in well-generated complex reflection groups

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joint work with Vic Reiner (Minneapolis) and Christian Stump (Berlin)

Motivation

- $\mathsf{NC}(n) := \{ w \in \mathfrak{S}_n \mid \ell_{\mathcal{T}}(w) + \ell_{\mathcal{T}}(w^{-1}c) = \ell_{\mathcal{T}}(c) \}, \text{ where }$
 - T := {all transpositions of 𝔅_n}, ℓ_T associated length function ("absolute length");
 - *c* is a long cycle (*n*-cycle).

NC(n) is

- equipped with a natural partial order ("absolute order"), and is a lattice;
- isomorphic to the poset of NonCrossing partitions of an *n*-gon ("noncrossing partition lattice"), so it is counted by the Catalan number $Cat(n) = \frac{1}{n+1} {\binom{2n}{n}}$.

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Generalization to finite Coxeter groups (or reflection groups):

- replace \mathfrak{S}_n with a Coxeter group W;
- replace T with $R := \{a \mid reflections of W\}$, and ℓ_T with ℓ_R ;
- replace c with a Coxeter element of W.

the W-noncrossing partition lattice

$$\mathsf{NC}(W,c) := \{ w \in W \mid \ell_R(w) + \ell_R(w^{-1}c) = \ell_R(c) \}$$

- also equipped with a "W-absolute order";
- counted by the W-Catalan number $\operatorname{Cat}(W) := \prod_{i=1}^{n} \frac{d_i+h}{d_i}$.

Cat(W) appears in other combinatorial objects attached to (W, c): cluster complexes, generalized associahedra, Cambrian fans and lattices, subword complexes...

 \rightsquigarrow "Coxeter-Catalan combinatorics".

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 \rightsquigarrow "Coxeter-Catalan combinatorics".

Outline

1 "Classical" definitions of a Coxeter elements

- ... for a Coxeter system (W, S)
- ... for a real reflection group
- ... for a complex reflection group

2 Extended definitions

- ... with alternative Coxeter structures
- ... with reflection automorphisms
- ... with other eigenvalues
- Main result and consequences on Coxeter-Catalan combinatorics

- Field of definition of W and Galois automorphisms
- Galois action on conjugacy classes of Coxeter elements

Galois automorphisms 00000000

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Coxeter element of a Coxeter system

Definition

A Coxeter system (W, S) is a group W equipped with a generating set S of involutions, such that W has a presentation of the form:

$$W=ig\langle S ig| \ s^2=1 \ (orall s\in S); \ (st)^{m_{s,t}}=1 \ (orall s
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angle \ ,$$

with $m_{s,t} \in \mathbb{N}_{\geq 2} \cup \{\infty\}$ for $s \neq t$.

Coxeter element, "Definition 0"

Write $S := \{s_1, \ldots, s_n\}$. A Coxeter element of (W, S) is a product of all the generators:

 $c = s_{\pi(1)} \dots s_{\pi(n)}$ for $\pi \in \mathfrak{S}_n$.

Fact: When W is **finite**, all Coxeter elements of (W, S) are **conjugate**. (ingredient: the Coxeter graph is a forest)

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Coxeter element of a real reflection group

- V real vector space of dimension n
- W finite subgroup of GL(V) generated by reflections

 \rightsquigarrow *W* admits a structure of **Coxeter system**:

- $\bullet\,$ fix a chamber ${\cal C}$ of the hyperplane arrangement of W
- take $S := \{ \text{reflections through the walls of } C \}$

Definition ("Classical definition")

Let W be a finite real reflection group. A Coxeter element of W is a product (in any order) of all the reflections through the walls of a chamber of W.

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Classical definitions

Extended definitions

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Complex reflection group

- V complex vector space of dimension n
- W finite subgroup of GL(V) generated by "reflections"
 (r ∈ GL(V) of finite order and fixing pointwise a hyperplane)

Finite *real* reflection groups can be seen as complex reflection groups.

But there are much more.

In general: no Coxeter structure, no privileged (natural, canonical) set of *n* generating reflections.

 \rightsquigarrow how to define a Coxeter element of W?

Digression: geometry of Coxeter elements in real groups

Assume W is real and irreducible.

Call h := Coxeter number = the order of a Coxeter element.

Fact: $h = d_n$, the highest invariant degree of W.

 $d_1 \leq \cdots \leq d_n$ degrees of homogeneous polynomials f_1, \ldots, f_n such that $\mathbb{C}[V]^W = \mathbb{C}[f_1, \ldots, f_n]$.

Proposition (Coxeter)

If c is a Coxeter element, then there exists a plane $P \subseteq V$ stable by c and on which c acts as a rotation of angle $\frac{2\pi}{h}$. In particular, c admits $e^{\frac{2i\pi}{h}}$ (and $e^{-\frac{2i\pi}{h}}$) as an eigenvalue. Digression: geometry of Coxeter elements in real groups

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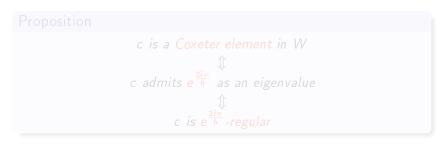
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Geometry of Coxeter elements in real groups

Better: c is $e^{\frac{2i\pi}{h}}$ -regular in the sense of Springer: it has a $e^{\frac{2i\pi}{h}}$ -eigenvector $v \in V_{\mathbb{C}}$, which does not lie in the reflecting hyperplanes.

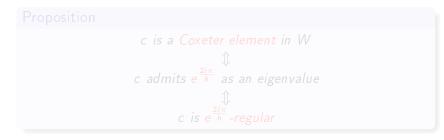
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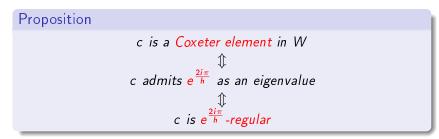
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Coxeter element in a complex reflection group

Now W is a well-generated, irreducible complex reflection group: W can be generated by $n = \dim V$ reflections. Define the Coxeter number h of W as the highest invariant degree: $h := d_n$.

The set of elements of W having $e^{rac{2i\pi}{h}}$ as eigenvalue

- is non-empty and forms a conjugacy class of W [Springer] ;
- = the set of elements having $e^{\frac{2i\pi}{h}}$ as eigenvalue.

Definition ("classical definition", Bessis '06)

Let W be a well-generated, irreducible complex reflection group. A Coxeter element of W is an element that admits $e^{\frac{2i\pi}{h}}$ as an eigenvalue.

Bessis' seminal work related to Coxeter-Catalan combinatorics and the dual braid monoid for complex groups uses this definition.

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Alternative Coxeter structures

In general a real reflection group does not have a unique Coxeter structure:

Example

Symmetry group of the regular hexagon $= l_2(6) \simeq A_1 \times A_2$

But unicity of the structure if "S must consist of reflections":

Rigidity Property (Observation/Folklore?)

Let W be a finite real reflection group, R the set of all reflections of W. Let $S, S' \subseteq R$ be such that (W, S) and (W, S') are both Coxeter systems. Then (W, S) and (W, S') are isomorphic Coxeter systems.

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(W, S) finite Coxeter system. $R := \bigcup_{w \in W} wSw^{-1}$. Let $S' \subseteq R$ be such that (W, S') is also a Coxeter system. Then (W, S') is isomorphic to (W, S).

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Classical definitions

Extended definitions

Galois automorphisms

New Coxeter elements

For a real reflection group W, one may be able to construct a set S of Coxeter generating reflections, which do not come from a chamber of the arrangement...

→ Isomorphic, but not conjugate structures!

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Example of I_2(5).
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Definition

We call generalized Coxeter element of W a product (in any order) of the elements of some set S, where S is such that:

- S consists of reflections;
- (W, S) is a Coxeter system.

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Reflection automorphisms

(W, S) and (W, S') are isomorphic Coxeter systems \implies there is an automorphism ψ of W mapping S to S'.

Fact: ψ is then a reflection automorphism of W, i.e., an automorphism of W stabilizing the set R of all reflections of W.

From the Rigidity Property we obtain:

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Proposition

Let W be a finite real reflection group.

c is a generalized Coxeter element of W

\updownarrow

c = \psi(c_0) with \psi reflection automorphism and c_0 classical Coxeter

element of W.
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Replace $e^{2i\pi/h}$ by another *h*-th root of unity

Definition ("Extended definition")

Let W be a well-generated, irreducible complex reflection group, and h its Coxeter number.

We call generalized Coxeter element an element of W that admits a primitive h-th root of unity as an eigenvalue.

Equivalently, c is a generalized Coxeter element if and only if $c = w^k$ where w is a *classical* Coxeter element and $k \wedge h = 1$

Is this definition compatible with the extended definition for real groups ?

Extended definitions

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Four definitions

	Classical definition	Extended definition
W real	Product of reflections through the walls of a chamber	$\prod_{s \in S} s, \text{ for some } S \subseteq R,$ with (W, S) Coxeter
W complex	$e^{\frac{2i\pi}{h}}$ is eigenvalue	$e^{rac{2ik\pi}{h}}$ is eigenvalue for some $k,\ k\wedge h=1$

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Compatibility of the extended definitions

Theorem (Reiner-R.-Stump)

Let $c \in W$. The following are equivalent:

- (i) c has an eigenvalue of order h;
- (ii) $c = \psi(w)$ where w is a classical Coxeter element and ψ is a reflection automorphism of W;
- (iii) c is a Springer-regular element of order h.

If W is real, this is also equivalent to:

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If W is real, this is also equivalent to:

Corollary

W well-gen., irred. c.r.g., R = Refs(W). Any property

• known for classical Coxeter elements, and

• "depending only on the combinatorics of the couple (W, R)", extends to generalized Coxeter elements. This applies in particular to Coxeter-Catalan combinatorics, e.g.:

the W-noncrossing partition lattices

$NC(W, c) := \{ w \in W \mid \ell_R(w) + \ell_R(w^{-1}c) = \ell_R(c) \}$

- the number of reduced R-decompositions of a generalized Coxeter element into reflections is num;;
- the Hurwitz action of the braid group B_n on reduced decompositions is transitive.

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- the Hurwitz action of the braid group B_n on reduced decompositions is transitive.

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• the W-noncrossing partition lattices

 $NC(W, c) := \{ w \in W \mid \ell_R(w) + \ell_R(w^{-1}c) = \ell_R(c) \}$

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Galois automorphisms

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Classical" definitions of a Coxeter elements

- ... for a Coxeter system (W, S)
- ... for a real reflection group
- ... for a complex reflection group

2 Extended definitions

- ... with alternative Coxeter structures
- ... with reflection automorphisms
- ... with other eigenvalues
- Main result and consequences on Coxeter-Catalan combinatorics

3 Galois automorphisms

- Field of definition of W and Galois automorphisms
- Galois action on conjugacy classes of Coxeter elements

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Extended definitions

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Field of definition of W

Definition

The field of definition K_W of W is

$$\mathcal{K}_{W} := \langle \operatorname{tr}_{V}(w), w \in W \rangle.$$

Fact: the representation V of W can be realized over K_W , so K_W is the smallest field over which one can write all matrices of W.

Examples

• $K_W = \mathbb{Q}$ iff W crystallographic (Weyl group)

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$$W=H_3$$
 or H_4 : $K_W=\mathbb{Q}(\sqrt{5})$

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$$W = l_2(m)$$
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Galois action on W

Let $\Gamma := \operatorname{Gal}(K_W/\mathbb{Q})$. For $\gamma \in \Gamma$ an $w \in W$, define $\gamma(w)$ by acting on the coefficients of the matrix of w written in K_W .

Problem: W is not necessarily preserved by the action of Γ .

But: $\gamma(W)$ is the "same" reflection group as W in the classification, so they are conjugate: $\gamma(W) = aWa^{-1}$, for $a \in GL(V)$.

$$W \xrightarrow{\gamma} \gamma(W) \xrightarrow{a^{-1}(-)a} a^{-1}\gamma(W)a = W$$

 \rightsquigarrow obtain a reflection automorphism ψ of W, associated to γ , defined modulo conjugation by an element of the normalizer $N_{GL(V)}(W)$.

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- The character of ψ (seen as a representation of W) is $w \mapsto \gamma(\operatorname{tr}_V(w))$.
- Any Galois automorphism of W is a reflection automorphism.
- Let ϕ be a reflection automorphism of W. Then ϕ is a Galois automorphism of W attached to $\gamma \in \Gamma$ if and only if ϕ satisfies

$$\forall w \in W, \operatorname{tr}_V(\phi(w)) = \gamma(\operatorname{tr}_V(w)).$$

Theorem (Marin-Michel '10)

Let W be an irreducible complex reflection group. Any reflection automorphism of W is a Galois automorphism (associated to some $\gamma \in \Gamma$).

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Simply transitive action of Γ

Via Galois automorphisms, $\Gamma = \text{Gal}(K_W/\mathbb{Q})$ does not act directly on W, but on $N_{\text{GL}(V)}(W)$ -conjugacy classes of W.

 \rightsquigarrow action of Γ on $Cox(W) := \{conjugacy classes of generalized Coxeter elements\}$ [Marin-Michel] and [RRS] \implies this action is transitive.

Theorem (Reiner-R.-Stump)

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 $\forall C, C' \in Cox(W), \exists ! \gamma \in \Gamma, C' = \gamma \cdot C.$

Consequence: $|Cox(W)| = [K_W : \mathbb{Q}].$

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Lemma

The number of conjugacy classes of generalized Coxeter elements is

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where m_1, \ldots, m_n are the exponents of W $(m_i = d_i - 1)$

2. Prove that $[K_W : \mathbb{Q}] = \frac{\varphi(h)}{\varphi_W(h)}$ (*) ... case-by-case. (*) is equivalent to Malle's characterization of K_W for W well-generated:

Theorem (Malle)

Let $\zeta = e^{2i\pi/h}$ and G_W be the setwise stabilizer of $\{\zeta^{m_1}, \ldots, \zeta^{m_n}\}$ in the Galois group $Gal(\mathbb{Q}(\zeta)/\mathbb{Q})$. Then K_W is equal to the fixed field $\mathbb{Q}(\zeta)^{G_W}$. Equivalently, K_W is generated by the coefficients of the characteristic polynomial of any Coxeter element of W.

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- The property of transitivity of reflection automorphisms on regular elements of order *h* extends to Springer's regular elements of arbitrary order.
- the characterization of generalized Coxeter elements for real groups extends to Shephard groups (those nicer complex groups with presentations "à la Coxeter").
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