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Number Theory — *Unlikely, likely and impossible intersections without algebraic groups*, by Zoé Chatzidakis, Dragos Ghioca, David Masser and Guillaume Maurin, communicated on 10 May 2013.

Abstract. — We formulate function field analogues for the Zilber-Pink Conjecture and for the Bounded Height Conjecture. The "special" varieties in our formulation are varieties defined over the constant field. We prove our function field Zilber-Pink Conjecture for all subvarieties, and we prove our function field Bounded Height Conjecture for a certain class of curves. We explain that for our problems the algebraic groups are no longer "special"; instead the relevant notion is the transcendence degree over the constant field of the field of definition for our varieties*.

Key words: Zilber-Pink conjecture, function field.

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1. Introduction

The Manin-Mumford Conjecture (proven by Raynaud [13, 14] in the abelian case and by Hindry [9] in the semiabelian case) asserts that if G is a semiabelian variety defined over the complex numbers $\mathbb{C}$, and V is an irreducible subvariety of G which is not a translate of an algebraic subgroup of G by a torsion point, then V does not contain a Zariski dense set of torsion points. If for each integer $m \geq 0$ we define $G^{[m]}$ as the union of all algebraic subgroups of G of codimension at least m, then the Manin-Mumford Conjecture states that $V \cap G^{[\dim G]}$ is not Zariski dense in V, as long as V is not a torsion translate of an algebraic subgroup of G.

Zilber and Pink (see [18] and [12]) generalized this by conjecturing that if V is not contained in a proper algebraic subgroup of G, then $V \cap G^{[\dim V+1]}$ is not Zariski dense in V (that Zilber-Pink implies Manin-Mumford is seen by a simple standard induction on the dimensions). By contrast rather little is known here; we mention only the key results for the multiplicative $G = \mathbb{G}_{\mathrm{m}}^n$. After earlier works by Bombieri, Zannier and the third author (see for example the first of these [1]), the fourth author (see [11]) succeeded in settling the case of curves V over $\overline{\mathbb{Q}}$ (see also [4] for a different proof). The case $\dim V = n-1$ amounts to Manin-Mumford, but in [2] the case $\dim V = n-2$ over $\overline{\mathbb{Q}}$ was settled. All this was extended to $\mathbb{C}$ in [3]. We may remark that despite the subsequent breakthroughs by Habegger the case of surfaces V in $\mathbb{G}_{\mathrm{m}}^5$ remains open. See Zannier's recent book [17] for more information on these and related topics.

* Presented by U. Zannier.

The main object of this paper is to formulate and prove function field versions of these statements, for the moment without any reference to group varieties. In the proofs of the above results over $\overline{\mathbb{Q}}$, a fundamental role is played by heights, notably an inequality of Vojta sharpened by Rémond [15] and the Bounded Height Theorem of Habegger [6]. Perhaps surprisingly we do not need such tools in our proofs. Nevertheless we formulate function field versions also here, and another small surprise is that their proofs do not seem at all straightforward, even for plane curves. We prove height boundedness for a class of "separated variable" curves defined by $F(x) = G(y)$.

We start by explaining what takes the place of algebraic subgroups in our version without algebraic groups. In the case $G = \mathbb{G}_{\mathrm{m}}^n$ the Manin-Mumford Conjecture enables us to characterize torsion translates of algebraic subgroups as the varieties containing a Zariski dense set of torsion points. Now torsion points are simply the points over $\overline{\mathbb{Q}}$ whose canonical height is zero. In our situation we postulate a function field K finitely generated over an algebraically closed field k of constants, and given a transcendence basis there is a canonical height on K which is zero precisely on k. If for simplicity we consider varieties in affine space $X = \mathbb{A}^n$, then we have a height on K^n and the points of zero height are precisely those in k^n. And the varieties containing a Zariski dense set of points in k^n are of course just those defined over k; that is, the *constant* varieties. Or in the standard terminology, these are our "special" varieties. Thus we should make the following

Definition 1.1. *For $X = \mathbb{A}^n$ and an integer $m \geq 0$ write $X^{(m)}$ for the union of all subvarieties of X of codimension at least m which are defined over k.*

This leads naturally to our analogue of the Zilber-Pink Conjecture; however we can actually prove it, even over arbitrary characteristic, as the following result about an "unlikely intersection".

Theorem 1.2. *Let V in $X = \mathbb{A}^n$ be an absolutely irreducible subvariety defined over K which is not contained in a proper subvariety of X defined over k. Then $V \cap X^{(\dim V+1)}$ is not Zariski dense in V.*

In fact it is sometimes just as easy to "descend" and obtain a finiteness result, and then we can even do this in an effective way. The following is a natural analogue of the definition in [2] (p. 25), but expressed in terms of the complement of V^{ta}.

Definition 1.3. *The constant-anomalous part Z_V^0 of a variety V in $X = \mathbb{A}^n$ is the union of all subvarieties W in V such that W is contained in some constant subvariety Y of X satisfying*

$$\dim W > \max\{0, \dim V + \dim Y - n\}.$$

The following is a quantitative version of the Torsion Openness and Torsion Finiteness Conjectures in [2] (p. 25). Recall that our height on K can be uniquely extended to the algebraic closure $\bar{K}$ and then to $\bar{K}^n$ (see references at the end of this section).

CONJECTURE 1.4. *There are constants c_0, κ, λ depending only on n, and c depending only on n and K, with the following properties. Let V in $X = \mathbb{A}^n$ be an absolutely irreducible subvariety defined over K. Then Z_V^0 is Zariski closed in V and*

$$F_V^0 = (V \backslash Z_V^0) \cap X^{(\dim V+1)}$$

is at most finite. Further suppose V is defined by the vanishing of polynomials of total degree at most $D \geq 1$ with coefficients in K of height at most $h \geq 1$. Then Z_V^0 is defined by the vanishing of polynomials of total degree at most $cD^\kappa h^\lambda$ with coefficients in $\bar{K}$ of height at most $cD^\kappa h^\lambda$, and

(a) *the cardinality of F_V^0 is at most $cD^\kappa h^\lambda$,*
(b) *the cardinality of F_V^0 is even at most $c_0 D^\kappa$,*
(c) *the points of F_V^0 over $\bar{K}$ have height at most $cD^\kappa h^\lambda$.*

One might even expect $\lambda = 1$. The analogous conjectures over $\bar{\mathbb{Q}}$ or $\mathbb{C}$ in $\mathbb{G}_{\mathrm{m}}^n$ do not seem to have been formally written down yet, perhaps because they are considered rather difficult to prove, but they are generally believed; see for example the discussion on pages 37, 38, 136, 137 of [17]. In view of Habegger's preprint [8], part (a) for curves defined over $\bar{\mathbb{Q}}$ may not be out of reach, but part (b) is known essentially only for plane curves. In our situation we can prove relatively easily everything for curves except (b); and something can be done for surfaces. Here we give the proof of (a) when the curve in $\mathbb{A}^n$ has a field of definition which for convenience is purely transcendental over k, as well as the very simple proof of (b) for plane curves.

Next we explain our results on bounded height. Bombieri, Zannier and the third author in [1] considered curves V over $\bar{\mathbb{Q}}$ in $G = \mathbb{G}_{\mathrm{m}}^n$, and they proved that if V is not contained in a translate of a proper algebraic subgroup then the points of $V \cap G^{[1]}$ over $\bar{\mathbb{Q}}$ have height bounded above. Again we have to get rid of algebraic subgroups in the function field analogue; but now the necessity of allowing translates by non-torsion points causes additional problems. Any translate of a proper algebraic subgroup of $\mathbb{G}_{\mathrm{m}}^n$ is contained in a variety which is the zero locus for an equation $x_1^{a_1} \cdots x_n^{a_n} = \alpha$ defined over $k(\alpha)$, which has transcendence degree over k at most equal to 1. Accordingly, in our setting, we propose the following definition, even for Y of arbitrary dimension (this is consistent with a remark of Zannier made to the third author).

DEFINITION 1.5. *The variety Y in $\mathbb{A}^n$ is* quasi-constant *if it is defined over a subfield of K which has transcendence degree over k at most equal to* 1.

We discuss in section 4 some further justification for this definition.

We propose the following preliminary version for curves, also over arbitrary characteristic.

CONJECTURE 1.6. *Let V in $X = \mathbb{A}^n$ be an absolutely irreducible curve defined over K which is not contained in a proper quasi-constant subvariety of $\mathbb{A}^n$. Then the points of $V \cap X^{(1)}$ over $\overline{K}$ have height bounded above.*

This gives a "likely intersection" because it is expected that there are infinitely many points in question. In fact the statement is independent of the choice of the transcendence basis defining the height.

We cannot prove Conjecture 1.6 even for plane curves $n = 2$, although the analogous result in $\mathbb{G}_m^2$ is not too difficult. But we provide some evidence for it in section 5.

The corresponding assertions for V of higher dimension are not so easy to state, because one must allow for exceptional sets W sitting inside V as in Definition 1.3. With our concept of quasi-constant the natural analogue of Definitions 1.1 and 1.2 of [2] is (but as before expressed in terms of the complement of V^{oa})

DEFINITION 1.7. *The anomalous part Z_V of a variety V in $X = \mathbb{A}^n$ is the union of all subvarieties W in V such that W is contained in some quasi-constant subvariety Y of X satisfying*

$$\dim W > \max\{0, \dim V + \dim Y - n\}.$$

In [2] the authors defined an analogue of Z_V for V in $G = \mathbb{G}_m^n$, proved that it is Zariski closed in V, and conjectured that the points of

$$(V \backslash Z_V) \cap G^{[\dim V]}$$

over $\overline{\mathbb{Q}}$ have height bounded above. This was proved by Habegger in [6] (see also [7]). Accordingly we propose, also over arbitrary characteristic,

CONJECTURE 1.8. *Let V in $X = \mathbb{A}^n$ be an absolutely irreducible subvariety defined over K. Then the points of*

$$(V \backslash Z_V) \cap X^{(\dim V)}$$

over $\overline{K}$ have height bounded above.

Unfortunately it is not always the case that our Z_V is Zariski closed, and we were unable to modify Definition 1.5 to rectify this. However an analogous phenomenon was noted in [2] over $\overline{\mathbb{Q}}$ in $\mathbb{G}_m^n$. Again we discuss such matters in detail in section 4.

One could formulate also a quantitative version along the lines of Conjecture 1.4, but in view of our extremely modest progress in proving even Conjecture 1.6 for curves we refrain from this. Here quite a lot is known over $\overline{\mathbb{Q}}$ in $\mathbb{G}_m^n$; see also [8].

Our paper is arranged as follows. In section 2 we prove Theorem 1.2 and we explain why the assumption on V is necessary. The proof is completely effective and to demonstrate this we give a simple example. We also explain why the whole of Z_V^0 has to be removed in Conjecture 1.4, and we prove this conjecture for curves V. Then in section 3 we give a family of examples where it can be proved that $V \cap X^{(\dim V+1)}$ is even empty. So we get an "impossible intersection" as in [10]. In fact it seems likely that this holds for "almost all" V in a suitable sense (as explained later), and we provide some evidence for this by considering lines in $\mathbb{A}^2$ and $\mathbb{A}^3$. No analogues of our results here are known in $\mathbb{G}_{\mathrm{m}}^n$.

In section 4 we justify further our definition of quasi-constant and discuss why the methods of [11] seem to fail in the setting of Conjecture 1.8. Then we focus on curves as in Conjecture 1.6. We explain the necessity of removing Z_V, and in section 5 we prove Conjecture 1.6 for a special class of plane curves.

Now we recall the construction of the Weil height for a function field. So, let k be an algebraically closed field (of arbitrary characteristic) and let K be a function field over k of finite transcendence degree. Let $t_1, \dots, t_e$ be a fixed transcendence basis for K/k and we fix a model of $\mathbb{P}^e$ whose function field is $k(t_1, \dots, t_e)$. We construct the set of all valuations on $k(t_1, \dots, t_e)$ which correspond to the irreducible divisors of $\mathbb{P}^e$; essentially a place of $k(t_1, \dots, t_e)$ either corresponds to an irreducible polynomial in $k[t_1, \dots, t_e]$ or it is the place at infinity corresponding to the (negative) total degree of a rational function. We let Ω be the set of all valuations of K which extend the places of $k(t_1, \dots, t_e)$ constructed above. By abuse of language, we will also call each $v \in \Omega$ a place of K; note that for each nonzero $x \in k$, we have $|x|_v = 1$, where $|\cdot|_v$ is the corresponding norm for the place v. Finally, we note that K admits a product formula with respect to the set of places Ω (see [16]); i.e., there exists a set of positive integers $\{N_v\}_{v \in \Omega}$ such that for all $x \in K^*$ we have

$$\prod_{v \in \Omega} |x|_v^{N_v} = 1. \tag{1.8.1}$$

Then we define the Weil height for points over $\overline{K}$ with respect to the set of places from Ω as in [16, Chapter 2].

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2. Unlikely intersections

In fact Theorem 1.2 is trivially implied by the following

Theorem 2.1. *Let V in $X = \mathbb{A}^n$ be an absolutely irreducible proper subvariety defined over K. Then there is a proper subvariety V_0 in X defined over k such that $V \cap X^{(\dim V+1)}$ is contained in V_0.*

Proof. The proof is by induction on $e = \operatorname{trdeg}_k K$, and without loss of generality we will suppose that K is the compositum of the field k and of the field of definition of V. We can also assume that V is not contained in a proper subvariety of X defined over k, for otherwise we could take this as V_0. Choose a finite tuple T such that $K = k(T)$.

Let P be a generic point for V (over K, in some algebraically closed field containing K). Then the transcendence degree $\operatorname{trdeg}_k k(P) = n$; otherwise P would belong to a proper subvariety of X defined over k. Moreover, our hypothesis on V also implies that $e \geq n - d$, where d is the dimension of V; otherwise, let Y be the algebraic locus of (P, T) over k; then $\dim Y = d + e < n$, which on projecting down to X would contradict our assumption on V.

Now choose a subset T_0 of T which is maximal algebraically independent over $\overline{k(P)}$. Then $|T_0| = e + d - n$, $\operatorname{trdeg}_{k(T_0)} k(P, T_0) = n$, and the elements of T lie in $\overline{k(P, T_0)}$. We now let Z be the algebraic locus of (P, T) over $K_0 := k(T_0)$, and $\pi : Z \to \mathbb{A}^n$ the projection on the first n coordinates. As P is a generic of $\mathbb{A}^n$ and $\pi^{-1}(P)$ is finite, the Fibre Dimension Theorem delivers a Zariski closed subset $\tilde{V}$ of $\mathbb{A}^n$, defined over K_0, and not containing P so not containing V, such that if $Q \in \mathbb{A}^n \setminus \tilde{V}$, then $\dim \pi^{-1}(Q) \leq 0$.

Let $Q \in (\mathbb{A}^n \setminus \tilde{V}) \cap V$; then $(Q, T) \in Z$, and therefore $\operatorname{trdeg}_k K_0(Q) \geq e$ since $T \in \overline{K_0(Q)}$; i.e.,

$$\operatorname{trdeg}_k k(Q) \geq e - |T_0| = n - d.$$

Such a Q cannot lie in a subvariety of $\mathbb{A}^n$ of codimension $\geq d + 1$. Hence

$$V \cap X^{(d+1)} \subset \tilde{V}.$$

Let W be an irreducible component of $\tilde{V} \cap V$; then $\dim W \leq d - 1$, and W is defined over K. It suffices to show that $W \cap X^{(d+1)}$ is contained in a proper subvariety of $\mathbb{A}^n$ which is defined over k. Let Q be a generic of W over K, let K_1 be a subfield of K with $\operatorname{trdeg}_k(K_1) = e - 1$, and let W_1 be the algebraic locus of Q over K_1. Then $\dim W_1 = \dim W + 1 \leq d$, and W_1 contains W. Hence

$$W \cap X^{(d+1)} \subseteq W_1 \cap X^{(d+1)} \subseteq W_1 \cap X^{(\dim W_1 + 1)};$$

by induction hypothesis applied to W_1, we obtain the desired result. $\square$

The hypothesis on V in Theorem 1.2 is essential as shown by the following example (similar examples exist in higher dimensions too).

Let $V \subset \mathbb{A}^3$ be the line given by the equations $x = 0$ and $y = tz$, where t is transcendental over k. Clearly, V has infinitely many points in common with the union of all curves in $\mathbb{A}^3$ defined over k. Indeed, for each positive integer n, the point $(0, t^{n+1}, t^n) \in V \cap C_n$, where $C_n \subset \mathbb{A}^3$ is given by the equations $x = 0$ and $y^n = z^{n+1}$.

As an example of the method of proof of Theorem 1.2 we consider the surface V in $X = \mathbb{A}^3$ given by

$$y^2 z = x(x - z)(x - tz),$$

the "projective generic Legendre elliptic curve", defined over $k(t)$ transcendental over k. Then π is from (x, y, z, w) in $\mathbb{A}^4$ defined by

$$0 = y^2 z - x(x-z)(x-wz) = y^2 - xz(x-z) - xz(x-z)w$$

to (x, y, z) in $\mathbb{A}^3$; and it is clear that $\pi^{-1}(x, y, z)$ is a single point if $xz(x-z) \neq 0$. Thus $V \cap X^{(3)}$ is contained in the three lines defined by

$$x = y = 0, \quad x = z = 0, \quad x - z = y = 0;$$

and it is plainly the set of the k-rational points on these lines.

As for Conjecture 1.4, we will now show that the whole of Z_V^0 must be removed from V to get the finiteness of the points inside $X^{(d+1)}$, where $d = \dim V$. More precisely, let W in V be any "constant-anomalous" irreducible subvariety, with $\dim W = \ell$, contained in some constant Y in $X = \mathbb{A}^n$, with $\dim Y = j$, such that $\ell > \max\{0, d + j - n\}$; we will show that $W \cap X^{(d+1)}$ is Zariski dense in W.

We choose such an irreducible Y of minimal dimension. As $W \subseteq Y$ we have $\ell \leq j$. If $\ell = j$ then $W = Y$, and hence it must be removed from V in order to obtain the finiteness of points inside $X^{(d+1)}$. Thus we may assume $\ell < j$; renumbering coordinates, we may assume that $x_1, \dots, x_\ell$ are independent on W and $x_1, \dots, x_\ell, x_{\ell+1}, \dots, x_j$ are independent on Y, and that the projection on the first j coordinates is finite on $Y \setminus Y_0$, where Y_0 is a proper subvariety of Y, which is defined over k. Note that by minimality of j, the variety Y_0 does not contain W; hence for a Zariski dense set of k-rational points $(\xi_1, \dots, \xi_\ell) \in \mathbb{A}^\ell$ we can find a point $P := (\xi_1, \dots, \xi_\ell, \dots, \xi_n) \in W$. We know nothing about $\xi_{\ell+1}, \dots, \xi_n$ (because we don't know the field of definition of W); but since Y is defined over k the $\xi_{j+1}, \dots, \xi_n$ are in $\overline{k(\xi_{l+1}, \dots, \xi_j)}$. Thus

$$\operatorname{trdeg}_k k(P) \leq j - \ell \leq n - (d+1).$$

Consequently P lies in $X^{(d+1)}$ as required. Since the projection on the first ℓ coordinates of the set S consisting of all such points $P \in W$ is Zariski dense, we conclude that the Zariski closure of S has dimension ℓ and thus it equals W.

As for Conjecture 1.4 for curves V, the set Z_V^0 is either V or empty, so we can forget about it.

We next prove part (b) for any plane curve C, where there is a single equation of degree at most D over K but not over k. If $C \cap X^{(2)}$ contains at least S distinct points (now defined over k), then we can find a curve C_0 defined over k of degree at most $D_0 = [\sqrt{2S}]$ passing through them, since $(D_0+1)(D_0+2)/2 > S$. So $C \cap C_0$ has at least S distinct points. As C cannot be a component of C_0 we see from Bezout that $S \leq DD_0 \leq D\sqrt{2S}$ and so $S \leq 2D^2$ as required in (b). Of course here the points of $C \cap X^{(2)}$ have zero height so (c) is trivial.

We now prove (a) for any curve V in $\mathbb{A}^n$ whose field of definition is a purely transcendental extension $K = k(T)$ of k for $T = \{t_1, \dots, t_e\}$. In general this restriction could be removed using the Chow Form to determine the exact field of definition, but we omit the (somewhat tedious) details here.

By linear algebra we can suppose that V is actually defined by $M \leq \binom{D+n}{n} \leq 2nD^n$ of the given equations defining it. Now any coefficient in these equations can be written as a rational function of T of degree at most h. By clearing common denominators (which would not arise with a better notion of height), we see that the locus $\mathscr{V}$ of (P, T) over k (with P as in the proof of Theorem 1.2) is defined by M equations in $x_1, \ldots, x_n$, $y_1, \ldots, y_e$ with total degree at most $D + \binom{D+n}{n}h \leq 3nhD^n$ (it would be better to use bidegree here). It follows that its degree is at most $3nhD^n M \leq 6n^2hD^{2n}$ (we are being very generous here). We can assume that $T_0 = \{t_1, \ldots, t_f\}$ with $f = e + 1 - n \geq 0$. Now for each $j = f+1, \ldots, e$ the $n + f + 1 > 1 + e = \dim_k \mathscr{V}$ coordinates $x_1, \ldots, x_n$, $y_1, \ldots, y_f, y_j$ are algebraically dependent over k, and since $\deg \pi(\mathscr{V}) \leq \deg \mathscr{V}$ we get a relation over k of total degree at most $6n^2hD^{2n}$. As the elements of T_0 are algebraically independent over $k(P)$ this relation must involve y_j; let g_j be its leading coefficient as a polynomial in y_j. Now we see that $\pi^{-1}(Q)$ is finite as long as $g(Q, T_0) \neq 0$ for $g = \prod g_j$. This defines $\tilde{V}$ of degree at most $6n^2hD^{2n}(e-f) \leq 6n^3hD^{2n}$. Intersecting it with V, we see that the cardinality of $V \cap X^{(2)}$ is at most $6n^3hD^{2n} \deg V \leq 6n^3hD^{3n}$ as required in (a) (we had to be a bit careful that the exponent of D did not involve the unknown e after all we were working in $\mathbb{A}^{n+e}$).

By Galois this suffices to bound the degrees over K of the points of this finite set. And using Arithmetic Bezout one could obtain a height estimate as in (c) of Conjecture 1.4. But as $k = \bar{k}$ is infinite all these with Northcott would not be quite enough to determine $V \cap X^{(2)}$ completely, even though there would be no trouble to do this for any particular V.

3. Impossible intersections

In this section we construct many surfaces V in $X = \mathbb{A}^4$ with $V \cap X^{(3)}$ not just finite but empty. Namely take t, s algebraically independent over k, sufficiently general polynomials f, g over k, and define V by

$$y = f(t, x), \quad w = g(s, z)$$

with coordinates x, y, z, w. We will deduce a contradiction from the existence of a point $(\xi, \eta, \zeta, \omega)$ on V lying in a constant curve C.

As x, y are algebraically dependent over k on C, and so also ξ, η, the additional relation $\eta = f(t, \xi)$ implies that both ξ, η are in $\overline{k(t)}$, provided say $f(T, X)$ really involves T. Similarly ζ is in $\overline{k(s)}$ with a similar proviso on g. If further f is such that $f(T, \xi_0)$ is never in k for any ξ_0 in k, then we can deduce that either ξ or η is not in k. Say ξ is not in k. Then the relation between ξ, ζ over k implies that ζ is in $\overline{k(t)}$. So ζ is in $\overline{k(t)} \cap \overline{k(s)} = k$. Similarly ω is in k. But then $\omega = g(s, \zeta)$ can be ruled out by imposing a condition on $g(S, \zeta_0)$ analogous to that above on $f(T, \xi_0)$.

Thus indeed $V \cap X^{(3)}$ is empty. An explicit example is

$$y = t^2x + t, \quad w = s^2z + s.$$

Or we may consider lines V in $X = \mathbb{A}^2$ of the form

$$f(t)x + g(t)y = h(t)$$

with polynomials f, g, h over k. It is immediate that $V \cap X^{(2)}$ (consisting of the points over k) is empty provided f, g, h are linearly independent over k. This clearly holds for "almost all" choices in any reasonable sense. For example we may introduce an integral parameter $d \geq 0$ and consider (f, g, h) in the space $\mathscr{X} = \mathbb{A}^{3(d+1)}$ defined by $\max\{\deg f, \deg g, \deg h\} \leq d$; then provided $d \geq 2$ the subset for which $V \cap X^{(2)}$ is not empty is proper Zariski closed in $\mathscr{X}$. Thus we have an example of "ubiquitous impossible intersections".

With more elaborate methods we can establish a similar result for lines in $\mathbb{A}^3$ of the form

$$f_1(t,s)x + g_1(t,s)y + h_1(t,s)z = \ell_1(t,s)$$
$$f_2(t,s)x + g_2(t,s)y + h_2(t,s)z = \ell_2(t,s)$$

with respect to total degree and $\mathbb{A}^{4(d+1)(d+2)}$. But we omit the proof.

4. Likely intersections and bounded height

First we discuss the concept of being quasi-constant (see Definition 1.5) for a subvariety Y of $\mathbb{A}^n$. As noted before, quasi-constant varieties are the function field analogue of the cosets of proper algebraic subgroups of $\mathbb{G}_{\mathrm{m}}^n$. So, working in parallel with the multiplicative group (since our ambient space is $\mathbb{A}^n$) one observes that $Y \subseteq \mathbb{G}_{\mathrm{m}}^n$ is contained in a coset of a proper algebraic subgroup if there exist $a_1, \dots, a_n \in \mathbb{Z}$, not all equal to 0, such that $x_1^{a_1} \cdots x_n^{a_n}$ is constant on Y. Hence one might expect that the proper definition of a subvariety Y of $\mathbb{A}^n$ contained in a "coset" for the function field case be that there exists a nonzero rational map $\psi \in k(X_1, \dots, X_n)$ which is constant when restricted to Y. However this notion would prove not sufficiently restrictive in order to guarantee a positive answer for Conjectures 1.8 and 1.6. Indeed, if one replaces our notion of a subvariety $Y \subseteq \mathbb{A}^n$ being "quasi-constant" with the stronger condition that Y is the intersection of finitely many hypersurfaces of the form $f(x_1, \dots, x_n) = \alpha g(x_1, \dots x_n)$ for various $\alpha \in K$ and polynomials f, g over k, then we have the following example which would contradict the corresponding strengthening of Conjecture 1.6.

Example 4.1. Let $K = k(t)$ transcendental over k, and let $V \subseteq \mathbb{A}^2$ be the line $x + ty = t^2$. It is immediate to see that there exists no rational function $\psi \in k(X, Y)$ whose restriction to V is constant. On the other hand, for each $m \in \mathbb{N}$ the point $(t^m, t - t^{m-1})$ lies both on V and on the rational curve in parametric form

$$x = \lambda^m, \quad y = \lambda - \lambda^{m-1},$$

which is defined over k. Therefore the points from the intersection of V with the union of all curves defined over k have unbounded heights.

However, we note that V is clearly quasi-constant and thus we do not have a counterexample to Conjecture 1.6.

We considered also the following perhaps more plausible substitute for "quasi-constant". Namely that Y should be the intersection of finitely many hypersurfaces of the form $f(\alpha, x_1, \ldots, x_n) = 0$ for various α in K and f in $k[X_0, X_1, \ldots, X_n]$. We will say that such a subvariety Y is *semi-constant.*

If Y is itself a hypersurface, then there is a single equation and so this is the same as quasi-constant. In particular, for any curve V, the assumption of Conjecture 1.6 is equivalent to the fact that V is not contained in any semi-constant subvariety of X. Also, an analogue of Conjecture 1.8 can be formulated by using semi-constant subvarieties instead of quasi-constant ones in the definition of Z_V. It provides a weaker bounded height conjecture. Indeed, the analogue of the quasi-constant anomalous locus Z_V is larger than Z_V itself. We will call it the *semi-constant anomalous locus.*

But now the following example shows that this substitute Z'_V for Z_V can be non-Zariski closed.

Example 4.2. Let t, s, u, v, w be algebraically independent over k, and let V in $\mathbb{A}^3$ be a generic plane

$$z - u = v(x - t) + w(y - s)$$

passing through $P = (t, s, u)$. It contains infinitely many lines defined by

$$z - vx = f(v), \quad y = \alpha_0$$

for $\alpha_0 = w^{-1}(f(v) - u + vt + ws)$ and any polynomial f over k. Clearly these have the required form.

We claim that there is no semi-constant curve C in V passing through P. Because V itself is not semi-constant, this will suffice to verify Z'_V is non-Zariski closed.

If the claim is false, then the various resulting equations $f(\alpha, t, s, u) = 0$ over k for C would show that all the α lie in $F = \overline{k(t, s, u)}$. So C is defined over F. We can easily pick a point $Q = (\xi, \eta, \zeta) \neq P$ in $C(F)$. As C lies in V we deduce $\zeta - u = v(\xi - t) + w(\eta - s)$, an absurdity because 1, v, w are linearly independent over F.

As mentioned, even our original Z_V need not be Zariski closed, as the following example shows.

Example 4.3. Let t, s, u be algebraically independent over k and let $V \subseteq \mathbb{A}^3$ be the plane defined by the equation $z = ty + s$ (so a product in $\mathbb{A} \times \mathbb{A}^2$). It contains the infinitely many lines L_c defined by

$$y = c, \quad z = tc + s$$

for any c in k, and clearly L_c is quasi-constant. Again V itself is not quasi-constant, and as in Example 4.2 it suffices to show that there is no quasi-constant curve C in V through some suitable point, say $P = (u, s, ts + s)$ in V. But C would be defined over a field k_1 of transcendence degree at most 1 over k, and so the transcendence degree of $F = k_1(P)$ over k_1 would be at most 1. Thus the transcendence degree of F over k would be at most 2, an obvious absurdity.

This example is perhaps not so disturbing as the previous one; already in the first two paragraphs of section 5 of [2] we restricted to particular cosets in $G = \mathbb{G}_{\mathrm{m}}^3$ involving various α but satisfying multiplicative relations. Specifically $x_1 = \alpha_1 x_3$, $x_2 = \alpha_2 x_3$ with α_1, α_2 multiplicatively dependent. We obtained a corresponding non-Zariski closed analogue of Z_V for V defined by $x_3 = x_1 + x_2$ (so now $\alpha_1 + \alpha_2 = 1$). Only on such particular cosets can one obtain points in $V \cap G^{[2]}$ of unbounded heights like $(\alpha_1^r, \alpha_2\alpha_1^r, \alpha_1^r)$ $(r \in \mathbb{Z})$.

Similarly in Example 4.3 each L_c contains points of unbounded height like $(f(tc+s), c, tc+s)$ $(f(T) \in k[T])$ in $V \cap X^{(2)}$, indicating the need to remove Z_V.

In fact all of Z_V must be removed in Conjecture 1.8. We see this just as for Z_V^0 in Conjecture 1.4. Namely with $\dim V = d$ let W in V be any irreducible "anomalous" subvariety, with $\dim W = \ell$, contained in some quasi-constant Y in $X = \mathbb{A}^n$, with $\dim Y = j$, such that $\ell > \max\{0, d + j - n\}$; we will show that $W \cap X^{(d)}$ contains a Zariski dense set of points of unbounded height.

We choose such an irreducible Y of minimal dimension, defined over some subfield k_1 of K with transcendence degree 1 over k. As $W \subseteq Y$ we have $\ell \leq j$. If $\ell = j$ then $W = Y$, and hence it must be removed from V in order to obtain the boundedness of height inside $X^{(d)}$. Thus we may assume $\ell < j$; renumbering coordinates, we may assume that $x_1, \ldots, x_\ell$ are independent on W and $x_1, \ldots, x_\ell, x_{\ell+1}, \ldots, x_j$ are independent on Y, and that the projection on the first j coordinates is finite on $Y \setminus Y_0$, where Y_0 is a proper subvariety of Y also defined over k_1. Note that by minimality of j, the variety Y_0 does not contain W. Fix t in k_1 not in k. We deduce that, for a Zariski dense set of points $(\xi_1, \ldots, \xi_\ell) \in \mathbb{A}^\ell$ where each $\xi_i = f_i(t)$ for polynomials $f_1, \ldots, f_\ell$ over k of arbitrarily large degree, there exist $\xi_{\ell+1}, \ldots, \xi_n$ such that $P := (\xi_1, \ldots, \xi_n) \in W$. We know nothing about $\xi_{\ell+1}, \ldots, \xi_j$; but since Y is defined over k_1 the $\xi_{j+1}, \ldots, \xi_n$ are in $\overline{k_1(\xi_{\ell+1}, \ldots, \xi_j)}$. Thus

$$\operatorname{trdeg}_k k(P) \leq 1 + j - \ell \leq n - d.$$

Consequently P lies in $X^{(d)}$ as required, and it has unbounded height as we vary the polynomials $f_1(t), \ldots, f_\ell(t)$. Also, since the set S of all such points $P = (\xi_1, \ldots, \xi_n) \in W$ for sufficiently large degree polynomials $\xi_i = f_i(t)$ projects dominantly onto $\mathbb{A}^l$ corresponding to the first ℓ coordinates, we conclude that the Zariski closure of S has dimension $\ell = \dim W$. Since W is irreducible we conclude that S is Zariski dense in W.

The rest of this section is restricted to zero characteristic.

In some respects, Example 4.1 shows that Conjecture 1.8 does not follow directly from any of the approaches that were used in [6] (see also [7] and [11]) to prove the classical Bounded Height Conjecture. We give more details below, focusing mainly on the methods of [11].

Comparing Conjecture 1.8 and the classical Bounded Height Conjecture, we see that both are of the following kind. For any integer $d \geq 1$, we specify a family $\mathscr{F}_d$ of dominant morphisms $\varphi : X \to \mathbb{A}_K^d$ and define a corresponding anomalous locus $Z_{V,\mathscr{F}} \subseteq V$ for any subvariety V of X. The problem is then to show that, for any subvariety V of dimension d, the height is bounded on an intersection of the form

$$(V \backslash Z_{V,\mathscr{F}}) \cap \bigcup_{\varphi \in \mathscr{F}_d} \varphi^{-1}(\xi),$$

where ξ has height zero.

If $X = \mathbb{G}_{\mathrm{m}}^n$ is a multiplicative torus, the Bounded Height Conjecture is of this type with $\xi = (1, \ldots, 1)$ and $\mathscr{F}_d$ defined as the family of all dominant homomorphisms $\mathbb{G}_{\mathrm{m}}^n \to \mathbb{G}_{\mathrm{m}}^d$. The corresponding anomalous locus $Z_{V,\mathscr{F}}$ was defined in [2]. It has the following remarkable property: for any subvariety V of dimension d,

$$Z_{V,\mathscr{F}} \neq V \Leftrightarrow \text{any } \varphi \in \mathscr{F}_d \text{ induces a dominant morphism on } V.$$

In particular, when we do have to prove that the height is bounded, i.e. when $V \backslash Z_{V,\mathscr{F}}$ is non-empty, V satisfies the extra assumption $\deg(\varphi|_V) > 0$ for any element $\varphi \in \mathscr{F}_d$. This is actually the only thing we need to know about V. The main step of the proof is then to derive finer numerical estimates from this property. For instance, we have a uniform lower bound on $\deg(\varphi|_V)$ when φ is varied among all "normalized" elements of $\mathscr{F}_d$ (see definition 7.1 in [11]). For simplicity, assume V is a curve so that any element of $\mathscr{F}_d$ is normalized. The lower bound then says that there is a real number $c > 0$ such that

$$\deg(\varphi|_V) \geq c\|\varphi\|, \tag{4.3.1}$$

for any $\varphi \in \mathscr{F}_d$, where $\|\varphi\|$ is the sum, in absolute value, of the exponents of the monomial φ (*cf.* proof of theorem 9.20 in [11]). Using diophantine approximation, we deduce that the following height inequality holds: there exist real numbers $c_1, c_2 > 0$ such that

$$h(\varphi(x)) \geq c_1\|\varphi\|h(x), \tag{4.3.2}$$

for any $\varphi \in \mathscr{F}_d$ and any $x \in V \backslash Z_{V,\mathscr{F}}$ of height at least c_2 (see for example theorem 9.30 in [11]). This readily proves that the height is bounded on the intersection of interest as any of its points satisfies an equation of the form $\varphi(x) = \xi$ for some $\varphi \in \mathscr{F}_d$; so either $h(x) < c_2$ or inequality (4.3.2) gives $h(x) \leq (1/c_1\|\varphi\|)h(\xi) = 0$.

Conjecture 1.8 is of the same form again, with $\xi = (0, \ldots, 0)$ and $\mathscr{F}_d$ given by the family of all dominant morphisms $X \to \mathbb{A}_K^d$ defined over k. Hence, it makes sense to use the same approach again. However, the anomalous locus $Z_{V,\mathscr{F}}$, as defined here, has a weaker relation with $\mathscr{F}_d$ than in the classical case. Indeed, both halves of the equivalence above are now false.

First, Example 4.3 shows there exist subvarieties V of X with $V \neq Z_{V,\mathscr{F}}$, such that $\varphi(V)$ is of lower dimension than V for some $\varphi \in \mathscr{F}_d$. Hence, one half of the equivalence is false, although we recover it by restricting to curves. Indeed, the conclusion follows from the stronger assumption that $V \neq Z'_{V,\mathscr{F}}$, where $Z'_{V,\mathscr{F}}$ is the semi-constant anomalous locus, which is equivalent to $V \neq Z_{V,\mathscr{F}}$ in the case of curves.

Then, there are examples such as 4.1 showing that some subvarieties V of X with $V = Z_{V,\mathscr{F}}$ have the property that any $\varphi \in \mathscr{F}_d$ induces a dominant morphism on V.

Actually, Example 4.1 even shows that the methods of [6] and [11] fail to give a proof of Conjecture 1.8. At least, these proofs don't translate directly into a proof of Conjecture 1.8. Indeed, for a curve V lying in X, the condition that any $\varphi \in \mathscr{F}_1$ induces a dominant morphism over V already implies a lower bound similar to (4.3.1) with $c = 1$ and $\|\varphi\|$ defined as the total degree of the polynomial φ. In particular, for any curve as in Example 4.1, the lower bound on the degree is satisfied, though the height is unbounded. It shows that, in the setting of Conjecture 1.8, height inequalities of the type (4.3.2) do not necessarily follow from lower bounds of the type (4.3.1). New ideas are thus needed to prove the conjecture along those lines.

5. Bounded height; an example

We can prove certain special cases of Conjecture 1.8; our strategy is tailored specifically to Conjecture 1.8 and revolves around properties of irreducibility similar to Bertini's Theorem.

As above let k be any algebraically closed field of any characteristic, let $K = k(t,s)$ for t, s algebraically independent over k, and let h be the Weil height function on $\overline{K}$ such that $h(t) = h(s) = 1$.

We need the following result, a direct consequence of Lemma 2.1 (p. 1053) of [5].

Lemma 5.1. *Supose ξ in $\overline{k(t,s)}$ satisfies an equation $F(\xi) = 0$ with $F(X) = \alpha_0 X^D + \cdots + \alpha_D$ in $k[X,t,s]$ irreducible over $k[t,s]$. Then $Dh(\xi) = \max\{\deg \alpha_0, \ldots, \deg \alpha_D\}$ for the total degrees.*

As a warm-up, we consider the line V in $X = \mathbb{A}^2$ defined by

$$y = tx + s$$

and we prove that

$$\max\{h(\xi), h(\eta)\} \leq 3$$

for any (ξ, η) in $V \cap X^{(1)}$ over $\overline{k(t,s)}$.

There is a non-zero polynomial f over k with $f(\xi, \eta) = 0$, and we can suppose f irreducible; let d be its total degree. We deduce $F(\xi) = 0$ for $F(X) = f(X, tX + s)$. We claim that F is irreducible over $k[t, s]$ (the intuition here is that V is a "generic" line with respect to the curve defined by f hence Bertini). For a non-trivial factorization $f(X, tX + s) = f_1(X, t, s)f_2(X, t, s)$ in $k[X, t, s]$ would imply

$$f(X, Y) = f_1(X, T, Y - TX)f_2(X, T, Y - TX).$$

Thus both factors here would have to be independent of T, say $\tilde{f}_1(X, Y)$ and $\tilde{f}_2(X, Y)$, leading to a non-trivial factorization of f. This proves the claim.

Now $F(X)$ has degree $D = d$ because t is transcendental over k, and so from Lemma 5.1 we see that $dh(\xi) \leq d$. So $h(\xi) \leq 1$ and finally

$$h(\eta) = h(t\xi + s) \leq h(t\xi) + h(s) \leq h(t) + h(\xi) + h(s) \leq 3.$$

Here we can easily see that $V \cap X^{(1)}$ is infinite, as suggested by the term "likely intersection"; for each positive integer r it contains (ξ, ξ^r) where $\xi = \xi_r$ satisfies $\xi^r = t\xi + s$ and so clearly has degree r over $k(t, s)$.

It may be instructive also to deduce a similar bound for points (ξ, η) on

$$sy^2 = x(x - 1)(x - t)$$

the "affine generic twisted Legendre elliptic curve" C contained in $\mathbb{A}^2$ defined over $k(t, s)$. We can write this as $z = ux + v$ for $z = \frac{y^2}{x(x-1)}$ and $u = \frac{1}{s}$, $v = -\frac{t}{s}$. Because the values of z, x stay algebraically dependent over k for any point $(x, y) = (\xi, \eta) \in C \cap X^{(1)}$ and $h(u) = h(v) = 1$ we see at once from the above that $h(\xi) \leq 1$ and $h\left(\frac{\eta^2}{\xi(\xi-1)}\right) \leq 3$, whence

$$2h(\eta) = h(\eta^2) \leq 3 + h(\xi(\xi - 1)) \leq 5.$$

But the main goal of the present section is to prove bounded height for curves V defined by

$$s\ell(y) = g(x, t),$$

where ℓ is a polynomial over k of degree greater than 0 and $g(X, T)$ is a polynomial over k having an irreducible factor involving both X and T.

First we proceed with a few reductions, as for twisted Legendre. We may assume $\ell(y) = y$ and also that all the irreducible factors of $g(X, T)$ involve both X and T. Indeed, if $g(X, T) = g_0(X)g_1(T)\tilde{g}(X, T)$ for g_0, g_1, $\tilde{g}$ over k then we may replace $\frac{y}{g_0(x)}$ by y and $\frac{s}{g_1(t)}$ by s. Thus we are now dealing with

$$sy = g(x, t)$$

and all the irreducible factors of $g(X, T)$ involve both X and T.

Pick (ξ, η) in $V \cap X^{(1)}$ over $\overline{k(t,s)}$. There is a non-zero polynomial f over k with $f(\xi, \eta) = 0$, and we can suppose f irreducible; let d be its total degree. We deduce $F(\xi) = 0$ now for

$$F(X) = s^{d_Y} f(X, g(X,t)/s)$$

in $k[X,t,s]$ with $d_Y = \deg_Y f(X,Y)$.

The case $d_Y = 0$ is easy, as from $\tilde{f}(\xi) = 0$ (where $\tilde{f}(X) = f(X,1)$) we get $h(\xi) = 0$ and then from $s\eta = g(\xi,t)$ an upper bound for $h(\eta)$ independent of f.

Thus we may assume $d_Y \geq 1$. We may also assume

$$d_X = \deg_X f(X,Y) \geq 1,$$

otherwise $h(\eta) = 0$ and then $g(\xi,t) = s\eta$ leads to a similar bound for $h(\xi)$. Namely, if $\eta \neq 0$ then $g(X,t) - s\eta$ is clearly irreducble in $k[t,s]$ and again we may use Lemma 5.1; and otherwise $g(\xi,t) = 0$ and there are at most finitely many possibilities for ξ.

Now write $F(X) = \alpha_0 X^D + \cdots + \alpha_D$ for $\alpha_0, \ldots, \alpha_D$ in $k[t,s]$ with $\alpha_0 \neq 0$. Put $e_T = \deg_T g(X,T)$, so that $e_T \geq 1$ by our hypotheses on g.

Claim 5.2. *We have* $\max\{\deg \alpha_0, \ldots, \deg \alpha_D\} \leq de_T$.

Proof. Each nonzero monomial from each α_i comes from a monomial $X^a Y^b$ of $f(X,Y)$ with $b \leq d_Y$, so in $F(X)$ we see only coefficients involving $s^{d_Y - b} t^{jb}$ for some $j \leq e_T$. Here the total degree is

$$d_Y + b(j-1) \leq d_Y + b(e_T - 1) \leq d_Y + d_Y(e_T - 1) = d_Y e_T \leq de_T$$

as desired. □

Claim 5.3. *We have* $D \geq d$.

Proof. A monomial $X^a Y^b$ in $f(X,Y)$ leads to terms $s^{d_Y - b} \tau X^{a+i}$ in $F(X)$, where τ is in $k[t]$ and $i \leq e_X b$ for $e_X = \deg_X g(X,T) \geq 1$. Taking $i = e_X b$ we get $M_0 = s^{d_Y - b} \tau_0 X^c$ for $\tau_0 \neq 0$ in $k[t]$ and $c = a + e_X b$. We choose (a,b) first to maximize c and second to maximize a. Now all other monomials $X^{a'} Y^{b'}$ in $f(X,Y)$ give rise to degree in X strictly less than c unless $a' + e_X b' = a + e_X b$. But then $a' < a$ so $b' > b$ and these have degree $d_Y - b' < d_Y - b$ in s, so they cannot annihilate M_0. Since $e_X \geq 1$ and there exist monomials $X^a Y^b$ with $a + b = d$, we are done. □

Claim 5.4. *The polynomial* $F(X)$ *is irreducible over* $k[t,s]$.

Proof. Assume the contrary and let $F(X) = f_1(X,t,s) f_2(X,t,s)$ for non-trivial f_1, f_2 over k. We first note that each $d_i = \deg_s f_i$ $(i = 1,2)$ is positive. Otherwise if say $d_1 = 0$ then by substituting $s = g(X,t)/Y$ we would get

$$g(X,t)^{d_Y} f(X,Y) = Y^{d_Y} \tilde{f}_1(X,t) f_2(X,t,g(X,t)/Y).$$

Now, using that f is irreducible, we obtain that $f(X,Y)$ and $\tilde{f}_1(X,t)$ are coprime unless $\tilde{f}_1(X,t)$ is in $k[X]$ but even this is impossible as $d_Y \geq 1$. Thus $\tilde{f}_1(X,t)$

divides $g(X,t)^{d_Y}$. So each irreducible factor $p(X,t)$ of $\tilde{f}_1(X,t)$ divides $g(X,t)$. But also $p(X,t)$ divides $F(X) = s^{d_Y} f(X, g(X,t)/s)$ so $p(X,t)$ divides $s^{d_Y} f(X,0)$. Therefore $p(X,t)$ lies in $k[X]$. As this holds for each p, we deduce that $\tilde{f}_1(X,t)$ lies in $k[X]$. But as we assumed that $g(X,t)$ has no non-constant factors in $k[X]$, this is a contradiction.

Hence indeed we may assume that $d_i \geq 1$ $(i = 1,2)$.

Also we may assume that f_i $(i = 1,2)$ is not divisible by s. Otherwise s divides $f_1 f_2 = F(X) = s^{d_Y} f(X, g(X,t)/s)$ contradicting the definition of d_Y. We note also that $d_1 + d_2 = \deg_s F = d_Y$.

Finally write

$$g_i(X,t,Y) = Y^{d_i} f_i(X,t,g(X,t)/Y) \quad (i = 1,2)$$

both involving Y. Thus

$$g(X,t)^{d_Y} f(X,Y) = g_1(X,t,Y) g_2(X,t,Y).$$

If we look at this in $k(X,t)[Y]$ then we get a non-trivial factorization of the irreducible $f(X,Y)$ and so the contradiction that proves Claim 5.4. $\square$

Using Claims 5.2, 5.3, 5.4 and Lemma 5.1 we conclude from $F(\xi) = 0$ that $h(\xi) \leq e_T$, and then we get an upper bound for $h(\eta)$ independent of f since $s\eta = g(\xi,t)$. This completes the proof of Conjecture 1.8 for all curves of the form $s\ell(y) = g(x,t)$.

References

[1] E. Bombieri - D. Masser - U. Zannier, *Intersecting a curve with algebraic subgroups of multiplicative groups*, Int. Math. Research Notices 20 (1999), 1119–1140.

[2] E. Bombieri - D. Masser - U. Zannier, *Anomalous subvarieties—structure theorems and applications*, Int. Math. Research Notices 19 (2007), 1–33.

[3] E. Bombieri - D. Masser - U. Zannier, *On unlikely intersections of complex varieties with tori*, Acta Arithmetica 133 (2008), 309–323.

[4] E. Bombieri - P. Habegger - D. Masser - U. Zannier, *A note on Maurin's Theorem*, Rend. Lincei Mat. Appl. 21 (2010), 251–260.

[5] H. Derksen - D. Masser, *Linear equations over multiplicative groups, recurrences, and mixing I*, Proc. London Math. Soc. 104 (2012), 1045–1083.

[6] P. Habegger, *On the bounded height conjecture*, Int. Math. Research Notices (2009), 860–886.

[7] P. Habegger, *Intersecting subvarieties of abelian varieties with algebraic subgroups of complementary dimension*, Inventiones Math. 176 (2009), 405–447.

[8] P. Habegger, *Effective height upper bounds on algebraic tori*, preprint (March 2012).

[9] M. Hindry, *Autour d'une conjecture de Serge Lang*, Inventiones Math. 94 (1988), 575–603.

[10] D. Masser - U. Zannier, *Bicyclotomic polynomials and impossible intersections* (submitted).

[11] G. Maurin, *Équations multiplicatives sur les sous-variétés des tores*, Int. Math. Research Notices (2011), 108 pages, doi:10.1093/imrn/rnq248.

[12] R. Pink, *A common generalization of the conjectures of André-Oort, Manin-Mumford, and Mordell-Lang*, preprint (2005), available at math.ethz.ch/~pink/ftp/AOMMML.pdf.
[13] M. Raynaud, *Courbes sur une variété abélienne et points de torsion*, Inventiones Math. 71 (1983), 207–233.
[14] M. Raynaud, *Sous-variétés d'une variété abélienne et points de torsion*, Arithmetic and geometry, Vol. I, 327–352, Progr. Math. 35 (1983), Birkhäuser Boston.
[15] G. Rémond, *Inégalité de Vojta généralisée*, Bull. Soc. Math. France 133 (2005), 459–495.
[16] J-P. Serre, *Lectures on the Mordell-Weil theorem*, Aspects of Mathematics E15, Friedr. Vieweg & Sohn, Braunschweig, 1989.
[17] U. Zannier, *Some problems of unlikely intersections in arithmetic and geometry*, Annals of Math. Studies 181, Princeton 2012.
[18] B. Zilber, *Exponential sums equations and the Schanuel conjecture*, J. London Math. Soc. 65 (2002), 27–44.

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