Course of June 15

Recall that a basis of the topology on $\mathbb{Q}_p$ is given by the open sets $B(a,\gamma) = \{b \mid v(b-a) > \gamma\}$, which are in fact open and closed. As $\mathbb{Z}_p$ is Hausdorff compact (it is the closed subset of $\prod_{n\in\mathbb{N}} \mathbb{Z}/p^n\mathbb{Z}$ consisting of sequences $(a_n)_n$, with $0 \le a_n < p^n$, and $a_{n+1} \equiv a_n$ modulo p^n.) So $\mathbb{Q}_p$ is locally compact, and can be equipped with a Haar measure, on the σ-algebra generated by the basic open sets. It is normalised by setting $\mu(\mathbb{Z}_p) = 1$; the fact that is stable under translation, i.e., that if B is in the σ-algebra, then $\mu(a+B) = \mu(B)$, means that it is uniquely defined on the basic open sets, and therefore on $\mathbb{Q}_p$. Indeed, $\mu(B(a,\gamma)) = \mu(B(0,\gamma)) = \mu(p^{\gamma+1}\mathbb{Z}_p) = p^{-\gamma-1}$ for $\gamma \in \mathbb{Z}$.
One also considers the product measure on $\mathbb{Q}_p^n$ (which is a Haar measure).
A valued field (K,v) whose value group is a subgroup of $\mathbb{R}$ can also be equipped with an *absolute value*. Choose a real r with $0 < r < 1$, and define $|x|_v = r^{v(x)}$. So the elements of $\mathcal{O}_v$ are the elements which satisfy $|x|_v \le 1$; and define $|0|_v = 0$. For $\mathbb{Q}_p$ it is customary to choose $r = p^{-1}$. For finite extensions of $\mathbb{Q}_p$ there are other "classical" choices for r. Note that infinite algebraic extensions of $\mathbb{Q}_p$ are not locally compact.

If $f : K^n \to K$, one can then consider $\int_B |f(x)|_p d\mu$. If B is contained in a compact (whence $\mu(B) < \infty$) and $|f(x)|_p$ is bounded on B, then this integral, if defined, will be finite.

The two Poincaré series

Let $f_1(X), \dots, f_r(X) \in \mathbb{Z}_p[X]$ (X an m-tuple). If $n \in \mathbb{N}^{>0}$, let

$$\tilde{N}_n = \mathrm{Card}\{a \in (\mathbb{Z}/p^n\mathbb{Z})^m \mid \mathbb{Z}/p^n\mathbb{Z} \models f_1(a) = f_2(a) = \cdots = f_r(a) = 0\}$$

and

$$N_n = \mathrm{Card}\{a \in (\mathbb{Z}/p^n\mathbb{Z})^m \mid \exists b \in \mathbb{Z}_p^n\ f_1(b) = f_2(b) = \cdots = f_r(b) = 0 \wedge \pi_n(b) = a\}$$

where π_n is the natural map $\mathbb{Z}_p^m \to (\mathbb{Z}/p^n\mathbb{Z})^m$. We then define

$$\tilde{P}(T) = \sum_{n=0}^{\infty} \tilde{N}_n T^n \quad \text{and} \quad P(T) = \sum_{i=0}^{\infty} N_n T^n.$$

So both series are in $\mathbb{Z}[[T]]$. Igusa and Meuser have shown that $P(T) \in \mathbb{Q}(T)$, and Denef, with another proof, that both of them are. Reference: J. Denef, *The rationality of the Poincaré series associated to the p-adic points on a variety*, Inventiones Math. 77 (1984), 1 – 23.

If $\varphi(x)$ is a formula of the language of rings, define $\tilde{N}_{\varphi,n} = \mathrm{Card}(\{a \in (\mathbb{Z}/p^n\mathbb{Z})^m \mid \mathbb{Z}/p^n\mathbb{Z} \models \varphi(a)\})$ and $N_{\varphi,n}$ the cardinality of the image by π_n of the subset of $\mathbb{Z}_p^m$ defined by φ. Again, one can consider the series

$$\tilde{P}_\varphi(T) = \sum_{n=0}^{\infty} \tilde{N}_{\varphi,n} T^n \quad \text{and} \quad P(T) = \sum_{i=0}^{\infty} N_{\varphi,n} T^n.$$

What makes things work is the following

Lemma 24. *Let $\varphi(x)$ be a formula, $P_\varphi(T)$ as above. Let*

$$D = \{(x,w) \in \mathbb{Z}_p^{m+1} \mid \exists y \in \mathbb{Z}_p^m \ v(x-y) \geq v(w) \wedge \varphi(y)\}.$$

For $s \in \mathbb{R}^{s>0}$, let

$$I(s) = \int_D |w|_p^s |dx||dw|.$$

Then

$$I(s) = \frac{p-1}{p} P_\varphi(p^{-m-1}p^{-s}).$$

The proof can be found in my notes, or in Denef's paper. It allows to compute $P_\varphi(T)$, if one can show that $I(s) = R(p^{-s})$ for some $R(T) \in \mathbb{Q}(T)$. Note that p^{-m-1} is a constant, and one then has $P_\varphi(T) = R(p^{m+1}T)$. A similar trick allows to compute $\tilde{P}_\varphi(T)$: note that $\pi_{v(w)}(z) = 0$ corresponds to $v(x) \geq v(w)$, and find a formula $\psi(x,w)$ (of the language of rings) such that for any $(a,b) \in \mathbb{Z}_p^{m+1}$, $\mathbb{Z}_p/b\mathbb{Z}_p \models \varphi(\pi_{v(b)}(a))$ if and only if $\mathbb{Z}_p \models \psi(a,b)$.

The rationality of all these series will then follow from

Theorem 25. *Let $S \subset \mathbb{Q}_p^m$ a definable set which is contained in a compact, and $h : S \to \mathbb{Q}_p$ a definable function such that $|h(x)|_p$ is bounded on S. Let $e \in \mathbb{N}^{\geq 1}$ be such that for all $x \in S$, $v(h(x)) \in e\mathbb{Z} \cup \{\infty\}$. Then*

$$Z(s) = \int_D |h(x)|^{s/e}|dx|$$

is a rational function of p^{-s} (for $s \in \mathbb{R}^{>0}$). I.e., there is $Q(T) \in \mathbb{Q}(T)$ such that $Z(s) = Q(p^{-s})$ for all $s \in \mathbb{R}^{>0}$.

The proof is long, and uses several ingredients. The one maybe most important, besides the quantifier elimination that we proved (recall: $\mathbb{Q}_p$ eliminates quantifiers in the language $\{+,-,\cdot,0,1,P_n\}_n$), is a cell decomposition theorem, which allows to "prepare" the functions. Also, another lemma which tells you what definable functions $K^m \to \Gamma = \mathbb{Z}$ (the value group of $\mathbb{Q}_p$) look like. (Since we are not unterested in the actual p-adic number $h(x)$, only in its absolute value). Here are the ingredients:

Lemma 26. *Let $S \subset \mathbb{Q}_p^m$ be definable, and $\theta : S \to \Gamma$ is definable. Then there is a finite partition of S into sets, such that on each set A of the partition there are an integer $e \geq 1$, and $f(x), g(x) \in \mathbb{Q}_p[x]$ such that for all $x \in A$,*

$$\theta(x) = \frac{1}{e}(v(f(x)) - v(g(x))).$$

This is not very hard to prove. It works for all the Henselian valued fiels we consider, basically by "stable embededness". What is much harder to prove, is the following:

Theorem 27. *Let $f_i(X,T) \in \mathbb{Q}_p[X,T]$, $i = 1, \dots, r$, T a single variable, and $n > 0$ an integer. Then there is a finite partition of $\mathbb{Q}_p^{m+1}$ into subsets A of the form*

$$\{(a,t) \in \mathbb{Q}_p^{m+1} \mid a \in C,\ v(a_1(a)) \prec_1 v(t - c(a)) \prec_2 v(a_2(a))\}, \tag{1}$$

where $C \subset \mathbb{Q}_p^m$ is definable, $\prec_1$ and $\prec_2$ are $\leq$ or $<$, the functions $a_1(x)$, $a_2(x)$ and $c(x)$ are definable : $\mathbb{Q}_p^m \to \mathbb{Q}_p$, and for all $(a,t) \in A$, and $i = 1, \dots, r$, we have

$$f_i(a,t) = u_i(a,t)^n h_i(a)(t - c(a))^{e_i} \qquad (2),$$

with $v(u_i(a,t)) = 0$, the functions h_i and u_i are definable, and $e_i \in \mathbb{N}$. We allow the left-hand-side condition to be empty, "$v(a_1(a)) = -\infty$".

The hard thing is to get the centers to be all the same. These are manipulations, with many cases, comparing values. And of course it uses (or redoes in a way) quantifer elimination. I will now "sketch" the proof. It is by induction on m, and for $m = 0$ there is nothing to prove. We assume the result proved for m, try to prove it for $m + 1$. First of all, by the Lemma, we may replace our definable function $|h(x,t)|_p^e$ by $|g_1(x)/g_2(x)|_p^{1/e'}$. Second, we look at S. By qe it can be expressed as a finite disjoint union of sets B defined by formulas of the form

$$x \in C \wedge \bigwedge_i f'_i(x,t) = 0 \wedge \bigwedge_i P_n^*(f_i(x,t))$$

where C is definable, the f'_i and f_i are polynomials over $\mathbb{Q}_p$, and $P_n*(x) \iff P_n(x) \wedge x \neq 0$. So that $x \neq 0 \iff P_n^*(x^n)$ holds. We are going to compute $\int_C (\int_{(x,t)\in B} |g_1(x,t)/g_2(x,t)|_p^{s/e'} |dt|)|dx|$. We may delete from S a set of measure 0, and therefore assume that each of the sets in our partition involves no equality, is of the form

$$x \in C \wedge \bigwedge_i P_n^*(f_i(x,t))$$

We then apply the cell decomposition theorem to the functions $f_1, \dots, f_r, g_1, g_2$ to get that on each cell A of the decomposition, we have

$$|g(a,t)|_p^{1/e} = |g_0(a)|_p^{1/e'} |t - c(a)|_p^{\nu/e'}, \tag{3}$$

where $\nu \in \mathbb{Z}$, and $g_0(x)$ is definable. The problem is that $A \cap B$ is probably not in the nice shape, so one needs to work a little further, and partition $A \cap B$ further to finally reach a finite partition, consisting of definable sets T of the form

$$T = \{(x,t) \in \mathbb{Q}_p^{m+1} \mid x \in D, v(a_1(x)) \prec_1 v(t - c(x)) \prec_2 v(a_2(x)), P_n^*(k^{-1}(t - c(x))\} \tag{5}$$

where $D \subset \mathbb{Q}_p^m$ is definable. But ... the proof and computation still takes at least one page. One finally gets

$$Z_T(s) = M \sum_{\ell \equiv v(k) \mod n} p^{-\ell(s\nu/e'+1)} \int_{F(\ell)} |g_0(x)|^{s/e'} |dx|, \tag{9}$$

where $F(\ell)$ est l'ensemble $\{x \in D,\ v(a_1(x)) \prec_1 \ell \prec_2 v(a_2(x))\}$. And, maybe by refining further, one shows that the values of ℓ are bounded below (by a function depending on x), so that using induction, we have computed the integral.