More topology on $\mathbb{O}$-minimal groups.

Let $(\mathbb{R}, +, -, 0, 1, <, \ldots)$ be $\mathbb{O}$-minimal, with $(\mathbb{R}, +, -, 0)$ an abelian group, and $1 > 0$.

We saw earlier two properties:

1. **Definable choice:** (1) If $S \subseteq \mathbb{R}^{m+n}$ is definable, then there is a definable $f: \pi^{-1}(S) \to \mathbb{R}^m$ such that $\Gamma(f) \subseteq S$.

2. **Definable selection:** (2) Each definable order relation on a definable set has a definable set of representatives.

**Curve selection:** If $a \in cl(X) \setminus X$, $X$ definable, then there is a definable continuous $f: (0, \varepsilon) \to X$ for some $\varepsilon > 0$, such that $\lim_{t \to 0} \gamma(t) = a$.

Both these results are false in the $\mathbb{O}$-minimal structure $(\mathbb{R}, <)$: no canonical way of choosing an element between $a$ & $b$ when $a < b$.

**Lemma:** Let $C \subseteq \mathbb{R}^m$ be a bounded cell, $\pi: \mathbb{R}^m \to \mathbb{R}^{m-1}$ the projection on the first $m-1$ coordinates. Then $\pi(cl(C)) = cl(\pi(C))$.

**Proof:** $D = \pi^{-1}(C)$. Assume $C = (f, g)_D$, where $f: D \to \mathbb{R}$ is continuous and $f < g$ on $D$. By continuity of $\pi$, we have $\pi(cl(C)) = cl(\pi(C))$. Let $a \in cl(\pi(C)) \setminus \pi(C)$. Then there is a continuous definable map $\gamma: (0, \varepsilon) \to \pi(C)$ such that $\lim_{t \to 0} \gamma(t) = a$.

Since $C$ is bounded, there is $r > 0$ such that $-r < f(x) < g(x) < r$ for all $x \in \pi(C)$. Define $\lambda: (0, \varepsilon) \to \mathbb{R}$ by $\lambda(t) = \frac{1}{2} (t + \gamma(t) + g(\gamma(t)))$.

Then $-r < \lambda(t) < r$, and by the monotonicity theorem, there is $S \subseteq \mathbb{R}$ such that $\lim_{t \to 0} \lambda(t) = S$. (Use that $\lambda$ is continuous on some $(0, \varepsilon)$.) So $\lim_{t \to 0} (\gamma(t), \lambda(t))$ is a continuous function $(0, \varepsilon) \to C$ with limit $(a, S)$ as $t \to 0$. So $(a, S) \in cl(C)$.

If $C$ was not bounded, we could have had $\lim_{t \to 0} \lambda(t) = \pm \infty$. 

(Continued...
If $C = \Gamma(f)$, by boundedness, if $a \in \text{cl}(\pi C)$, and $f$ as before, then we get that ($a, \lim_{t \to 0} f \circ g(t)$) $\in \text{cl}(C)$.

Note $f: X \to Y$ continuous, $Y$ Hausdorff imply $\Gamma(f)$ closed in $X \times Y$.

**Lemma** Let $X \subset \mathbb{R}^m$ be closed and bounded, $f: X \to \mathbb{R}^n$ be definable and continuous. Then $\Gamma(f)$ is bounded in $\mathbb{R}^n$.

**Proof** Suppose we may assume $n = 1$.

Suppose by contradiction that for every $t \in \mathbb{R}$ there is some $x \in X$ with $|f(x)| > t$. By definable choice there is a definable map $g: \mathbb{R}^+ \to X$ such that $|f \circ g(t)| > t$ for all $t \in \mathbb{R}^+$. But $X$ is bounded, and by monotonicity, there is $r > 0$ at $g$ is continuous on $(r, +\infty)$.

So $\lim_{t \to +\infty} g(x) \in X$, and because $X$ is closed, $e X$.

This contradict the fact that $\lim_{t \to +\infty} |f \circ g(t)| = +\infty$, since $f$ is continuous on $X$.

**Proposition** Let $f: X \to \mathbb{R}^m$, $X \subset \mathbb{R}^m$, be definable and continuous, and assume that $X$ is closed and bounded. Then $f(X)$ is also closed and bounded.

**Proof** We already know that $f(X)$ is bounded, we need to show it is closed. Let $Y = \{f(\pi(x)), x \in X \}$ and write it as $Y = \bigcup_{i=1}^{k} \pi(C_i)$ where the $C_i$'s are cells. As $Y$ is closed, we have $Y = \text{cl}(\pi(C_1)) \cup \ldots \cup \text{cl}(\pi(C_k))$.

As $Y$ is bounded so are the $C_i$'s. If $\pi: \mathbb{R}^{m+n} \to \mathbb{R}^n$ is the projection on the first $n$ coordinate, then $\text{cl}(\pi(C_i)) = \pi \text{cl}(C_i)$ [use the lemma with $m = 1$, induction on $m$].

Hence $\pi Y = \pi(\text{cl}(C_1) \cup \ldots \cup \text{cl}(C_k)) = \text{cl}(\pi C_1) \cup \ldots \cup \text{cl}(\pi C_k)$ is closed. And $\pi(Y) = f(X)$. 


Corollary \( f : X \rightarrow \mathbb{R} \) continuous, definable, \( X \) closed, bounded non-empty. Then \( f \) attains its minimum and its maximum value.

Corollary \( f : X \rightarrow \mathbb{R}^n \) a continuous, injective definable map on a closed, bounded \( X \subseteq \mathbb{R}^m \), then \( f \) is a homeomorphism between \( X \) and \( f(X) \).

Indeed \( f \) maps closed, definable subsets of \( X \) to closed, definable subsets of \( X \), and therefore maps open, definable subsets of \( X \) to open, definable subsets of \( X \). But every open subset of \( X \) is a union of definable open subsets, hence \( f \) is open.

Cor: \( f : X \rightarrow \mathbb{R}^n \) continuous, definable, \( X \) closed, bounded \( \subseteq \mathbb{R}^m \), let \( Y = f(X) \). Then

(a) A definable set \( S \subseteq Y \) is closed iff \( f^{-1}(S) \) is closed.

(b) A definable map \( g : Y \rightarrow \mathbb{R}^p \) is continuous if and only if \( g \circ f : X \rightarrow \mathbb{R}^p \) is continuous.

Clean.

Proposition: Let \( X \subseteq \mathbb{R}^m \) be closed, bounded and definable, let \( A \subseteq X \times Y \subseteq \mathbb{R}^m \times \mathbb{R}^n \) definable. Then the projection map \( q : X \times Y \rightarrow Y \) maps definable, closed subsets of \( X \times Y \) onto definable, closed subsets of \( Y \).

Let \( A \subseteq X \times Y \) be definable and closed in \( X \times Y \), let \( y \in cl_Y(q(A)) \). So for each \( t > 0 \) there is \( a \in A \) such that \( |q(a) - y| < t \).

Notation: \(|(x_1, \ldots, x_n)| = \sup_{i} x_i^{+} \quad i = 1, \ldots, n\).
By definable choice + monotonicity, there is a definable continuous map \( \alpha : (0, \varepsilon) \to A \) such that
\[
|q \circ \alpha(t) - y| < t \text{ for all } t \in (0, \varepsilon) .
\]
Write \( \alpha(t) = (\beta(t), \gamma(t)) \). Then \( \lim_{t \to 0} \gamma(t) = y \).

As \( X \) is bounded it follows that \( \lim_{t \to 0} \beta(t) \) exists in \( R^n \),
and is in \( X \) because \( X \) is closed. So \( (x, y) \in X \times Y \),
\( \lim_{t \to 0} \alpha(t) = (x, y) \), and therefore \( (x, y) \in A \) because \( A \) is closed. So \( y \in q(A) \).

**Definable paths**: \((R^+, 0, 1, < ....)\)

Let \( X \subseteq R^n \). A definable path in \( X \) is a definable continuous map \( \gamma : [a, b] \to X \subseteq R^n \), where \( a, b \in R, a < b \).

We say that \( \gamma \) connects the points \( \gamma(a) \) and \( \gamma(b) \).

If \( \gamma : [a, b] \to X \) and \( \delta : [b, c] \to X \) are definable paths with \( \gamma(b) = \delta(b) \), then we may concatenate them.

To obtain \( \gamma \circ \delta : [a, c] \to X \), a path as well.

**Proposition**: Let \( X \subseteq R^n \) be definably connected (i.e., not the union of two disjoint non-empty definable open sets). Then any two points, can be connected by a definable path in \( X \).

**Pf**: Assume first that \( X \) is a cell. The proof is by induction on \( m \), and for \( m = 1 \), this is clear. For \( m > 1 \), let \( \pi \) be the projection on the first \( m-1 \) coordinates. \( D = \pi(X) \).

If \( X = \Gamma(f) \) then this is clear, as \( D \) is also connected.
If $X = (f,g)_D$, let $(y,r)$ and $(g,s)$ be two points in $X$, with $y,s \in D$. We will assume $f$ and $g$ take their values in $R$.

Let $\gamma_1$ be a path connecting $(y,r)$ to $(y, f(y) + g(y)/2)$.
If $r \leq s$, let $\gamma_2$ be the path between $(y, f(y) + g(y)/2)$ and $(s, f(s) + g(s)/2)$ given by
$$(s(t)(f \circ s(t) + g \circ s(t))/2),$$
where $s$ is a path connecting $y$ to $s$. (If $y = s$, do nothing). Finally, let $\gamma_3$ be a path connecting $(s, g(s) + f(s)/2)$ to $(s,s)$. Take $\gamma = \gamma_1 \circ \gamma_2 \circ \gamma_3$.

Using the group law, we may assume that the end point of $\text{dom}(\gamma_i)$ is the starting point of $\text{dom}(\gamma_{i+1})$ for $i = 1,2$.

In the general case, we take a decomposition of $X$ into cells $C_1, \ldots, C_k$, such that for $1 \leq i < k$, either $C_i \cap \text{cl}(C_{i+1}) = \emptyset$, or $C_{i+1} \cap \text{cl}(C_i) = \emptyset$.

Let $\mathcal{I} = \{1,\ldots,k\}$ be maximal such that any two points of $\bigcup_{i \in \mathcal{I}} C_i$ can be connected by a path. We claim that $\mathcal{I} = \{1,\ldots,k\}$. Otherwise, let $C_\mathcal{I} = \bigcup_{i \in \mathcal{I}} C_i$.

**Case 1.** $\text{cl}(C_\mathcal{I}) \cap D_\mathcal{I} = \emptyset$.

Say $a \in \text{cl}(C_\mathcal{I}) \cap C_j$, $j \not\in \mathcal{I}$. Let $b$ be a point in $C_j$.

If a point in $C_j$, then $a$ and $c$ are connected by a path, and so are $a$ and $b$: since $b \in \text{cl}(C_\mathcal{I})$, there is $\gamma : (0,\varepsilon) \to C_\mathcal{I}$, with $\lim_{t \to 0} \gamma(t) = b$. Consequently

**Case 2.** $\text{cl}(C_\mathcal{I}) \cap D_\mathcal{I} = \emptyset$. So $C_\mathcal{I}$ is closed in $X$, and therefore $D_\mathcal{I}$ is open in $X$, but not closed.

Repeating the procedure with any $i \not\in \mathcal{I}$, we may
write $X = \bigcup_{i \in I} C_i$ as a disjoint union of $C_{ij}$, $I_j \subseteq \{1, \ldots, k\}$, each $I_j$ maximal with the property that $C_{ij} = \bigcup_{i \in I_j} C_i$ is "part connected". So by the above, each $C_{ij}$ is closed, and therefore also open, which contradicts the fact that $X$ was connected.

Corollary: Let $X$ and $Y \subseteq \mathbb{R}^n$ be definable, with $X$ definably connected, $X \cap \text{bd}(Y) = \emptyset$. Then either $X \subseteq Y$ or $X \cap Y = \emptyset$.

Proof: Else, we can find $x \in X \setminus Y$, $y \in X \cap Y$.
Let $\gamma: [a, b] \rightarrow \mathbb{R}^n$ connect $x$ and $y$ inside $X$.
Let $c = \inf \{t \in [a, b] \mid \gamma(t) \in Y\}$. Then $\gamma(c) \in X \cap \text{bd}(Y)$.

Some separation properties.

Lemma: Let $A \subseteq \mathbb{R}^m$ definable, with $A$ closed in $B$.
Then there is a definable function $f: B \rightarrow [0, 1]$ with $A = f^{-1}(0)$.

Proof: We assume $A \neq \emptyset$. Let $d(x, A) = \inf \{\|x - a\| : a \in A\}$ and define $f(x) = \min\{1, d(x, A)\}$.
Since $A$ is closed, $A = f^{-1}(0)$.

Lemma: Let $A_1$ and $A_2$ be disjoint definable closed subsets of the definable set $B \subseteq \mathbb{R}^m$. Then there are disjoint definable open subsets $U_1$, $U_2$ of $B$ with $A_i \subseteq U_i$ for $i = 1, 2$.

Proof: Let $f_1, f_2: B \rightarrow [0, 1]$ be continuous with $A_i = f_i^{-1}(0)$.
Let $U_1 = \{x \in B \mid f_1(x) < f_2(x)\}$, $U_2 = \{x \in B \mid f_2(x) < f_1(x)\}$.
Corollary (Shrinking of open coverings)

Let the definable set \( B \subseteq \mathbb{R}^m \) be the union of \( U_1, \ldots, U_n \), where the \( U_i \)'s are definable open \( \mu^B \). Then \( B = V_1 \cup \cdots \cup V_m \), where the \( V_i \)'s are definable open, and \( \text{cl}_B(V_i) \subseteq U_i \).

Proof: Assume \( V_1, \ldots, V_k \) have been defined s.t. \( V_1 \cup \cdots \cup V_k \cup V_{k+1} \cup \cdots \cup V_m = B \). Apply the previous lemma to

\[
A_k = B \setminus U_{k+1} \\
A_2 = B \setminus \left( \bigcup_{i=1}^{k} V_i \cup \bigcup_{j=k+2}^m U_j \right)
\]

\( A_k \) is open \( W_1, W_2 \), \( W_1 \cap W_2 = \emptyset \), \( A_i \subseteq W_i \).

Let \( V_{k+1} = B \setminus W_1 \); this is closed, contained in the open set \( U_{k+1} \). We know that \( V_{k+1} \cup (B \setminus W_2) = B \).

Hence, as \( (B \setminus W_2) \) is closed, \( \text{int}_B V_{k+1} \cup B \setminus W_2 = B \).

Here is another application of the definable choice. Shrinking, but impractical unless you know what are the definable functions.

**Def.** Let \((R, \ldots)\) be a structure.

We consider the set \( \mathcal{F} \) of all \( \mu \)-definable functions \( X \to R, X \subseteq R^m \), \( m \geq 0 \). By \( \mu \)-definable, I mean that the graph of the function is defined by a formula \( \phi(x, y) \) with no parameters (other than the constants of the language).

We define \( \text{dcl}(A) \) to be the closure of \( A \) under the elements of \( \mathcal{F} \) (restricted to \( A^m \)). So in particular \( A \) will be a substructure of \( R \).

**Exercise.** Show that \( \text{dcl}(\text{dcl}(A)) = \text{dcl}(A) \).

**Note.** One does not need any hypothesis on \( R \) to make their definition; it is valid in any structure.
Recall Tarski's criterion: Let $M,N$ be structures in a language $L$. To show $M \preceq N$ (i.e., every formula $\varphi(\bar{a})$, if $\bar{a} \in M$, then $M \models \varphi(\bar{a})$ if and only if $N \models \varphi(\bar{a})$) is without parameters, it suffices to show that whenever $\varphi(\bar{x},y)$ is a formula (without parameters), $\bar{a} \in M$, and $b \in N$ is such that

$$N \models \varphi(\bar{a},b)$$

then there is $\bar{d} \in M$ such that

$$N \models \varphi(\bar{a},b)$$

Then let $(R,+,\cdot,0,1,\leq,\ldots)$ be $\omega$-minimal, let $A \subseteq R$. Then $\text{dcl}(A) \subseteq R$.

Proof: Use definable choice and Tarski's criterion.