Notes on motivic integration

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Abstract

This set of notes presents some of the results of the paper *Motivic integration in all residue field characteristics for Henselian discretely valued fields of characteristic zero* by Raf Cluckers and François Loeser, posted on Arxiv, `http://arxiv.org/abs/1102.3832`. Most of the proofs are omitted.

1 Preliminary results and definitions

1.1. Setting, notation. We study valued fields, and aim at defining motivic integration on discretely valued fields, i.e., fields whose value group is isomorphic to $\mathbb{Z}$. We will denote by: v the valuation of the field K; by π a uniformizer, i.e., an element whose value is 1; by $\mathcal{O}_K$ the valuation ring of K; by $\mathcal{M}_K$ the maximal ideal, which equals $\pi\mathcal{O}_K$; and by R_n the residue ring $\mathcal{O}_K/\pi^n\mathcal{O}_K$, and res_n the natural projection $\mathcal{O}_K \to R_n$.

Any valued field K has a natural topology, the valuation topology, with basic open sets given by the (closed) balls $B(a;\gamma) = \{x \in K \mid v(x-a) \geq \gamma\}$. Cartesian powers of K are endowed with the product topology.

Given a real number $q > 1$, and assuming that $v(K)$ is archimedean, one defines an absolute value on K by setting

$$|a| = q^{-v(a)}.$$

1.2. Haar measure. Let K be a locally compact valued field: either $\mathbb{Q}_p$, $\mathbb{F}_p((t))$, or a finite algebraic extension of one of these fields. Then K has a unique measure μ which is defined on the σ-algebra generated by the compact subsets of K, and which satisfies the following conditions:

- $\mu(\mathcal{O}_K) = 1$, where $\mathcal{O}_K$ is the valuation ring of K, $\{a \in K \mid v(a) \geq 0\}$,
- μ is translation invariant, i.e., if A is measurable and $a \in K$, then $\mu(a + A) = \mu(A)$,
- if $(A_n)_{n\in\mathbb{N}}$ is a sequence of pairwise disjoint measurable sets, then $\mu(\bigcup_n A_n) = \sum_n \mu(A_n)$,

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- if A is measurable, then for all $\varepsilon > 0$, there are an open set U and a closed set F such that $F \subseteq A \subseteq U$ and $\mu(U \setminus F) < \varepsilon$.

1.3. Some observations.
– This definition works for any topological group which is locally compact (except for the condition on $\mathcal{O}_K$). The measure is then unique up to a multiple, i.e., if one fixes the measure of a compact subgroup, it is unique.
– If the group G is profinite, then this coincides with the usual (normalized) Haar measure on G. In particular, when $K = \mathbb{Q}_p$, then $\mathcal{O}_K = \mathbb{Z}_p$, and the measure μ on $\mathbb{Z}_p$ is the normalized Haar measure.
– If q is the cardinality of the residue field of K, let us consider the absolute value on K defined by $|a| = q^{-v(a)}$. Then for any measurable set A and $a \in K$, one has $\mu(aA) = |a|\mu(A)$.
Proof. For the last item, assume first that $A = \mathcal{O}_K$, and $v(a) \geq 0$. Then $a\mathcal{O}_K$ has $q^{v(a)}$ cosets in $\mathcal{O}_K$; by translation invariance, these cosets all have the same measure; by the additivity of μ, they must have measure $q^{-v(a)}$.

One proves in the same way that if B is a ball of valuative radius $n \in \mathbb{Z}$, $B = B(b; n) = \{x \in K \mid v(x - b) \geq n\}$, then $\mu(B) = q^{-n}$, so that if $a \in K$, then $aB = B(ab; n + v(a))$ has measure $q^{-n-v(a)} = |a|\mu(B)$. The result now follows for an arbitrary A, using the fact that balls form a basis (of clopen sets) for the topology, and using the last condition on the measure.

1.4. As usual, one extends the measure by adjoining to the algebra of measurable sets all subsets of sets of measure 0, and extending the measure to this larger algebra in the obvious fashion.

On K^n, one just takes the product measure.

1.5. p-adic integration. Let K be a discretely valued field with finite residue field, and let $f : K^n \to K$ be a function. One can then try to compute the integral

$$\int_{K^n} |f(x)||dx|$$

where $|dx|$ denotes the Haar measure. One computes such an integral in the usual way, by partitioning the domain into smaller and smaller balls B_i, choosing a point $b_i \in B_i$, and computing $\sum_i |f(b_i)|\mu(B_i)$. If all these numbers converge to a common limit, then the integral is defined.

1.6. Early results of Igusa, Denef, Meuser, Macintyre, Pas. Let $K = \mathbb{Q}_p$, $f_1(X), \dots, f_r(X) \in \mathbb{Z}_p[X]$, $X = (X_1, \dots, X_m)$, and define for $n \geq 1$,

$$N_n = \#\{\bar{a} \in (\mathbb{Z}_p/p^n\mathbb{Z}_p)^m \mid f_1(\bar{a}) = \cdots = f_r(\bar{a}) = 0\},$$
$$\tilde{N}_n = \#\mathrm{res}_n(Y), \text{ where } Y = \{\bar{a} \in \mathbb{Z}_p^m \mid f_1(\bar{a}) = \cdots = f_r(\bar{a}) = 0\}.$$

(In the definition of N_n, we look at the polynomials modulo p^n. Observe that $\mathbb{Z}_p/p^n\mathbb{Z}_p \simeq \mathbb{Z}/p^n\mathbb{Z}$).

One then defines the *Poincaré series* $P(T) = \sum_{n=0}^{\infty} N_n T^n$ and $\tilde{P}(T) = \sum \tilde{N}_n T^n$.

1.7. Theorem (Igusa, Denef). $P(T), \tilde{P}(T) \in \mathbb{Q}(T)$.
Igusa's proof uses resolution of singularities. Denef's proof uses quantifier eliminations for the field of p-adic numbers. The result was later extend to finite extensions of $\mathbb{Q}_p$, and uniformity results were obtained (Meuser, Macintyre, Pas). The proof of the theorem is the consequence of the following result:

1.8. Theorem (Denef). Let S be a definable subset of $\mathbb{Q}_p^m$, which is contained in a compact set, let $h : S \to \mathbb{Q}_p$ be a definable function, with $|h(x)|$ bounded on S. Assume that for every $x \in S$, $h(x)$ is divisible by the integer $e > 1$. Then there is $R(T) \in \mathbb{Q}(T)$ such that

$$Z(s) = \int_S |h(x)|^{s/e} |dx|$$

exists, and equals $R(p^{-s})$, for all $s \in \mathbb{R}^{>0}$.

This results allows to show the rationality of the two Poincaré series mentioned above, and more generally of the two Poincaré series associated to a formula φ of the language of rings: define $N(\varphi, n)$ to be the number of m-tuples in the ring $\mathbb{Z}/p^n\mathbb{Z}$ which satisfy φ, and $\tilde{N}(\varphi, n)$ to be the size of the image by res_n of the set of m-tuples of $\mathbb{Z}_p$ satisfying φ. Then set $P_\varphi(T) = \sum N(\varphi, n)T^n$, $\tilde{P}_\varphi(T) = \sum \tilde{N}(\varphi, n)T^n$.

1.9. Lemma. Let $D = \{(x, w) \in \mathbb{Z}_p^m \times \mathbb{Z}_p \mid \exists y \in \mathbb{Z}_p^m \; v(x-y) \geq v(w) \wedge f_1(y) = \cdots = f_r(y) = 0\}$. [Here x and y are m-tuples, and $v(x-y) \geq v(w)$ is an abbreviation for $\bigwedge v(x_i - y_i) \geq v(w)$.] For $s \in \mathbb{R}^{>0}$, define

$$I(s) = \int_D |w|^s |dx||dw|.$$

Then $I(s) = \frac{p-1}{p} \tilde{P}(p^{-n-1}p^{-s})$.
Proof.

$$\begin{aligned}
I(s) &= \sum_{n=0}^{\infty} \int_{D, v(w)=n} p^{-ns} |dx||dw| \\
&= \sum_{n=0}^{\infty} p^{-ns} \int_{v(w)=n} (\int_{(x,p^n)\in D} |dx|)|dw| \\
&= \sum_{n=0}^{\infty} p^{-ns} (\int_{v(w)=n} |dw|)(\int_{(x,p^n)\in D} |dx|) \\
&= \sum_{n=0}^{\infty} p^{-ns} \frac{p-1}{p^{n+1}} \tilde{N}_n p^{-nm} \\
&= \frac{p-1}{p} \sum_{n=0}^{\infty} \tilde{N}_n (p^{-s}p^{-m-1})^n.
\end{aligned}$$

1.10. The lemma will imply that $\tilde{P}(T)$ is in $\mathbb{Q}(T)$: Denef's result and the lemma tell us that $I(s) = R(p^{-s}) = \frac{p-1}{p}\tilde{P}(p^{m-1}p^{-s})$ for all $s > 0$; this implies that the functions $R(T)$ and $\frac{p-1}{p}\tilde{P}(p^{m-1}T)$ are equal on the real interval $(0, 1)$. Hence, $\tilde{P}(T)$ is a rational function over $\mathbb{Q}$.

2 The structures, and their language

2.1. Definition. Let p be a prime or 0, and e an integer. A $(0,p,e)$*-field* is a Henselian valued field K of characteristic 0, with residue field of characteristic p, with value group isomorphic to $\mathbb{Z}$, with uniformizer π with $v(\pi) = 1$; if $p > 0$ we furthermore require that $v(p) = e$.

2.2. The language $\mathcal{L}_{\text{high}}$. We will study these fields in the many-sorted language $\mathcal{L}_{\text{high}}$, which is described as follows, for K a $(0,p,e)$-field:

(1) The valued field sort, also called the *main sort*, in the language of rings with a constant symbol for π: $(K, +, -, \cdot, 0, 1, \pi)$.

(2) For each $m > 0$, a sort for $R_m = \mathcal{O}_K/\pi^m\mathcal{O}_K$, called an *auxiliary ring sort*, in the language of rings with a constant symbol for π modulo π^m:: $(R_m, +, -, \cdot, 0, 1, \pi)$.

(3) The value group sort (an other *auxiliary sort*), interpreted by the value group $\Gamma = \mathbb{Z}$ to which one adjoins the singleton $\{\infty\}$, in the Presburger language for ordered Z-groups, with two new constant symbols for 1 and ∞: $(\mathbb{Z} \cup \{\infty\}, +, -, \leq, 0, 1, \equiv_n \ (n \in \mathbb{N}^{>0})$. For any $a \in \mathbb{Z}$, we have $a + \infty = \infty$, $a < \infty$, and $-\infty = \infty$.

In addition, we have some functions connecting the sorts: the valuation map $v : K^\times \to \Gamma$; the residual maps $\text{res}_m : K \to R_m$ and the *angular component maps* $\overline{\text{ac}} : K \to R_m$ for $m > 0$, defined by $\overline{\text{ac}}(x) = \pi^{-v(x)}\text{res}_m(x)$ (to be discussed below); for $n < m$, the natural projection maps $p_{m,n} : R_m \to R_n$. The many-sorted structure will sometimes be denoted by $\mathcal{K}$, or simply by K.

2.3. The Presburger language. We interpret the binary relation $\equiv_n$ on $\mathbb{Z}$ by: $x \equiv_n y$ if and only if $x - y \equiv 0 \mod n$. This relation is first-order definable in the language of abelian groups, and it is a result of Presburger that the theory of $\mathbb{Z}$ in the language $\{+, -, 0, 1, \leq, \equiv_n \ (n \in \mathbb{N})\}$ eliminates quantifiers. Its models are called ordered Z-groups.

2.4. Angular component maps - definition. They can be defined for an arbitrary valued field K, with value group having a smallest positive element 1 and uniformizer π, as a map having the following properties:

– On $K^\times$, $\overline{\text{ac}}_m$ is a multiplicative morphism $K^\times \to R_m^\times$. (Recall that ${R_m}^\times$ denotes the set of invertible elements of the ring R_m).

– On $\mathcal{O}_K^\times$, $\overline{\text{ac}}_m = \text{res}_m$,

– If $n < m$, then $\overline{\text{ac}}_n = p_{m,n}\overline{\text{ac}}_m$.

2.5. Angular component maps - existence. In case K has value group isomorphic to $\mathbb{Z}$, we saw that K can be equipped canonically with angular component maps. In fact, as soon as the valuation $v : K^\times \to \Gamma$ has a cross-section s (i.e., s is a group homomorphism $\Gamma \to K^\times$ satisfying $v \circ s = id_\Gamma$), then one can define $\overline{\text{ac}}_m(x) = \text{res}_m(xs(x)^{-1})$.

It may happen that a given valued field K does not have angular component maps. However, every valued field K has an elementary extension K^* such that the valuation on K^* has a cross-section, and therefore K^* will have angular component maps.

2.6. Angular component maps - definability. In general, the angular component maps are not definable in the original valued field language. For instance, adding angular components to the valued field $\mathbb{C}((t))$ does add structure.

In the case of finite residue field $\mathbb{F}_q$ however, one sees that they are definable (possibly adding finitely many parameters). The key observation is that the invertible elements of R_m satisfy the equation $x^{q^{m-1}(q-1)} = 1$, and therefore the kernel of the map $\overline{\mathrm{ac}}_m$ will contain $K^{\times N}$ for $N = q^{m-1}(q-1)$. If K is a finite extension of $\mathbb{Q}_p$ or of $\mathbb{F}_p((t))$, then $[K^\times : K^{\times N}]$ is finite (this is easily checked using Hensel's lemma and the finiteness of the residue rings), and therefore it suffices to describe $\overline{\mathrm{ac}}_m$ on the finitely many cosets of $K^{\times N}$ inside $K^\times$.

2.7. [theories]**The theories**. We let $\mathcal{T}(0, p, e)$ be the $\mathcal{L}_{\text{high}}$-theory which axiomatises all the properties of a $(0, p, e)$-field described above: henselian of characteristic 0, value group elementarily equivalent to $\mathbb{Z}$ in the Presburger language, ramification e when the residue field is of positive characteristic, properties of the angular component maps, etc.

We will consider languages $\mathcal{L}$ with the same sorts as $\mathcal{L}_{\text{high}}$, but which may be larger, and theories T which contain $\mathcal{T}(0, p, e)$ and have a model which is a $(0, p, e)$-field (i.e., with value group equal to $\mathbb{Z}$). A model of T which is a $(0, p, e)$-field is called a *T-field.*

2.8. [ex1]**Interesting expansions**. Here are some languages and theories to which the results will apply:

• (harmless) Let R be a subring of a $(0, p, e)$-field, and consider the language $\mathcal{L}_{\text{high}}(R)$ obtained from $\mathcal{L}_{\text{high}}$ by adding a new constant symbol for each element of R; we let $\mathcal{T}(0, p, e)(R)$ be the theory obtained by adjoining to $\mathcal{T}(0, p, e)$ the quantifier-free diagram of the ring R. Thus any model of $\mathcal{T}(0, p, e)(R)$ will contain a copy of the valued ring R (and also of its fraction field).

• One can put extra structure on the value group $\mathbb{Z}$, as well as on finite products of auxiliary ring sorts.

• Analytic structure. Let K be a $(0, p, e)$-field which is complete. For $X = (X_1, \dots, X_n)$ we consider the ring $K\{X\}$ of power series $\sum_{\nu \in \mathbb{N}^n} a_\nu X^\nu$ of series indexed by multiindices $\nu = (\nu_1, \dots, \nu_n)$ and satisfying $|a_\nu| \to 0$ as $|\nu| := \nu_1 + \cdots + \nu_n$ tends to ∞. (Here $X^\nu := \prod_i X_i^{\nu_i}$)

Each element of $K\{X\}$ defines a function $\mathcal{O}_K^n \to K$, $b \mapsto \sum_\nu a_\nu b^\nu$. See the paper by Cluckers and Lipshitz [1].

3 Axiomatic properties of the fields considered

I will present a series of properties, which are enjoyed by the examples mentioned above. These properties are at the heart of the proofs of Cluckers and Loeser.

3.1. Jacobian property of a function. Let $K \models \mathcal{T}(0, p, e)$, let B, B' be balls ($\subseteq K$) and $F : B \to B'$ be a definable function. We say that F has the *Jacobian property* on B if the following properties hold: F is bijection, which is differentiable on B with non-zero derivative F'; the valuation of F' is constant on B (we denote by $v(F')$ this constant value); and if $x \neq y \in B$, then $v(F') + v(x - y) = v(F(x) - F(y))$.

If $m > 0$, we say that F *has the m-Jacobian property* if in addition $\overline{\mathrm{ac}}_m \circ F'$ is constant on B (with value denoted by $\overline{\mathrm{ac}}_m(F')$), and for $x, y \in B$, we have $\overline{\mathrm{ac}}_m(F')\overline{\mathrm{ac}}_m(x-y) = \overline{\mathrm{ac}}_m(F(x)-F(y))$.

Note that if F has the Jacobian property on B, then F is injective on B. Note also that the derivative of a definable function is also definable: one uses the fact that the topology is uniformly definable.

3.2. Jacobian property of a theory. Let T as above be a theory. *The Jacobian property holds for the theory* T if for any model $\mathcal{K}$, the following holds: for any finite subset A of $\mathcal{K}$, and A-definable function $F : K \to K$, there is an A-definable function

$$f : A \to S$$

with S a finite product of auxiliary sorts, such that for any $s \in S(K)$, if $f^{-1}(s)$ is infinite, then $f^{-1}(s)$ is a ball B, on which F is either constant, or has the m-Jacobian property for all $m > 0$.

3.3. Split. The theory T is *split* if in any $\mathcal{K} \models T$,

(i) Any $\mathcal{K}$-definable subset of Γ^r is Γ-definable in the language of ordered abelian groups,

(ii) If $A \subset \mathcal{K}$, then any A-definable subset of $\Gamma^r \times S$, where S is a finite product of auxiliary ring sorts, is a finite disjoint union of sets of the form $Y \times Z$, where $Y \subseteq \Gamma^r$ is A-definable, and $Z \subseteq S$ is A-definable.

Model-theoretically, item (i) corresponds to Γ being *stably embedded* in $\mathcal{K}$ (i.e., any $\mathcal{K}$-definable subset of any cartesian power of Γ is definable with parameters in Γ), and the induced structure is the structure of an abelian ordered group. Item (ii) is called *orthogonality*, or *full orthogonality.*

3.4. Finite b-minimality. T is finitely b-minimal if the following holds for any model $\mathcal{K}$ of T:

(i) Each locally constant $\mathcal{K}$-definable function $g : K^\times \to K$, has finite image.

(ii) For any A-definable $X \subset K$, there is n, an A-definable $f : X \to S$, where S is a finite product of auxiliary sorts, and an A-definable function $c : S \to K$, such that for all $s \in S$, if $f^{-1}(s) \neq \emptyset$, then either $f^{-1}(s) = \{c(s)\}$, or $f^{-1}(s) = \{x \in K \mid v(x - c(s)) = \xi(s), \overline{\mathrm{ac}}_n(x - c(s)) = z(s)\}$ for some functions $\xi : S \to \Gamma$ and $z : S \to R_n$ (Note that the functions ξ and z will also be A-definable).

Recall that a function is locally constant if and only if any point of its domain is contained in an open set on which the function is constant.

3.5. Lemma 1. Assume T finitely b-minimal, and let $\mathcal{K} \models T$.

(i) Then any definable function from a product of auxiliary sorts to K has finite image.

(ii) Any definable locally constant function from $X \subseteq K^n$ to K has finite image.

Proof. (i) Write $S \subseteq \prod_{i=1}^n S_n$, where each S_i is either Γ or an auxiliary ring sort, and let $F : S \to K$. The proof is by induction on n.

Assume $n = 1$, $S_n = \Gamma$, and define $g : K^\times \to K$ by

$$g(x) = \begin{cases} F \circ v(x) & \text{if } v(x) \in S, \\ 0 & \text{otherwise.} \end{cases}$$

Then g is locally constant and has finite image. Whence F has finite image.

If $S_n = R_m$ for some m, one defines $g : K \to K$ by

$$g(x) = \begin{cases} F \circ \mathrm{res}_m(x) & \text{if } \mathrm{res}_m(x) \in S, \\ 0 & \text{otherwise.} \end{cases}$$

Again, g is locally constant, whence g and therefore F have finite image.

Suppose now $n > 1$ and that the result holds for $n - 1$. Let $p : \prod_{i=1}^n S_i \to \prod_{i<n} S_i$ be the natural projection. By induction, the fibers of p have finite image under F. By compactness, $\#F(p^{-1}(s))$ is bounded, and therefore, partitionning S and working on each piece of the partition, we may assume that $\#F(p^{-1}(s)) = t$ for some $t \in \mathbb{N}^{>0}$ and all $s \in S$.

We want to define a locally constant function $\prod_{i<n} S_i \to K$; at the moment, we have a locally constant function which takes its values in the set of t-element subsets of K, namely $s \mapsto F(p^{-1}(s))$ for $s \in p(S)$.

Recall that given a set $A = \{a_1, \dots, a_t\}$ with t distinct elements, the coefficients of the polynomial $\prod_{i=1}^t (X - a_i)$ are stable under all permutations of A. Writing this polynomial as $X^t + \sum_{i<n} b_i T^i$, the tuple $((-1)^t b_0, \dots, -b_{t-1})$ is called the tuple of symmetric functions in A, and the map which to A associates this tuple is definable in the language of fields. Let us denote by sym_i the function $A \mapsto b_i$. We then obtain t definable maps $g_i : S \to K$, defined as follows:

$$g(s) = \begin{cases} sym_i F(p^{-1}(s)) & \text{if } s \in p(S), \\ 0 & \text{otherwise,} \end{cases}$$

and by induction each of these definable maps has finite image. This finishes the proof of (i).

(ii) For $n = 1$, this is given by the definition of finite b-minimality. We let $p : K^n \to K^{n-1}$ be the projection on the first $n - 1$ coordinates. By induction hypothesis, if $y \in K^{n-1}$, the function g_y defined by

$$g_y(x) = \begin{cases} F(y, x) & \text{if} (y, x) \in S, \\ 0 & \text{otherwise,} \end{cases}$$

has finite image (it is clearly locally constant). As in (i), partitioning K^n, we may assume that the size of the image is always equal to some integer t, and we conclude as in (i).

3.6. b-minimality. The concept of b-minimality, a very important concept, was introduced by Cluckers and Loeser in [2]. It generalises several notions such as C-, p-, o-minimality, and one can hope to use it in positive characteristic as well. It also allows for some discrete phenomena, e.g., the theory of the abelian (ordered) group $\mathbb{Z}$ is b-minimal, as is the theory of the real field $\mathbb{R}$ endowed with a predicate for the set $2^{\mathbb{Z}}$. As the name suggests, it is implied by finite b-minimality. Here are the axioms of b-minimality: A theory T is b-minimal if in any model $\mathcal{K}$ of T, if $X \subset K$ is A-definable, then

(b1) There exists an A-definable function $f : X \to S$, S a finite product of auxiliary sorts, such that for every $s \in S$, $f^{-1}(s)$ is either a singleton or a ball.

(b2) If g is a definable function from a product of auxiliary sorts to a ball, then g is not surjective.

(b3) Given an A-definable function $F : X \to K$, there exists an A-definable function $f : X \to S$, with S a product of auxiliary sorts, such that for every $s \in f(X)$, the restriction of F to $f^{-1}(s)$ is either constant, or injective.

In our case (sufficiently many constants in the languages of the auxiliary sorts), (b3) can be replaced by

(b3') Given an A-definable function $F : X \to K$, there exists a finite A-definable partition (X_i) of X, such that the restriction of F to each X_i satisfies (b3).

While clearly finite b-minimality implies (b1) and (b2), the implication of (b3) needs one further result, proved in [2] (Proposition 2.6), that in the presence of (b1) and (b2), axiom (b3) is equivalent to

($*$) If F is a definable surjection from some definable subset X of K onto a ball $B \subset K$, then not all fibers of F contain balls.

3.7. Examples. All theories listed in 2.8 are finitely b-minimal. One can also show that they have the Jacobian property and are split.

4 Definable sets - notation

In this section, we introduce some notation. Definable sets are defined without parameters, and are viewed as functors, from a category of models of our theory T to the category of sets. T and $\mathcal{L}$ are as in 2.7.

First of all, the basic definable sets are the universes of the sorts, or the cartesian products of those. If $m = (m_1, \dots, m_s)$ is a finite tuple of integers, $n, r \geq 0$, and $\mathcal{K} = (K, (R_n)_{n \in \mathbb{N}}, \Gamma) \models T$, then

$$h[n, m, r](K) = K^n \times \prod_{i=1}^{s} R_{m_i} \times \Gamma^r.$$

If X is a definable set, then we abbreviate $S \times h[n, m, r]$ by $S[n, m, r]$.

We write Def(T) or Def for the category of $\emptyset$-definable sets. Morphisms are maps whose graph is in Def.

If Z is definable, then Def_Z denotes the category of *definable sets over* Z, i.e., with objects $(f : X \to Z)$, where $X \in \mathrm{Def}$, f is definable. A morphism between two objects $(f : X \to Z)$ and $(g : Y \to Z)$ of Def_Z is simply a morphism $F : X \to Y$ such that $f = g \circ F$.

4.1. Cartesian products. In the category Def, cartesian products are defined the usual way. In the category Def_Z, they are defined by the fibered product, and denoted with a $\otimes$: if $(f : X \to Z)$ and $(g : Y \to Z)$ are in Def_Z, then, for $\mathcal{K} \models T$,

$$X \otimes_Z Y(\mathcal{K}) = \{(a, b) \in X(\mathcal{K}) \times Y(\mathcal{K}) \mid f(x) = g(y)\},$$

endowed with the map $(f \times g)$. Note that there are natural maps $(X \otimes_Z Y \to Z)$ to $(X \to Z)$ and $(Y \to Z)$ induced by the projections of $X \times Y$ onto X and onto Y.

4.2. Induced maps. Let $f : Z \to Z'$ be a definable map. We then obtain

$$f_! : \mathrm{Def}_Z \to \mathrm{Def}_{Z'}, \qquad (g : X \to Z) \mapsto (f \circ g : X \to Z'),$$

and

$$f^* : \mathrm{Def}_{Z'} \to \mathrm{Def}_Z, \qquad (g : X \to Z') \mapsto (h : X \otimes_{Z'} Z \mapsto Z),$$

where the map h is the map induced by the projection $X \times Z \to Z$.

4.3. Points. A *point in the definable set* X is a pair (x, K) where K is a T-field, and $x \in X(K)$. (Strictly speaking, all my K should be $\mathcal{K}$).

The collection of all points is written $|X|$.

4.4. Dimension of definable sets. Let $X \subset h[n, m, r]$ be definable, and K a T-field. One then defines $\dim(X)$ to be -1 if $X(K) = \emptyset$, and otherwise the largest integer ℓ such that there exists a coordinate projection $X(K) \to K^\ell$ whose image contains a product of ℓ balls (i.e., has non-empty interior).

One can show that in a T-field K, a definable set $X(K)$ as above has dimension 0 if and only if its image under the natural projection $h[n, m, r](K) \to K^n$ is finite. This is proved using the fact that in a b-minimal theory, definable sets have a *decomposition into finitely many cells.*

So, to compute the dimension of a set, we only take into account the valued field part, and totally ignore the auxiliary sorts. This can be also compared to the results of Van den Dries ([5]), who shows that all Henselian valued fields of characteristic 0 are algebraically bounded (see below).

4.5. An aside: algebraically bounded fields. A field K (in a language containing the language of rings) is *algebraically bounded* if given any formula $\varphi(x, \bar{y})$, there are polynomials $f_1(X, \bar{Y}), \ldots, f_r(X, \bar{Y}) \in \mathbb{Z}[X, \bar{Y}]$ such that for all tuple $\bar{b}$ in K, if $\varphi(K, \bar{b}) := \{a \in K \mid K \models \varphi(a, \bar{b})\}$ is finite, then for some i such that $f_i(X, \bar{b})$ is not identically 0, it is contained in the zero-set $Z(f_i(X, \bar{b}))$ of $f_i(X, \bar{b})$.

Algebraic boundedness of K has the following consequences, in any model L of $\mathrm{Th}(K)$:
– If $\varphi(x, \bar{y})$ is a formula, then there is N such that for any $\bar{b}$ in L, if $\varphi(x, \bar{b})$ is finite, then it is of size $\leq N$.
– Define the algebraic dimension of a definable set $S \subset L^n$ as the algebraic dimension of its Zariski closure (= intersection of all Zariski closed sets containing S). Then this dimension is well-behaved, and is definable. The main property is that if $f : S_1 \to S_2$ is definable and surjective, and all fibers have the same dimension d, then $\dim(S_1) = \dim(S_2) + d$. Also, a set has dimension 0 if and only if it is finite.

However the algebraic boundedness of Henselian fields in the language of valued fields does not quite give the result above (since in general the angular component maps add structure to the valued field). Nevertheless, Van den Dries' result gives a good intuition of why this result should be true.

5 Integration over the value group

5.1. Assumption. Throughout this section and the next, we will work in a theory T in a language $\mathcal{L}$ with the same sorts as $\mathcal{L}_{\mathrm{high}}$, which contains $\mathcal{T}(0, p, e)$ and has a model with value group $\mathbb{Z}$. We will furthermore assume that the theory T is finitely b-minimal, has the Jacobian property and is split.

5.2. The ring $\mathbb{A}$. Consider the ring

$$\mathbb{A} = \mathbb{Z}[\mathbb{L}, \mathbb{L}^{-1}, \frac{1}{1-\mathbb{L}^{-i}}(i \in \mathbb{N}^{>0})] \ \subset \mathbb{Q}(\mathbb{L}),$$

where $\mathbb{L}$ is an indeterminate. If $q \in \mathbb{R}^{>1}$, we define

$$\theta_q : \mathbb{A} \to \mathbb{R}, \qquad r(\mathbb{L}) \mapsto r(q).$$

5.3. Let S be a definable set, and $\alpha : S \to h[0, 0, 1]$ a definable function. We will be working in T-fields, and so identify $h[0, 0, 1]$ with the subring $\mathbb{Z}$ of $\mathbb{A}$. Thus, α gives rise to a definable function $\alpha : |S| \to \mathbb{A}$, $(s, K) \in |S| \mapsto \alpha_K(s) \in \mathbb{Z} \subset \mathbb{A}$.

In addition, α will also give rise to the function $\mathbb{L}^\alpha : |S| \to \mathbb{A}$, $(s, K) \in |S| \mapsto \mathbb{L}^{\alpha_K(s)}$.

We let $\mathcal{P}(S)$, the ring of *constructible Presburger functions* on S, be the subring of $\mathbb{A}^{|S|}$ generated by
– all constant functions $|S| \to \mathbb{A}$,
– all definable $\alpha : |S| \to \mathbb{Z}$ as above, and
– all $\mathbb{L}^\alpha$, for α as above.

Thus an element of $\mathcal{P}(S)$ is a finite sum of monomials of the form $a\mathbb{L}^\beta \prod_{i=1}^r \alpha_i$, where the α_i and β are definable functions $|S| \to h[0, 0, 1]$.

5.4. Given a T-field K, $S \in \mathrm{Def}$, $f \in \mathcal{P}(S)$, and a real $q > 1$, we then define

$$\theta_{q,K}(f) : S(K) \to \mathbb{R}, \qquad s \mapsto \theta_q(f(s, K)).$$

We define a partial ordering $\geq$ on $\mathbb{A}$ and on $\mathcal{P}(S)$ by setting, for $a \in \mathbb{A}$ and $f \in \mathcal{P}(S)$, $a \geq 0 \iff \theta_q(a) \geq 0$ for all real $q > 1$; and $f \geq 0 \iff \theta_q(f(s,K)) \geq 0$ for all $(s,K) \in |S|$. We then set $\mathcal{P}_+(S) = \{f \in \mathcal{P}(S) \mid f \geq 0\}$, $\mathbb{A}_+ = \{a \in \mathbb{A} \mid a \geq 0\}$. Both are semi-rings (closed under addition and multiplication, but not necessarily subtraction). One can also describe $\mathcal{P}_+(S)$ as the set of elements of $\mathcal{P}(S)$ taking their values in $\mathbb{A}_+$.

5.5. Summable families. Let V be a vector space with a norm, and which is complete. Let $(u_i)_{i\in I}$ a family of elements of V. We say that $(u_i)_{i\in I}$ is *summable* if there is $U \in V$ such that for every positive real ε there is a finite subset I_0 of I such that for any finite subset J of I containing I_0, one has

$$|U - \sum_{i\in J} u_i| < \varepsilon.$$

The element U, if it exists, is called the *sum* of the family, and is denoted $\sum_{i\in I} u_i$.

Some properties: If $(I_j)_{j\in J}$ is a partition of I, and $(u_i)_I$ is summable, then each subfamily $(u_i)_{I_j}$ is summable, and if $U_j = \sum_{I_j} u_i$, then $(U_j)_J$ is summable, with sum $\sum_J U_j = \sum_I u_i$.

The converse is in general false. There are certain cases where it holds, e.g., if $V = \mathbb{R}$, and all u_i's are ≥ 0; or if J is finite.

If $V = \mathbb{C}$, then one can show that $(u_i)_I$ is summable if and only if $(|u_i|)_I$ is summable, if and only if $(\Re(u_i))_I$ and $(\Im(u_i))_I$ are summable.

If $V = \mathbb{R}$, the above will imply for instance a version of Fubini:

$$\sum_{(i,j)\in\mathbb{Z}^2} u_{i,j} = \sum_{i\in\mathbb{Z}} \sum_{j\in\mathbb{Z}} u_{i,j} = \sum_{j\in\mathbb{Z}} \sum_{i\in\mathbb{Z}} u_{i,j},$$

if any of these three numbers is defined, then all three are defined and equal. This follows from the remark about positive families and finite partitions.

5.6. Definition. We say that $\varphi \in \mathcal{P}(h[0,0,r])$ is *integrable* if for any T-field $\mathcal{K}$ and real $q > 1$, the family $(\theta_{q,K}(\varphi)(i))_{i\in\mathbb{Z}^r}$ is summable.

If $S \in \mathrm{Def}$ and $\varphi \in \mathcal{P}(S[0,0,r])$, we say that φ *is integrable over* S, or S*-integrable* if for any T-field $\mathcal{K}$, real $q > 1$ and $s \in S(K)$, the family $(\theta_{q,K}(\varphi(s,i)))_{i\in\mathbb{Z}^r}$ is summable.

We let $I_S\mathcal{P}(S[0,0,r])$ denote the set of S-integrable functions, and $I_S\mathcal{P}_+(S[0,0,r])$ its intersection with $\mathcal{P}_+(S[0,0,r])$. Note that if $a \in \mathcal{P}(S)$, then for any $(s,K) \in |S|$, the function $a(s)$ is constant on $\mathbb{Z}^r$. Hence, if $a \in \mathcal{P}(S)$ and $\varphi \in I_S\mathcal{P}(S[0,0,r])$ then $a\varphi \in I_S\mathcal{P}(S[0,0,r])$. Thus $I_S\mathcal{P}(S[0,0,r])$ is a $\mathcal{P}(S)$-module, and $I_S\mathcal{P}_+(S[0,0,r])$ a semi-$\mathcal{P}_+(S)$-module.

5.7. Comment. Being able to integrate *over a set* will be useful to compute integrals: we will simply compute iterated integrals. Then Fubini will tell us that the result (if it exists) does not depend on the order of integration.

5.8. Theorem/Definition 5. If $\varphi \in I_S\mathcal{P}(S[0,0,r])$, there exists a unique function $\psi := \mu_{/S}(\varphi)$ in $\mathcal{P}(S)$ such that for all $q > 1$, T-field $\mathcal{K}$ and $s \in S(K)$ one has

$$\theta_{q,K}(\psi)(s) = \sum_{i\in\mathbb{Z}^r} \theta_{q,K}(\varphi(s,i)).$$

Moreover, $\varphi \mapsto \mu_{/S}(\varphi)$ is a morphism of $\mathcal{P}(S)$-modules, which sends $I_S\mathcal{P}_+(S[0,0,r])$ to $\mathcal{P}_+(S)$.

Proof. Using Fubini (see discussion in the paragraph on summable families), and induction, one may reduce to the case $r = 1$. See the proof of Theorem 4.5.1 in [3]. The proof uses finite b-minimality, the fact that T is split, and basic results about Presburger sets and functions (see section 4.4 in [3]), and explicit calculations on geometric series. (Such as $\sum_{i\in\mathbb{N}} \mathbb{L}^{-in} = \frac{1}{1-\mathbb{L}^{-n}}$)

5.9. Notation. If $Y \subset S$ is definable, then 1_Y, the characteristic function of Y, belongs to $\mathcal{P}_+(S)$. We let $\mathcal{P}^0(S)$ be the subring of $\mathcal{P}(S)$ generated by all 1_Y, $Y \subset S$ definable, and by $\mathbb{L} - 1$. Similarly $\mathcal{P}^0_+(S)$ is the subsemiring of $\mathcal{P}_+(S)$ generated by all 1_Y, $Y \subset S$ definable, and by $\mathbb{L} - 1$.

If $f : X \to S$ is definable, then we get a map $f^* : \mathcal{P}(S) \to \mathcal{P}(X)$, $\varphi \mapsto f \circ \varphi$.

6 Integration over the residue ring

6.1. Let $Z \in \mathrm{Def}$. We define the semi-group $\mathcal{Q}_+(Z)$ as follows. Consider first the free semi-group generated by symbols $[Y]$, where $Y \in \mathrm{Def}_Z$ is a subset of $Z[0,m,0]$ for some finite tuple m in $\mathbb{N}$ and the map $Y \to Z$ is the one induced by the natural projection $Z[0,m,0] \to Z$. We now quotient this semi-ring $\oplus_Y \mathbb{N}[Y]$ by the following equivalences:

- $[\emptyset] = 0$,
- $[Y] = [Y']$ if there is an isomorphism between Y and Y' (in the category Def_Z),
- $[Y_1 \cup Y_2] + [Y_1 \cap Y_2] = [Y_1] + [Y_2]$,
- Let m and m' be tuples in $\mathbb{N}$ of the same length, $Y \subset Z[0,m,0]$ definable, and $p : Z[0,m+m',0] \to Z[0,m,0]$ the natural projection. If $Y' = p^{-1}(Y)$, then $[Y'] = [Y[0,m',0]]$.

6.2. Comments on these equivalences. The first three are fairly standard, the fourth one is more interesting. Notice that as a consequence we identify $R_1 \times R_1$ with R_2. Why is this legitimate? First of all, if the residue field of the valued field K is finite, then these two sets have the same size. Also, if $K = k((t))$, then $R_1 \simeq k$, and $R_2 \simeq k \oplus kT$.

In fact, more is true. If the residue field is of positive characteristic and is perfect, then for any $m = (m_1, \ldots, m_\ell)$, if $|m| := \sum m_i$, then there is a definable bijection between $\prod_i R_{m_i}$ and $R_{|m|}$. This is proved by induction on m and ℓ, the two essential points being the following (let p be the characteristic, π a uniformizer):

- The map $\mathrm{Frob} : x \mapsto x^p$ induces a bijection f between R_1 and the set P of p-powers in R_2.
- Every element of R_2 can be written uniquely as the product of an element of P and an element of πR_2.

Hence, if $g : R_1 \to R_2$ is the map induced by multiplication by π, we obtain that (f, g) defines a bijection between $R_1 \times R_1$ and R_2.

6.3. Multiplication on $\mathcal{Q}_+(Z)$. If $Y_1 \in Z[0, m, 0]$, $Y_2 \in Z[0, m', 0]$, then define $[Y_1][Y_2] = [Y_1 \otimes_Z Y_2]$.

The element corresponding to 1 (=neutral element for the multiplication) in $\mathcal{Q}_+(Z)$ will be $[Z \times \{0\}] \subset [Z \times R_1]$, with the natural projection. In fact, any point of R_1, or of R_m, would have worked as well. If $f : Z_1 \to Z_2$ is definable, one obtains a map $f^* : \mathcal{Q}_+(Z_2) \to \mathcal{Q}_+(Z_1)$ (induced by $f^* : \mathrm{Def}_{Z_2} \to \mathrm{Def}_{Z_1}$).

We write $\mathbb{L}$ for $[Z[0, 1, 0]]$ ($Z[0, 1, 0]$ with the natural projection). By the above equivalence, if $m = (m_1, \ldots, m_\ell)$, then $[Z[0, m, 0]] = \mathbb{L}^{|m|}$.

One can also show that any $a \in \mathcal{Q}_+(Z)$ can be written as $[Y]$ for some definable $Y \subset [0, m, 0]$ for some tuple m. For instance, given $[Y_1], [Y_2] \in \mathcal{Q}_+(Z)$, let us find $[Y_3]$ such that $[Y_1]+[Y_2] = [Y_3]$. First, using the equivalences above, we may assume that both Y_1 and Y_2 are definable subsets of $Z \times \pi R_m$ for some integer m. We then replace Y_2 by its image $[Y_2']$ under the translation $f(z, r) = (z, r + 1)$: we now have that $Y_1 \cap Y_2 = \emptyset$, so that $[Y_1] + [Y_2] = [Y_1 \cup Y_2']$.

6.4. Integral on fibers. Let $Z = X[0, k, 0]$ for some tuple k of integers, p the natural projection $Z \to X$ and let $Y \in \mathcal{Q}_+(Z)$. The map p^* allows us to consider Y as an object of $\mathcal{Q}_+(X)$, and this is what we will do: define

$$\mu_X := p^* : \mathcal{Q}_+(Z) \to \mathcal{Q}_+(X).$$

Note that this map is not necessarily injective: there are more isomorphisms in Def_X than in Def_Z. We call this map the *formal integral in the fibers of p.*

7 Putting $\mathcal{P}_+$ and $\mathcal{Q}_+$ together

We have two canonical semi-ring morphisms of $\mathcal{P}_+^0(Z)$ into $\mathcal{P}_+(Z)$ and into $\mathcal{Q}_+(Z)$. The first one is the natural inclusion. The second one is obtained by sending 1_Y (for a definable $Y \subset Z$) to $[Y]$, and sending $\mathbb{L}-1$ to $[Z \times R_1^\times]$. Note that this is consistent with our notation $\mathbb{L} = [Z[0, 1, 0]]$ since we saw that 1 could be identified with $[Z \times \{0\}]$.

We now define the semi-ring $\mathcal{C}_+(Z)$ as the tensor product $\mathcal{P}_+(Z) \otimes_{\mathcal{P}_+^0(Z)} \mathcal{Q}_+(Z)$. In other words, it is the free semigroup on all elementary tensors $a \otimes c$, quotiented by the equivalence relation generated by $(a + b) \otimes c = a \otimes c + b \otimes c$, $a \otimes (c + d) = a \otimes c + a \otimes d$, $(ea \otimes c) = a \otimes ec$ for $a, b \in \mathcal{P}_+(Z)$, $c, d \in \mathcal{Q}_+(Z)$ and $e \in \mathcal{P}_+^0(Z)$.

7.1. As before, if $f : Z \to X$ is definable, then we get $f^* : \mathcal{C}_+(X) \to \mathcal{C}_+(Z)$, induced by $a \otimes c \mapsto f^*(a) \otimes f^*(c)$.

7.2. Interpretation in non-archimedean local fields. Let $X \subset h[n, m, r]$ be definable, $\varphi \in \mathcal{C}_+(X)$, and let $\mathcal{Q}$ be a T-field with residue field $\mathbb{F}_q$. Then φ gives rise to a function $\varphi_K : X(K) \to \mathbb{Q}^{\geq 0}$ as follows:

- if $a \in \mathcal{P}_+(X)$, replace $\mathbb{L}$ by q to get $a_K : X(K) \to \mathbb{Q}^{\geq 0}$,

- ($b \in \mathcal{Q}_+(X)$) if $b = [Y]$ where $Y \subset X[0, m', 0]$, let $p : Y \to X$ be the projection, and let $b_K : X(K) \to \mathbb{N}$ be defined by $b(x) = \#(p^{-1}(x))$ (i.e., number of points lying in the fiber of Y above x). We know $b(x)$ is finite, in fact bounded above by $q^{|m'|}$.
- if $\varphi = \sum_i a_i \otimes b_i$, ($a_i \in \mathcal{P}_+(X)$, $b_i \in \mathcal{Q}_+(X)$) then define $\varphi_K = \sum_i a_{i,K} \otimes b_{i,K}$.

An important observation is that this does not depend on the choice of the presentation of φ as $\sum_i a_i \otimes b_i$.

7.3. Proposition 6. $S \in \mathrm{Def}$. The canonical morphism

$$t : \mathcal{P}_+(S[0, 0, r]) \otimes_{\mathcal{P}^0_+(S)} \mathcal{Q}_+(S[0, m, 0]) \to \mathcal{C}_+(S[0, m, r]$$

is an isomorphism. Here the morphisms $\mathcal{P}^0_+(S) \to \mathcal{P}_+(S[0, 0, r])$ and $\mathcal{P}^0_+(S) \to \mathcal{Q}_+(S[0, m, 0])$ are induced by the pull-back of the natural projections $S[0, 0, r] \to S$ and $S[0, m, 0] \to S$. [t is defined by $t(a \otimes b) = p_1^*(a) \otimes p_2^*(b)$ for $a \in \mathcal{P}_+(S[0, 0, r])$, $b \in \mathcal{Q}_+(S[0, m, 0])$, where $p_1 : S[0, m, r] \to S[0, 0, r]$ and $p_2 : S[0, m, r] \to S[0, m, 0]$ are the natural projections.]

Proof. The proof uses that T is split.

7.4. Lemma/Definition 7. Let $\varphi \in \mathcal{C}_+(Z)$, $Z \subset S[0, m, r]$. Say that φ is S-integrable if it can be written as $\sum_{i=1}^{\ell} a_i \otimes b_i$, with $a_i \in I_S\mathcal{P}_+(Z)$, $b_i \in \mathcal{Q}_+(Z)$. If this happens, we define

$$\mu_{/S}(\varphi) = \sum_{i=1}^{\ell} \mu_{/S}(a_i) \otimes \mu_{/S}(b_i) \ (\in \mathcal{C}_+(S)).$$

This definition does not depend on the choice of the a_i's and b_i's. We call $\mu_{/S}(\varphi)$ the *integral of φ in the fibers of the projection $Z \to S$.*

Proof. We need to show that the map from the free semi-group W on $I_S\mathcal{P}_+(Z) \otimes \mathcal{Q}_+(Z)$, $\sum_i (a_i, b_i) \mapsto \sum_i \mu_{/S}(a_i) \otimes \mu_{/S}(b_i)$ factors through the tensor product. The bi-additivity is clear. Let $c \in \mathcal{P}^0_+(Z)$, $a \in I_S\mathcal{P}_+(Z)$, $b \in \mathcal{Q}_+(Z)$. We need to show that $\mu_{/S}(a, cb) = \mu_{/S}(ca, b)$. Let K be a T-field, and $s \in S(K)$. We will first do the case where $c = 1_Y$, where $Y \subset S$ is definable.

Then one verifies that if $p : Z \to S$ is the projection, both functions agree with $\mu_{/S}(a) \otimes \mu_{/S}(b)$ on $p^{-1}(Y)$, and are 0 on its complement.

When $c = \mathbb{L} - 1$, the equality is clear. The full result then follows.

7.5. Lemma 8. Let $S \in \mathrm{Def}$, $Z = S[0, m, r]$, $\varphi \in \mathcal{C}_+(Z)$, S-integrable, $\psi \in \mathcal{C}_+(S)$ and $p : Z \to S$. Then $p^*(\psi)\varphi$ is S-integrable, and

$$\mu_{/S}(p^*(\psi)\varphi = \psi\mu_{/S}(\varphi).$$

We know that $I_S\mathcal{P}_+(Z)$ is a $\mathcal{P}_+(S)$-module under p^*.

7.6. Lemma (Interpretation in non-archimedean local fields). Let Z, S, φ be as above, and K a T-field with residue field $\mathbb{F}_q$, and consider φ_K. If φ is S-integrable, and $\mu_{/S}(\varphi) = \psi$, then for all $s \in S(K)$,

$$\varphi_K(s, -) : z \mapsto \varphi_K(s, z)$$

is integrable for the counting measure, with integral $\psi_K(s)$ ($= \sum_{z \in p^{-1}(s)} \varphi_K(s, z)$; since the integral with respect to the counting measure is simply the sum).

8 Integration over the valued field

8.1. An easy case. We have $Z \subset S[1, 0, 0]$, $S \in \mathrm{Def}$, $\varphi \in \mathcal{P}_+(Z)$, and we would like to integrate φ along the fiber $Z \to S$. Assume that φ factors through $\varphi_0 \in \mathcal{P}_+(S[0, 0, 1])$. I.e., the value of φ at a point (s, y) only depends on $(s, v(y))$.

Let $q > 1$, and define the *q-volume of a ball B* of radius n to be q^{-n}. Then, with that measure on K, we would like to compute, for each $s \in S(K)$, the integral of $\theta_q \circ \varphi(s, x)$ over K.

We look at all partitions of K into balls, say $K = \cup B_i$, pick a point x_i in each ball, and compute $\sum_I \mu_q(B_i)\theta_q \circ \varphi(s, x_i)$; then we want to take the limit of such sums. In our particular case, where φ factors through Γ, this boils down to compute, for every $q > 1$ and $s \in |S|$,

$$\sum_{n \in \mathbb{Z}} (q - 1)q^{-n}\theta_q \circ \varphi_0(s, n),$$

i.e.,

$$\sum_{n \in \mathbb{Z}} \theta_q((\mathbb{L} - 1)\mathbb{L}^{-n}\varphi_0(s, n)).$$

Setting $\varphi_1(s, n) = (\mathbb{L} - 1)\mathbb{L}^{-n}\varphi_0(s, n)$, we then say that φ *is S-integrable* just in case φ_1 is, and define $\mu_{/S}(\varphi) = \mu_{/S}(\varphi_1)$.

The general case is more complicated, even though we know fairly well how to describe definable functions $K \to \Gamma$.

8.2. Cluckers and Loeser define a notion of *step function*, and *step domain*. In fact, the important property of these functions is that they factor through Γ, so that the easy case above takes care of them. Let us now go to the general case.

8.3. Lemma/Definition 9. $S \in \mathrm{Def}$, $Z = S[1, 0, 0]$, $\varphi \in \mathcal{P}_+(Z)$. We say that φ *is S-integrable* if for some tuple m there is $\psi \in \mathcal{P}_+(Z[0, m, 0])$ such that $\mu_{/Z} = \varphi$, and ψ is $S[0, m, 0]$-integrable, as defined above. If this is the case, then we set

$$\mu_{/S}(\varphi) = \mu_{/S}(\mu_{/S[0,m,0]}(\psi)).$$

This does not depend on the choice of ψ.

Vague idea of the proof. If $\psi_1 \in \mathcal{P}_+(Z[0, m_1, 0])$ is another function satisfying the same hypotheses as ψ, then ψ and ψ_1 have a common refinement, i.e., there is some $m_2 \geq m_1, m$

and $\psi_2 \in \mathcal{P}_+(Z[0, m_2, 0])$ which also satisfies the requirements, and with $p_!\psi_2 = \psi$, $p_{1!}\psi_2 = \psi_1$, where p and p_1 are the natural projections $Z[0, m_2, 0] \to Z[0, m, 0]$ and $Z[0, m_2, 0] \to Z[0, m_1, 0]$. Hence we may assume that ψ_1 refines ψ, and may assume that $m = 0$. I.e., we need to show that

$$\mu_{/S}(\psi)\mu_{/S}(\mu_{/S[0,m_1,0]}\psi_1).$$

One reduces to the case when $\psi(s, -)$ is constant on a ball B_s, and 0 outside. One uses finite b-minimality, and our famous equivalence on $\mathcal{Q}_+$.

For more details, see 8.2 in [4]. See also section 9 of [3].

8.4. Lemma/Definition 10. $S \in \mathrm{Def}$, $Z = S[m, n, r]$, $\varphi \in \mathcal{C}_+(Z)$. We say that φ *is S-integrable* if there exists a definable $Z' \subset Z$, with complement $Z \setminus Z'$ having relative dimension $< n$ over X (i.e., for any $(x, K) \in |X|$, if p is the natural projection $Z \to X$, then the dimension of $p^{-1}(x) \cap (Z \setminus Z')$ is $< n$ – observe that n is the dimension of the fibers of p), and an ordering of the valued field coordinates on $X[m, n, r]$ such that $\varphi' := 1_{Z'}\varphi$ is $X[n-1, m, r]$-integrable, and $\mu_{/X[n-1,m,r]}(\varphi')$ is X-integrable. If this holds, then

$$\mu_{/X}(\varphi) = \mu_{/X}(\mu_{/X[n-1,m,r]}(\varphi')) \ \in \mathcal{C}_+(X)$$

does no depend on the choices, and is called *the integral in the fibers of* $p : Z \to X$.

8.5. This definition of course generalises to a definable $Z \subset S[m, n, r]$ and $\varphi \in \mathcal{C}_+(Z)$: say that φ is S-integrable if $\tilde{\varphi} \in \mathcal{P}_+(S[m, n, r])$ is S-integrable, where $\tilde{\varphi}$ equals φ on Z and 0 outside. One then sets $\mu_{/S}(\varphi) = \mu_{/S}(\tilde{\varphi})$.

8.6. Vague idea of the proof. The proof is by induction on n, and changing the order of integration, one may assume that $n = 2$, $m = r = 0$. Let $p_1, p_2 : S[2, 0, 0] \to S[1, 0, 0]$ be the two natural projections. By Fubini on summable families, if it is S-integrable, then the order of integration does not matter, and there is some definable $Z' \subset Z$, of relative codimension < 2, such that $\varphi' = 1_{Z'}\varphi$ is integrable in the fibers of p_1 and of p_2. Replace φ by φ'.

Partitioning S, and replacing it by some $S[0, m', r']$, we may assume that above any $(s, K) \in |S|$, the fiber $Z_s(K)$ has the form

$$\{(t_1, t_2) \in K^2 \mid t_1 \in B_s, t_2 \in B_{s,t_1}\}$$

where B_s is a ball only depending on s, and B_{s,t_1} is a ball of the form

$$\{t_2 \in K \mid \overline{\mathrm{ac}}_n(t_2 - c(s, t_1)) = \xi \wedge \ v(t_2 - c(s, t_1)) = z\}$$

for some $\xi \in R_n^\times, z \in \mathbb{Z}$ and $n < N$. Here one uses finite b-minimality, the fact that a definable function from K^2 to a product of auxiliary sorts is locally constant, and compactness. Partitioning further, we may assume that the functions $c(s, -)$ have the Jacobian property on each ball B_s. There are three cases to consider.

If the function $c(s, -)$ is constant, then $Z_s(K)$ is the product of two balls which only depend on s. The situation is symmetric in the two variables. Moreover, the fact that

8.7. Application to local fields. Let $\varphi \in X[n,m,r]$ for some $X \in \mathrm{Def}$ and n,m,r tuple of integers. If φ is X-integrable, then for every T-field K and $x \in X(K)$, one has that $\varphi_K(x,-)$ is integrable (in the standard measure theoretic sense). If $\psi = \mu_{/X}(\varphi)$, then for all $x \in X(K)$,

$$\psi_K(x) = \int_{y \in h[n,m,r]} \varphi_K(x,y),$$

where the integral is computed with respect to the Haar measure on K, and the counting measure on $\mathbb{Z}$ and on the residue rings.

8.8. Some properties.

Projection. Let $\varphi \in \mathcal{C}_+(Z)$, where $Z \subset X[n,m,r]$, $X \in \mathrm{Def}$, and $\psi \in \mathcal{C}_+(X)$, $p : Z \to X$ the canonical projection. If φ is X-integrable, then $p^*(\psi)$ is X-integrable, and

$$\mu_{/X}(p^*(\psi)\varphi) = \psi \mu_{/X}(\varphi).$$

Definition of the Jacobian. Let $f : A \to K^n$ be definable, where $A \subset K^n$. Let $\mathrm{Jac}(f)$ be defined as the determinant of the Jacobian matrix of f at x if x belongs to the interior of A, and 0 otherwise.
There exists also a relative version of this definition for a function in Def_X, and which is denoted by $\mathrm{Jac}_{/X}(f)$.

Change of variable formula. Let $X \in \mathrm{Def}$, and $F : Z \to Z'$ be a definable isomorphism, where $Z, Z' \subset X[n,0,0]$, and let $\varphi \in \mathcal{C}_+(Z)$. Assume that there is $Y \subset Z$, such that the relative dimension of $(Z \setminus Y)$ over X, $\dim_{/X}(Z \setminus Y)$, is strictly less than $\dim_{/X}(Z) = n$, and such that $\mathrm{Jac}_{/X}(F)$ does not vanish on Y. If $\varphi' \in \mathcal{C}_+(Z')$ is defined by $F^*\varphi$, then

$$\varphi' \text{ is } X\text{-integrable} \iff \varphi \mathbb{L}^{-v\mathrm{Jac}_{/X}F} \text{ is } X\text{-integrable},$$

and then,

$$\mu_X(\varphi \mathbb{L}^{-v\mathrm{Jac}_{/X}(F)}) = \mu_{/X}(\varphi').$$

[By convention, $\mathbb{L}^{-\infty} = 0$.]

Fubini-Tonnelli. Let $X \in \mathrm{Def}$, $Z \subset X[n,m,r]$ be definable, and $p : X[n,m,r] \to X[n-n', m-m', r-r']$ be the natural projection (where $n' < n$, $m' < m$ and $r' < r$). Let $\varphi \in \mathcal{C}_+(Z)$. Then φ is X-integrable, if and only if there is some definable Y of small relative codimension over X, such that, replacing φ by $\varphi' = 1_Y \varphi$, we have that φ' is $X[n-n', m-m', r-r']$-integrable, and $\mu_{/X[n-n',m-m',r-r']}(\varphi')$ is X-integrable. If this is the case, then

$$\mu_{/X}(\mu_{/X[n-n',m-m',r-r']}(\varphi')) = \mu_{/X}(\varphi).$$

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