Recall the definition of a cell $C \subseteq \mathbb{R}^n$: $\dim(C)$ is the maximal $r$ s.t. there is a projection $\pi: \mathbb{R}^n \to \mathbb{R}^r$ s.t. $\pi(C)$ is an open cell.

Alternatively, if $C$ is an $(i_1, \ldots, i_m)$-cell, then $\dim(C) = i_1 + \ldots + i_m$.

Let $X \subseteq \mathbb{R}^n$ be definable. The dimension of $X$, $\dim(X)$, is the maximal dimension of a cell $C \subseteq X$. If $X = \emptyset$, we set $\dim(X) = -\infty$.

**Lemma:** If $A \subseteq \mathbb{R}^m$ is an open cell, and $f: A \to \mathbb{R}^m$ is an injective definable map then $f(A)$ contains an open cell.

*Proof:* Clear if $m = 1$. Assume it holds for values $< m$.

Take a decomposition $\mathcal{D}$ of $\mathbb{R}^m$ which partitions $f(A)$, we get $f(A) = C_1 \cup \ldots \cup C_k$, $C_i$ cells in $\mathbb{R}^m$.

and $A = f^{-1}(C_1) \cup \ldots \cup f^{-1}(C_k)$.

Refining $\mathcal{D}$, we may assume that if $f(C_i) \subseteq A$ then $f|_{f^{-1}(C_i)}$ is continuous. But then $f^{-1}(C_i)$ contains an open box $B_i$, on which $f$ is continuous. The corresponding $C_i$ is a cell, which contains $f(B)$. If we can show that $f(B)$ has dimension $m$ then we will also have $\dim(C_i) = m$, i.e., $C_i$ is open.

Otherwise, if $\dim C_i < m$, there is a definable homeomorphism $\pi$ of $C_i$ with an open cell of $\mathbb{R}^{\dim C_i}$ and we get a continuous injective map $g: B \to \mathbb{R}^{\dim C_i}$.

Write $B = B' \times (a, b)$, let $c \in (a, b)$,

$h: B' \to \mathbb{R}^{m-1}$ defined by $h(x) = g(x, c)$.

By IH, $h(B') \supseteq D$, $D$ an open box in $\mathbb{R}^{m-1}$. 

Let \( y \in D, x \in D' \) with \( f(x) = y \).

By continuity, if \( C' \) is sufficiently close to \( C \), then \( g(x, C') \in D \). This contradicts the injectivity of \( f \).

**Proposition**

1. If \( x \leq y \leq \mu^m \) with \( x, y \) definable, then \( \dim(x) \leq \dim(y) \).

2. If \( x \in \mathbb{R}^m, y \in \mathbb{R}^n \) are definable, and there is a definable bijection \( f: x \to y \), then \( \dim(x) = \dim(y) \).

3. If \( x, y \in \mathbb{R}^m \) are definable, then \( \dim(x \cup y) = \max \{ \dim x, \dim y \} \).

**Proof**

1. It is clear from the definition of \( \dim \).

2. Let \( D \) be a decomposition of \( \mathbb{R}^m \) partitioning \( x \) and \( x \subseteq D \), \( C \subseteq x \) implies \( f|_C \) is a homeomorphism. Let \( C \in D \), \( C \subseteq x \) be of dimension \( d = \dim(x) \).

Let \( \pi: \mathbb{R}^m \to \mathbb{R}^d \) be such that \( f|_C \) is a homeomorphism with an open cell of \( \mathbb{R}^d \). Then \( f|_C \) defines a homeomorphism between the open cell \( D \) and \( f(C) \). Writing \( f(C) \) as a union of cells in \( \mathbb{R}^n \), the proof of the lemma shows that necessarily one of those cells has \( \dim \geq d \). Since there is no injective map from an open box \( B \subseteq D \subseteq \mathbb{R}^d \) into an \( \mathbb{R}^k, k < d \). Hence \( \dim y \geq d \).

3. Let \( d = \dim(x \cup y) \), \( A \subseteq x \cup y \) a cell of dimension \( d \). Let \( p: \mathbb{R}^m \to \mathbb{R}^d \) s.t. \( p|A \) is a homeomorphism.

So \( p(A) = p(A \cap X) \cup p(A \cap Y) \), and one of them
Prop. Let \( S \subseteq \mathbb{R}^m \times \mathbb{R}^n \) be definable. For \( d \in \{-\infty, 0, 1, \ldots, \infty\} \) define
\[
S(d) = \{ a \in \mathbb{R}^m \mid \dim S_a = d \}.
\]
Then \( S(d) \) is definable, and
\[
\dim((S(d) \times \mathbb{R}^n) \cap S) = \dim S(d) + d.
\]

Let \( D \) be a decomposition of \( \mathbb{R}^{m+n} \) partitioning \( S \). Let \( C \in D \), \( \pi C \subseteq \mathbb{R}^m \). If \( C \) is an \((i_1, \ldots, i_{m+n})\)-cell, then \( \pi C \) is an \((i_1, \ldots, i_m)\)-cell, and for every \( a \in \pi C \), \( C_a \) is an \((i_{m+1}, \ldots, i_{m+n})\)-cell. So \( \dim(C) = \dim \pi C + \dim C_a \), \( \forall a \in \pi C \).

If \( A \) is a cell in \( \pi D \), and \( C_1, \ldots, C_k \in D \) are the cells above \( A \), which are contained in \( S \), then
\[
S_A = (C_1)_a \cup \cdots \cup (C_k)_a,
\]
so
\[
\dim S_A = \sup \dim(C_i)_a = \sup \dim C_i - \dim(A)
\]
by \( \ref{prop} \).

Note \( \dim S_A \) ind of \( a \in A \), whence \( A \subseteq S(d) \) if \( \dim S_a = d \).

So \( S(d) = \) union of cells, is definable.

Note \( d = \sup (\dim C_i) - \dim(A) \)
\[
= \dim(\bigcup_{i=1}^{k} C_i) - \dim(A)
\]
\[
= \dim(\bigcup_{a \in A} a \times S_a) - \dim A.
\]

i.e.
\[
\dim(\bigcup_{a \in A} a \times S_a) = \dim A + d
\]
\[
\dim((S(d) \times \mathbb{R}^n) \cap S) = \dim S(d) + d.
\]
\[ \dim S = \max_{0 \leq d \leq n} (\dim S(d) + d) \]

(2) Let \( X \subseteq \mathbb{R}^n \) be definable, \( \vdash X \to \mathbb{R}^m \) a def. map. Then for each \( d \in \{0, \ldots, n\} \), the set \( S_f(d) = \{ a \in \mathbb{R}^m \mid \dim f^{-1}(a) = d \} \) is definable, and
\[ \dim f^{-1}(S_f(d)) = d + \dim S_f(d). \]

(3) \( \dim (A \times B) = \dim(A) + \dim(B) \), \( A, B \) def.

Results on the boundary and the frontier of a definable set \( S \subseteq \mathbb{R}^m \).
\[ \partial S = \text{frontier of } S = \overline{\partial S} \]
\[ \text{bd}(S) = \partial S \setminus \text{int}(S) = \text{boundary of } S. \]

Aim: \( \dim \partial S < \dim S \) if \( S \neq \emptyset \).

This result holds in all 0-minimal structures.
The proof in the general case is quite long.
We will give a shorter proof, which uses a curve selection lemma, a lemma only true if the 0-minimal structure has a group law.
We work in an $\mathcal{O}$-minimal expansion $(\mathbb{R}, +, -, 0, <, \cdots)$ of an ordered abelian group (divisible). We set

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0. \end{cases}$$

**Curve selection**

Let $X \subseteq \mathbb{R}$ be definable and non-empty. We will choose an element in $X$, using the same parameters over which $X$ is defined, and maybe an additional parameter, denoted $1$, which satisfies $1 > 0$.

If $X$ has a least element, then we take this least element. If not, then $X$ can be written as $I \cup Y$, where $I$ is an interval, $Y$ is definable, $Y \cap I = \emptyset$.

- **Case 1:** $I = (-\infty, a)$. Let $a' \in \mathbb{R}$ be maximal such that $(-\infty, a') \subseteq X$, and pick $a' - 1$.
- **Case 2:** $I = (a, b)$, $a, b \in \mathbb{R}$. Pick $\frac{a + b}{2}$.
- **Case 3:** $I = (a, +\infty)$. Let $a' \in \mathbb{R}$ be minimal such that $(a', +\infty) \subseteq X$ and choose $a' + 1$.

**Case 4**

$I = (-\infty, +\infty)$, Pick 0.

---

**Important:** This choice function only depends on $X$, not on the way it is described.

**Proposition (Definable choice)** $(\mathbb{R}, +, -, 0, 1, <, \cdots)$

(1) If $S \subseteq R^{m+n}$ is definable non-empty, $\pi : R^{m+n} \to R^m$. Then there is a definable map $f : \pi(S) \to R^m$ such that $f(f) \subseteq S$. 

---
(2) Each definable equivalence relation on a definable set \( X \) has a definable set of representatives.

\[ \begin{align*}
\text{Pf:} \quad (1) & \text{ When } n = 1, \text{ we saw how to define the function } f. \text{ We use induction on } n: \\
& \text{if } S \subseteq R^{n+m+1}, \text{ consider } \pi_1 : R^{n+m+1} \to R^{n+m}. \\
& \text{We have a definable } f_1 : f_1(\pi_1(S)) \to R, \text{ such that } f_1(\pi_1(S)) \subseteq S. \text{ Then } f_1(\pi_1(S)) \text{ is definable, as well as } \pi_1(\pi_1(S)) \in R^{n+m}. \\
& \text{By IH, there is a definable map } f_2 : \pi_1(\pi_1(S)) \to R^n \text{ such that } f_2(\pi_1(S)) \subseteq \pi_1(\pi_1(S)). \text{ We then define } \\
& f : \pi_1(S) \to R^{n+m+1} \text{ by } f(x) = (f_2(x), x_1, x_2(x)) \text{.}
\end{align*} \]

(2) Consider the set \( \{ e(A) \mid A \text{ an equivalence class } \} \), where \( e \) is a choice function for \( S \subseteq X^2 \to X \), with \( S_a = \text{equivalence class of } a \) for \( a \in X \).

**Theorem (Curve selection lemma).** Let \( S \subseteq R^n \) be definable, and \( b \in cl(S) \). There is a continuous definable map \( \gamma : [0, r) \to S \) for some \( r \in R^{>0} \), such that \( \gamma(0, r) \subseteq S \) and \( \gamma(0, r) = b \).

\[ \text{Pf: Let } X = \{ (t, x) \in R \times R^n \mid x \in S, \sup |x_i - b_i| < t \} \]

\[ (x = (x_1, \ldots, x_n), b = (b_1, \ldots, b_n)) \]

Let \( \pi : R^{1+n} \to R \) be the projection on the first coordinate. Because \( b \in cl(S) \setminus S \), \( \pi(X) = R^{>0} \). Applying the definable choice, we get a definable function \( \gamma : R^{>0} \to S \). By continuity, there is some \( r > 0 \) such that \( \gamma|([0, r)) = \gamma(0, r) \). Extend \( \gamma \) to \([0, r)\) by setting \( \gamma(0) = b \).

i.e., each coordinate function is continuous.
Lemma. Let $A \subseteq \mathbb{R}^m \times \mathbb{R}^p$ be definable. Let

$$M = \{ x \in \mathbb{R}^m \mid \text{cl}(A_x) \neq (\text{cl}(A))_x \}.$$ 

Then $M$ is definable, of dimension $< m$. If $m = 1$, then $M$ is finite.

Proof. We always have $\text{cl}(A_x) \subseteq (\text{cl}(A))_x$, because $(\text{cl}(A))_x$ is closed, contains $A_x$.

Suppose by way of contradiction that $M$ has dimension $m$. Hence it contains an open cell $C$.

For every $x \in C$, we can find a box $B = \prod_{i=1}^p (a_i, b_i)$ such that $B \cap (\text{cl}(A))_x \neq \emptyset$, $B \cap A_x = \emptyset$.

Using definable choice and definable continuity (and shrinking $C$), we may assume that we have

continuous functions $a_i, b_i : C \to \mathbb{R}$,

let $U = \{ (x, y) \in C \times \mathbb{R}^p \mid a_i(x) < y < b_i(x), i = 1 \ldots p \}$. Then $U$ is open in $\mathbb{R}^m \times \mathbb{R}^p$, disjoint from $A$, but intersects $\text{cl}(A)$: this is impossible.

Hence $\dim(M) < m$.

Note: we used the "definable choice", which only holds when we have a group. The result with $m = 1$ holds without this assumption, but the proof is much longer.

Theorem. Let $A \subseteq \mathbb{R}^m$ be definable.

Then $\dim(\text{cl}(A) - A) < \dim(A)$.
The proof is by induction on $n$. It is obvious for $n = 1$: if \( \dim(A) = 0 \) then \( \delta(A) = \emptyset \) and if \( \dim(A) = 1 \), \( \delta(A) \) consists of points or is empty.

We assume $n > 1$, and that the result holds for $n-1$. Let \( \text{cl}_i(A) = \{ x \in \mathbb{R}^n \mid x \in \text{cl}(A \cap \{ y_i = x_i \}) \} \) for $i = 1, \ldots, n$.

**Step 1** \( \dim(\text{cl}(A \setminus A) \leq \sup_i \dim(\text{cl}_i(A) \setminus A) \)

Indeed, \( \text{cl}_i(A) \subseteq \text{cl}(A) \). Apply the previous lemma, and obtain that \( \text{cl}(A \setminus \text{cl}_i(A) \) is contained in a finite union of hyperplanes \( \{ x_i = a_{i,j} \} \) \( j = 1, \ldots, l_i \).

Indeed we are looking at the set \( \{ x_i \in \mathbb{R} \mid \text{cl}(Ax_i) \neq \text{cl}(A)x_i \} \)

which is finite.

Permute the coordinates, we get that for $i = 2, \ldots, n$,

\[
\text{cl}(A) \setminus \text{cl}(A) \leq \bigcup_{j=1}^{l_i} \{ x_i = a_{i,j} \}
\]

Hence \( \text{cl}(A) \setminus \bigcup_{i=1}^{n} \text{cl}_i(A) \) is contained in the finite set of points \( \{ a_{11}, \ldots, a_{1n}, a_{21}, \ldots, a_{2n}, \ldots, a_{nn} \} \times \cdots \times \{ a_{n1}, \ldots, a_{nn} \} \).

In particular, \( \dim(\text{cl}(A \setminus A) \leq \sup_i \dim(\text{cl}_i(A) \setminus A) \)

**Step 2** Show \( \dim(\text{cl}_i(A) \setminus A) < \dim(A) \), for any $i$.

Let $a = (a_1, \ldots, a_n) \in A$. As \( \dim(\{ x_i = a_i \} = n-1 \), the IH gives that either \( \text{cl}(A \setminus \{ x_i = a_i \} \setminus A \setminus \{ x_i = a_i \} \) is empty, or has dimension $< \dim(\{ x_i = a_i \} \). The result then follows from the following exercise:

**Exercise** Let $A \subseteq B \subseteq \mathbb{R}^m$ be definable, $A \neq \emptyset$. Assume that for every $x \in \mathbb{R}^m$, \( \dim(A_x) < \dim(B_x) \), or $A_x = \emptyset$.

Then \( \dim(A) < \dim(B) \).
Indeed, one applies it to \( \text{cl}_i(A) \setminus A \subseteq \text{cl}_i(A) \).

The induction hypothesis tells us that \( \dim \text{cl}_i(A)_{\alpha_i} = \dim(A)_{\alpha_i} \).

**Corollary** Let \( \phi \neq S \subseteq T \subseteq \mathbb{R}^m \) be definable, with \( \dim(S) = \dim(T) \). Then \( S \) has non-empty interior \( \text{int}_T(S) \) in \( T \), and

\[
\dim(S \setminus \text{int}_T(S)) < \dim(S).
\]

By letting \( \text{cl}_T \) denote closure in \( T \), then

\[
S \setminus \text{int}_T(S) = S \cap \overline{T \setminus S} = \text{cl}_T(T \setminus S) \setminus (T \setminus S)
\]

\( a \in S \setminus \text{int}_T(S) \): Here is a small box containing a set \( B \cap T \neq S \), i.e. \( B \cap (T \setminus S) \neq \emptyset \).

By the thm: \( \dim \text{cl}_T(T \setminus S) \setminus (T \setminus S) < \dim(T \setminus S) \leq \dim(S) \).

**Corollary** Let \( S \subseteq \mathbb{R}^m \) be definable. Then

\[
\dim(\text{cl}(S) \setminus \text{int}(S)) < m.
\]

**Proof** If \( \dim(S) < m \), then we know \( \dim(\text{cl}(S)) < m \), so this is clear. Assume \( \dim(S) = m \).

So \( \dim(\text{cl}(S) \setminus S) < m \), \( \dim(S \setminus \text{int}_{\mathbb{R}^m}(S)) < m \),

which gives the conclusion.