Finiteness lemma

Let $A \subseteq \mathbb{R}^2$ be definable, and suppose that for each $x \in \mathbb{R}$, the fiber $A_x = \{ y \in \mathbb{R} \mid (x, y) \in A \}$ is finite. Then there is $N \in \mathbb{N}$ such that $|A_x| \leq N$ for all $x \in \mathbb{R}$.

Proof. A point $(a, b) \in \mathbb{R}^2$ is “normal” if there is a box $I \times J$ around $(a, b)$ s.t. either $I \times J \cap A = \emptyset$ or $(a, b) \in A$ and $I \times J \cap A = \Gamma(f)$, for some continuous function $f$.

The sets $\{ (a, b) \in \mathbb{R}^2 \mid (a, b) \text{ normal} \}$ is definable.

Say $(a, -\infty)$ is “normal” if there is a box $I \times J$ with $I \times J \cap A = \emptyset$, $a \in I$, and $J = (-\infty, b)$ for some $b \in \mathbb{R}$.

Finally $(a, +\infty)$ is normal if there is a box $I \times J$ with $I \times J \cap A = \emptyset$, $a \in I$, and $J = (b, +\infty)$ for some $b$. Let $(a, -\infty)$ normal and $(a, +\infty)$ normal.

Define functions $f_1, f_2, \ldots$ by $(n \in \mathbb{N})$

$$\text{dom } f_n = \{ x \in \mathbb{R} \mid |A_x| \geq n \}$$

$$f_n(x) = \text{ the } n\text{-th element of } A_x$$

Note: For all $a \in \mathbb{R}$ there is $n \in \mathbb{N}$ s.t. $a \notin \text{dom}(f_n)$.

Let $a \in \mathbb{R}$, take $n^0$ maximal s.t. $f_1, \ldots, f_n$ are defined and continuous on an interval containing $a$.

Say $a$ is good if $a \notin \text{cl} \text{(dom } f_{n+1})$

bad if $a \in \text{cl} \text{(dom } f_{n+1})$.

Let $B = \text{ set of bad points } \cup \mathcal{G} = \text{ set of good points}$.
If \( a \) is good, with \( n \) as above, then \( \text{dom} \ f_{n+1} \) is disjoint from some open interval containing \( a \), and on which \( f_1, \ldots, f_n \) are defined and continuous.

In particular,

(2) \( |\mathcal{A}_a| \) is constant on an interval around \( a \).

(3) \((a, b)\) is normal \( \forall b \in \mathbb{R}_+ \times \mathbb{R}_+ \).

(note that the graphs of \( f_1, \ldots, f_n \) must be disjoint around \( a \)).

We want to show \( A, B \) are definable.

A priori infinite union of def. sets.

We will show:

(4) If \( a \) is bad, there is at least \( b \in \mathbb{R}_+ \times \mathbb{R}_+ \) such that \((a, b)\) is not normal.

Let \( \lambda(a, -) = \lim_{x \to a^-} f_{n+1}(x) \) if \( f_{n+1} \) is defined on some \( t, a \),

\( \lambda(a, 0) = f_{n+1}(a) \) if \( a \in \text{dom}(f_{n+1}) \)

\( = +\infty \) otherwise

\( \lambda(a, +) = \lim_{x \to a^+} f_{n+1}(x) \) if \( f_{n+1} \) is defined on some \( a, t \)

\( = +\infty \) otherwise

Take \( \beta(a) = b = \min \{ \lambda(a, -), \lambda(a, 0), \lambda(a, +) \} \).

\((a, b)\) not normal: if \( I \times J \) contains \((a, b)\) then \( (I \times J) \cap A \) is not the graph of a continuous function and is not empty.

So this means that it must intersect \( \gamma(f_m) \) for some \( m > n \). The smallest such \( b \) is \( m = n+1 \).
When \( (a, -\infty) \) not normal: if \( I \) is an interval containing \( a \), then \( \lim_{x \to a^-} f(x) = -\infty \), or \( \lim_{x \to a^+} f(x) = -\infty \).

If \( B \) is finite, say \( B = \{a_1, \ldots, a_k\} \), with \( a_1 < \ldots < a_k \), \( a_0 = -\infty, a_{k+1} = +\infty \). Then

**Claim**: \( |A_x| \) is constant on each \((a_i, a_{i+1})\).

Use (2): \( \{x \in (a_i, a_{i+1}) \mid |A_x| = m \} \) is open.

Hence only one of them is non-empty.

Suppose \( B \) infinite. We will derive a contradiction.

Let \( B_- = \{a \in B \mid \exists y \ (y < \beta(a) \land (a, y) \in A)\} \)

\( B_+ = \{a \in B \mid \exists y \ (y > \beta(a) \land (a, y) \in A)\} \).

\( \beta_- : B_- \to R, \ \ \beta_+ : B_+ \to R \)

\( \beta_-(a) = \max \{y \mid y < \beta(a), (a, y) \in A\} \)

\( \beta_+(a) = \min \{y \mid y > \beta(a), (a, y) \in A\} \)

\( B \) is infinite; so one of \( B_- \cap B_+, (B_- \setminus B_+) \), \( B_+ \setminus B_- \), \( B \setminus (B_+ \cup B_-) \) is infinite.

\( B_- \cap B_+ \) infinite.

By (4) the function \( \beta \) is definable: if \( a \in B \), \( \beta(a) = -\infty \) (possibly with \( \beta(a) = -\infty \) if \((a, b)\) normal for all \( b \).

And so are \( \beta_+ \), \( \beta_- \) on their domain.

We have \( \beta_-(a) < \beta(a) < \beta_+(a) \).

\( B_- \cap B_+ \) contains an \( I \), on which \( \beta_-, \beta, \beta_+ \) are defined & continuous.

\( I = \{x \in I \mid (x, \beta(x)) \in A\} \cup \{x \in X \mid (x, \beta(x)) \notin A\} \)
So one of these sets contains an interval $J$, and so either $\Gamma(\beta|J) \cap A = \emptyset$ or $\Gamma(\beta|J) \subseteq A$. But then $\Gamma(\beta|J)$ consists of normal points because $\beta_-, \beta_+, \beta_0$ are continuous on $J$. But .... $\alpha(\beta(\alpha))$ is never normal. Contradiction.

**Definition.** Let $(R, <, ...)$ be $0$-minimal.

(Simple version). We define by induction cells and their dimension. These are nice definable subsets of $\mathbb{R}^n$.

(i) $(n = 1)$. $X \subseteq R$ is a cell if

- $X = \{a, b\}$, then $\dim(X) = 0$.
- $X = (a, b)$ for some $a, b \in R \cup \{+\infty\}$. Then $\dim X = 1$.

(ii) We assume that $X \subseteq \mathbb{R}^n$ is a cell, of dimension $k$. There are two kinds of cells in $\mathbb{R}^{n+1}$ which are associated to $X$:

- Let $f : X \to R$ be definable and continuous. Then $X_1 = \text{graph}(f)$ is a cell in $\mathbb{R}^{n+1}$, $\dim X_1 = k$.

- Let $f, g : X \to R$ be definable, continuous functions and assume that for all $\bar{x} \in X$, $f(\bar{x}) < g(\bar{x})$. Then $X_2 = \{(\bar{x}, y) \in X \times R \mid f(\bar{x}) < y < g(\bar{x})\}$ is a cell, and $\dim X_2 = k + 1$. We allow $f = -\infty$ and $g = +\infty$. Nothing else is a cell. Note that they are definable (with parameters) and that if $X \subseteq \mathbb{R}^n$ is a cell and has dimension $n$, then $X$ is open. And $\dim X = 0$ iff $X$ is a singleton.

When $n = 0$, we consider $\mathbb{R}^0$ as a $\emptyset$-cell.
Corollary to the finiteness theorem

Let $A \subseteq \mathbb{R}^2$ be definable, with $A_x$ finite for every $x \in \mathbb{R}$. Then there are $a_1 < \ldots < a_k$ in $\mathbb{R}$ such that the intersection of $A$ with each $(a_i, a_{i+1}) \times \mathbb{R}$ has the form $\text{graph}(f_{i+1}) \cap \text{graph}(f_i)$, with $f_{i+1}(x) < \ldots < f_i(x)$ for $x \in (a_i, a_{i+1})$, with $f_{ij}$ definable, continuous, (strictly monotone or constant).
Remark. Van den Dries has a more precise notion of dimension, which keeps track of the various steps of the construction. At each stage of the construction, one attaches a number 0 or 1, giving the dimension of the "fiber". In the notation above, $X_2$ is a 0-cell, $(a,b)$ a 1-cell, $X_1$ a $(i_1, \ldots, i_m, 0)$-cell, $X_2$ a $(i_1, \ldots, i_m, 1)$-cell if $X$ was a $(i_1, \ldots, i_m)$-cell. This amount of precision is sometimes useful, as below:

X

**Remarks**

1. An $(i_1, \ldots, i_m)$-cell has dimension $i_1 + i_2 + \cdots + i_m$.

It is definably homeomorphic to an open cell.

Let $\lambda_1 < \lambda_2 < \cdots < \lambda_k$ be the indices with $\lambda_j = 1$ (so $k = \dim X$), and consider the projection

$$
\pi : \mathbb{R}^n \to \mathbb{R}^k
$$

$$(x_1, \ldots, x_n) \mapsto (x_{\lambda_1}, \ldots, x_{\lambda_k}).$$

One then shows by induction that $\pi|_X$ is a homeomorphism, and that $\pi(X)$ is open.

2. If $X \subset \mathbb{R}^{n+1}$ is a cell, and $\pi : \mathbb{R}^{n+1} \to \mathbb{R}^n$ is the projection on the first $n$ coordinates, then $\pi(X)$ is a cell.

This is obvious by the inductive definition of cells.

**Proof of (1)**. By induction on $n$, for $n = 1$ there is nothing to prove. Assume the result proved for $X$, with projection $\pi$, and let $Y \subset \mathbb{R}^{n+1}$ be a cell, with $\pi(Y) = X$.

If $\dim(Y) = \dim(X)$, then $X$ and $Y$ are homeomorphic via $\pi$, and so the result holds (the projection of the graph of a continuous function onto its domain is a homeomorphism).

If $\dim(Y) = \dim(X) + 1$, let $f < g$ be the two continuous functions involved in the definition of $Y$. Then $f|_X \equiv id$ define a homeomorphism between $Y$ and
The cell $Z = \{(x, y) \mid x \in \pi(Y), f \circ p(x) < y < g \circ p(x)\}$

Here, $p : \pi(Y) \to Y$ coincides with $\pi^{-1}$.

Because $p$ is a homeomorphism, $f, g$ are continuous, $Z$ is indeed a cell.

Proposition Every cell is definably connected.

A set $X \subseteq \mathbb{R}^n$ is definably connected if $X$ is definable and cannot be written as $X = X_1 \cup X_2$ where $X_1 \cap X_2 = \emptyset$, $X_1, X_2$ are definable and open with respect to the induced topology.

Proof: Clear when $n = 1$.

Induction for $n > 1$: assume the result proved for the cell $X \subseteq \mathbb{R}^n$. If $X \subseteq X' \times \mathbb{R}$ is the graph of a function this is clear; if $Y \subseteq X' \times \mathbb{R}$ has dimension $\dim(X) + 1$, then each fiber is a (non-empty) open interval, and is definably connected.

Remarks

1. The union of finitely many non-open cells has empty interior. In fact, if $X \subseteq \mathbb{R}^m$ is a cell then $\dim(X) = m$ implies $X$ has non-empty interior.

2. A cell $X$ is open in its (topo.) closure $\text{cl}(X)$.

Proof. By induction on $n$ if $X \subseteq \mathbb{R}^n$. For $n = 1$, clear.

If $p(X) = B$ ($p : \mathbb{R}^{n+1} \to \mathbb{R}^n$), $B$ open in its closure, i.e. $\text{cl}(B) \setminus B$ closed. If $X = \text{graph}(f)$ then $\text{cl}(X) \subseteq (\text{cl}(B) \setminus B) \times \mathbb{R}$, which is closed, so $X$ open in $\mathbb{R}^{n+1}$.

If $X = (f, g)$, with $f < g : B \to \mathbb{R}$ univ.
the \( cl(X) \subseteq X \cup \text{graph}(f) \cup \text{graph}(g) \cup ((cl(B) \setminus A) \times R) \)

since any point in \( cl(B) \times R \) not in that set is contained in a set of the form \( U \times I \), with \( I \) an open interval, \( U \times I \cap cl(X) = \emptyset \), or \( U \times I \subseteq (X) \), \( U \) open in \( cl(B) \).

**Definition:** A decomposition of \( \mathbb{R}^m \) is a partition of \( \mathbb{R}^m \) into finitely many cells, satisfying some additional conditions:

1. A decomposition of \( R^1 = \mathbb{R} \) is a collection
   \[ \{(-\infty, a_1), (a_1, a_2), \ldots, (a_k, +\infty), \{a_1\}, \{a_2\}, \ldots, \{a_k\} \} \]
   where \( a_1 < a_2 < \ldots < a_k \) are elements of \( \mathbb{R} \).
2. A decomposition of \( \mathbb{R}^{m+1} \) is a finite partition of \( \mathbb{R}^{m+1} \) into cells \( A \) such the set of projections \( \pi(A) \) is a decomposition of \( \mathbb{R}^m \) (\( \pi : \mathbb{R}^{m+1} \rightarrow \mathbb{R}^m \) standard proj.)

So typically in (ii), we have the following:

A decomposition \( A_1, \ldots, A_k \) of \( \mathbb{R}^m \), for each \( 1 \leq i \leq k \),

continuous, definable functions \( f_1 < \ldots < f_{m(i)} : A_i \rightarrow \mathbb{R} \)

Consider \( D_i : \{(-\infty, f_1), (f_1, f_2), \ldots, (f_{m(i)}, +\infty), \Gamma(f_1) \ldots, \Gamma(f_{m(i)}) \} \)

is a decomposition of \( A_i \times \mathbb{R} \) into cells.

And \( \mathcal{D} = \bigcup D_i \) is a decomposition of \( \mathbb{R}^{m+1} \).

**Def:** A decomposition \( \mathcal{D} \) of \( \mathbb{R}^m \) partitions a (definable) set \( S \subseteq \mathbb{R}^m \) if and only if each cell in \( \mathcal{D} \) is either contained in \( S \) or disjoint from \( S \).