Content of the beginning of the course

Definition of valuations on fields: a valued field is a field K, together with a map $v : K \to \Gamma \cup \{\infty\}$, where Γ is an ordered abelian group, and v satisfies: $v(x) = \infty$ if and only if $x = 0$; $v(ab) = v(a) + v(b)$; $v(a + b) \geq \inf\{v(a), v(b)\}$.
$\mathcal{O}_v = \{a \in K \mid v(a) \geq 0\}$ is a subring of K, with maximal ideal $\mathcal{M}_v = \{a \in K \mid v(a) > 0\}$. We define k_v or k_K, the *residue field* to be $\mathcal{O}_v/\mathcal{M}_v$, and let $\mathrm{res} : \mathcal{O}_v \to k_v$ be the natural projection.
$\mathcal{O}_v$ is a *valuation ring*: if $a \in K \setminus \mathcal{O}_v$ then $a^{-1} \in \mathcal{O}_v$.

Define $a|b \iff v(a) \leq v(b)$ ($\iff b/a \in \mathcal{O}_v$). Then from $|$ (pronounced "div"), or from $\mathcal{O}_v$, one can recover the valuation (up to equivalence). Indeed,

$$K^\times/\mathcal{O}_v^\times \simeq \Gamma,$$

(Here $\mathcal{O}_v^\times$ denotes the invertible elements of $\mathcal{O}_v$), and the ordering on the quotient is defined by $a\mathcal{O}_v^\times \leq b\mathcal{O}_v^\times$ if and only if $b/a \in \mathcal{O}_v$, if and only if $a|b$. Noting that $a \in \mathcal{O}_v$ if and only if $a|1$, one gets the result.

Examples of valued fields.
The valuations on $\mathbb{Q}$ (trivial, or p-adic);
The valuations on $k(T)$ which are trivial on the field K;
If Γ is an ordered abelian group, define $k[\Gamma]$ to be the set of formal sums $f = \sum_{g\in\Gamma} a_g g$, with $\mathrm{supp}(f) = \{g \mid a_g \neq 0\}$ finite. Then $k[\Gamma]$ is a domain, let $k(\Gamma)$ be its field of fractions. Define $v(f) = \min \mathrm{supp}(f)$, for $f \in k[\Gamma]$, and extend to $k(\Gamma)$.

Definition of well-ordered set. Some words about ordinals and cardinals, and their operations. Transfinite induction.

Fields of power series $k((T))$; If Γ is an ordered abelian group, k a field, we define the *field of generalised power series* $k((t^\Gamma))$ as the set of formal sums $f = \sum_{g\in\Gamma} a_g t^g$, with $\mathrm{supp}(g)$ **well-ordered**.
So, $k((t)) = k((t^{\mathbb{Z}}))$. However, $k((t^{\mathbb{Q}}))$ strictly contains $\bigcup_{n\in\mathbb{N}} k((t^{1/n}))$.

Definition of the topology on the valued field (K, v): given $a \in K$ and γ in Γ define the "open ball" $B(a, \gamma)$ ["closed ball" $\bar{B}(a, \gamma)$] of center a and (valuative) radius γ by

$$B(a, \gamma) = \{x \in K \mid v(x - a) > \gamma\}, \quad \bar{B}(a, \gamma) = \{x \in K \mid v(x - a) \geq \gamma\}.$$

Note that any two balls are either disjoint, or one contains the others. A closed ball is the union of the open balls it contains, and is therefore also open (despite its name). In fact .... all balls are **open and closed**. They can also be described as $B(a, \gamma) = a+c\mathcal{M}_v$ and $\bar{B}(a, \gamma) = a+c\mathcal{O}_v$, where c is any element with $v(c) = \gamma$. They form the basis of open sets for a topology on K: the open sets of K are the unions of balls.

Extensions of valuations. Let (K, v) be a valued field, L an overfield of K. Then v extends to a valuation on L. If L is a Galois extension of K, then all valuations extending v are conjugate: if w_1, w_2 extend v, then there is $\sigma \in Gal(L/K)$ such that $w_1 = w_2 \circ \sigma$. Equivalently,

$\mathcal{O}_{w_1} = \sigma^{-1}(\mathcal{O}_{w_2})$.
Furthermore, if R is the *integral closure of* $\mathcal{O}_v$ *in* L (the set of elements of L which satisfy some monic polynomial equation over $\mathcal{O}_v$, which is a subring), then the extension w of v is determined by $P := \mathcal{M}_w \cap R$, and one has $\mathcal{O}_w = R_P$, $\mathcal{M}_w = PR_P$.

Theorem. *Consider the language* $\mathcal{L}_{\text{div}} = \{+, -, \cdot, 0, 1, |\}$, *and the theory ACVF of non-trivially valued algebraically closed fields. Then ACVF eliminates quantifiers.*

Thus the complete theories extending ACVF are obtained by specifying the characteristic of the field and of its residue field. One writes $\text{ACVF}_{0,0}$, $\text{ACVF}_{0,p}$ and $\text{ACVF}_{p,p}$.

Assume L/K finite algebraic, (K, v) a valued field.
Let w extend v. Then $e(L/K, w) := [w(L^\times) : v(K^\times)]$ is finite (index of *ramification*), $f(L/K, w) := [k_L : k_K]$ is finite, and $e(L/K)f(L/K) \leq [L : K]$.
Assume L/K Galois over K, let $G = Gal(L/K)$, fix a valuation w on L extending v, and define $G_{dec} = \{\sigma \in G \mid \sigma(\mathcal{O}_w) = \mathcal{O}_w\}$ (the *decomposition subgroup of* w), and let L^{dec} be the subfield of L fixed by G_{dec}. Then w is the only extension of $w|_{L^{dec}}$ to L. One shows that L^{dec} is an *immediate* extension of K (i.e., neither the value group nor the residue field increase). Furthermore, a theorem of Ostrowski tells you (in case L/K is finite) that

$$[L : K] = [G : G_{dec}]e(L/K)f(L/K)\chi^d,$$

where $\chi = 1$ if the residue field has characteristic 0, and equals the characteristic of the residue field otherwise.
Henselian fields: a valued field K is *Henselian* if whenever $f(T) \in \mathcal{O}_v[T]$ and $a \in \mathcal{O}_v$ are such that $v(f(a)) > 0 = v(f'(a))$, then there is $b \in \mathcal{O}_v$, with $v(b - a) = v(f(a))$, such that $f(b) = 0$. The condition on $v(b - a) = v(f(a))$ can be replaced by $v(b - a) > 0$. Looking at the Taylor expansion (see below) will give that necessarily one will have $v(b - a) = v(f(a))$.

Theorem. Complete valued fields with an archimedean value group are Henselian.

Theorem. Let (K, v) be a valued field. The following conditions are equivalent:

(1) K is Henselian.

(2) The valuation v has a unique extension to K^{sep} (the *separable closure* of K, i.e., the elements of an algebraic closure of K whose minimal polynomial is separable).

(3) Let $f(T) \in \mathcal{O}_v[T]$ be monic, and assume that there are $g(T), h(T) \in \mathcal{O}_v[T]$ such that $\text{res}(f)(T) = \text{res}(g)(T)\text{res}(h)(T)$, and $\text{res}(g)$, $\text{res}(h)$ are relatively prime. Then there are $\tilde{g}(T), \tilde{h}(T) \in \mathcal{O}_v[T]$ such that $f = \tilde{g}\tilde{h}$.

(4) (Hensel-Rychlik) Let $f(T) \in \mathcal{O}[T]$ and assume that $v(f(0)) > 2v(f'(0))$. Then there is a with $f(a) = 0$ and $v(a) = v(f(0)) - v(f'(0))$.

Corollary. An algebraic extension of a Henselian field is Henselian.
If (K, v) is a valued field, choose an extension w of c to K^{sep}, let G_{dec} be the decomposition

group of w, and K^h the subfield of K^{sep} fixed by G_{dec}. Then K^h is Henselian, and in fact every Henselian field extending (K, v) contains a k-isomorphic copy of K^h. One calls K^h the *henselization* of K.

Taylor expansion: if $f(X)$ is a polynomial of degree n, define the polynomials $D_i(f)(X)$, $0 \leq i \leq n$ by

$$f(X+Y) = \sum_{i=0}^{n} D_i(f)(X)Y^i.$$

When the characteristic is 0, $D_i(f)$ is simply the i-th derivative of f divided by $i!$. Not so in positive characteristic p: $(X+Y)^p = X^p + Y^p$.
So, this gives: $f(b) = f(a) + f'(a)(b-a) + \sum_{i\geq 2} D_i(f)(a)(b-a)^i$, the *Taylor expansion of f near a.*

References

[1] Z. Chatzidakis, notes in French of a course taught in 2008.
`http://www.math.ens.fr/~zchatzid/papiers/cours08.pdf`

[2] Antonio J. Engler, Alexander Prestel, *Valued fields*, Springer monographs in Mathematics, Berlin (2005).

[3] I. Kaplansky, Maximal fields with valuations, Duke Math. J. 9, (1942), 303 – 321.

Course of June 2

Some results on arithmetic of cardinals: if κ, λ are infinite cardinals then:
$\kappa + \lambda := \mathrm{card}(\kappa \dot\bigcup \lambda) = \sup\{\kappa, \lambda\}$
$\kappa \times \lambda := \mathrm{card}(\kappa \times \lambda) = \sup\{\kappa, \lambda\}$
$\kappa < 2^\kappa = \kappa^\kappa$; where 2^κ is the cardinality of the set of maps $\kappa \to \{0, 1\}$.

If $\mathrm{card}(A) = \kappa$, then the set $\mathcal{P}_f(A)$ of finite subsets of A has cardinality κ, as does the set of finite sequences of elements of A.
So, in particular, if $\mathcal{L}$ is a countable language, then there are $\aleph_0$ formulas, and the size of an $\mathcal{L}$-structure generated by a subset A has cardinality $\aleph_0 + \mathrm{card}(A)$.
Similarly, if $\mathcal{L}$ has cardinality κ, then there are κ $\mathcal{L}$-formulas.

Definition 1. Let $\mathcal{L}$ be a countable language, M be an $\mathcal{L}$-structure, $A \subset M$, $n \in \mathbb{N}$ and κ an infinite cardinal.

(1) Recall that $\mathcal{L}(A)$ is obtained by adding new constant symbols for all elements of A.

(2) A *(complete) n-type over A* is a set $\Sigma(x)$ of $\mathcal{L}(A)$-formulas in the variables $x = (x_1, \ldots, x_n)$ (for instance), which is finitely *satisfiable* in M (if Σ_0 is a finite subset of Σ, then $M \models \exists x \bigwedge_{\varphi \in \Sigma_0} \varphi(x)$) and maximal such.
Example: let a be an n-tuple in M, and define the *type of a over A*,

$$tp(a/A) = \{\varphi(x) \in \mathcal{L}(A) \mid M \models \varphi(a).\}$$

(3) The model M is *κ-saturated* if whenever $A \subset M$ has cardinality $< \kappa$, then any n-type over A is realised in M, i.e., there is an n-tuple a in M which satisfies all the formulas in the type.

Theorem 2. *Let κ be a cardinal. Every $\mathcal{L}$-structure has an elementary extension which is κ-saturated.*

Theorem 3. *Let $\mathcal{U}$ be a non-principal ultrafilter on the countable set I, and for each $i \in I$, let M_i be an $\mathcal{L}$-structure, where $\mathcal{L}$ is countable. Then $M^* := \prod_{i \in I} M_i/\mathcal{U}$ is $\aleph_1$-saturated.*
Observe that if the M_i's are countable and the continuum hypothesis holds (i.e.: $2^{\aleph_0} = \aleph_1$, where $\aleph_1$ denotes the smallest cardinal larger than $\aleph_0 = \mathrm{card}(\mathbb{N})$), then also $\mathrm{card}(M^*) = \aleph_1$.

The three-sorted language of Johan Pas. Recall that to a valued field (K, v), we associate the 3-sorted structure $\mathcal{K} = (K, \Gamma_K, k_K)$: a structure with three sorts: the Valued Field sort (VF), the Value Group sort (VG) and the Residue Field sort (RF). The two fields come equipped with the language of rings $\{+, -, \cdot, 0, 1\}$, the value group with the language $\{+, 0, <, \infty\}$. In addition we have the valuation $v : K \to \Gamma_K \cup \{\infty\}$, and an *angular component map* $\underline{\mathrm{ac}} : K \to k_K$, which is multiplicative on $K^\times$, sends 0 to 0, and on $\mathcal{O}_v^\times$ coincides with the residue map. I will denote this language by $\mathcal{L}_{\mathrm{Pas}}$. (Since not all fields have an angular component map, sometimes one instead considers the partially defined map $\mathrm{res} : \mathcal{O}_v \to k$.)

Lemma 4. *Let (K, v) be an $\aleph_1$-saturated valued field. Then there is a group homomorphism $s : \Gamma_K \to K^\times$ such that $v \circ s = \mathrm{id}$.*
Hence, the field can be equipped with an angular component map, by definining $\underline{\mathrm{ac}}(0) = 0$, and for $x \neq 0$, $\underline{\mathrm{ac}}(x) = x \cdot \mathrm{res}(xs(v(x))^{-1})$.

Let $T_{0,0}$ be the theory in $\mathcal{L}_{\mathrm{Pas}}$ which expresses the following property of the structure $\mathcal{K} = (K, \Gamma_K, k_K)$:
K and k_K are fields of characteristic 0, Γ is an ordered abelian group, and ∞ is larger than all elements of Γ_K, with $a + \infty = \infty = \infty + \infty$ for $a \in \Gamma_K$. Furthermore, v defines a (non-trivial) Henselian valuation on K, with value group Γ_K. The map $\underline{\mathrm{ac}}$ satisfies $\underline{\mathrm{ac}}(0) = 0$, and defines a multiplicative homomorphism from $K^\times$ onto $k_K^\times$, which on $\mathcal{O}^\times$ coincides with res: if $a, b, a + b \in \mathcal{O}^\times$, then $\underline{\mathrm{ac}}(a + b) = \underline{\mathrm{ac}}(a) + \underline{\mathrm{ac}}(b)$; if $u \in \mathcal{M}_v$, then $\underline{\mathrm{ac}}(1 + u) = 1$ (is it enough?).

Theorem 5. *The theory $T_{0,0}$ eliminates the quantifiers of sort VF: every formula of the language is equivalent modulo $T_{0,0}$ to a formula without any quantification on the variables of the Valued Field sort.*

Corollary 6. *Let (K, v) and (L, v) be Henselian valued fields of residual characteristic 0. Then*

$$K \equiv L \iff k_K \equiv k_L \quad \textit{and } \Gamma_K \equiv \Gamma_L.$$

If furthermore $(K, v) \subset (L, w)$ then

$$K \prec L \iff k_K \prec k_L \quad \textit{and } \Gamma_K \prec \Gamma_L.$$

We consider the set Δ of $\mathcal{L}_{\mathrm{Pas}}$-formulas, defined as the smallest set closed under $\wedge$, $\vee$ and $\neg$, (and logical equivalence), which contains all the quantifier-free formulas, and all the formulas of the form $\varphi(v(t(x), \xi)$, $\psi(\underline{\mathrm{ac}}(t(x)), \bar{y})$ where x is a tuple of variables of sort VF, ξ a tuple of variables of sort VG and $\bar{y}$ a tuple of variables of sort RF; t is a tuple of terms (=polynomials) in x, and φ is a formula of the VG-language $\{+, 0, < \infty\}$, ψ a formula of the language of rings.

So, we need to show that every formula is equivalent, modulo $T_{0,0}$, to a formula in Δ.

Criterion for quantifier-elimination down to Δ.
Whenever we have $\mathcal{M} = (M, \Gamma_M, k_M)$ and $\mathcal{N} = (N, \Gamma_N, k_N)$ two $\aleph_1$-saturated models of $T_{0,0}$, $\mathcal{A} \subset \mathcal{M}$, and $\mathcal{B} \subset \mathcal{N}$ two countable substructures, and an $\mathcal{L}_{\mathrm{Pas}}$-isomorphism $f : \mathcal{A} \to \mathcal{B}$ which respects the formulas of Δ, then for $a \in M$, there is an $\mathcal{L}_{\mathrm{Pas}}$-isomorphism g which extends f, has domain containing a, and image contained in $\mathcal{N}$, and which respects the formulas in Δ.