Course of June 7

Proof of Theorem 5.

Recall what we have to prove:
We are given $\mathcal{M} = (M, \Gamma_M, k_M)$ and $\mathcal{N} = (N, \Gamma_N, k_N)$ two $\aleph_1$-saturated models of $T_{0,0}$, $\mathcal{A} \subset \mathcal{M}$, and $\mathcal{B} \subset \mathcal{N}$ two countable substructures, and an $\mathcal{L}_{\text{Pas}}$-isomorphism $f : \mathcal{A} \to \mathcal{B}$ which respects the formulas of Δ, then for $a \in M$, there is an $\mathcal{L}_{\text{Pas}}$-isomorphism g which extends f, has domain containing a, and image contained in $\mathcal{N}$, and which respects the formulas in Δ.
THe partial isomorphism f between the substructure $\mathcal{A} = (A, \Gamma_A, k_A)$ of $\mathcal{M}$ and the substructure $|calb = (B, \Gamma_B, k_B)$ of $\mathcal{N}$, means a triple $(f, f_\Gamma, \bar{f})$ of isomorphisms $f : A \to B$, $f_\Gamma : \Gamma_A \to \Gamma_B$ and $\bar{f} : k_A \to k_B$ with f_Γ and $\bar{f}$ elementary, i.e., if α is a tuple in Γ_A and φ a formula of the language of VG, then

$$\Gamma_M \models \varphi(\alpha) \iff \gamma_N \models \varphi(f_\Gamma(\alpha)),$$

and similarly for k_A, k_M. We let $\mathcal{C} = (C, \Gamma_C, k_C)$ be a countable elementary substructure of $\mathcal{M}$ which contains $\mathcal{A}$ and a. We will extend f to $\mathcal{C}$. The proof consists of several steps.

Step 0. We may assume that A and k_A are fields, and Γ_A is a subgroup of Γ_C.
The map f extends uniquely to $\mathrm{Frac}(A)$, f_Γ extends uniquely to the subgroup generated by Γ_A, which equals $\Gamma_A - \Gamma_A$, and $\bar{f}$ extends uniquely to $\mathrm{Frac}(k_A)$. A few minutes of thought show that this induces an $\mathcal{L}_{\text{Pas}}$-isomorphism, and that the new f_Γ and $\bar{f}$ are indeed elementary.

Step 1. We may assume $k_A = k_C$.
We use the following facts: the map $\bar{f}$ is elementary; $\mathcal{N}$, whence k_N, are $\aleph_1$-saturated; k_C is countable. Hence $\bar{f}$ extends to an elementary map $k_C \to k_N$, which I will also denote $\bar{f}$.

Step 2 We may assume $\Gamma_A = \Gamma_C$.
Same reasoning as in step 2.

Step 3. We may extend f to the henselization A^h of A.
Because N is Henselian, and the henselization of a valued field is unique up to isomorphism, we know that f extends to an embedding of valued fields $A^h \to N$. To check that it respects $\underline{\mathrm{ac}}$, use the fact that A^h/A is immediate, and therefore, given $a \in A^h$ there is $b \in A$ such that $v(a - b) > v(a) = v(b)$, whence $\underline{\mathrm{ac}}(a) = \underline{\mathrm{ac}}(b)$.

Step 4. Extend f to a subfield D of C such that $\mathcal{O}_D$ has residue field k_C.
The proof is adding one element at a time. Let k'_A denote the residue field of A. Let $\bar{a} \in k_C \setminus k'_A$; if $\bar{a}$ is transcendental over k'_A, then take any $a \in C$ with $\mathrm{res}(a) = \bar{a}$, let $b \in N$ be such that $\mathrm{res}(b) = \bar{f}(\bar{a})$, and extend f by sending a to b. Because the monomials a^i, $i \in \mathbb{N}$, are $k'_{\underline{A}}$-linearly independent, and the monomials b^i, $i \in \mathbb{N}$ are k'_B-linearly independent, we know that $\bar{f}$ respects the valuation.
If $\bar{a}$ is algebraic over k'_A, lift its minimal monic polynomial to some $P(T) \in \mathcal{O}_A[T]$ of the same degree, and find $a \in A$ with $P(a) = 0$, $\mathrm{res}(a) = \bar{a}$. Send it to some $b \in \mathcal{O}_N$, root of $f(P)(T) = 0$ satisfying $\mathrm{res}(b) = \bar{f}(\bar{a})$. Reason as above to deduce that it is an isomorphism of valued field.

Step 5. Extend f to a subfield E of C such that $v(E^\times) = \Gamma_C$.

Do this one at a time. Let $\alpha \in \Gamma_C, \notin \Gamma'_A = v(A^\times)$. Again, using the Lemma below and the fact that $k'_A = k_C$ by step 4, we may assume that $\alpha = v(a)$ for some $a \in C$ with $\underline{\mathrm{ac}}(a) = 1$; furthermore, if $n\alpha \in \Gamma'_A$ for some $n > 0$, we may assume that $a^n \in A$. Send a to some $b \in N$ with the same properties, and with $w(\beta) = f_\Gamma(\alpha)$.

Step 6. We extend f to all of C.

We know that C/A is immediate. We first use step 2 to extend f to A^h. Then we do this step by step: add an element, go to the henselization, etc.

Let $a \in C \setminus A$. Consider $I = \{v(a - a') \mid a' \in A\}$. Then, reason as in the proof of the quantifier elimination for ACVF in $\mathcal{L}_{div}$: the set I is an initial segment of $\Gamma_C = v(A^\times)$ (by steps 4 and 5). Again, I need an additional lemma (see below): for any $P(T) \in A[T]$, the value $v(P(a'))$ becomes constant for all a' such that $v(a - a') > \alpha_0$, some $\alpha_0 \in I$. Thus, the same is true on the other side: if $b \in N$ is such that for all $a' \in A$, $w(b - f(a')) = f_\Gamma(v(a - a'))$, then the isomorphism $A(a) \to B(b)$ which extends f and sends a to b, respects the valuation. As in Step 3, it also respects $\underline{\mathrm{ac}}$.

Lemma 7. *Let $(E, v) \subset (F, w)$ be valued field of characteristic 0, with F Henselian. If the characteristic of the residue field is $p > 0$, then assume in addition that the value group of F has a smallest positive element, 1, which is also the smallest element of E, and that $v(p)$ is a finite multiple of 1.*

(1) *Let $u \in F$ be such that $w(u) > w(n)$. Then there is $v \in F$ such that $v^n = (1 + u)$.*

(2) *Let $\gamma \in w(F^\times)$, and assume that there is some $n > 0$ such that $n\gamma \in v(E^\times)$, and that n is minimal such. Then there is some $a \in F$ with $v(a) = \gamma$ and $a^n \in E$.*