# HENSEL FIELDS IN EQUAL CHARACTERISTIC $p > 0$

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This talk is an attempt to be a survey of the problem of Hensel fields in equal characteristic. No paper has appeared since the works of Ershov [E] and Ax - Kochen ([AK],[KO]). The only positive result in char. $p > 0$ is due to Ershov, based on an important algebraic work of Kaplansky [Ka] . Our talk doesn't countain new results but gives many examples and counter-examples which show the limits and difficulties of a possible generalization.

1.- We consider in this talk <u>valued fields</u>, that is fields with a surjection "the valuation" $K^* \to G$, where $G$ is an ordered group, extended in $0$ by putting $val(0) = \infty$, an infinite element adjoined to $G$, and with the usual properties

$$val(xy) = val(x) + val(y)$$

and the stronger form of triangular inequality

$$val(x + y) \geq Min\,[val(x), val(y)]\ .$$

In this field we define the valuation ring

$$A = \{x \in K \ ; \ val(x) \geq 0\}$$

which is a local ring, with maximal ideal

$$I = \{x \in K \ ; \ val(x) > 0\}$$

and the residue field $\overline{K} = A/I$. In our cas K and $\overline{K}$ have the same characteristic.

The Hensel property for the valued field K is the following : If $f(x) \in A[x]$ has a simple residual root, then it has a root in A, whose residue agrees with the root of f. It is a first-order property.

The important result of Ax-Kochen and Ershow is the following :

Proposition.- If K is a henselian field of equal characteristic 0, then $\mathcal{H} \cup Th(\overline{K}) \cup Th(\text{val } K) \vdash Th(K)$; where $\mathcal{H}$ are the axioms saying that K is henselian.

It is well known that this result doesn't generalize to the case of characteristic p. The following counter-example is often given :

Definition.- If k is a field and G an ordered group, we define the field of generalized power series with coefficients in k and exponents in G :

$$k((T^g\ ;\ g \in G)) = \{\Sigma\, a_i\, T^i\ ;\ a_i \in k, (i) \text{ is a well-ordered subset of } G\}$$

The operations are the usual over the series, multiplication being possible by the condition of well-ordered support :

$$(\sum_{k_0} a_k)(\sum_{\ell_0} b_\ell) = (\sum_{k+\ell=i} a_k . b_\ell)$$

then when k increases from $k_0$ to $i - \ell_0$, $\ell$ decreases from $\ell_0$ to $i - k_0$ ans so takes only a finite number of values.

Example.- Let k be an algebraically closed field with characteristic $p > 0$. It is known that $k((T^g\ ;\ g \in \mathbb{Q}))$ is an algebraically closed field. Let us now look at the subfield :

$$K = \bigcup_{n \in \mathbb{N}} k((T^{\frac{1}{n!}}))$$

(K is generalization of the Puisieux series over $\mathbb{C}$) ; K is not algebraically closed as we see it by looking at the Artin Schreier equation $x^p - x + \frac{1}{T} = 0$ whose solutions are

$$x_i = \frac{1}{T} + \frac{1}{T^p} + \dots + \frac{1}{T^{p^n}} + \dots + i, \quad i = 0,1,\dots,p-1.$$

The fields $k((T^g\ ;\ g \in \mathbb{Q}))$ and $K$ are then two Hensel fields with same residue field $k$ and valuation group $\mathbb{Q}$, but they are not elementary equivalent.

We draw out of Ax and Kochen's proof two facts which are true for characteristic 0 but false for characteristic p :

1°.- We give first some definitions :

If $K \subset L$ are two valued fields, we can look at the residue extension and the group extension. We say that the extension is <u>immediate</u> when $\overline{K} = \overline{L}$ and val K = val L. It is the case for $k(T)$ and $k((T))$, and more generally for a field and its completion. A field is said to be <u>maximal</u> when it has no immediate proper extension ; an example is $k((T))$ or all generalized power series fields.

The valuation is an homomorphism $(K^*,.) \to (\text{val } K,+)$ ; a <u>cross-section</u> is a section of this mapping. It allows us to see the valuation group as included in K.

For the characteristic 0, we have an isomorphism theorem : two maximal fields with cross-section and having same residue field and same valuation group are isomorphic.

For the characteristic p, Kaplansky in [Ka] has studied the uniqueness of the immediate maximal extension of a valued field K ; he has given conditions on K, called Kaplansky conditions or conditions A guaranteeing the uniqueness :

- val K is p-divisible
- for all $a_{n-1},\dots,a_0,\ b \in \overline{K}$ the equation $x^{p^n} + a_{n-1}x^{p^{n-1}} + \dots + a_1x^p + a_0x + b = 0$ has a solution in $\overline{K}$.

With the same conditions the isomorphism theorem is true.

2°.- An henselian field of equal characteristic 0 doesn't admit any immediate algebraic extension. In the example of the Puiseux serie, at the opposite extreme, $K[x_0]$ where $x_0 = \sum_{i \in \mathbb{N}} \frac{1}{T^{\frac{1}{p^i}}}$ is an immediate extension of K.

This difference is easy to be taken up : we remark that the property for a valued field of having no immediate algebraic proper extension - K is then called <u>algebraically maximal</u> - is first order ; K has only to satisfy for all integer n the sentence : "For all polynomial $P(X) \in K[X]$ of degree n

$$\{\forall v\ [\ \exists x(\text{val}\ P(x) = v)] \rightarrow [\ \exists x'(\text{val}\ P(x') > v)]\}$$

$$\rightarrow \{\exists y(P(y) = 0)\}\ \text{"}$$

Here we use again the work of Kaplansky ; if there is in K a pseudo-convergent sequence $(u_\alpha)_{\alpha<\alpha_0}$ and a polynomial P(X) (of minimal degree) such that val $(P(u_\alpha))$ is not eventually constant, then K has an immediate extension K[u] where P(u) = 0. With the terminology of Kaplansky, K is algebraically maximal iff all pseudo-convergent sequence of algebraic type have a pseudo-limit in K.

A direct proof allows us to avoid the reference to Kaplansky : algebraic maximality is equivalent to the sentences which say :

- "K is henselian" ; we then have uniqueness of the extension of the valuation in all algebraic extension of K ; if c is algebraic over K with minimal polynomial $x^m + c_{m-1}x^{m-1} + \ldots + c_1x + c_0$, we must have $\text{val}(c) = \frac{1}{m}\text{val}(c_0)$.

- For all n : "for each polynomial $P(X) \in K[X]$ of degree n, there is $a \in K(x) = K[X]/P(X)$ such that $\text{val}(a) \notin \text{val}\ K$ or $[\text{val}(a) = 0$ and $\bar{a} \in \bar{K}]$. "

Now by elimination of x between P(x) and the decomposition of a in K(x), we know how to characterize a by its minimal polynomial over K. Hence the expression inverted commas is equivalent to :

"There exists $A(X) = X^r + a_{r-1}X^{r-1} + \ldots + a_1X + a_0$ minimal polynomial of an $a \in K(x)$ such that

$$[\bigwedge_{i=0}^{r-1} \text{val}(a_i) \geq 0 \wedge \forall y\ \bar{A}(\bar{y}) \neq \bar{0}] \vee [r \nmid \text{val}\ (a_0)]\text{"}$$

As far as we know, this notion of algebraic maximality is only studied in [Z].

With these two precisions, the same proof as for char. 0 works and gives the following result :

<u>Proposition</u>.- (Ershov) : <u>When</u> $K_1$ <u>and</u> $K_2$ <u>are two algebraically maximal Kaplansky fields</u>, <u>we have</u> $K_1 \equiv K_2$ <u>iff</u> $\overline{K}_1 \equiv \overline{K}_2$ <u>and</u> val $K_1 \equiv$ val $K_2$.

2.- <u>On the contrary</u>, <u>if</u> K <u>is not Kaplansky</u>, <u>the system</u> <u>Th($\overline{K}$) ∪ Th(val K) ∪ ("K is algebraically maximal") is in general not complete</u> ; we shall give a counterexample which shows other interesting facts.

<u>Proposition</u>.- <u>Let</u> k <u>be a field</u>, char. $k = p > 0$, <u>and</u> $a = a_o^p + T.a_1^p \in k((T))$ <u>be such that</u> $a_o$, $a_1 \in k((T))$ <u>are algebraically independant over</u> $k(T)$ ; K <u>is the relative algebraic closure for</u> $k(T,a)$ <u>in</u> $k((T))$.

Then K is algebraically maximal and $K \not\equiv k((T))$.

<u>Proof</u>.- The field K is valued in $\mathbb{Z}$ and hence its completion is its unique immediate maximal extension. It is easy to see that a valued field is algebraically maximal iff it is relatively algebraically closed in each immediate maximal extension ; therefore K is algebraically maximal by construction. The first-order property which distinguishes the theories of $k((T))$ and K is the algebraic completeness, notion introduced by Ershov : a valued field K is <u>algebraically complete</u> iff

1) it is henselian

2) each finite algebraic extension $L \supset K$ satisfies $[L : K] = [\overline{L} : \overline{K}]$ (val L : val K).

(To be sure this property is first order the reader can use the same kind of proof that we gave directly for the algebraic maximality ). The field $k((T))$ is algebraically complete as it is a maximal field (see for example [R]). On the other hand the extension $L = K[T^{\frac{1}{p}} ; a^{\frac{1}{p}}]$ of K satisfies $[L : K] = p^2$, $\overline{L} = \overline{K}$, (val L : val K) = p.

<u>Remarks</u> : 1°) We draw as a lesson from this example the fact that an algebraic extension, even a finite one, of an algebraically maximal field is not necessarly algebraically maximal. So $a^{1/p}$ is immediate over $K[T^{\frac{1}{p}}]$ but is not in this field.

2°) We see the limits of the algebraic maximality : we have definitely the implication

$$\left\{\begin{array}{l} K \text{ alg. maximal} \subset L \\ \text{val } K \text{ pure subgroup of val } L \\ \overline{K} \text{ relat. alg. closed in } \overline{L} \end{array}\right.$$

$$\Rightarrow K \text{ relativ alg. closed in } L,$$

but nothing works for transcendental extensions ; for example $L = K[a_o, a_1]$ is an immediate but inseparable extension of K (it is known that an elementary extension is separable).

3°) The previous remark gives another first-order property distinguishing K and k((x)). In this particular case (where the valuation group is $\mathbb{Z}$) , the following sentences express the fact that there is no immediate inseparable extension :

$$(\text{for all } n) \quad \text{"}\forall\, k_1,\ldots,k_n\ [\ \forall\, v\ \exists\, x_1,\ldots,x_n\ (\text{val}(\Sigma k_i x_i^p) > v)]$$

$$\rightarrow [\ \exists\, y_1,\ldots,y_n\ (\Sigma k_i y_i^p = 0)]\text{"}.$$

3.- We have defined different properties of maximal fields. If we refine the algebraic maximality into separable or inseparable alg. max., we have the implications :

alg. complete $\Rightarrow$ alg. max. $\Rightarrow$ sep. alg. max. $\Rightarrow$ henselian

with coïncidence of all there notions in char. 0, of the two first for Kaplansky fields and of the two last when the valuation group is $\mathbb{Z}$ :

<u>Proposition</u>.- <u>Let</u> K <u>be a field valued in</u> $\mathbb{Z}$ , <u>then</u> K <u>is henselian iff it is separably algebraically maximal</u>.

<u>Proof.</u>- One of the implications is obvious. Conservely let K be a field valued in $\mathbb{Z}$ with an immediate algebraic extension K(a), where the minimal polynomial A of a over K is separable ; a is then a limit of a Cauchy sequence $(a_\alpha)_{\alpha<\alpha_o}$ in K, such that $\text{val}(P(a_\alpha))$ increases with

$\alpha$ with no limit. Now the only initial segment ($\neq \emptyset$) of $\mathbb{Z}$ without supremum is $\mathbb{Z}$ ; hence A admits a root approached to each order. In particular, since $A'(a) \neq 0$ we have eventually

$$\mathrm{val}(A(a_\alpha)) > 2 \,\mathrm{val}\, A'(a) = 2\,\mathrm{val}\, A'(a_\alpha)$$

and then, by the strong form of Hensel lemma, A has a root in K.

Q.E.D.

This equivalence doesn't generalize when the valuation group is finitely generated or has the same theory as $\mathbb{Z}$ .

Example of a Hensel field valued in a $\mathbb{Z}$-group, with an immediate Artin Schreier extension

$$K = \bigcup_{i \in \mathbb{N}} k((T^g \; ; \; g \in \mathbb{Z}[\tfrac{\alpha}{i!}])) \subset k((T^g \; ; \; g \in \mathbb{Z}^*))$$

where k is a field with characteristic p, $\mathbb{Z}^*$ is a non-standard model of $\mathbb{Z}$ and $\alpha \in \mathbb{Z}^* - \mathbb{Z}$ is positive and divisible by all standard integers ; K is an henselian field as it is an increasing union of henselian fields ; val K is the divisible envelope of $\mathbb{Z}[\alpha]$ in $\mathbb{Z}^*$ and is hence a $\mathbb{Z}$-group. Now the root $\sum_{i \in \mathbb{N}} T^{-\frac{\alpha}{p^i}}$ of the equation $x^p - x + T^{-1} = 0$ is not in K.

Example of a Hensel field valued in $\mathbb{Z}[\beta]$, with an immediate Artin-Schreier extension

Let K be the henselisation of $k(T, T^\beta, a^p)$, where k is a field of characteristic $p, \beta$ a non-standard positive integer, p doesn't divide $\beta$ and $a = a'.T^{-\beta}$ , with $a' \in k((T))$ transcendental series over $k(T)$. The solutions of the equation $x^p - x + a^p = 0$ are

$$x_i = a + a^{\frac{1}{p}} + \ldots + a^{\frac{1}{p^n}} + \ldots + i, \; i = 0, 1, \ldots, p-1 \;;$$

this notation is valid because in the decomposition of each term $a^{\frac{1}{p^n}} = T^{-\frac{\beta}{p^n}} . a'^{\frac{1}{p^n}}$ as a series in T with exponents in $\mathbb{Z}[\frac{\beta}{p^n} \; ; \; n \in \mathbb{N}]$

all the monomials have valuation at standard distance from $-\beta p^{-n}$ (the reader will note that the support of the series $x_i$ has order-type $\omega^2$). Because of the inclusion $K \subset k((T^g, g \in \mathbb{Z}[\beta]))$ $x_i$ is not in K ; to have that $x_i$ is immediate over K, it is enough to note that a is not in K, as it is not in $k(T)(T^\beta)(a^p)$ and as it is radical over this field.

4.- We can ask ourselves whether algebraic completeness is the good first-order characterization of completeness or not. The fact that we do not know of two algebraically complete fields K and L satisfying $\overline{K} \equiv \overline{L}$, val K $\equiv$ val L and $K \not\equiv L$ may tempt us to give a positive answer. On the other hand we have the following result (unpublished result of Kochen and Jacob ; see for example a proof in [BDL]) :

Proposition.- In the language of valued fields with cross-section, $\mathbb{F}_p((T))$ is undecidable.

So in this enriched language, not only is the system "$\overline{K} \equiv \mathbb{F}_p$" $\cup$ "val K $\equiv \mathbb{Z}$" $\cup$ "K is algebraically complete" incomplete but but so also are all systems obtained by replacing algebraic maximality by a recursively enumerable system of axioms.

Along the same lines, we may have two maximal fields with the same residual and valuational theories but not elementary equivalent in this language :

Proposition.- In the language of valued fields with cross-section, if char. $k = p > 0$, $k((T)) \not\equiv k((T^g \ ; \ g \in \mathbb{Z}^*))$

Proof.- Let $\pi$ be the cross-section, let us consider the sentence

$$\exists\, a\ [\ \exists\, b\ a = \pi(b)] \wedge [\mathrm{val}(a) < 0] \wedge [\ \exists\, x(x^p - x + a = 0)]$$

which is false in $k((T))$ but true in $k((T^g \ ; \ g \in \mathbb{Z}^*))$. We have only to take $a = T^{-\alpha}$ where $\alpha$ is a positive non-standard integer, infinitely divisible by p.

The question is now to determine the importance of cross-section in the language. But it remains true that in characteristic 0, even if we adjoin it in the language, $k((X))$ is decidable iff k is.

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