}(n, \mathbb{K})$ acts on $R(E)$ by precomposing $\varphi$. This gives $R(E)$ the structure of a principal $\text{GL}(n, \mathbb{K})$-bundle, and one can verify that

$$R(E) \times_{\text{GL}(n, \mathbb{K})} \mathbb{K}^n = E.$$  

More generally, if a Lie group $G$ acts linearly on a vector space $V$, then the bundle $P \times_G V$ associated to a principal bundle $P$ on $M$ can be seen as a vector bundle on $M$ with “additional structure” coming from $G$-invariant objects on $V$.

**Example 1.4.7.** Let $G$ be the Lie group $\text{SL}(n, \mathbb{R})$ acting on $V = \mathbb{R}^n$. Then a $(G, V)$-bundle is rank $n$ vector bundle $E$ with a *volume form*, i.e. a nowhere vanishing section of $\Lambda^n E^*$. 
CHAPTER 1. LIE GROUPS AND SYMMETRIC SPACES

Example 1.4.8. Let $G$ be the Lie group $O(n, \mathbb{R})$ acting on $V = \mathbb{R}^n$. Then a $(G, V)$-bundle is a rank $n$ vector bundle with a metric, i.e. a scalar product on each fiber.

Example 1.4.9. Let $G$ be a connected Lie group of dimension $n$. Let $P$ be a principal $G$-bundle over $M$. Then the bundle $\text{Ad}(P)$ is a vector bundle $E$ of rank $n$ together with a section of $\Lambda^2 E^* \otimes E$ (i.e. an antisymmetric bilinear map $[\cdot, \cdot]_x : E_x \times E_x \to E_x$ at each point $x$), which satisfies the Jacobi identity fiberwise, and such that $(E_x, [\cdot, \cdot]_x)$ is isomorphic to $\text{Lie}(G)$ for each $x$.

Extension and reduction of structure group

Let $G$ be a Lie group and $H$ a Lie subgroup of $G$. Then $H$ acts on $G$ by left translation and this action. Thus, if $P$ is a principal $H$-bundle over $M$, one can define an associated bundle $P \times_H G$. Since the left multiplication of $H$ on $G$ commutes with the right multiplication of $G$ on itself the associated bundle $P \times_H G$ carries a natural structure of $G$-bundle. We say that $P \times_H G$ is obtained from $P$ by extension of structure group. The reverse operation is called reduction of structure group. Let us be more precise.

Let $P$ be a principal $H$-bundle and let $P'$ be the associated bundle $P \times_H G$. Then

- the right action of $G$ on $P \times G$ given by
  $$(x, y) \cdot g = (x, yg)$$
  induces a structure of a principal $G$-bundle on $P'$,

- the composition of $x \mapsto (x, 1_G)$ with the projection from $P \times G$ to $P'$ defines a $H$-equivariant map from $\varphi : P \to P'$,

- quotienting by the action of $H$, one obtains a section of the $G/H$-bundle $P'/H$ associated to $P'$.

Proposition 1.4.10. Let $P'$ be a principal $G$-bundle over $M$. Then the following objects are equivalent:

- a principal $H$-bundle $P$ and an isomorphism $P \times_H G \cong P'$,

- a principal $H$-bundle $P$ and an $H$-equivariant map from $P$ to $P'$ lifting the identity on $M$ (i.e. mapping the fiber $P_x$ to $P'_x$),

- a section of $P'/H$.

Any such object is called a reduction of structure group $P'$ to $H$.

Example 1.4.11. Let $P$ be a principal $\text{GL}(n, \mathbb{R})$ bundle and $E$ the associated vector bundle. Then the bundle $P/\text{SO}(n, \mathbb{R})$ can be seen as the bundle of metrics of $E$, in the sense that the fiber of $P/\text{O}(n, \mathbb{R})$ at a point $x$ identifies with the space of scalar products on $E_x$. Thus, a reduction of structure group of $P$ to $\text{O}(n, \mathbb{R})$ is the same as a metric on $E$. 
1.4.2 Connections

If $E$ is a vector bundle over $M$, we denote by $\Gamma(E)$ its space of (smooth) sections and more generally by $\Omega^k(E)$ the space of $k$-forms on $M$ with values in $E$, i.e. sections of $\Lambda^k T^*M \otimes E$.

**Principal connections**  In this paragraph, let us fix a principal $G$-bundle $P$ over a manifold $M$ and denote by $\pi : P \to M$ the fibration.

All the fibers are morally "copies of $G". However there is no standard way to identify two different fibers. Sometimes, one needs a way to "follow the fiber" over a path in $M$ in a canonical way. This way is given by the parallel transport of a connection.

Let $V \subset TP$ denote the subbundle consisting of vectors tangent to the fibers of the fibration $\pi : P \to M$.

**Definition 1.4.12.** A principal connection on $P$ is a the data of a a subbundle $H$ of $TP$ which is preserved by the right action of $G$, and such that

$$TP = V \oplus H .$$

**Example 1.4.13.** The trivial bundle $M \times G$ has a trivial principal connection given by the kernel of the projection to the second factor.

Since the action of $G$ on each fiber is simply transitive, the tangent space to the fiber at every point $x \in P$ can be identified with the Lie algebra $g$ of the group $G$. More precisely, if we define

$$r_x : G \to P$$

$$g \mapsto x \cdot g ,$$

then there is a well-defined section $\alpha$ of the bundle $\text{Hom}(V, g)$ such that for all $x \in P$ and all $u \in g$,

$$\alpha_x (\text{d}r_x (u)) = u .$$

The form section $\alpha$ satisfies the following equivariance property:

$$g^* \alpha = \text{Ad}_{g^{-1}} \alpha$$

for every $g \in G$.

Now let $H$ be a principal connection on $P$. let $p_H : TP \to V$ be the projection transversally to $H$. Then $\omega = \alpha \circ p$ is a 1-form on $P$ with values in $g$.

**Definition 1.4.14.** The form $\omega = \alpha \circ p$ associated to a principal connection $H$ is called the connection form of $H$. 

Since $H$ is $G$-invariant, the connection form also satisfies

$$g^*\omega = \text{Ad}_{g^{-1}}\omega.$$  

Conversely, one can recover the horizontal distribution $H$ from the connection form:

**Proposition 1.4.15.** Let $\omega$ be a 1-form on $P$ with values in $\mathfrak{g}$ such that:

1. $g^*\omega = \text{Ad}_{g^{-1}}\omega$ for all $g \in G$,
2. $\omega|_F = \alpha$.

Then $\ker\omega$ is the horizontal distribution of a principal connection on $P$.

From now on, we consider alternatively a connection as the data of a horizontal distribution or the data of a connection form.

**Proposition 1.4.16.** Let $\omega_1, \ldots, \omega_n$ be $n$ connection forms on $P$ and $f_1, \ldots, f_n$ be $n$ smooth functions on $M$ such that $f_1 + \ldots + f_n = 1$. Then

$$\omega = \sum_{i=1}^{n} f_i \circ \pi \omega_i$$

is a connection form on $P$.

**Proof.** Since the functions $f_i \circ \pi$ are by definition $G$-invariant functions on $P$, the equivariance property (2) for $\omega$ follows from that of the $\omega_i$. Since $f_1 + \ldots + f_n = 1$, we have $\omega|_F = \alpha$.

**Corollary 1.4.17.** Every principal bundle admits a principal connection.

**Proof.** One can construct connection forms locally by taking arbitrary trivializations, and glue them together using partitions of unity.

**Proposition 1.4.18.** The space of connection forms on $P$ is an affine space over $\Omega^1(\text{Ad}(P))$.

**Proof.** Let $\omega_1$ and $\omega_2$ be two connection forms on $P$. Then $\omega_1 - \omega_2$ satisfies

$$g^*(\omega_1 - \omega_2) = \text{Ad}_{g^{-1}}(\omega_1 - \omega_2)$$

and vanishes on $F$. This is equivalent to $\omega_1 - \omega_2$ being the pullback of a 1-form on $M$ with values in $\text{Ad}(P)$. 

Principal connections are related to linear connections on vector bundles. Linear connections can be viewed either as a way to parallel transport the fibers of a vector bundle, or as a way to derive sections.

**Definition 1.4.19.** Let $E$ be a vector bundle over $M$. A **linear connection** or **covariant derivative** is an $\mathbb{R}$-linear differential operator $\nabla : \Gamma(E) \to \Omega^1(E)$ satisfying the *Leibniz rule*:

$$\nabla(fs) = (df)s + f \nabla s$$

for all $s \in \Gamma(E)$ and all $f \in C^\infty(M)$.

**Proposition 1.4.20.** A linear connection $\nabla$ on $E$ extends uniquely to a family of differential operators $\nabla^k : \Omega^k(E) \to \Omega^{k+1}(E)$ satisfying the *generalized Leibniz rule*:

$$\nabla(\alpha \wedge s) = (d\alpha) \wedge s + (-1)^p \alpha \wedge ds$$

for all $\alpha \in \Omega^p(M)$ and all $s \in \Omega^q(E)$.

**Proposition 1.4.21.** The space of connections on $E$ is an affine space over $\Omega^1(\text{End}E)$.

**Proof.** If $\nabla^1$ and $\nabla^2$ are two linear connections on $E$. Then $\nabla^1 - \nabla^2$ is $C^\infty$-linear operator, i.e.

$$(\nabla^1 - \nabla^2)s = f(\nabla^1 - \nabla^2)s.$$  

This implies that $(\nabla^1 - \nabla^2)s$ at a point $x$ only depends on the value of $s$ at $x$, and thus that $\nabla^1 - \nabla^2$ acts as an element of $\Omega^1(\text{End}E)$.

Conversely, if $\nabla$ is a connection and $A \in \Omega^1(\text{End}E)$, then $\nabla + A$ still satisfies the Leibniz rule and is thus a connection. \qed

Let now $P$ be a principal $G$-bundle, $\rho : G \to \text{GL}(n, \mathbb{R})$ a linear representation of $G$ and $E = P \times_G \mathbb{R}^n$ the associated vector bundle of rank $n$. We still denote by $\rho$ the induced representation of Lie algebras from $\mathfrak{g}$ to $\text{End}(\mathbb{R}^n)$. Note that $\rho$ induces a bundle map from $\text{Ad}(P)$ to $\text{End}(E)$ that we still denote by $\rho$.

**Proposition 1.4.22.** Let $D$ denote the trivial linear connection on $P \times \mathbb{R}^n$. If $\omega$ is a connection form on $P$, then the connection $D + \rho(\omega)$ on $P \times \mathbb{R}^n$ that induces a linear connection on $P \times \mathbb{R}^n$. We denote it by $\nabla^\omega$.  

---

**linear connections**
Proof. Let $\bar{s}$ be a section of $P \times_G \mathbb{R}^n$. Then $s = \bar{s} \circ \pi$ is a function from $P$ to $\mathbb{R}^n$ that which is $G$-equivariant, i.e. such that
\[ s(x \cdot g) = \rho(g^{-1})s(x) \]

Let $s$ be such a section. Deriving the above relation, we get that
\[ Ds(u) = -\rho(\omega(u))s \]

for $u \in V$. Thus the 1-form $(D + \rho(\omega))s$ vanishes along the fibers of $P$. Moreover,
\[ g^*(D + \rho(\omega))s = D(s \circ g) + \rho(g^*\omega)s \circ g = \rho(g^{-1})Ds + \rho(\text{Ad}_{g^{-1}}\omega)\rho(g^{-1})s = \rho(g^{-1})Ds + \rho(g^{-1})\rho(\omega)s . \]

Hence $(D + \rho(\omega))s$ is $G$-equivariant. It thus descends to a one form $\nabla^G \bar{s} \in \Omega^1(P \times_G \mathbb{R}^n)$. One easily verifies that $\nabla^G$ is linear and satisfies the Leibniz rule.

Corollary 1.4.23. Let $E$ be a vector bundle over $M$. Then the map $\omega \mapsto \nabla^G \omega$ is a bijection between the space of principal connections on the frame bundle $R(E)$ of $E$ and linear connections on $E$.

Proof. Both spaces are affine spaces over $\Omega^1(\text{End}(E))$. To be more precise, one easily verifies from the construction of $\nabla^G$ that if $A \in \Omega^1(\text{End}E)$, then
\[ \nabla^G + A = \nabla^G + A . \]

For other structure groups, one can refine the bijective correspondance by restricting to linear connections that “preserve some structure”. For instance, let $E$ be a vector bundle and $q$ a section of $\text{Sym}^2(E^*)$ which is non degenerate at every point.

Definition 1.4.24. A linear connection on $E$ preserves $q$ if
\[ dq(s_1, s_2) = q(\nabla s_1, s_2) + q(s_1, \nabla s_2) \]

for all $1, s_2 \in \Gamma(E)$.

In this situation, $E$ is the $\mathbb{R}^{k,l}$-bundle associated to a certain $O(k,l)$ principal bundle $P$ (where $(k,l)$ denotes the signature of $q$) and we have

Proposition 1.4.25. The map
\[ \omega \mapsto \nabla^G \omega \]

is a bijection between principal connections on $P$ and linear connections on $E$ preserving $q$. 
1.4.3 Curvature

Let $\alpha$ and $\beta$ be 1-forms on a manifold $M$ with values in a Lie algebra $\mathfrak{g}$. We define $[\alpha, \beta] \in \Omega^2(M, \mathfrak{g})$ by

$$[\alpha, \beta](u, v) = [\alpha(u), \beta(v)] - [\alpha(v), \beta(u)].$$

In particular, form $\alpha \in \Omega^1(M, \mathfrak{g})$, we have

$$[\alpha, \alpha](u, v) = 2[\alpha(u), \alpha(v)].$$

Let $P$ be a principal $G$-bundle over $M$ and $\omega$ a connection form on $P$.

**Proposition 1.4.26.** Then the form

$$\hat{F}_\omega = d\omega + \frac{1}{2}[\omega, \omega] \in \Omega^2(M, \mathfrak{g})$$

satisfies the following properties:

- the vertical directions are in the kernel of $\hat{F}_\omega$, i.e. $\hat{F}_\omega(u, v) = 0$ for all $u \in V$ and all $v \in TP$,
- $g^*\hat{F}_\omega = \text{Ad}_g^{-1}\hat{F}_\omega$.

The form $\hat{F}_\omega$ thus induces a 2-form on $M$ with values in $\text{Ad}(P)$.

**Definition 1.4.27.** The form $F_\omega \in \Omega^2(M, \text{Ad}(P))$ induced by $\hat{F}_\omega$ is called the curvature form of $\omega$.

The form $\hat{F}_\omega$ on $P$ measures the non-integrability of the horizontal distribution $H = \ker \omega$. Indeed, if $X$ and $Y$ are two vector fields on $P$ belonging to $H$, then we have

$$\hat{F}_\omega(X, Y) = -\omega([X, Y]),$$

which vanishes if and only if $[X, Y]$ also belongs to $H$.

In the case of a linear connection, the curvature form has another interpretation:

**Proposition 1.4.28.** Let $P$ be a $\text{GL}(n, \mathbb{R})$ bundle over $M$ and $E$ the associated vector bundle of rank $n$. Let $\omega$ be a principal connection on $P$ and $\nabla$ the associated linear connection on $E$. Then for every section $s$ of $E$, we have

$$F_\omega s = \nabla(\nabla s) \in \Omega^2(E).$$
1.4.4 Levi–Civita connection

Let $M$ be a manifold. Recall that $TM$ denotes the tangent bundle of $M$.

**Definition 1.4.29.** Let $\nabla$ be a linear connection on $TM$. The **torsion** of $\nabla$ is the unique tensor $T \in T^*M \otimes T M$ such that

$$T(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y]$$

for all vector fields $X, Y$ on $M$.

We say that $\nabla$ is **torsion free** if its torsion vanishes identically.

Let now $q$ be a pseudo-Riemannian metric on $M$.

**Proposition-Definition 1.4.30.** There exists a unique connection $\nabla^q$ on $TM$ which is torsion free and preserves the metric $q$. This connection is called the **Levi–Civita connection** of $q$.

We define the **curvature tensor** $R^q$ of a pseudo-Riemannian metric $q$ has the curvature tensor of its Levi–Civita connection. The curvature tensor of a pseudo-Riemannian metric $q$. Finally, if $W$ is a 2-dimensional subspace of the tangent space of $M$ at some point $x$ and if $q$ is non degenerate on $W$, we define the **sectional curvature** of $W$ by

$$S_q(W) = \frac{q(R_q(u,v)v, u)}{q(u,u)q(v,v) - q(u,v)^2}$$

for any basis $(u, v)$ of $W$.

We say that $(M, q)$ has **constant sectional curvature** if the sectional curvature is the same for all non degenerate 2-planes in $TM$.

1.4.5 Curvature of symmetric spaces

Let $G$ be a semisimple Lie group and $\theta$ an involution of $G$. Denote by $\mathfrak{g}$ the Lie algebra of $G$, by $K$ the subgroup fixed by $\theta$, by $\mathfrak{k}$ its Lie algebra, and by $\mathfrak{p}$ the orthogonal of $\mathfrak{k}$ in $\mathfrak{g}$ with respect to the Killing form. Finally let $X$ the pseudo-Riemannian symmetric space $G/K$ with the metric induced by $\kappa_{\mathfrak{g}|\mathfrak{p}}$, and $o \in X$ the basepoint $[1_G]$.

Recall that the adjoint action of $K$ preserves the decomposition

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}.$$ 

We thus have

$$[\mathfrak{k}, \mathfrak{p}] \subset \mathfrak{p}.$$ 

Moreover, we have

$$[\mathfrak{p}, \mathfrak{p}] \subset \mathfrak{k}.$$
since 
\[ \theta([u,v]) = [\theta(u), \theta(v)] = [-u, -v] = [u,v] \]
for all \( u,v \in \mathfrak{p} \).

Note that \( G \) with the action of \( K \) by right multiplication can be seen as a principal \( K \) bundle over \( X \), let us denote \( G \) by \( P \) when we think of it as a principal \( H \)-bundle. The distribution \( V \subset TG \) tangent to the fibers of \( P \) is the left-invariant distribution such that \( V_{1_G} = \mathfrak{k} \).

Let \( H \subset TG \) be the \( G \)-invariant distribution such that \( H_{1_G} = \mathfrak{p} \). Then \( V \oplus H = TG \). Moreover, since \( \mathfrak{p} \) is preserved by the adjoint action of \( K \), the distribution \( H \) is also invariant by right multiplication by \( K \). Thus \( H \) defines a principal connection on \( P \).

We can now describe the Levi–Civita connection of \( X \):

**Proposition 1.4.31.** The tangent bundle \( TX \) is \( G \)-equivariantly isomorphic to the associated bundle \( P \times_K \mathfrak{p} \). With respect to this identification, the Levi–Civita connection of \( X \) is the linear connection associated to \( H \).

**Corollary 1.4.32.** The curvature tensor of the Levi–Civita connection on \( X \) at the point \( o \) is given by
\[ R(u,v)w = -[[u,v],w] \]
for all \( u,v,w \in \mathfrak{p} \cong T_oX \).

**Corollary 1.4.33.** If \( \theta \) is a Cartan involution of \( G \), then \( X \) has non positive sectional curvature.

**Proof.** Let \( u,v \) be two vectors in \( T_oX \cong \mathfrak{p} \). Then
\[
\kappa_g(R(u,v)v,u) = -\kappa_g([[u,v],v],u) = \kappa_g([u,v],[u,v]) \leq 0
\]
since \([u,v] \in \mathfrak{k}\) and \( \kappa |_{\mathfrak{k}} \) is negative definite. \( \square \)
Chapter 2

Discrete subgroups of Lie groups

The goal of this chapter is to introduce various discrete subgroups of semisimple Lie groups and discuss to which extent we can see them “in a natural way” as fundamental groups of manifolds.

2.1 Finitely generated groups, presentations

Let us start with discussing “abstract discrete groups”, i.e. finitely generated group.

Definition 2.1.1. Let $\Gamma$ be a group and $S$ a subset of $G$. The subgroup spanned by $F$, denoted $\langle F \rangle$, is the smallest subgroup of $\Gamma$ containing $F$. A subset $F$ generates $\Gamma$ if $\langle F \rangle = \Gamma$. The group $\Gamma$ is finitely generated if its admits a finite generating set.

The subgroup spanned by $F$ is easily seen to be the subgroup of elements of $\Gamma$ that can be written as a product of finitely many elements of $F$ or their inverses. In particular, a finitely generated group is countable.

2.1.1 The free group of rank $n$

The main example of a finitely generated group is the free group of rank $n$, denoted $F_n$, which has $n$ generators and “no relations between them”. Though the concept is relatively intuitive, constructing $F_n$ is not that easy.

Consider $n$ “letters” $s_1, \ldots, s_n$. We say that a word $w$ on the alphabet \{s_1, \ldots, s_n, s_1^{-1}, \ldots, s_n^{-1}\} is reduced if the sequences $s_is_i^{-1}$ or $s_i^{-1}s_i$ do not appear in $w$. If a word $w$ is not reduced, there is a canonical procedure to reduce it, by recursively deleting all the appearances of $s_is_i^{-1}$ or $s_i^{-1}s_i$. We denote this reduced word by $\overline{w}$. We can then make the following definition:
Definition 2.1.2. The free group of rank \( n \) is the set \( F_n \) of reduced words on the alphabet \( \{s_1, \ldots, s_n, s_1^{-1}, \ldots, s_n^{-1}\} \) (including the empty word \( e \)) with the multiplication law given by
\[
ww' = w \ast w',
\]
where \( \ast \) denotes the concatenation of words.

The free group of rank \( n \) is characterized by the following universal property:

Proposition 2.1.3. Let \( G \) be a group and \( g_1, \ldots, g_n \) elements of \( G \). Then there is a unique morphism from the free group \( F_n \) in \( n \) generators \( s_1, \ldots, s_n \) mapping \( s_i \) to \( g_i \).

Finally, let us mention another more geometric point of view on \( F_n \):

Proposition 2.1.4. Let \( B_n \) be the bouquet of \( n \) circles, i.e. the graph with one vertex and \( n \) loops at that vertex. Then the fundamental group of \( B_n \) is isomorphic to \( F_n \).

This proposition follows from Van Kampen’s theorem, which describe how the fundamental group of topological spaces behaves under gluing. Interestingly, one could also take it as a definition of the free group and work the theory from there.

2.1.2 Presentations of a finitely generated group

Let \( \Gamma \) be a group generated by \( n \) elements \( g_1, \ldots, g_n \). Then there is a surjective morphism from \( F_n \) to \( \Gamma \) sending the generators \( s_i \) of \( F_n \) to \( g_i \). The group \( \Gamma \) is characterized by the kernel of this morphism.

Definition 2.1.5. Let \( \{r_j, j \in J\} \) be a set of reduced words in \( \{s_i, s_i^{-1}, 1 \leq i \leq n\} \). The group of presentation \( \langle s_1, \ldots, s_n \mid r_j, j \in J \rangle \) (or simply the group \( \langle s_1, \ldots, s_n \mid r_j, j \in J \rangle \)) is the quotient of the free group generated by \( s_1, \ldots, s_n \) by the smallest normal subgroup containing \( \{r_j, j \in J\} \).

This defines abstractly a group by a set of generators and some relations. If we started with a given group \( \Gamma \), we define a presentation of \( \Gamma \) as the data of a set of generators \( g_1, \ldots, g_n \) and a set relations \( \{r_j, j \in J\} \) such that \( \Gamma \) is isomorphic to \( \langle s_1, \ldots, s_n \mid r_j, j \in J \rangle \) (or, more precisely, such that the morphism from the free group generated by \( g_1, \ldots, g_n \) to \( \Gamma \) factors through an isomorphism from \( \langle s_1, \ldots, s_n \mid r_j, j \in J \rangle \) to \( \Gamma \)).

Definition 2.1.6. We say that a group \( \Gamma \) is finitely presented if \( \Gamma \) admits a presentation of the form \( \langle s_1, \ldots, s_n \mid r_j, j \in J \rangle \) with \( J \) finite.
Remark 2.1.7. Beware that, even if $\Gamma$ is finitely presented, the kernel of the map from $F_n$ to $\Gamma$ is not finitely generated. Indeed, this kernel is generated by the relations and all their conjugate.

The group of presentation $\langle s_1, \ldots, s_n \mid r_j, j \in J \rangle$ is characterized by a universal property:

**Proposition 2.1.8.** Let $G$ be a group and $g_1, \ldots, g_n$ elements such that $r_j(g_1, \ldots, g_n) = 1_G$ for all $j \in J$ (where $r_j(g_1, \ldots, g_n)$ is the element of $G$ obtained by substituting $g_i$ to $s_i$). Then there exists a unique morphism from $\langle s_1, \ldots, s_n \mid r_j, j \in J \rangle$ to $G$ mapping $s_i$ to $g_i$.

**Example 2.1.9.** The group of presentation $\langle s_0, \ldots, s_n \mid \prod_{i=0}^n s_i \rangle$ is isomorphic to $F_n$! In particular, $F_n$ has an automorphism of order $n + 1$.

This simple example shows that presentations of a group are far from canonical. In fact, known a presentation of a group gives little intuition in general. For instance, determining whether a given presentation yields the trivial group is an algorithmic problem that is impossible to solve in general.

**Coxeter groups** Coxeter groups are a class of groups abstractly given by a finite presentation, but for which this presentation brings an unusually good insight on their geometric properties.

**Definition 2.1.10.** A Coxeter diagram as a symmetric matrix $M = (m_{ij})_{1 \leq i,j \leq n}$ of size $n$ with diagonal coefficients equal to 1 and other coefficients in $\{2, 3, \ldots, \infty\}$. The Coxeter group with Coxeter diagram $M$ is the finitely presented group with presentation

$$C(M) = \langle s_1, \ldots, s_n \mid (s_is_j)^{m_{ij}} \text{ for } m_{ij} \neq \infty \rangle.$$

Note that for $i = j$, we get that $s_i^2 = 1$ in $C(M)$. One should think of $s_i$ as a reflection along a face of an abstract polytope. The relations $(s_is_j)^{m_{ij}} = 1$ intuitively mean that for $m_{ij} \neq \infty$ the faces of the polytope corresponding to $s_i$ and $s_j$ meet at an angle $\frac{\pi}{m_{ij}}$.

We will see later on that such a polytope can be realized in a certain geometry. We will deduce that $C(M)$ embeds naturally as a discrete linear group generated by reflections.

**Example 2.1.11.** Let $T$ be an equilateral triangle in the plane. Then the group of euclidean isometries generated by reflections along the sides of $T$ is isomorphic to the Coxeter group $C(M)$ with

$$M = \begin{pmatrix} 1 & 3 & 3 \\ 3 & 1 & 3 \\ 3 & 3 & 1 \end{pmatrix}.$$
2.1.3 Finitely presented groups as fundamental groups

Finitely presented groups are essentially the same as fundamental groups of closed manifolds. Indeed, we have the following:

Proposition 2.1.12. The fundamental group of a compact manifold with boundary is finitely presented.

Proof. This is true more generally of any $CW$-complex whose 2-skeleton is finite. Indeed, by Van Kampen’s theorem, the fundamental group of a finite graph is a finitely generated free group. Attaching a 2-cell to this graph creates one relation (by “killing” the element of $F_n$ given by the boundary of this 2-cell), so attaching finitely many 2-cells yields a finitely presented group. Finally, attaching higher dimensional cells does not modify the fundamental group. 

Conversely:

Proposition 2.1.13. Every finitely presented group is the fundamental group of a closed manifold of dimension 4.

Proof. Let $\langle s_1, \ldots, s_n \mid r_j, j \in J \rangle$ be a finitely presented group. Let $M_0$ be the connected sum of $n$ copies of $S^1 \times S^3$, whose fundamental group is $F_n$.

We want to do some surgeries on $M_0$ to “add relations” in the fundamental group. For this, let $\gamma$ be a curve in $M_0$ representing some $r_j$ in $\pi_1(M_0)$. We can always deform $\gamma$ so that it is embedded (here we use that the dimension is higher than 2). Removing a tubular neighborhood of $\gamma$ does not change the fundamental group of $M_0$ (here, we use that the dimension is higher than 4).

The tubular neighborhood is diffeomorphic $S^1 \times \mathbb{D}^3$ and its boundary to $S^1 \times S^2$, with the $S^1$ realizing our element $r_j$ in $\pi_1(M_0)$. Replacing the tubular neighborhood by $\mathbb{D}^2 \times S^2$ (which has the same boundary) kills the elements $r_j$ in the fundamental group. Applying this to all the relations gives the desired manifold.

This last proposition can be useful by allowing us to think of any finitely presented group as the fundamental group of a surface. However, it is also a negative result, showing that there is no chance of classifying closed 4-manifolds. Indeed, this would be at least as complicated as classifying finitely presented groups, which is an impossible task.

In regard of this, you may know that Thurston’s geometrization conjecture (which was proved by Perelman) implies that every closed 3-manifold decomposes into “geometric pieces” in a canonical way. This has strong implications on the algebraic structure of fundamental groups of 3-manifolds.

Let us cite one corollary of the previous proposition which can be useful:
2.1. FINITELY GENERATED GROUPS, PRESENTATIONS

**Corollary 2.1.14.** Let $\Gamma$ be a finitely presented group and $\Gamma'$ a finite index subgroup of $\Gamma$ then $\Gamma'$ is finitely presented.

*Proof.* Let $M$ be a closed manifold with fundamental group $\Gamma$. Then $\Gamma'$ is the fundamental group of a finite cover of $M$, which is still a closed manifold. \qed

### 2.1.4 Finitely generated linear groups

In this section, we prove two fundamental results for finitely generated subgroups of linear groups, namely that they are residually finite and virtually torsion-free.

**Definition 2.1.15.** A group $\Gamma$ is residually finite if for every $g \in \Gamma \setminus \{1\}$, there exists finite index subgroup of $\Gamma$ that does not contain $g$.

**Definition 2.1.16.** A group $\Gamma$ is torsion-free if for every $g \in \Gamma \setminus \{1\}$, we have $g^k \neq 1$ for all $k \neq 0$. It is virtually torsion-free if it admits a torsion-free subgroup of finite index.

**Theorem 2.1.17** (Malcev). Every finitely generated subgroup of $\text{GL}(n, \mathbb{R})$ is residually finite.

**Theorem 2.1.18** (Selberg). Every finitely generated subgroup of $\text{GL}(n, \mathbb{R})$ is virtually torsion-free.

The choice of the field $\mathbb{R}$ is not very relevant, and the theorems are actually true over any field of characteristic $0$. Actually, the first step of the proof consists in reducing the domain of definition.

**Lemma 2.1.19.** Let $\Gamma$ be a finitely generated subgroup of $\text{GL}(n, \mathbb{R})$. Then $\Gamma$ embeds in $\text{GL}(k, \mathbb{Z}[X_1, \ldots, X_s, \frac{1}{D}])$ for some $f \in \mathbb{Z}[X_1, \ldots, X_s]$.

*Proof.* Let $\mathbb{K}$ be the extension of $\mathbb{Q}$ generated by all the coefficients of all the matrices in a finite generating set of $\Gamma$. Then $\mathbb{K}$ is an extension of finite type of $\mathbb{Q}$ and $\Gamma \subset \text{GL}(n, \mathbb{K})$. By property of extensions of $\mathbb{Q}$, the field $\mathbb{K}$ is a finite extension of the field $= \mathbb{Q}(X_1, \ldots, X_s)$. Let $d$ be the degree of the extension $\mathbb{K}/\mathbb{Q}$. Then $\text{GL}(n, \mathbb{K})$ naturally embeds in $\text{GL}(nd, \mathbb{K})$.

Let us now see $\Gamma$ inside $\text{GL}(nd, \mathbb{K})$. All the elements of can be written as ratios of polynomials with integral coefficients. Let $f \in \mathbb{Z}[X_1, \ldots, X_s]$ be a multiple of all the denominators of all the coefficients of a finite generating set of $\Gamma$. Then $\Gamma$ is contained in $\text{GL}(k, \mathbb{Z}[X_1, \ldots, X_s, \frac{1}{D}])$. \qed

**Proof of Malcev’s theorem.** By Lemma 2.1.19, we can restrict to the case where $\Gamma = \text{GL}(k, \mathbb{Z}[X_1, \ldots, X_s, \frac{1}{D}])$.

Let $g$ be an element in $\text{GL}(k, \mathbb{Z}[X_1, \ldots, X_s, \frac{1}{D}]) \setminus \{1_k\}$. If $g$ is not diagonal, let $s$ be a non-diagonal coefficient of $g$ which is non-zero. Write $s = \frac{P}{D}$ for some $P \in \mathbb{Z}[X_1, \ldots, X_s] \setminus \{0\}$ and some $l \in \mathbb{Z}$. Let $\mathbb{F}$ be a finite field in which
there exist \(a_1, \ldots, a_s\) such that \(D(a_1, \ldots, a_s) \neq 0\) and \(P(a_1, \ldots, a_s) \neq 0\). If \(g\) is diagonal, choose similarly a diagonal coefficient \(s \neq 1\), write \(s = \frac{P}{D}\) and choose \(F\) and \(a_1, \ldots, a_s\) such that \(D(a_1, \ldots, a_s) \neq 0\) and \(P(a_1, \ldots, a_s) \neq D(a_1, \ldots, a_s)!\).

Then the ring morphism

\[
\begin{align*}
&: \ Z[X_1, \ldots, X_s, \frac{1}{D}] \rightarrow F \\
& X_i \mapsto a_i
\end{align*}
\]

induces a morphism

\[
\varphi : \text{GL}(k, Z[X_1, \ldots, X_s, \frac{1}{D}]) \rightarrow \text{GL}(k, F)
\]

such that \(\varphi(g) \neq I_k\). The kernel of \(\varphi\) has finite index in \(\text{GL}(k, Z[X_1, \ldots, X_s, \frac{1}{D}])\) and does not contain \(g\).

Proof of Selberg’s lemma. By Lemma 2.1.19, we can restrict to the case where \(\Gamma = \text{GL}(k, Z[X_1, \ldots, X_s, \frac{1}{D}])\).

Let \(p\) be a prime number strictly larger than \(2k\) and that does not divide all the coefficients of \(D\). Let \(F\) be a finite field of characteristic \(p\) in which there exist \(a_1, \ldots, a_s\) such that \(D(a_1, \ldots, a_s) \neq 0\), let \(\varphi : \text{GL}(k, Z[X_1, \ldots, X_s, \frac{1}{D}]) \rightarrow \text{GL}(k, F)\) be the morphism induced by

\[
\begin{align*}
&: \ Z[X_1, \ldots, X_s, \frac{1}{D}] \rightarrow F \\
& X_i \mapsto a_i
\end{align*}
\]

and let \(\Gamma'\) be the kernel of \(\varphi\). Let us prove that \(\Gamma'\) is torsion-free.

Let \(\gamma \in \Gamma'\) be an element such that \(\gamma^n = I_k\) for some \(n \neq 0\). Then the eigenvalues of \(\gamma\) in the algebraic closure \(\overline{L}\) of \(L = \mathbb{Q}(X_1, \ldots, X_s)\) are roots of unity. In particular, they are algebraic integers in \(\overline{Q}\). Since the algebraic integers form a ring, \(\text{Tr}\gamma\) is an algebraic integer. But \(\text{Tr}\gamma\) belongs to \(\mathbb{Q}(X_1, \ldots, X_s)\). It follows that \(\text{Tr}\gamma\) belongs to \(Z\).

Finally, since \(\text{Tr}\gamma\) is a sum of roots of unity, we have

\[
|\text{Tr}\gamma| \leq k,
\]

with equality if and only if \(\gamma = \pm I_k\). On the other side,

\[
\text{Tr}\gamma \equiv k \mod p
\]

since \(\varphi(\gamma) = I_k\) and \(F\) has characteristic \(p\). Since \(p > 2k\), we conclude that \(\text{Tr}\gamma = k\) and \(\gamma = I_k\).
2.2 Discrete subgroups of Lie groups

Let $G$ be a Lie group and $\Gamma$ a subgroup of $G$. Then $\Gamma$ is called discrete if one (hence all) of the following equivalent properties hold:

- every point of $\Gamma$ is isolated (i.e. admits a neighborhood in which it is the only point of $\Gamma$),
- $1_G$ is isolated in $\Gamma$,
- $\Gamma$ is a Lie subgroup of $G$ dimension 0,
- $\Gamma$ acts properly discontinuously on $G$ by left multiplication,
- $\Gamma$ acts properly discontinuously on $G/K$ for any compact subgroup $K$ of $G$.

2.2.1 Discrete subgroups as fundamental groups of locally symmetric spaces

If $\Gamma$ is a discrete group of $G$, there are more geometric ways of thinking of $\Gamma$ as the fundamental group of a manifold. For instance, if $G$ is connected and simply connected, then $\Gamma$ is the fundamental group of $\Gamma \backslash G$.

Beware that, if $G$ is not simply connected, then the inclusion of $\Gamma$ into $G$ does not lift in general to a morphism to the universal cover $\tilde{G}$. The only thing one can say is that, if $\tilde{\Gamma}$ is the preimage of $\Gamma$ by the covering map $\tilde{G} \to G$, then $\tilde{\Gamma}$ is a central extension of $\Gamma$, and

$$\tilde{\Gamma} \simeq \pi_1(\tilde{\Gamma} \backslash \tilde{G}).$$

If $G$ is connected and semisimple, another natural construction would be to quotient the symmetric space $X$ of $G$ by $\Gamma$, which is contractible. The following proposition states that this is possible as soon as $\Gamma$ is torsion-free.

**Proposition 2.2.1.** Let $\Gamma$ be a discrete subgroup of a semisimple Lie group $G$ with trivial center. Then $\Gamma$ acts freely on the symmetric space of $G$ if and only if $\Gamma$ is torsion-free.

By Selberg’s lemma, if $\Gamma$ is moreover finitely generated, then one often reduce to that case up to taking a finite index subgroup.

One of the many interesting aspects of $\Gamma \backslash X$ is that (when $\Gamma$ is torsion-free) it is a classifying space for $\Gamma$ (i.e. a metric space with fundamental group $\Gamma$ and contractible universal cover). In particular, the homology of the group $\Gamma$ is isomorphic (or can be defined as) the homology of $\Gamma \backslash X$.

Let us now describe examples of discrete subgroups of Lie groups.
2.2.2 Schottky groups

Here, we prove the following:

Proposition 2.2.2. There discrete and faithful representations of the free group into \( \text{PSL}(2, \mathbb{R}) \).

Note that representations of the free group into \( \text{PSL}(2, \mathbb{R}) \) do lift to representations into \( \tilde{\text{PSL}}(2, \mathbb{R}) \) by the universal property of the free group. Moreover, the structure theory of semisimple Lie algebras implies that every non-compact semisimple Lie group contains a subgroup isomorphic to a cover of \( \text{PSL}(2, \mathbb{R}) \).

Corollary 2.2.3. There discrete and faithful representations of the free group into any semisimple Lie group.

The most widely used argument to prove that a group generated \( n \) transformations is free is the so-called ping-pong lemma.

Lemma 2.2.4. Let \( g_1, \ldots, g_n \) be homeomorphisms of a topological space \( X \). Assume that there exist disjoint open subset \( \{ U_1, V_1, \ldots, U_n, V_n \} \) of \( X \) such that

\[ g_i(X \setminus U_i) \subset V_i \]

for all \( i \in \{1, \ldots, n\} \). Then the morphism from \( F_n \) to \( \text{Homeo}(X) \) sending the \( i \)-th generator to \( g_i \) is faithful and discrete.

Proof. To simplify the proof, assume that there exists a point \( x \in \text{int}(X \setminus \bigcup_{i=1}^n U_i \cup V_i) \). Note that \( g_i(X \setminus U_i) \subset V_i \) implies that \( g_i^k(X \setminus U_i) \subset V_i \) for all \( k > 0 \) and \( g_i^k(X \setminus V_i) \subset U_i \) for all \( k < 0 \). In particular, setting \( W_i = U_i \cap V_i \), we have that \( g_i^k(X \setminus W_i) \subset W_i \) for all \( k \neq 0 \).

Let \( g \) be an element of the group generated by \( g_1, \ldots, g_n \) and write \( g = g_{i_1}^{n_1} \cdots g_{i_k}^{n_k} \), with \( n_j \neq 0 \) and \( i_j \neq i_{j+1} \). Assume that \( k \geq 1 \). Then \( g_{i_1}^{n_1} \) maps \( x \) into \( W_{i_1} \), which is contained in \( X \setminus W_{i_2} \). An easy induction shows that \( gx \in W_{i_1} \). In particular, \( gx \neq x \), showing that \( g \neq \text{Id}_X \). Moreover, \( gx \) is outside a neighbourhood of \( x \), showing that \( g \) is "far from" the identity in the compact open topology. Thus the group generated by \( g_1, \ldots, g_n \) is a free group of rank \( n \) embedded discretely in \( \text{Homeo}(X) \). \( \square \)

Proof of Proposition 2.2.2. Recall that \( \text{PSL}(2, \mathbb{R}) \) is the group of orientation preserving isometries of the hyperbolic plane \( \mathbb{H}^2 \).

Let \( g_1, \ldots, g_n \) be hyperbolic isometries of \( \mathbb{H}^2 \) with disjoint axes and sufficiently large translation length. Then the action of \( g_1, \ldots, g_n \) on \( \mathbb{H}^2 \) satisfies the conditions of the ping-pong lemma. \( \square \)

Building (a lot) on this simple example, Tits proved the following:

Theorem 2.2.5 (Tits alternative). Let \( \Gamma \) be a subgroup of \( \text{SL}(n, \mathbb{R}) \). Then exactly one of the following holds:
• $\Gamma$ contains a solvable subgroup of finite index,

• $\Gamma$ contains a free group in $n$ generators for all $n$.

The general strategy of the proof is to find sufficiently long elements in $\Gamma$ so that their action on (some boundary of) the symmetric space of $\text{SL}(n, \mathbb{R})$ satisfies the hypotheses of the ping-pong lemma. However, this has no chance to if $\Gamma$ is contained in a compact group! For such $\Gamma$ one needs to change the field and see $\Gamma$ inside $\text{SL}(n, \mathbb{K})$ for some non-archimedean field $\mathbb{K}$ (such as $\mathbb{Q}_p$). There, the Bruhat–Tits building provides an analog of the symmetric space.

### 2.2.3 Surface groups

Let $\Gamma_g$ be the fundamental group of a closed oriented surface of genus $g \geq 2$.

**Proposition 2.2.6.** There exists a discrete and faithful representation of $\Gamma_g$ into $\text{PSL}(2, \mathbb{R})$.

### 2.2.4 Coxeter groups

Coxeter groups are naturally represented as discrete isometry groups the sphere, the Euclidean or the hyperbolic space when they are associated to an actual polytope in the corresponding geometry.

**Definition 2.2.7.** A polytope in a sphere, Euclidean or hyperbolic space is the intersection of finitely many closed half-spaces.

Let $P$ be a polytope with $n$ faces in some hyperbolic space $\mathbb{H}^k$ (resp. some Euclidean space $\mathbb{E}^k$, some round sphere $\mathbb{S}^k$). Label the faces by $f_1, \ldots, f_n$. Assume that when the faces $f_i$ and $f_j$ intersect in a codimension 2 cell of $P$, they do so at an angle $\frac{\pi}{m_{ij}}$ for some integer $m_{ij} \geq 2$. Set $m_{ij} = \infty$ if $f_i$ and $f_j$ do not intersect in codimension 2. Then $(m_{ij})_{1 \leq i, j \leq n}$ is a Coxeter diagram. We call $C(P) = C(M)$ the reflection group associated to $P$. Clearly, there is a representation

$$\rho : C(P) \rightarrow \text{Isom}(\mathbb{H}^k)$$

sending the generator $s_i$ to the reflection along the face $f_i$.

**Theorem 2.2.8 (Poincaré).** The set of all $\rho(\gamma)P$, $\gamma \in C(P)$ tiles $\mathbb{H}^k$ (resp. $\mathbb{E}^k$, $\mathbb{S}^k$), i.e. $\rho(\gamma)\hat{P} \cap \hat{P} = \emptyset$ for $\gamma \neq 1_{\Gamma}$ and $\bigcup_{\gamma \in \Gamma} \rho(\gamma)P = \mathbb{H}^k$ (resp. $\mathbb{E}^k$, $\mathbb{S}^k$). In particular, $\rho$ is discrete and faithful.

**Sketch of the proof.** One constructs the expected tiling by putting one tile at each point of $\Gamma$ and gluing this tiles in the natural way. More precisely.
Let $X$ be the quotient of the space $\Gamma \times P$ by the relations $(g, x) \simeq (gs_i, x)$ if $x \in f_i$. The group $\Gamma$ acts on $X$ in the following way:

$\gamma \cdot (g, x) = (\gamma g, x)$.

The hyperbolic metric on $P$ induces a “singular” hyperbolic structure on $H^n$. Moreover, there is a natural $\rho$-equivariant map $dev : X \to \mathbb{H}^k$ sending $(g, x)$ to $\rho(g)x$.

To show that $dev$ is a diffeomorphism, we first need to prove that the hyperbolic structure on $X$ is not singular. Let $x$ be a point in $X$. If $x$ is in the interior of a tile, the hyperbolic structure is not singular at $x$. If $x$ is in the interior of a face of a tile, then two pieces of hyperbolic space with totally geodesic boundary are glued at $x$, yielding a smooth hyperbolic metric.

If $x$ is in the interior of $f_i \cap f_j$, then one verifies that $2m_{ij}$ tiles are glued at $x$, each with dihedral angle $\frac{\pi}{m_{ij}}$. Again the resulting hyperbolic structure is smooth.

Finally, let $x$ belong to a cell $a$ of codimension $l \geq 2$. Let $P_x$ denote the intersection $P \cap H_l \cap S_\varepsilon(x)$, where $H_l$ is the $l$-dimensional hyperbolic space orthogonal to $a$ at $x$ and $S_\varepsilon(x)$ the sphere of radius $\varepsilon$ at $x$. Then $P_x$ is a spherical Coxeter polytope in the $l-1$-dimensional sphere (i.e. a polytope whose faces intersect at angles dividing $\pi$) and the way the tiles glue together at $x$ is encoded in the abstract tiling associated to this polytope. To prove that the hyperbolic structure is regular at $x$, one needs to prove that this tiling forms a sphere of dimension $S_{l-1}$. This is proved by invoking the theorem for this sphere of smaller dimension (the proof is thus an induction on $k$).

Once we know that the hyperbolic structure on $X$ is smooth, we obtain that $dev$ is a local isometry. We now need to prove that the hyperbolic metric on $X$ is complete. For that, we want to find an $\varepsilon$ such that for every $x$ in $X$, the ball of center $x$ and radius $\varepsilon$ is mapped surjectively to the ball of radius $dev(x)$ and radius $\varepsilon$. We will then deduce that dev has the path lifting property. It is thus a covering map, hence a diffeomorphism, which will conclude the proof.

It is easy to find a uniform $\varepsilon$ for $x$ in the compact part of a tile. In particular, if $P$ is compact, then the proof is finished. If $P$ is not compact, however, there could be points in $X$ whose ball of radius $\varepsilon$ intersect arbitrarily many copies of $P$ (this only happens for hyperbolic geometry). To deal with this issue one needs to take a point $x$ “at infinity” and intersect our tiling $X$ with a horosphere at $x$. One will see the tiling associated to a euclidean Coxeter polytope in dimension $k-1$. Applying the theorem in dimension $k-1$, one deduces that this horosphere in $X$ is mapped surjectively to a horosphere in $\mathbb{H}^k$. The completeness eventually follows.

In general, an abstract Coxeter diagram $M$ need not be associated to a hyperbolic, spherical or Euclidean Coxeter polytope. Yet one can construct
a discrete and faithful representation of $C(M)$.

**Definition 2.2.9.** The Tits form associated to $M$ is the bilinear form on $\mathbb{R}^n$ given in the canonical basis by

$$q_M(e_i, e_j) = -\cos\left(\frac{\pi}{m_{ij}}\right).$$

The Tits form is constructed so that the vectors $e_i$ and $e_j$ form an angle $\pi \left(1 - \frac{1}{m_{ij}}\right)$. Hence the hyperplanes $e_i^\perp$ and $e_j^\perp$ form an angle $\frac{\pi}{m_{ij}}$ and the reflections

$$\sigma_i : v \mapsto v - 2q_M(v, e_i)e_i$$

with respect to $e_i^\perp$ satisfy the relation

$$(\sigma_i \sigma_j)^{m_{ij}} = I_n.$$

**Definition 2.2.10.** The Tits representation of $C(M)$ is the representation

$$\rho : C(M) \to O(q_M)$$

$$s_i \mapsto \sigma_i.$$

**Theorem 2.2.11** (Tits, Vinberg). The Tits representation $\rho$ is discrete and faithful.

For a proof of this theorem, one can consult Yves Benoist’s lecture notes [?]. Let us simply mention that, compared to Poincaré’s theorem, a new difficulty arises. We have no control on the signature of $q_M$, and the Riemannian geometric arguments used there in the proof are not available here.

### 2.2.5 Lattices

Let $G$ be a Lie group. Then $G$ admits a left invariant volume form (unique up to a scalar). This volume induces a volume form on any quotient of $G$.

**Definition 2.2.12.** A discrete subgroup $\Gamma$ of $G$ is a lattice if $\Gamma \backslash G$ has finite volume. It is a uniform (or cocompact) lattice if $\Gamma \backslash G$ is compact.

**Theorem 2.2.13** (Kazhdan, Ragnunathan). Every lattice is finitely presented.

We have already seen a few lattices. For instance, surface groups discretely embedded in $\text{PSL}(2, \mathbb{R})$ are uniform lattices. In higher dimension, if $P$ is a finite volume hyperbolic Coxeter polytope, then the Coxeter group of $P$ is a hyperbolic lattice (i.e. a lattice in the isometry group of the hyperbolic space). It is uniform if and only if $P$ is compact.

These constructions, however, are far from producing lattices in any semisimple Lie group. Vinberg proved for instance that hyperbolic spaces of
sufficiently high dimension do not contain any finite volume Coxeter polytope.

The only general construction of lattices in semisimple Lie groups is arithmetic and was carried out by Borel and Harish-Chandra.

**Theorem 2.2.14.** Let $G$ be a semisimple algebraic subgroup of $\text{GL}(n, \mathbb{R})$ defined over $\mathbb{Q}$ (i.e. defined by polynomial equations with rational coefficients). Then the group $G_\mathbb{Z} = G \cap \text{GL}(n, \mathbb{Z})$ is a lattice in $G$.

The first example, due to Minkowski is $\text{SL}(n, \mathbb{Z})$ inside $\text{SL}(n, \mathbb{R})$. Borel and Harish-Chandra also have a criterion for such a lattice to be uniform. After a careful study of rational forms of semisimple linear algebraic groups, Borel and Harish-Chandra obtain the following:

**Theorem 2.2.15.** Every semisimple Lie group admits both uniform and non-uniform lattices.

By Selberg’s lemma, one can moreover assume that such lattices are torsion-free. Quotienting the symmetric space by such lattices gives the following corollary:

**Corollary 2.2.16.** For every Riemannian symmetric space $X$, there exist closed Riemannian manifolds locally isometric to $X$. If moreover $X$ has a factor of non-compact type, then there also exists complete non-compact Riemannian manifolds of finite volume locally isometric to $X$.

Let us prove a particular case of Borel and Harish-Chandra’s result, due to Siegel:

**Theorem 2.2.17 (Siegel).** Let $q$ be a non-degenerate quadratic form on $\mathbb{R}^n$ with integral coefficients. If $q$ never vanishes on $\mathbb{Z}^n \setminus \{0\}$, then $\text{O}(q, \mathbb{Z}) = \text{O}(q) \cap \text{GL}(n, \mathbb{Z})$ is a uniform lattice in $\text{O}(q)$.

The proof consists in viewing $\text{O}(q)/\text{O}(q, \mathbb{Z})$ as a subset of the space of lattices in $\mathbb{R}^n$.

More precisely, let $\text{Latt}(\mathbb{R}^n)$ denote the space of lattices in $\mathbb{R}^n$, i.e. the space of all discrete subgroups of $\mathbb{R}^n$ isomorphic to $\mathbb{Z}^n$. Then $\text{GL}(n, \mathbb{R})$ acts transitively on $\text{Latt}(\mathbb{R}^n)$ and the stabilizer of the standard lattice $\mathbb{Z}^n \subset \mathbb{R}^n$ is the subgroup $\text{GL}(n, \mathbb{Z})$. Hence $\text{Latt}(\mathbb{R}^n)$ is isomorphic to $\text{GL}(n, \mathbb{R})/\text{GL}(n, \mathbb{Z})$ as a $\text{GL}(n, \mathbb{R})$-homogeneous space. In $\text{Latt}(\mathbb{R}^n)$, we have the following characterization of relatively compact sets:

**Lemma 2.2.18 (Mahler’s criterion).** Let $A$ be a subset of $\text{Latt}(\mathbb{R}^n)$. Then $A$ is relatively compact if and only if there exists $\varepsilon > 0$ and $C$ such that

- $\text{Vol}(\mathbb{R}^n/\Lambda) \leq C$ for all $\Lambda \in A$
- $\Lambda \cap B(0, \varepsilon) = \{0\}$ for all $\Lambda \in A$. 

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For short, this criterion says that a sequence of lattices leaving every compact sets must either have bigger and bigger covolume or contain a smaller and smaller vector.

Proof of Theorem 2.2.17. The map
\[
\varphi : \text{O}(q) \rightarrow \text{Latt}(\mathbb{R}^n)
\]
\[
g \mapsto g \cdot \mathbb{Z}^n
\]
induces a continuous injective map from \(\text{O}(q)/\text{O}(q, \mathbb{Z})\) to \(\text{Latt}(\mathbb{R}^n)\) since \(\text{O}(q, \mathbb{Z})\) is exactly the stabilizer of \(\mathbb{Z}^n\) in \(\text{O}(q)\). We thus need to prove that the image of \(\varphi\) is relatively compact and closed.

Relative compactness. Since matrices of \(\text{O}(q)\) have determinant \(\pm 1\), every lattice in the image of \(\varphi\) has covolume 1. Assume by contradiction that there exists a sequence \(g_n \in \text{O}(q)\) such that \(g_n \cdot \mathbb{Z}^n\) contains smaller and smaller non-zero vectors. Let \(v_n\) be a sequence in \(\mathbb{Z}^n \setminus \{0\}\) such that \(g_n \cdot v_n\) converges to 0. In particular,
\[
q(g_nv_n) = q(v_n) \xrightarrow{n \to +\infty} 0.
\]
But \(q(v_n)\) belongs to \(\mathbb{Z}\) since \(q\) has integral coefficients. Thus \(q(v_n) = 0\) for \(n\) large enough, contradicting the hypothesis on \(q\). Therefore, such a sequence \(g_n\) does not exist. By Mahler’s criterion, we obtain that the image of \(\varphi\) is relatively compact.

Closedness. Let \(g_n\) be a sequence in \(\text{O}(q)\) such that \(g_n \cdot \mathbb{Z}^n\) converges to \(h \cdot \mathbb{Z}^n\) for some \(h \in \text{GL}(n, \mathbb{Z})\). Then there exists a sequence \(h_n \in \text{GL}(n, \mathbb{Z})\) such that \(g_nh_n\) converges to \(h\). For every \(v \in \mathbb{Z}^n\), we have
\[
q(g_nh_nv) \xrightarrow{n \to +\infty} q(hv).
\]
But \(q(g_nh_nv) = q(h_nv) \in \mathbb{Z}\). We deduce that \(q(h_nv) = q(hv)\). Applying this to sufficiently many vectors at once, we deduce that \(hh_n^{-1}\) preserves \(q\) for \(n\) large enough. Thus \(h \cdot \mathbb{Z}^n = hh_n^{-1} \cdot \mathbb{Z}^n = \varphi(hh_n^{-1})\).

Example 2.2.19. Define
\[
q(x, y, z, t) = x^2 + y^2 + z^2 - 7t^2.
\]
Then \(\text{O}(q, \mathbb{Z})\) is a uniform lattice in \(\text{O}(q) \simeq \text{Isom}(\mathbb{H}^3)\).

Unfortunately, Siegel’s theorem does not provide so many uniform lattices. Indeed, as a corollary of the Hasse–Minkowski theorem, we have:

Theorem 2.2.20. For every indefinite quadratic form \(q\) with integral coefficients in dimension \(n \geq 5\), the equation \(q = 0\) has a non-zero solution in \(\mathbb{Z}^n\).
To construct uniform lattices in higher dimensional semisimple Lie groups, one should extend the study to quadratic forms with coefficients in a number field. Let $K$ be a finite extension of $\mathbb{Q}$, let $O_K$ be the ring of integers in $K$ and let $q$ be a quadratic form on $K^n$. Let $\rho_1, \ldots, \rho_r : K \to \mathbb{R}$, $\sigma_1, \sigma_1', \ldots, \sigma_s, \sigma_s' : K \to \mathbb{C}$ denote respectively the real and complex embeddings of $K$ and let $q^{\rho_1}$ and $q^{\sigma_i}$ denote respectively the quadratic forms on $\mathbb{R}^n$ and $\mathbb{C}^n$ corresponding to the embeddings $\rho_i : K \to \mathbb{R}$ and $\sigma_i : K \to \mathbb{C}$. Then the group

$$G = \prod_{i=1}^r O(q^{\rho_i}) \times \prod_{j=1}^s O(q^{\sigma_j})$$

is defined over $\mathbb{Q}$ and the subgroup of integral points $G_{\mathbb{Z}}$ is the diagonal embedding of $O(q) \cap GL(O_K)$:

$$G_{\mathbb{Z}} = \{(\rho_1(g), \ldots, \rho_r(g), \sigma_1(g), \ldots, \sigma_s(g)), g \in O(q) \cap GL(n, O_K)\}.$$ 

As a particular case of Borel–Harish-Chandra’s theorem, we have the following:

**Theorem 2.2.21.** The group $G_{\mathbb{Z}}$ is a lattice in $G$. If, moreover, one of the $q^{\rho_i}$ is definite (positive or negative), this lattice is uniform.

In particular, if all the embeddings of $K$ are real, and if $q^{\rho_2}, \ldots, q^{\rho_r}$ are all definite, then $O(q) \cap GL(n, O_K)$ is a uniform lattice in $O(q^{\rho_1})$.

**Example 2.2.22.** Let $q$ be the quadratic form

$$q(x) = x_1^2 + \ldots + x_p^2 - \sqrt{2}(x_{p+1}^2 + \ldots + x_{p+q}^2).$$

Since the image of $q$ by the non-trivial Galois automorphism of $\mathbb{Q}[\sqrt{2}]$ is positive definite, we obtain that $O(q) \cap GL(n, \mathbb{Z}[\sqrt{2}])$ is a uniform lattice in $O(q)$. This provides uniform lattices in all the groups $O(p,q)$.

### 2.3 Rigidity

#### 2.3.1 Rank of a semisimple Lie group

Let $\mathfrak{g}$ be a semisimple Lie algebra and $\theta$ a Cartan involution. Denote by $\mathfrak{k}$ the subalgebra fixed by $\theta$ and by $\mathfrak{p}$ the orthogonal of $\mathfrak{k}$ with respect to the Killing form.

**Definition 2.3.1.** The (real) rank of $\mathfrak{g}$ is the maximal dimension of a (necessarily Abelian) subalgebra of $\mathfrak{p}$. The rank of a semisimple Lie group $G$ is the rank of its Lie algebra.

Let now $G$ the a semisimple Lie group with Lie algebra $\mathfrak{g}$ and let $(X, o)$ be the symmetric space $G/G^\theta$. Recall that the tangent space to $X$ at $o$ is identified with $\mathfrak{p}$ and that, if $R$ denotes the curvature tensor of the Riemannian metric $q$ at $o$, we have

$$q(R(u, v)w, u) = \kappa_{\mathfrak{g}}([u, v], [u, v]).$$
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Proposition 2.3.2. Let \( a \) be an abelian subalgebra of \( \mathfrak{p} \). Then \( \exp(\mathfrak{a}) \cdot o \) is a totally geodesic submanifold of \( X \) of curvature 0 (i.e. isometric to a Euclidean space). Conversely, if \( P \) is flat a totally geodesic submanifold of \( X \) containing \( o \), then \( \mathfrak{a} = T_0P \subset \mathfrak{p} \) is an Abelian subalgebra of \( \mathfrak{p} \).

The rank of \( G \) is thus the maximal dimension of a flat totally geodesic subspace of its symmetric space. In particular, \( G \) has rank 1 if and only if the sectional curvature of \( X \) is strictly negative.

Example 2.3.3. The group \( \text{SL}(n, \mathbb{R}) \) has rank \( n - 1 \). An Abelian subalgebra of \( \mathfrak{p} = \{ S \mid S^T = S \} \) of maximal dimension is the algebra of diagonal matrices with trace 0.

Example 2.3.4. If \( p \geq q \), the group \( \text{O}(p, q) \) has rank \( q \). Let \( \mathfrak{q} \) be the quadratic form

\[
\mathfrak{q}(\mathbf{x}) = x_1x_2 + x_3x_4 + \ldots + x_{2q-1}x_{2q} + x_{2q+1}^2 + \ldots + x_{p+q}^2,
\]

of signature \((p, q)\). Then the space of diagonal matrices

\[
\begin{pmatrix}
\lambda_1 & -\lambda_1 & \cdots & \cdots & -\lambda_p & -\lambda_p \\
& & & \ddots & & \\
& & \ddots & & & \\
& & & & 0
\end{pmatrix}
\]

is a maximal abelian subalgebra of \( \mathfrak{p} \) (for some suitable choice of Cartan involution).

Lie groups of rank 1 Here is the list of all quasi-simple Lie groups of rank 1 (up to covering) and their symmetric space:

- The group \( \text{O}(n, 1) \), whose symmetric space is the hyperbolic space \( \mathbb{H}^n \),
- The group \( \text{SU}(n, 1) = \{ g \in \text{SL}(n + 1, \mathbb{C}) \mid \overline{g}^T I_{n+1} g = I_{n+1} \} \), whose symmetric space is the complex hyperbolic space \( \mathbb{H}^n_{\mathbb{C}} \),
- Their quaternionic analog \( \text{Sp}(n, 1) \), whose symmetric space is the quaternionic hyperbolic space (of real dimension \( 4n \)),
- The exceptional group \( F_{4}^{-20} \), whose symmetric space is in some sense the octonionic hyperbolic plane (of real dimension 16).
2.3.2 Local rigidity

Let $\Gamma$ be a finitely generated group and $G$ a Lie group. Let $S$ be a finite generating set. The map

$$\varphi : \text{Hom}(\Gamma, G) \to G^S$$

$$\rho \mapsto (\rho(s))_{s \in S}$$

embeds $\text{Hom}(\Gamma, G)$ as a closed subset of $G^S$.

The group $G$ acts continuously on $\text{Hom}(\Gamma, G)$ by

$$g \cdot \rho : \gamma \mapsto g\rho(\gamma)g^{-1}.$$ 

Definition 2.3.5. A representation $\rho : \Gamma \to G$ is called locally rigid if for every $\rho'$ in a neighbourhood of $\rho$, there exists $g \in G$ such that $\rho' = g \cdot \rho$.

It turns out that most lattices are rigid in the following sense:

Theorem 2.3.6 (Selberg, Calabi, Weil). Let $G$ be a simple Lie group which is not isomorphic to $\text{PSL}(2, \mathbb{R})$ and $\Gamma$ a uniform lattice in $G$. Then the inclusion $i : \Gamma \to G$ is locally rigid in $\text{Hom}(\Gamma, G)$.

Remark 2.3.7. The theorem generalizes to a lattice $\Gamma$ in a semisimple Lie group $G$, but the precise statement requires extra care. One needs to exclude two things: $G$ could be of the form $G' \times K$ with $K$ compact, in which case a lattice $\Gamma$ in $G'$ can be “twisted” by any representation from $\Gamma \to K$, and $G$ could be of the form $\text{PSL}(2, \mathbb{R}) \times G'$ and $\Gamma$ of the form $\Gamma_1 \times \Gamma_2$ with $\Gamma_1$ a lattice in $\text{PSL}(2, \mathbb{R})$ that can be deformed.

The case of non-uniform hyperbolic lattices in rank 1 was later settled by Garland and Raghunathan.

Theorem 2.3.8 (Garland–Ragunathan). Let $G$ be a simple Lie group of rank 1 and $\Gamma$ a (not necessarily uniform) lattice in $G$. Then the inclusion $i : \Gamma \to G$ is locally rigid in $\text{Hom}(\Gamma, G)$.

Finally, the case of non-uniform lattices in general was settled by Margulis, as a particular case of his super-rigidity theorem, whose local version states that every representation of a higher rank lattice is locally rigid.

Margulis’s super-rigidity theorem (local version). Let $G$ be a simple Lie group of rank at least 2, $\Gamma$ a lattice in $G$ and $\rho$ a representation of $\Gamma$ into a Lie group $H$. Then $\rho$ is locally rigid in $\text{Hom}(\Gamma, H)$.

Finally, Corlette proved that super-rigidity also holds for lattices in the rank 1 Lie groups $\text{Sp}(n, 1)$ and $F_4 - 20$.

Theorem 2.3.9 (Corlette). The previous theorem holds for $G = \text{Sp}(n, 1)$ and $F_4 - 20$. 

2.3.3 Global rigidity

The above local rigidity theorems also have global counterparts, roughly stating that certain representations of lattices extends to representations of the whole Lie group. We state those results here for completeness.

**Theorem 2.3.10** (Mostow, Prasad). Let $G$ be a simple Lie group which is not isomorphic to $\text{PSL}(2, \mathbb{R})$, and $\Gamma_1, \Gamma_2$ two lattices in $G$. If $\rho : \Gamma_1 \to \Gamma_2$ is an isomorphism, then $\rho$ extends to an automorphism of $G$.

In fact for $G$ of rank at least 2, Margulis’s super-rigidity theorem states that most representations of a lattice extend to the whole group.

**Margulis’s super-rigidity theorem** (Global version). Let $G$ be a simple Lie group of rank at least 2 and $\Gamma$ a lattice in $G$. Let $\tilde{G}$ denote the universal cover of $G$ and $\pi : \tilde{G} \to G$ the covering morphism.

Let $H$ be another Lie group. Then, for every representation $\rho : \Gamma \to H$ with unbounded image, there exists $\tilde{\rho} : \tilde{G} \to H$ such that $\tilde{\rho}|_{\pi^{-1}(\Gamma)} = \rho \circ \pi$.

Again, this theorem also holds for $G = \text{Sp}(n, 1)$ and $F_4^{-20}$.

**Theorem 2.3.11** (Corlette). The above theorem holds for $G = \text{Sp}(n, 1)$ and $F_4^{-20}$.

Actually, Margulis’s super-rigidity theorem also holds for representations into $p$-adic Lie groups (such as $\text{GL}(n, \mathbb{Q}_p)$). A striking consequence of that fact is **Margulis’s arithmeticity theorem**, which we do not state precisely here:

**Theorem 2.3.12** (Margulis). Let $G$ be a simple Lie group of rank at least 2. Then every lattice $\Gamma$ in $G$ is arithmetic (i.e. comes from an arithmetic construction).

The $p$-adic version of Corlette’s super-rigidity theorem was proven by Gromov and Schoen, who obtained as a consequence:

**Theorem 2.3.13** (Gromov–Schoen). Lattices in $\text{Sp}(n, 1)$ and $F_4^{-20}$ are arithmetic.

2.4 Flexibility

In conclusion of the previous section, lattices in semisimple Lie groups cannot be deformed except possibly in the following cases:

1. Lattices in $\text{PSL}(2, \mathbb{R})$ could be deformed inside $\text{PSL}(2, \mathbb{R})$,

2. Non-uniform lattices in $\text{PSL}(2, \mathbb{C}) \simeq \text{Isom}_+ (\mathbb{H}^3)$ could be deformed into $\text{PSL}(2, \mathbb{C})$, though those deformations will not remain discrete and faithful,
3. Lattices in $G = \text{O}(n, 1)$ and $\text{SU}(n, 1)$, $n \geq 2$ could be deformed inside Lie groups containing strictly $G$.

Let us discuss more precisely those situations.

1. Up to taking a finite index subgroup, lattices in $\text{PSL}(2, \mathbb{R})$ are either free groups of surface groups. Free groups easily deform and surface groups discretely embedded also have a non-trivial deformation space which identifies with the Teichmüller space of the surface.

2. Let $\Gamma$ be a torsion-free non-uniform lattice in $\text{PSL}(2, \mathbb{C}) \simeq \text{Isom}_+ (\mathbb{H}^3)$. Then the inclusion $i: \Gamma \to \text{PSL}(2, \mathbb{C})$ admits deformations. More precisely, the complex dimension of the character variety of $\Gamma$ near the point $i$ equals the number of cusps of $\Gamma \setminus \mathbb{H}^3$. Using those deformations, Thurston constructs hyperbolic structures on closed manifolds obtained from $\Gamma \setminus \mathbb{H}^3$ via Dehn surgeries.

3. Some lattices in $G = \text{SO}(n, 1)$, $n \geq 3$ and $\text{SU}(n, 1)$, $n \geq 2$ surject onto free groups and surface groups. Such lattices can be deformed into $G \times H$ by simply deforming the trival representation to $H$ into representations that factor through a free group.

However, lattices in $\text{SU}(n, 1)$ are fairly rigid according to results of Raghunathan, Goldman–Millson, Kim–Pansu, Klingler... I don’t know if there are other examples of deformations of such lattices.

The purpose of the rest of this section is to describe other deformations of real hyperbolic lattices with more geometric meaning. The key ingredient of these constructions will be that some hyperbolic manifolds (in dimension 2, all of them) can be “cut” along a totally geodesic hypersurface.

### 2.4.1 Hyperbolic manifolds with totally geodesic hypersurfaces

The starting point of the construction of these deformations is a closed hyperbolic manifold $M$ containing a totally geodesic hypersurface $N$. In dimension 2, these exist for any $M$, thanks to the following lemma:

**Lemma 2.4.1.** Let $\Sigma$ be a closed hyperbolic surface and $c$ a non-homotopically trivial simple closed curve (i.e. embedded circle) in $\Sigma$. Then $c$ is freely homotopic to a simple closed geodesic.

**Sketch of the proof.** The closed minimizing the length in the free homotopy class of $c$ is a geodesic $\gamma$. The difficulty is to see that this geodesic does not self-intersect.

Let $\pi : \mathbb{H}^2 \simeq \tilde{\Sigma} \to \Sigma$ be the universal covering. The curve $c$ defines a conjugacy class $[g]$ in $\pi_1(\Sigma) \subset \text{Isom}(\mathbb{H}^2)$, and $\pi^{-1}(\gamma)$ consists in the reunion
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of all the axes of $h$ for $h \in \gamma$. If $\gamma$ intersects itself, then one obtains $h_1$ and $h_2 \in \gamma$ such that the axes if $h_1$ and $h_2$ intersect. This implies that the endpoints of these axes are intertwined, i.e. in the cyclic order $(h^-_1, h^-_2, h^+_1, h^+_2)$. But $\pi^{-1}(c)$ contains two arcs going respectively from $h^-_1$ to $h^+_1$ and from $h^-_2$ to $h^+_2$. These two arcs must intersect, contradicting the fact that $c$ was embedded.

In higher dimension, finding totally geodesic hypersurfaces is harder, and such hypersurfaces may not exist in general. However, they tend to exist for quotients of $\mathbb{H}^n$ by arithmetic lattices, after taking a suitable cover, thanks to the following:

**Proposition 2.4.2.** Let $\Gamma_n$ be a uniform lattice in $\text{Isom}(\mathbb{H}^n)$ such that $\Gamma_n \cap \text{Stab}(\mathbb{H}^{n-1})$ is a uniform lattice in $\text{Stab}(\mathbb{H}^{n-1})$. Then there exists a finite index subgroup $\Gamma'$ of $\Gamma_n$ which is torsion-free and such that the map

$$
\Gamma' \cap \text{Stab}(\mathbb{H}^{n-1}) \backslash \mathbb{H}^{n-1} \to \Gamma \backslash \mathbb{H}^n
$$

is an embedding.

**Example 2.4.3.** Let $q_n$ be the quadratic form

$$
q_n(x_0, \ldots, x_n) = -\sqrt{2} x_0^2 + x_1^2 + \cdots + x_n^2
$$
on $\mathbb{R}^{n+1}$, and let $\Gamma_n$ be the uniform lattice $O(q_n) \cap \text{GL}(n+1, \mathbb{Z}[$$\sqrt{2}$]) in $O(q_n)$. View $\mathbb{H}^n$ as the space of lines in $\mathbb{R}^{n+1}$ in restriction to which $q_n$ is negative, and let $\mathbb{H}^{n-1} \subset \mathbb{H}^n$ be the subset of those lines contained in $\mathbb{R}^n = \{(x_0, \ldots, x_{n-1}, 0)\}$.

Then we have

$$
\Gamma_n \cap \text{Stab}(\mathbb{H}^{n-1}) = \left\{ \begin{pmatrix} g & 0 \\ 0 & \pm 1 \end{pmatrix} : g \in O(q_n) \right\},
$$

which is a lattice in $\text{Stab}(\mathbb{H}^{n-1}) \cong O(q_n) \times \mathbb{Z}/2\mathbb{Z}$.

This example shows that the hypotheses of Proposition 2.4.2 can be satisfied. Thus Proposition 2.4.2 does construct closed hyperbolic manifolds with embedded totally geodesic hypersurfaces.

To prove Proposition 2.4.2, recall first that Selberg’s lemma allows us to assume that $\Gamma_n$ is torsion-free, which we do from now on.

Let us introduce the set

$$
A = \{ g \in \Gamma_n \mid g \cdot \mathbb{H}^{n-1} \cap \mathbb{H}^{n-1} \neq 0 \}.
$$

Note that $A$ contains $\Gamma_{n-1} = \Gamma_n \cap \text{Stab}(\mathbb{H}^{n-1})$ and that $A = \Gamma_{n-1}$ if and only if the immersion $\Gamma_{n-1} \backslash \mathbb{H}^{n-1} \to \Gamma_n \backslash \mathbb{H}^n$ is injective.

Note also that $A$ is invariant by multiplication by $\Gamma_{n-1}$ on the right and on the left.
Lemma 2.4.4. The double coset space $\Gamma_{n-1} \backslash A \backslash \Gamma_{n-1}$ is finite, i.e. there exists finitely many elements $g_0, \ldots, g_k \in A$ such that every element of $A$ has the form $h_1g_ih_2$ for some $0 \leq i \leq k$ and $h_1, h_2 \in \Gamma_{n-1}$.

Proof. Since $\Gamma_{n-1}$ acts cocompactly on $\mathbb{H}^{n-1}$, we can choose a compact subset $\mathcal{K}$ of $\mathbb{H}^{n-1}$ such that $\bigcup_{h \in \Gamma_{n-1}} h \cdot \mathcal{K} = \mathbb{H}^{n-1}$.

Since $\Gamma_n$ acts properly discontinuously on $\mathbb{H}^n$, there are only finitely many elements $\{g_0, \ldots, g_k\}$ of $\Gamma_n$ such that $g_i \cdot \mathcal{K} \cap \mathcal{K} \neq \emptyset$. Note that those elements belong to $A$.

Now let $g$ be any element of $A$. Let $x$ and $y$ be points in $\mathbb{H}^{n-1}$ such that $g \cdot x = y$. By cocompactness of the action of $\Gamma_{n-1}$ on $\mathbb{H}^{n-1}$, we can find $h_1$ and $h_2$ such that $h_1 \cdot x$ and $h_2 \cdot y$ belong to $\mathcal{K}$. We have

$$h_2gh_1^{-1} \cdot (h_1 \cdot x) = h_2 \cdot y.$$  

Thus $h_2gh_1^{-1}$ is one of the $g_i$. □

Let us assume that $g_0 = \text{Id}$ and that $g_i$ does not belong to $\text{Stab}(\mathbb{H}^{n-1})$ for $i \geq 1$. If we could find a finite index subgroup $\Gamma'$ of $\Gamma_n$ containing $\Gamma_{n-1}$ but not the $g_i$, $i \geq 1$, then we would have $\Gamma' \cap A \subset \text{Stab}(\mathbb{H}^{n-1})$, which would conclude the proof. This property is called separability of the subgroup $\Gamma_{n-1}$. Note that it is a stronger requirement than simply asking for a finite index subgroup that does not contain the $g_i$, $i \geq 1$. Nevertheless, a clever use of Malcev’s theorem will allow us to find our required finite index subgroup.

Proof of Proposition 2.4.2. Let $\sigma$ denote the reflection with respect to $\mathbb{H}^{n-1}$. Then $\text{Stab}(\mathbb{H}^{n-1})$ is exactly the centralizer of $\sigma$:

$$\text{Stab}(\mathbb{H}^{n-1}) = \{ g \in \text{Isom}(\mathbb{H}^n) \mid g\sigma = \sigma g \}.$$  

In particular, $[\sigma, g_i] \neq \text{Id}$ for $i \geq 1$.

Let $H$ denote the group spanned by $\Gamma_n$ and $\sigma$. Since $H$ is finitely generated, we can use Malcev’s theorem to find a finite index subgroup $H'$ of $H$ that does not contain $[\sigma, g_i]$ for $i \geq 1$. We can moreover assume that $H'$ is normal in $H$.\footnote{It is a classical fact that any finite index subgroup contains a finite index normal subgroup} Take $\Gamma' = \Gamma_n \cap H'$. We want to prove that $\Gamma' \cap A \subset \text{Stab}(\mathbb{H}^{n-1})$.

Let $g$ be an element in $\Gamma' \cap A$ and write $g = h_1g_1h_2$ for some $h_1, h_2 \in \Gamma_{n-1}$. Since $H'$ is normal in $H$, we have that $[\sigma, g]$ belongs to $H'$. Since $\sigma$ commutes with $h_1$ and $h_2$, we have

$$[\sigma, g] = \sigma h_1g_1h_2\sigma h_2^{-1}g_i^{-1}h_1^{-1} = h_1[\sigma, g_i]h_1^{-1}.$$  

Using again that $H'$ is normal, we deduce that $[\sigma, g_i]$ belongs to $H'$. By construction of $H'$, we thus have $i = 0$ and $g \in \Gamma_{n-1}$. □
2.4. FLEXIBILITY

2.4.2 Van Kampen’s theorem and applications

Let $M$ be a closed connected oriented manifold, and $H$ a closed connected embedded hypersurface. Classical theorems in algebraic topology describe the relations between the fundamental groups of $M$, $H$ and $M\setminus H$. There are two cases, depending on whether $H$ disconnects $M$ or not.

Amalgamated products In the case where $M\setminus H$ is the union of two connected components $M_1$ and $M_2$, the fundamental group of $M$ has the structure of an amalgamated product.

**Definition 2.4.5.** Let $\Gamma_1$, $\Gamma_2$ and $N$ be three groups and $i_1, i_2$ morphisms from $N$ to $\Gamma_1$ and $\Gamma_2$ respectively. The amalgamated product $\Gamma_1 \ast_N \Gamma_2$ is the quotient of the free product $\Gamma_1 \ast \Gamma_2$ by the relations

$$i_1(g) = i_2(g), \quad g \in N.\]

The amalgamated product is characterized by the following universal property:

**Proposition 2.4.6.** There are morphisms $m_1 : \Gamma_1 \to \Gamma_1 \ast_N \Gamma_2$ and $m_2 : \Gamma_2 \to \Gamma_1 \ast_N \Gamma_2$ such that $m_1 \circ i_1 = m_2 \circ i_2$. Moreover, for any group $G$ and any morphisms $\rho_1 : \Gamma_1 \to G$ and $\rho_2 : \Gamma_2 \to G$ such that $\rho_1 \circ i_1 = \rho_2 \circ i_2$, there exists a unique morphism $\rho : \Gamma_1 \ast_N \Gamma_2 \to G$ such that $\rho \circ m_1 = \rho_1$ and $\rho \circ m_2 = \rho_2$.

Consider $M_1$ and $M_2$ as manifolds with boundary identified with $H$, and take fundamental groups of $M$, $M_1$, $M_2$ and $H$ with respect to a base point in $H$. Then inclusions define the following commuting diagram of morphisms:

\[
\begin{array}{ccc}
\pi_1(H) & \xrightarrow{i_1} & \pi_1(M_1) \\
\downarrow & & \downarrow m_1 \\
\pi_1(M) & \xleftarrow{m_2} & \pi_1(M_2) \\
\downarrow & & \downarrow i_2 \\
\pi_1(H) & \xleftarrow{i_1} & \pi_1(M_1) \\
\end{array}
\]

**Theorem 2.4.7 (Van Kampen).** The fundamental group of $M$ is isomorphic to the amalgamated product

$$\pi_1(M_1) \ast_{\pi_1(H)} \pi_1(M_2).$$

HNN Extensions When $M' = M\setminus H$ is connected the fundamental group of $M$ has the structure of an HNN extension (after Higman, Bernhard Neumann and Hanna Neumann).
Definition 2.4.8. Let $\Gamma, N$ be groups and $i_1, i_2$ two morphisms from $N$ to $\Gamma$. Then the **HNN extension** $\Gamma *_N$ is the quotient of the free product $\Gamma * Z$ by the relations
\[ i_2(g) = t i_1(g) t^{-1}, \quad g \in N, \]
where $t$ denotes a generator of $Z$.

Remark 2.4.9. The usual definition of an HNN extension requires $i_1$ and $i_2$ to be injective, but I think the above definition is better suited to the description of the fundamental group of a manifold glued to itself along the boundary.

The HNN extension is characterized by the following universal property:

**Proposition 2.4.10.** There is a morphism $m : \Gamma \to \Gamma *_N$ such that
\[ m \circ i_2(g) = t m \circ i_1(g) t^{-1} \]
for all $g \in N$. Moreover, for any group $G$ and any morphism $\rho \Gamma \to G$ for which there exists $h \in G$ such that
\[ \rho \circ i_2(g) = h \rho \circ i_1(g) h^{-1}, \]
there exists a unique morphism $\rho' : \Gamma *_N \to G$ such that $\rho' \circ m = \rho$ and $\rho'(t) = h$.

Consider $M' = M \setminus H$ as manifold with two boundary components $H_-$ and $H_+$ that are both identified with $H$. Take a basepoint $x$ in $H$ and let $x_-$ and $x_+$ denote its lifts to $H_-$ and $H_+$ respectively. Then the isomorphisms between $H$ and $H_-$ and $H_+$ respectively define morphisms $i_1$ and $i_2$ from $\pi_1(H, x)$ to $\pi_1(M, x_-)$ and $\pi_1(M, x_+)$. Let $t$ be a math from $x_-$ to $x_+$. Then $t$ defines an isomorphism between $\pi_1(M, x_-)$ and $\pi_1(M, x_+)$, and we can thus see $i_1$ and $i_2$ as morphisms from $\pi_1(H, x)$ to $\pi_1(M', x_-)$. Finally, let $m$ denote the morphism from $\pi_1(M', x_-)$ to $\pi_1(M, x)$. Then $m \circ i_1 : \pi_1(H, x) \to \pi_1(M, x)$ is the morphism associated to the inclusion of $H$, while $m \circ i_2 = C_t \circ m \circ i_1$, where $C_t$ denotes the conjugation by $t$ (seen as a loop in $M$).

**Theorem 2.4.11** (Higman, Neumann, Neumann). The fundamental group of $M$ is isomorphic to the HNN extension
\[ \pi_1(M') *_{\pi_1(H)} \]

2.4.3 Deformation of hyperbolic lattices

Let $M$ be a closed connected oriented manifold, $H$ a closed connected oriented hypersurface, $G$ a Lie group and $\rho$ a representation of $\pi_1(M)$ into $G$.

Assume there exists an element $h \in G$ that commutes with $\rho(\pi_1(H))$. Then one defines a new representation $T_h \rho$ in the following way:
• If $H$ separates $M$, write $\pi_1(M) = \pi_1(M_1) \ast_{\pi_1(H)} \pi_1(M_2)$ and define $T_h \rho$ as the representation whose restriction of $\pi_1(M_1)$ is $\rho$ and whose restriction to $\pi_1(M_2)$ is $\rho$ conjugated by $h$.

• If $H$ does not separate $M$, write $\pi_1(M) = \pi_1(M') \ast_{\pi_1(H)}$ and define $T_h \rho$ as the representation whose restriction to $\pi_1(M')$ is $\rho$ and that sends $t$ to $\rho(t)h$.

In both cases, one can verify that the appropriate relations are satisfied so that $T_h \rho$ is well-defined.

The representation $T_h \rho$ may be conjugate to $\rho$. For instance, in the first case, it happens when $h$ commutes with $\rho(\pi_1(M_1))$ or $\rho(\pi_1(M_2))$. But there is a good chance to get non-trivial deformations when the centralizer of $\rho(\pi_1(H))$ is significantly larger than the centralizer of $\rho$. 
Bibliography


