SOME REMARKS ABOUT THE POSSIBLE BLOW-UP FOR THE NAVIER-STOKES EQUATIONS

JEAN-YVES CHEMIN, ISABELLE GALLAGHER, AND PING ZHANG

Abstract. In this work we investigate the question of preventing the three-dimensional, incompressible Navier-Stokes equations from developing singularities, by controlling one component of the velocity field only, in space-time scale invariant norms. In particular we prove that it is not possible for one component of the velocity field to tend to 0 too fast near blow up. We also introduce a space "almost" invariant under the action of the scaling such that if one component of the velocity field measured in this space remains small enough, then there is no blow up.

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1. Introduction

The purpose of this paper is the investigation of the possible behaviour of a solution of the incompressible Navier-Stokes equation in $\mathbb{R}^3$ near the (possible) blow up time. Let us recall the form of the incompressible Navier-Stokes equation

$$(NS) \qquad \begin{cases} \partial_t v + \operatorname{div}(v \otimes v) - \Delta v + \nabla p = 0\,, \\ \operatorname{div} v = 0 \quad \text{and} \quad v|_{t=0} = v_0\,, \end{cases}$$

where the unknowns $v = (v^1, v^2, v^3)$ and p stand respectively for the velocity of the fluid and its pressure.

It is well known that the system has two main properties related to its physical origin:

- the scaling invariance, which states that if $v(t,x)$ is a solution on $[0,T] \times \mathbb{R}^3$ then for any positive real number λ, the rescaled vector field $v_\lambda(t,x) \stackrel{\text{def}}{=} \lambda v(\lambda^2 t, \lambda x)$ is also a solution of (NS) on $[0, \lambda^{-2}T] \times \mathbb{R}^3$;
- the dissipation of energy which writes

$$\frac{1}{2}\|v(t)\|_{L^2}^2 + \int_0^t \|\nabla v(t')\|_{L^2}^2 dt' \leq \frac{1}{2}\|v_0\|_{L^2}^2\,. \tag{1.1}$$

The first type of results which describe the behaviour of a (regular) solution just before blow up are those which are a consequence of an existence theorem for initial data in spaces more regular than the scaling. The seminal text [18] of J. Leray already pointed out in 1934 that the life span $T^\star(v_0)$ of the regular solution associated with an initial data in the Sobolev space $H^1(\mathbb{R}^3)$ is greater than $c\|\nabla v_0\|_{L^2}^{-4}$; applying this result with $v(t)$ as an initial data gives immediately that if $T^\star(v_0)$ is finite, then

$$\|\nabla v(t)\|_{L^2}^4 \geq \frac{c}{T^\star(v_0) - t}\,, \quad \text{which implies that} \quad \int_0^{T^\star(v_0)} \|\nabla v(t)\|_{L^2}^4 dt = \infty\,. \tag{1.2}$$

More generally, it is a classical result that for any γ in $]0, 1/2[$ there holds

$$T^\star(v_0) \geq c_\gamma \|v_0\|_{\dot H^{\frac12 + 2\gamma}}^{-\frac{1}{\gamma}} \quad \text{which leads to} \quad \|v(t)\|_{\dot H^{\frac12+2\gamma}} \geq \frac{c_\gamma}{(T^\star(v_0) - t)^\gamma}\,. \tag{1.3}$$

The definition of homogeneous Sobolev spaces is recalled in the Appendix. Let us notice that the above formula is scaling invariant and comes from the resolution of (NS) with a fixed point argument following the Kato method [15] and [28]. Moreover, E. Poulon proved in [25] that if a regular initial data exists, the associate solution of which blows up at a finite time, then an initial data $\underline{v}_0$ exists in the unit sphere of $\dot H^{\frac12+\gamma}$ such that

$$T^\star(\underline{v}_0) = \inf\bigl\{T^\star(u_0)\,,\ u_0 \in \dot H^{\frac12+\gamma}\,,\ \|u_0\|_{\dot H^{\frac12+\gamma}} = 1\bigr\}\,.$$

Assertion (1.3) can be generalized to the norm associated with the greatest space which is translation invariant, continuously included the space of tempered distributions $\mathcal{S}'(\mathbb{R}^3)$ and whose norm has the same scaling as $\dot H^{\frac12+2\gamma}$. As pointed out by Y. Meyer in Lemma 9 of [19], this space is the Besov space $\dot B^{-1+2\gamma}_{\infty,\infty}$ which can be defined as the space of distributions such that

$$\|u\|_{\dot B^{-1+2\gamma}_{\infty,\infty}} \stackrel{\rm def}{=} \sup_{t>0} t^{\frac12-\gamma}\|e^{t\Delta}u\|_{L^\infty} \tag{1.4}$$

is finite. The generalization of the bound given by (1.3) to this norms is not difficult (see for instance Theorem 1.3 of [8]). Let us recall the statement.

Theorem 1.1 ([8]). *For any γ in the interval $]0,1/2[$, a constant c_γ exists such that for any regular initial data v_0, its life span $T^\star(v_0)$ satisfies*

$$T^\star(v_0) \geq c_\gamma \|v_0\|^{-\frac1\gamma}_{\dot B^{-1+2\gamma}_{\infty,\infty}} \quad \textit{which leads to} \quad \|v(t)\|_{\dot B^{-1+2\gamma}_{\infty,\infty}} \geq \frac{c_\gamma}{(T^\star(v_0)-t)^\gamma}\,\cdot \tag{1.5}$$

This result has an analogue as a global regularity result under a smallness condition, which is the Koch and Tataru theorem (see [16]) which claims that an initial data which has a small norm in the space $BMO^{-1}(\mathbb{R}^3)$ generates a global unique solution (which turns out to be as regular as the initial data). The space $BMO^{-1}(\mathbb{R}^3)$ is a very slightly larger space than $\dot B^{-1}_{\infty,2}(\mathbb{R}^3)$ defined by

$$\|u\|^2_{\dot B^{-1}_{\infty,2}} \stackrel{\rm def}{=} \int_0^\infty \|e^{t\Delta}u\|^2_{L^\infty}dt < \infty \tag{1.6}$$

and very slightly smaller than the space $\dot B^{-1}_{\infty,\infty}$. Let us notice that the classical spaces $\dot H^{\frac12}(\mathbb{R}^3)$ and $L^3(\mathbb{R}^3)$ are continuously embedded in $BMO^{-1}(\mathbb{R}^3)$.

Let us point out that the proof of all these results does not use the special structure of (NS) and in particular they are true for any system of the type

$$(GNS) \quad \partial_t v - \Delta v + \sum_{i,j} A_{i,j}(D)(v^iv^j) = 0$$

where $A_{i,j}(D)$ are smooth homogenenous Fourier multipliers of order 1. The problem investigated here is to improve the description of the behavior of the solution near a possible blow up with the help of the special structure of the non linear term of the Navier-Stokes equation.

One major achievement in this field is the work [12] by L. Escauriaza, G. Seregin and V. Sverák which proves that

$$T^\star(v_0) < \infty \Longrightarrow \limsup_{t\to T^\star(v_0)} \|v(t)\|_{L^3} = \infty\,. \tag{1.7}$$

This was extended to the full limit in time in $\dot H^{\frac12}$ and not only the upper limit by G. Seregin in [26] .

A different context consists in formulating a condition which involves only one component of the velocity field. The first result in that direction was obtained in a pioneering work by

J. Neustupa and P. Penel (see [21]) but the norm involved was not scaling invariant. A lot of works (see [4, 5, 14, 17, 20, 23, 24, 27, 29]) establish conditions of the type

$$\int_0^{T^\star} \|v^3(t,\cdot)\|_{L^q}^p dt = \infty \quad \text{or} \quad \int_0^{T^\star} \|\partial_j v^3(t,\cdot)\|_{L^q}^p dt = \infty$$

with relations on p and q which however still fail to make these quantities scaling invariant.

The first result in that direction involving a scaling invariant condition was proved by the first and the third author in [9] . It claims that for any regular intial data with gradient in $L^{\frac{3}{2}}(\mathbb{R}^3)$ and for any unit vector $\sigma \in \mathbb{S}^2$, there holds

$$T^\star < \infty \Longrightarrow \int_0^{T^\star} \|v(t)\cdot\sigma\|^p_{\dot H^{\frac12+\frac2p}}\, dt = \infty\,, \tag{1.8}$$

for any p in the interval $]4,6[$. This was extended by Z. Zhang and the first and the third author to any p greater than 4 in [11] . This is the analogue of the integral condition of (1.2) for one component only.

The first result of this paper is the analogue of (1.8) in the case when $p=2$. More precisely, we prove the following theorem.

Theorem 1.2. *Let v be a maximal solution of (NS) in $C([0,T^\star[;\dot H^1)$. If $T^\star$ is finite, then*

$$\forall \sigma \in \mathbb{S}^2\,,\ \int_0^{T^\star} \|v(t)\cdot\sigma\|^2_{\dot H^{\frac32}}\, dt = \infty\,.$$

The proof we present here differs from the proofs in [9] and [11] . It is simpler but seems to be specific to the case when $p=2$.

The rest of the paper is devoted to the study of what happens to the above criteria in the case when p is infinite. In other words, is it possible to extend the L. Escauriaza, G. Seregin and V. Sverák criterion (1.7) for one component only? This question sems too ambitious for the time being. Indeed, following the work [13] by G. Koch, F. Planchon and the second author[1] one way to understand the work of L. Escauriaza, G. Seregin and V. Sverák is the following: assume that a solution exists such that the $\dot H^{\frac12}$ norm remains bounded near the blow up time. The first step consists in proving that the solution tends weakly to 0 when t tends to the blow up time. The second step consists in proving a backward uniqueness result which implies that the solution is 0, which of course contradicts the fact that it blows up in finite time. The first step relies in particular on the fact that the Navier-Stokes system (NS) is globally wellposed for small data in $\dot H^{\frac12}$. In our context, the equivalent statement would be that if $\|v_0\cdot\sigma\|_{\dot H^{\frac12}}$ is small enough for some unit vector σ of $\mathbb{R}^3$, then there is a global regular solution. Such a result, assuming it is true, seems out of reach for the time being.

The result we prove in this paper is that if there is blow up, then it is not possible for one component of the velocity field to tend to 0 too fast. More precisely, we are going to prove the following theorem.

Theorem 1.3. *A positive constant c_0 exists such that for any initial data v_0 in $H^1(\mathbb{R}^3)$ with associate solution v of (NS) blowing up at a finite time $T^\star$, for any unit vector σ of $\mathbb{S}^2$, there holds*

$$\forall t < T^\star\,,\quad \sup_{t'\in[t,T^\star[} \|v(t')\cdot\sigma\|_{\dot H^{\frac12}} \geq c_0 \log^{-\frac12}\Big(e + \frac{\|v(t)\|^4_{L^2}}{T^\star - t}\Big)\cdot$$

The other result we prove here requires reinforcing slightly the $\dot H^{\frac12}$ norm, while remaining (almost) scaling invariant.

[1]See also [6] for a more elementary approach

Definition 1.1. *Let E be a positive real number and σ an element of the unit sphere $\mathbb{S}^2$. We define $\dot H^{\frac12}_{\log_\sigma,E}$ the space of distributions a in the homogeneous space $\dot H^{\frac12}(\mathbb{R}^3)$ such that*

$$\|a\|^2_{\dot H^{\frac12}_{\log_\sigma,E}} \stackrel{\rm def}{=} \int_{\mathbb{R}^3} |\xi| \log\bigl(|\xi_\sigma|E+e\bigr)|\widehat a(\xi)|^2 d\xi < \infty \quad \textit{with} \quad \xi_\sigma \stackrel{\rm def}{=} \xi-(\xi\cdot\sigma)\sigma\,.$$

Our theorem is the following.

Theorem 1.4. *A positive constant c_0 exists which satisfies the following. If v is a maximal solution of (NS) in $C([0,T^\star[;H^1)$ and if $T^\star$ is finite, then for any positive real number E,*

$$\forall \sigma\in\mathbb{S}^2\,,\quad \limsup_{t\to T^\star} \|v(t)\cdot\sigma\|_{\dot H^{\frac12}_{\log_\sigma,\mathrm{E}}} \geq c_0\,.$$

The structure of the paper is the following: in Section 2, we reduce the proof of the three theorems to the proofs of three lemmas. The basic idea in this section consists in estimating the L^2 norm of the horizontal derivatives of the solution. Let us point out that the standard L^2 energy estimate plays an important role. These ideas are common to the proof of the three theorems.

Section 3 is devoted to the proof of the three lemmas. The one relative to Theorem 1.4 uses paradifferential calculus.

2. Proof of the theorems

Following an idea of [5], we perform an L^2 scalar product on the momentum equation of (NS) with $-\Delta_{\rm h}v$. This can be interpreted as a $\dot H^1$ energy estimate for the horizontal variables. Recalling that

$$\nabla_{\rm h} v \stackrel{\rm def}{=} (\partial_1 v,\partial_2 v)\,,$$

we have

$$\frac12\frac{d}{dt}\|\nabla_{\rm h}v\|^2_{L^2} + \|\nabla_{\rm h}v\|^2_{\dot H^1} = \sum_{j=1}^4 \mathcal{E}_j(v) \quad \text{with}$$

$$\begin{aligned}
\mathcal{E}_1(v) &\stackrel{\rm def}{=} -\sum_{i=1}^2 \bigl(\partial_i v^{\rm h}\cdot\nabla_{\rm h}v^{\rm h}\big|\partial_i v^{\rm h}\bigr)_{L^2}\,,\\
\mathcal{E}_2(v) &\stackrel{\rm def}{=} -\sum_{i=1}^2 \bigl(\partial_i v^{\rm h}\cdot\nabla_{\rm h}v^3\big|\partial_i v^3\bigr)_{L^2}\,,\\
\mathcal{E}_3(v) &\stackrel{\rm def}{=} -\sum_{i=1}^2 (\partial_i v^3\partial_3 v^{\rm h}\big|\partial_i v^{\rm h})_{L^2} \quad \text{and}\\
\mathcal{E}_4(v) &\stackrel{\rm def}{=} -\sum_{i=1}^2 (\partial_i v^3\partial_3 v^3\big|\partial_i v^3)_{L^2}\,.
\end{aligned} \tag{2.1}$$

Let $\mathrm{div}_{\rm h}\, v^{\rm h} \stackrel{\rm def}{=} \partial_1 v^1+\partial_2 v^2$. A direct computation shows that

$$\mathcal{E}_1(v) = -\int_{\mathbb{R}^3} \mathrm{div}_{\rm h}\, v^{\rm h}\Bigl(\sum_{i,j=1}^2 (\partial_i v^j)^2 + \partial_1 v^2\partial_2 v^1 - \partial_1 v^1\partial_2 v^2\Bigr)\,dx\,,$$

which, together with $\mathrm{div}\, v=0$, ensure that

$$\mathcal{E}_1(v) = \int_{\mathbb{R}^3} \partial_3 v^3\Bigl(\sum_{i,j=1}^2 (\partial_i v^j)^2 + \partial_1 v^2\partial_2 v^1 - \partial_1 v^1\partial_2 v^2\Bigr)\,dx\,.$$

Then let us observe that the three terms $\mathcal{E}_1(v)$, $\mathcal{E}_2(v)$ and $\mathcal{E}_4(v)$ are sums of terms of the form

$$I(v) \stackrel{\text{def}}{=} \int_{\mathbb{R}^3} \partial_i v^3(x) \partial_j v^k(x) \partial_\ell v^m(x) dx \tag{2.2}$$

with (j,ℓ) in $\{1,2\}^2$ and (i,k,m) in $\{1,2,3\}^3$.

Without loss of generality, we shall always take $\sigma = e_3$ in the rest of this paper.

Proof of Theorem 1.2. Hölder's inequality implies that

$$|I(v)| \leq \|\nabla v^3\|_{L^3} \|\nabla_{\rm h} v\|_{L^3}^2 \,.$$

The Sobolev embedding $\dot{H}^{\frac12} \hookrightarrow L^3$ and an interpolation inequality between L^2 and $\dot{H}^1$ imply

$$|I(v)| \lesssim \|v^3\|_{\dot{H}^{\frac32}} \|\nabla_{\rm h} v\|_{L^2} \|\nabla_h v\|_{\dot{H}^1} \,.$$

A convexity inequality then gives

$$|I(v)| \leq \frac{1}{100} \|\nabla_h v\|_{\dot{H}^1}^2 + C\|v^3\|_{\dot{H}^{\frac32}}^2 \|\nabla_{\rm h} v\|_{L^2}^2 \,. \tag{2.3}$$

In order to estimate $\mathcal{E}_3(v)$, we have to study terms of the type

$$J_{i\ell}(v,v^3) \stackrel{\text{def}}{=} \int_{\mathbb{R}^3} \partial_i v^3 \partial_3 v^\ell \partial_i v^\ell \, dx, \tag{2.4}$$

with $(i,\ell) \in \{1,2\}^2$. This is achieved through the following lemma, which will be proved in Section 3 .

Lemma 2.1. *A constant C exists such that, for any positive real number E, we have*

$$\big|J_{i\ell}(v,v^3)\big| \leq \frac{1}{10}\|\nabla_{\rm h} v\|_{\dot{H}^1}^2 + C\Big(\log\big(\|\nabla_{\rm h} v\|_{L^2}^2 E + e\big) \|v^3\|_{\dot{H}^{\frac32}}^2 + \frac{C}{E}\|\nabla v\|_{L^2}^2\Big) \|\nabla_{\rm h} v\|_{L^2}^2 \,.$$

Continuation of the proof of Theorem 1.2 *.* Using (2.1), (2.3) and Lemma 2.1, we infer that

$$\begin{aligned}\frac12 \frac{d}{dt}\|\nabla_{\rm h} v\|_{L^2}^2 + \|\nabla_{\rm h} v\|_{\dot{H}^1}^2 &\leq \frac12 \|\nabla_{\rm h} v\|_{\dot{H}^1}^2 \\ &\quad + C\Big(\log\big(\|\nabla_{\rm h} v\|_{L^2}^2 E + e\big)\|v^3\|_{\dot{H}^{\frac32}}^2 + \frac1E \|\nabla v\|_{L^2}^2\Big)\|\nabla_{\rm h} v\|_{L^2}^2\end{aligned} \tag{2.5}$$

which implies that

$$\frac{d}{dt} \log\big(\|\nabla_{\rm h} v\|_{L^2}^2 E + e\big) \lesssim \|v^3\|_{\dot{H}^{\frac32}}^2 \log\big(\|\nabla_{\rm h} v\|_{L^2}^2 E + e\big) + \frac1E \|\nabla v\|_{L^2}^2 \,.$$

Gronwall's lemma implies that

$$\log\big(\|\nabla_{\rm h} v(t)\|_{L^2}^2 E + e\big) \leq \big(\log(\|\nabla_{\rm h} v_0\|_{L^2}^2 E + e) + \frac{C}{E}\int_0^t \|\nabla v(t')\|_{L^2}^2 dt'\big) \exp\Big(C\int_0^t \|v^3(t')\|_{\dot{H}^{\frac32}}^2 dt'\Big).$$

The energy estimate (1.1) then provides

$$\log\big(\|\nabla_{\rm h} v(t)\|_{L^2}^2 E + e\big) \leq \big(\log(\|\nabla_{\rm h} v_0\|_{L^2}^2 E + e) + \frac{C\|v_0\|_{L^2}^2}{E}\big) \exp\Big(C\int_0^t \|v^3(t')\|_{\dot{H}^{\frac32}}^2 dt'\Big) \cdot$$

Thus if u is a $C([0,T[;\dot{H}^1)$ solution of (NS) and if $\displaystyle\int_0^T \|v^3(t)\|_{\dot{H}^{\frac32}}^2 dt$ is finite, then $\nabla_{\rm h} v$ is in $L^\infty([0,T[;L^2)$. Plugging this in (2.5) implies also that $\nabla_{\rm h} v$ is in $L^2([0,T[;\dot{H}^1)$.

At this stage we can invoke Theorem 1.4 of [9] to conclude, because the vertical component v^3 of the solution remains bounded in the inhomogeneous space H^1 on the time interval $[0,T[$. For the reader's convenience, we present here a elementary and self contained proof, inspired by a method introduced in [7] . Differentiating (NS) with respect to the vertical variable and taking the L^2 scalar product of this system with $\partial_3 v$ gives, thaks to the divergence free condition

$$
\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|\partial_3 v(t)\|_{L^2}^2 + \|\partial_3 v(t)\|_{\dot H^1}^2 &= \sum_{k=1}^{3}\Big(\sum_{j=1}^{2}\int_{\mathbb{R}^3}\partial_3 v^j\partial_j v^k\partial_3 v^k dx + \int_{\mathbb{R}^3}\partial_3 v^3\partial_3 v^k\partial_3 v^k dx\Big)\\
&= \sum_{k=1}^{3}\Big(\sum_{j=1}^{2}\int_{\mathbb{R}^3}\partial_3 v^j\partial_j v^k\partial_3 v^k dx - \int_{\mathbb{R}^3}\mathrm{div}_{\mathrm{h}}\, v^{\mathrm{h}}\partial_3 v^k\partial_3 v^k dx\Big).
\end{aligned}
$$

All the terms on the right-hand side of the above equality can be estimated by

$$
\|\partial_3 v\|_{L^6}\|\nabla_{\mathrm{h}} v\|_{L^3}\|\partial_3 v\|_{L^2}
$$

Sobolev embeddings and an interpolation between L^2 and $\dot H^1$, along with the convexity inequality imply that

$$
\frac{1}{2}\frac{d}{dt}\|\partial_3 v(t)\|_{L^2}^2 + \|\partial_3 v(t)\|_{\dot H^1}^2 \leq \frac{1}{2}\|\partial_3 v(t)\|_{\dot H^1}^2 + C\|\nabla_{\mathrm{h}} v(t)\|_{L^2}\|\nabla_{\mathrm{h}} v(t)\|_{\dot H^1}\|\partial_3 v(t)\|_{L^2}^2 .
$$

As we have

$$
\sup_{t\in[0,T[}\|\nabla_{\mathrm{h}} v(t)\|_{L^2}^2 + \int_0^T \|\nabla_{\mathrm{h}} v(t)\|_{\dot H^1}^2 dt < \infty ,
$$

the solution v remains bounded in $\dot H^1$ and thus T cannot be the maximal time of existence. Theorem 1.2 is proved. □

Proof of Theorem 1.3. We restart from (2.1) and (2.2). Laws of product in three dimensional Sobolev spaces ensure that

$$
\begin{aligned}
I(v) &\leq \|\partial_i v^3\|_{\dot H^{-\frac12}}\big\|\partial_j v^k\partial_\ell v^m\big\|_{\dot H^{\frac12}}\\
&\lesssim \|v^3\|_{\dot H^{\frac12}}\|\nabla_{\mathrm{h}} v\|_{\dot H^1}^2.
\end{aligned}
\tag{2.6}
$$

Now let us turn to the estimate of J defined in (2.4). Then the proof relies on the following lemma, which we shall prove in Section 3.

Lemma 2.2. *For any positive ε, a constant C_ε exists such that for any positive constant E there holds*

$$
\big|J_{i\ell}(v,v^3)\big| \leq \Big(\varepsilon + C_\varepsilon\|v^3\|_{\dot H^{\frac12}}\sqrt{\log\big(e+E\|\nabla_{\mathrm{h}} v\|_{L^2}^2\big)}\ \Big)\|\nabla_{\mathrm{h}} v\|_{\dot H^1}^2 + C_\varepsilon\|v^3\|_{\dot H^{\frac12}}^2\frac{\|\partial_3 v\|_{L^2}^2}{E^2} .
$$

Continuation of the proof of Theorem 1.3. Considering that $\mathcal{E}_3(v)$ is a sum of expressions of the type $J_{i\ell}(v,v^3)$, let us apply this lemma along with Inequality (2.6). Plugging those results into (2.1) gives

$$
\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|\nabla_{\mathrm{h}} v\|_{L^2}^2 + \|\nabla_{\mathrm{h}} v\|_{\dot H^1}^2 \leq \Big(\frac14 + C\|v^3\|_{\dot H^{\frac12}}\sqrt{\log\big(e+E\|\nabla_{\mathrm{h}} v\|_{L^2}^2\big)}\ \Big)\|\nabla_{\mathrm{h}} v\|_{\dot H^1}^2&\\
+ C\|v^3\|_{\dot H^{\frac12}}^2\frac{\|\partial_3 v\|_{L^2}^2}{E^2}& .
\end{aligned}
\tag{2.7}
$$

Let us define, for $T\leq T^\star$,

$$
\begin{aligned}
m(T) &\stackrel{\mathrm{def}}{=} \sup_{t\in[0,T[}\|v^3(t)\|_{\dot H^{\frac12}} \quad\text{and}\\
\underline{T} &\stackrel{\mathrm{def}}{=} \sup\big\{T'\leq T\,/\ \|\nabla_{\mathrm{h}} v\|_{L^\infty([0,T'];L^2)}^2 \leq 2\|\nabla_{\mathrm{h}} v_0\|_{L^2}^2\big\} .
\end{aligned}
\tag{2.8}
$$

Let us note that for any divergence free vector field in $\dot H^1$, we have

$$\|\nabla w^3\|_{L^2}^2 = \|\nabla_{\rm h} w^3\|_{L^2}^2 + \|\partial_3 w^3\|_{L^2}^2 = \|\nabla_{\rm h} w^3\|_{L^2}^2 + \|\operatorname{div}_{\rm h} w^{\rm h}\|_{L^2}^2$$

which ensures that

$$\|\nabla w^3\|_{L^2}^2 \leq 2\|\nabla_{\rm h} w\|_{L^2}^2\,. \tag{2.9}$$

Then for $t \leq \underline{T}$, Inequality (2.7) becomes

$$\frac{1}{2}\frac{d}{dt}\|\nabla_{\rm h} v\|_{L^2}^2 + \|\nabla_{\rm h} v\|_{\dot H^1}^2 \leq \Bigl(\frac{1}{4} + C_0 m(T)\sqrt{\log\bigl(e + E\|\nabla_{\rm h} v_0\|_{L^2}^2\bigr)}\,\Bigr)\|\nabla_{\rm h} v\|_{\dot H^1}^2 + Cm^2(T)\frac{\|\partial_3 v\|_{L^2}^2}{E^2}\,\cdot$$

Let us choose E equal to $\frac{\|v_0\|_{L^2}}{\|\nabla_h v_0\|_{L^2}}$ and let us assume that

$$m(T)\sqrt{\log\bigl(e + \|v_0\|_{L^2}\|\nabla_{\rm h} v_0\|_{L^2}\bigr)} \leq c_0 \tag{2.10}$$

with small enough c_0 (less than $1/4C_0$ for the time being). Then we get

$$\frac{d}{dt}\|\nabla_{\rm h} v\|_{L^2}^2 + \|\nabla_{\rm h} v\|_{\dot H^1}^2 \lesssim m^2(T)\|\nabla_{\rm h} v_0\|_{L^2}^2\frac{\|\partial_3 v(t)\|_{L^2}^2}{\|v_0\|_{L^2}^2}\,\cdot$$

By time integration and using the energy inequality (1.1), we infer that for any $t \leq \underline{T}$,

$$\|\nabla_{\rm h} v(t)\|_{L^2}^2 \leq \|\nabla_{\rm h} v_0\|_{L^2}^2\bigl(1 + C_1 m^2(T)\bigr). \tag{2.11}$$

Then the argument used at the end of the proof of Theorem 1.2 can be repeated. So by contraposition, we infer that

$$m(T^\star)\sqrt{\log\bigl(e + \|v_0\|_{L^2}\|\nabla_{\rm h} v_0\|_{L^2}\bigr)} \geq c_0\,.$$

Now let us translate in time this assertion. Defining

$$M(t) \stackrel{\rm def}{=} \sup_{t'\in[t,T^\star]} \|v^3(t')\|_{\dot H^{\frac12}},$$

the above assertion claims that

$$M(t) \geq \frac{c_0}{\sqrt{\log\bigl(e + \|v(t)\|_{L^2}\|\nabla_{\rm h} v(t)\|_{L^2}\bigr)}}\,\cdot$$

If $M(t)$ is infinite there is nothing to prove. Let us assume $M(t)$ (which is a non-increasing function) is finite. Then the above inequality can be written

$$\forall t' \in [t, T^\star]\,,\ \|v(t')\|_{L^2}^2\|\nabla_{\rm h} v(t')\|_{L^2}^2 \geq \exp\Bigl(\frac{2c_0^2}{M^2(t')}\Bigr)\,\cdot$$

Because of the decay of the kinetic energy and the monotonicity of M, this can be written

$$\forall t' \in [t, T^\star]\,,\ \|v(t)\|_{L^2}^2\|\nabla_{\rm h} v(t')\|_{L^2}^2 \geq \exp\Bigl(\frac{2c_0^2}{M^2(t)}\Bigr)\,\cdot$$

By integration of this inequality in the interval $[t, T^\star]$, we infer that

$$\|v(t)\|_{L^2}^2\int_t^{T^\star}\|\nabla_{\rm h} v(t')\|_{L^2}^2 dt' \geq (T^\star - t)\exp\Bigl(\frac{2c_0^2}{M^2(t)}\Bigr)\,\cdot$$

The energy estimate implies that

$$\frac{1}{2}\|v(t)\|_{L^2}^4 \geq (T^\star - t)\exp\Bigl(\frac{2c_0^2}{M^2(t)}\Bigr)$$

which concludes the proof of Theorem 1.3. □

Proof of Theorem 1.4. Using (2.1), (2.2) and (2.6), we get, recalling notation (2.4),

$$\frac{1}{2}\frac{d}{dt}\|\nabla_{\rm h}v\|_{L^2}^2+\|\nabla_{\rm h}v\|_{\dot H^1}^2\lesssim\|v^3\|_{\dot H^{\frac12}}\|\nabla_{\rm h}v\|_{\dot H^1}^2+\sum_{(i,\ell)\in\{1,2\}^2}\big|J_{i\ell}(v,v^3)\big|\,.\tag{2.12}$$

Then the proof relies on the following lemma.

Lemma 2.3. *Let us define*

$$\|a\|^2_{\dot H^{\frac12}_{\log_{\rm h},E}}\stackrel{\rm def}{=}\int_{\mathbb{R}^3}|\xi|\,|\widehat a(\xi)|^2\log(|\xi_{\rm h}|E+e)d\xi\,.$$

A constant C exists such that, for any positive E, we have

$$\big|J_{i\ell}(v,v^3)\big|\leq\Big(\frac{1}{10}+C\|v^3\|_{\dot H^{\frac12}_{\log_{\rm h},{\rm E}}}\Big)\|\nabla_{\rm h}v^{\rm h}\|_{\dot H^1}^2+C\|v^3\|_{\dot H^{\frac12}}^2\frac{\|\partial_3v^{\rm h}\|_{L^2}^2}{E^2}\,\cdot\tag{2.13}$$

Conclusion of the proof of Theorem 1.4. Let us plug this lemma into (2.12). This gives

$$\frac{1}{2}\frac{d}{dt}\|\nabla_{\rm h}v(t)\|_{L^2}^2+\|\nabla_{\rm h}v\|_{\dot H^1}^2\leq\Big(\frac14+C\|v^3\|_{\dot H^{\frac12}_{\log_{\rm h},{\rm E}}}\Big)\|\nabla_{\rm h}v^{\rm h}\|_{\dot H^1}^2+C\|v^3\|_{\dot H^{\frac12}}^2\frac{\|\partial_3v^{\rm h}\|_{L^2}^2}{E^2}\,\cdot\tag{2.14}$$

Then by time integration and thanks to the energy estimate we find that as long as

$$t\leq T_*\stackrel{\rm def}{=}\sup\Big\{\,T\in]0,T^\star[\,/\sup_{t\in[0,T]}\|v^3(t)\|_{\dot H^{\frac12}_{\log_{\rm h},{\rm E}}}\leq\frac{1}{4C}\,\Big\}\,,$$

there holds

$$\begin{aligned}\|\nabla_{\rm h}v(t)\|_{L^2}^2+\int_0^t\|\nabla_{\rm h}v(t')\|_{\dot H^1}^2\,dt'&\leq\|\nabla_{\rm h}v_0\|_{L^2}^2+\frac{1}{2E^2}\int_0^t\|\partial_3v^{\rm h}(t')\|_{L^2}^2\,dt'\\&\leq\|\nabla_{\rm h}v_0\|_{L^2}^2+\frac{\|v_0\|_{L^2}^2}{E^2}\,\cdot\end{aligned}$$

Then to conclude we use the same arguments as in the conclusion of the previous two theorems. Theorem 1.4 is proved. □

3. Proof of the three lemmas

In this section we prove Lemmas 2.1, 2.2 and 2.3 . We shall use the following notation: $L^p_{\rm h}$ will denote the space $L^p(\mathbb{R}^2)$ in the horizontal variables $x_{\rm h}\stackrel{\rm def}{=}(x_1,x_2)$ (and we shall write $\mathbb{R}^2_{\rm h}$ to specify the space $x_{\rm h}$ belongs to), and $L^p_{\rm v}$ will denote the space $L^p(\mathbb{R})$ in the vertical variable (and we shall write $\mathbb{R}_{\rm v}$ to specify the space x_3 belongs to).

The main problem is that the control of $\nabla_{\rm h}v$ in $L^2_{\rm h}$ does not imply any control on v, simply because the Sobolev space $\dot H^1(\mathbb{R}^2_{\rm h})$ is not continuously included in $\mathcal{S}'(\mathbb{R}^2_{\rm h})$. In order to overcome this problem, the idea consists in decomposing v into a term containing only low horizontal frequencies, a term containing only medium horizontal frequencies and a term containing only high horizontal frequencies, and in estimating each of those three terms differently. More precisely, for a couple of positive real numbers (λ,Λ) such that $\lambda\leq\Lambda$, let us define

$$\begin{aligned}&a_{\flat,\lambda}\stackrel{\rm def}{=}\mathcal{F}^{-1}(\mathbf{1}_{B_{\rm h}(0,\lambda)}\widehat a)\,,\ a_{\natural,\lambda,\Lambda}\stackrel{\rm def}{=}\mathcal{F}^{-1}\big((\mathbf{1}_{B_{\rm h}(0,\Lambda)}-\mathbf{1}_{B_{\rm h}(0,\lambda)})\widehat a\big)\quad\text{and}\\&a_{\sharp,\Lambda}\stackrel{\rm def}{=}\mathcal{F}^{-1}(\mathbf{1}_{B^c_{\rm h}(0,\Lambda)}\widehat a)\,.\end{aligned}\tag{3.1}$$

Proof of Lemma 2.1. Let us study first low horizontal frequencies. We start by writing

$$\begin{aligned}J^\flat_\lambda&\stackrel{\rm def}{=}\Big|\int_{\mathbb{R}^3}\partial_3v^\ell_{\flat,\lambda}\partial_iv^3\partial_iv^\ell dx\Big|\\&\leq\|\partial_3v^\ell_{\flat,\lambda}\|_{L^4_{\rm h}(L^2_{\rm v})}\|\partial_iv^3\|_{L^4_{\rm h}(L^2_{\rm v})}\|\partial_iv^\ell\|_{L^2_{\rm h}(L^\infty_{\rm v})}\,.\end{aligned}$$

Using Bernstein and Gagliardo-Nirenberg inequalities in the horizontal variable (see the Appendix for anisotropic Bernstein inequalities), we infer that

$$\begin{aligned} J^\flat_\lambda &\lesssim \lambda^{\frac12}\|\partial_3 v^\ell\|_{L^2}\|\partial_i v^3\|_{L^2}^{\frac12}\|\partial_i\nabla_{\rm h} v^3\|_{L^2}^{\frac12}\|\partial_i v^\ell\|_{L^2}^{\frac12}\|\partial_3\partial_i v^\ell\|_{L^2}^{\frac12}\\ &\lesssim \lambda^{\frac12}\|\partial_3 v^\ell\|_{L^2}\|\nabla_{\rm h} v\|_{L^2}\|\nabla_{\rm h} v\|_{\dot H^1}\,. \end{aligned}$$

By convexity we infer that

$$J^\flat_\lambda \leq \frac{1}{100}\|\nabla_{\rm h} v\|_{\dot H^1}^2 + C\lambda\|\partial_3 v^\ell\|_{L^2}^2\|\nabla_{\rm h} v\|_{L^2}^2\,. \tag{3.2}$$

Let us now estimate the high horizontal frequency term. Using Bernstein (inverse) and Gagliardo-Nirenberg inequalities in the horizontal variable, we infer that

$$\begin{aligned} J^\sharp_\Lambda &\stackrel{\rm def}{=} \Big|\int_{\mathbb{R}^3}\partial_3 v^\ell_{\sharp,\Lambda}\partial_i v^3\partial_i v^\ell dx\Big|\\ &\leq \|\partial_3 v^\ell_{\sharp,\Lambda}\|_{L^4_{\rm h}(L^2_{\rm v})}\|\partial_i v^3\|_{L^4_{\rm h}(L^2_{\rm v})}\|\partial_i v^\ell\|_{L^2_{\rm h}(L^\infty_{\rm v})}\\ &\lesssim \Lambda^{-\frac12}\|\nabla_{\rm h}\partial_3 v^\ell\|_{L^2}\|\partial_i v^3\|_{L^2}^{\frac12}\|\partial_i\nabla_{\rm h} v^3\|_{L^2}^{\frac12}\|\partial_i v^\ell\|_{L^2}^{\frac12}\|\partial_3\partial_i v^\ell\|_{L^2}^{\frac12}\\ &\lesssim \Lambda^{-\frac12}\|\nabla_{\rm h} v\|_{L^2}\|\nabla_{\rm h} v\|_{\dot H^1}^2\,. \end{aligned} \tag{3.3}$$

In order to estimate the term concerning the medium horizontal frequencies, let us write that

$$\begin{aligned} J^\natural_{\lambda,\Lambda} &\stackrel{\rm def}{=} \Big|\int_{\mathbb{R}^3}\partial_3 v^\ell_{\natural,\lambda,\Lambda}\partial_i v^3\partial_i v^\ell dx\Big|\\ &\leq \|\partial_3 v^\ell_{\natural,\lambda,\Lambda}\|_{L^2_{\rm v}(L^\infty_{\rm h})}\|\partial_i v^3\|_{L^\infty_{\rm v}(L^2_{\rm h})}\|\partial_i v^\ell\|_{L^2}\,. \end{aligned} \tag{3.4}$$

Using Bernstein inequalities in the horizontal variables we get, for any x_3 in $\mathbb{R}$,

$$\begin{aligned} \|\partial_3 v^\ell_{\natural,\lambda,\Lambda}(\cdot,x_3)\|_{L^\infty_{\rm h}} &\leq \sum_{\lambda\lesssim 2^k\lesssim\Lambda}\|\partial_3\Delta^{\rm h}_k v^\ell(\cdot,x_3)\|_{L^\infty_{\rm h}}\\ &\leq \sum_{\lambda\lesssim 2^k\lesssim\Lambda} 2^k\|\partial_3\Delta^{\rm h}_k v^\ell(\cdot,x_3)\|_{L^2_{\rm h}}\,. \end{aligned}$$

By definition of the Sobolev norm in terms of Littlewood-Paley theory (see the Appendix), we infer

$$2^k\|\partial_3\Delta^{\rm h}_k v^\ell(\cdot,x_3)\|_{L^2_{\rm h}} \lesssim c_k(x_3)\|\nabla_{\rm h}\partial_3 v(\cdot,x_3)\|_{L^2_{\rm h}} \quad\text{with}\quad \sum_k c_k^2(x_3)=1\,.$$

Then using the Cauchy-Schwarz inequality, we infer that

$$\begin{aligned} \|\partial_3 v^\ell_{\natural,\lambda,\Lambda}(\cdot,x_3)\|_{L^\infty_{\rm h}} &\lesssim \|\nabla_{\rm h}\partial_3 v(\cdot,x_3)\|_{L^2_{\rm h}}\sum_{\lambda\lesssim 2^k\lesssim\Lambda} c_k(x_3)\\ &\lesssim \sqrt{\log\Big(\frac{\Lambda}{\lambda}\Big)}\,\|\nabla_{\rm h}\partial_3 v(\cdot,x_3)\|_{L^2_{\rm h}}\,. \end{aligned}$$

Taking the L^2 norm with respect to the variable x_3 gives

$$\|\partial_3 v^\ell_{\natural,\lambda,\Lambda}\|_{L^2_{\rm v}(L^\infty_{\rm h})} \lesssim \sqrt{\log\Big(\frac{\Lambda}{\lambda}\Big)}\|\nabla_{\rm h} v\|_{\dot H^1}\,. \tag{3.5}$$

Now let us observe that thanks to Lemma A.2

$$\|\partial_i v^3\|_{L^\infty_{\rm v}(L^2_{\rm h})} \leq \|(-\Delta_{\rm h})^{\frac14} v^3\|_{L^\infty_{\rm v}(\dot H^{\frac12}_{\rm h})} \lesssim \|v^3\|_{\dot H^{\frac32}}\,.$$

Plugging this inequality and (3.5) into (3.4) gives

$$J^\natural_{\lambda,\Lambda} \lesssim \sqrt{\log\Big(\frac{\Lambda}{\lambda}\Big)}\|\nabla_{\rm h} v\|_{\dot H^1}\|v^3\|_{\dot H^{\frac32}}\|\nabla_{\rm h} v\|_{L^2}\,.$$

By convexity we get

$$J^\natural_{\lambda,\Lambda} \leq \frac{1}{100}\|\nabla_{\rm h} v\|^2_{\dot H^1} + C\log\Bigl(\frac{\Lambda}{\lambda}\Bigr)\|v^3\|^2_{\dot H^{\frac32}}\|\nabla_{\rm h} v\|^2_{L^2}\,.$$

Together with (3.2) and (3.3), we infer that, for any couple of positive real numbers λ and Λ such that λ is less than Λ, there holds

$$\begin{aligned}\Bigl|\int_{\mathbb{R}^3}\partial_3 v^\ell\partial_i v^3\partial_i v^\ell dx\Bigr| \leq \frac{1}{50}\|\nabla_{\rm h} v\|^2_{\dot H^1} + C\Lambda^{-\frac12}\|\nabla_{\rm h} v\|_{L^2}\|\nabla_{\rm h} v\|^2_{\dot H^1}&\\ + C\Bigl(\lambda\|\nabla v\|^2_{L^2} + \|v^3\|^2_{\dot H^{\frac32}}\log\Bigl(\frac{\Lambda}{\lambda}\Bigr)\Bigr)\|\nabla_{\rm h} v\|^2_{L^2}\,.&\end{aligned}$$

Choosing

$$\Lambda = (50C)^2\|\nabla_{\rm h} v\|^2_{L^2} + \frac{e}{E} \quad\text{and}\quad \lambda = \frac{1}{E} \tag{3.6}$$

ensures the result. □

Proof of Lemma 2.2. Let us focus on the estimate of the term $J^\natural_{\lambda,\Lambda}$. Applying Lemma A.3, we infer that

$$\begin{aligned}J^\natural_{\lambda,\Lambda} &= \int_{\mathbb{R}_{\rm v}}\Bigl(\int_{\mathbb{R}^2_{\rm h}}\partial_i v^3(x_h,x_3)\partial_3 v^\ell_{\natural,\lambda,\Lambda}(x_h,x_3)\partial_i v^\ell(x_h,x_3)dx_{\rm h}\Bigr)dx_3\\ &\leq \int_{\mathbb{R}_{\rm v}}\|\partial_i v^3(\cdot,x_3)\|_{\dot H^{-\frac12}_{\rm h}}\bigl\|\partial_3 v^\ell_{\natural,\lambda,\Lambda}(\cdot,x_3)\partial_i v^\ell(\cdot,x_3)\bigr\|_{\dot H^{\frac12}_{\rm h}}dx_3\\ &\lesssim \int_{\mathbb{R}_{\rm v}}\|\partial_i v^3(\cdot,x_3)\|_{\dot H^{-\frac12}_{\rm h}}\bigl(\|\partial_3 v^\ell_{\natural,\lambda,\Lambda}(\cdot,x_3)\|_{L^\infty_{\rm h}} + \|\partial_3 v^\ell_{\natural,\lambda,\Lambda}(\cdot,x_3)\|_{\dot H^1_{\rm h}}\bigr)\|\partial_i v^\ell(\cdot,x_3)\|_{\dot H^{\frac12}_{\rm h}}dx_3\\ &\lesssim \|\nabla_{\rm h} v\|_{L^\infty_{\rm v}(\dot H^{\frac12}_{\rm h})}\int_{\mathbb{R}_{\rm v}}\|\partial_i v^3(\cdot,x_3)\|_{\dot H^{-\frac12}_{\rm h}}\bigl(\|\partial_3 v^\ell_{\natural,\lambda,\Lambda}(\cdot,x_3)\|_{L^\infty_{\rm h}} + \|\partial_3 v^\ell_{\natural,\lambda,\Lambda}(\cdot,x_3)\|_{\dot H^1_{\rm h}}\bigr)dx_3\,.\end{aligned}$$

As we obviously have

$$\|\partial_3 v^\ell_{\natural,\lambda,\Lambda}\|_{L^2_{\rm v}(\dot H^1_{\rm h})} \leq \|\nabla_{\rm h} v\|_{\dot H^1} \quad\text{and}\quad \|\partial_i v^3\|_{L^2_{\rm v}(\dot H^{-\frac12}_{\rm h})} \leq \|v^3\|_{\dot H^{\frac12}}\,,$$

we infer, using (3.5) and the Cauchy-Schwarz inequality, that

$$J^\natural_{\lambda,\Lambda} \lesssim \|\nabla_{\rm h} v\|_{L^\infty_{\rm v}(\dot H^{\frac12}_{\rm h})}\|\nabla_{\rm h} v\|_{\dot H^1}\|v^3\|_{\dot H^{\frac12}}\sqrt{\log\Bigl(\frac{\Lambda}{\lambda}\Bigr)}\,. \tag{3.7}$$

Let us observe that thanks to Lemma A.2 there holds

$$\|\nabla_{\rm h} v\|_{L^\infty_{\rm v}(\dot H^{\frac12}_{\rm h})} \lesssim \|\nabla_{\rm h} v\|_{\dot H^1}\,.$$

Along the same lines, we get, by using the product law $(\dot B^1_{2,1})_{\rm h}\times \dot H^{\frac12}_{\rm h}\subset \dot H^{\frac12}_{\rm h}$ (see the appendix for the definition of $(\dot B^1_{2,1})_{\rm h}$), that

$$\begin{aligned}J^\flat_\lambda &= \Bigl|\int_{\mathbb{R}^3}\partial_3 v^\ell_{\flat,\lambda}\partial_i v^3\partial_i v^\ell dx\Bigr|\\ &\lesssim \|v^3\|_{\dot H^{\frac12}}\|\partial_3 v^\ell_{\flat,\lambda}\partial_i v^\ell\|_{L^2_{\rm v}(\dot H^{\frac12}_{\rm h})}\\ &\lesssim \|v^3\|_{\dot H^{\frac12}}\|\partial_3 v_{\flat,\lambda}\|_{L^2_{\rm v}(\dot B^1_{2,1})_{\rm h}}\|\partial_i v^\ell\|_{L^\infty_{\rm v}(\dot H^{\frac12}_{\rm h})}\,.\end{aligned}$$

But according to the definition of $v_{\flat,\lambda}$, there holds

$$\begin{aligned}\|\partial_3 v_{\flat,\lambda}\|_{L^2_{\rm v}(\dot B^1_{2,1})_{\rm h}} &\leq \sum_{2^k\leq\lambda}2^k\|\Delta^{\rm h}_k\partial_3 v_{\flat,\lambda}\|_{L^2}\\ &\lesssim \lambda\|\partial_3 v\|_{L^2}\,,\end{aligned}$$

so

$$
\begin{aligned}
J_\lambda^\flat &\le C\lambda \|v^3\|_{\dot H^{\frac12}} \|\partial_3 v\|_{L^2} \|\nabla_{\rm h} v\|_{\dot H^1}\\
&\le \frac{1}{100}\|\nabla_{\rm h} v\|_{\dot H^1}^2 + C\lambda^2 \|v^3\|^2_{\dot H^{\frac12}} \|\partial_3 v\|^2_{L^2}\,.
\end{aligned}
\tag{3.8}
$$

Then with the choice of λ and Λ made in (3.6), the lemma is proved using (3.3), (3.7) and (3.8). $\square$

Proof of Lemma 2.3. The main point consists in estimating

$$
J_E^\sharp \stackrel{\rm def}{=} \int_{\mathbb{R}^3} \partial_i v^3 \partial_3 v^\ell_{\sharp, E^{-1}} \partial_i v^\ell dx\,.
$$

Using Bony's decomposition in the horizontal variable introduced in the proof of Lemma A.3, let us write that

$$
\begin{aligned}
&J_E^\sharp = J_E^{\sharp,1} + J_E^{\sharp,2} \quad \text{with}\\
&J_E^{\sharp,1} \stackrel{\rm def}{=} \int_{\mathbb{R}_{\rm v}} \Big(\int_{\mathbb{R}^2_h} \partial_i v^3(x_h,x_3) \widetilde T^{\rm h}_{\partial_i v^\ell(\cdot,x_3)} \partial_3 v^\ell_{\sharp,E^{-1}}(\cdot,x_3) dx_{\rm h}\Big) dx_3 \quad \text{and}\\
&J_E^{\sharp,2} \stackrel{\rm def}{=} \int_{\mathbb{R}_{\rm v}} \Big(\sum_{k\in\mathbb{Z}} \int_{\mathbb{R}^2_h} \Delta_k^{\rm h} \partial_i v^3(\cdot,x_3) \widetilde \Delta_k^{\rm h} T^{\rm h}_{\partial_3 v^\ell_{\sharp,E^{-1}}(\cdot,x_3)} \partial_i v^\ell(\cdot,x_3) dx_{\rm h}\Big) dx_3\,.
\end{aligned}
\tag{3.9}
$$

Let us estimate $J_E^{\sharp,1}$. The control of this term does not use the fact that $v^\ell_{\sharp,E^{-1}}$ contains only high horizontal frequencies. Using (A.4), we can write

$$
\begin{aligned}
J_E^{\sharp,1} &\le \int_{\mathbb{R}_{\rm v}} \|\nabla_{\rm h} v^3(\cdot,x_3)\|_{\dot H^{-\frac12}(\mathbb{R}^2_{\rm h})} \|\widetilde T^{\rm h}_{\partial_i v^\ell(\cdot,x_3)} \partial_3 v^\ell_{\sharp,E^{-1}}(\cdot,x_3)\|_{\dot H^{\frac12}(\mathbb{R}^2_{\rm h})} dx_3\\
&\lesssim \|\nabla_{\rm h} v\|_{L^\infty_{\rm v}(\dot H^{\frac12}(\mathbb{R}^2_{\rm h}))} \int_{\mathbb{R}_{\rm v}} \|v^3(\cdot,x_3)\|_{\dot H^{\frac12}(\mathbb{R}^2_{\rm h})} \|\partial_3 v^\ell(\cdot,x_3)\|_{\dot H^1(\mathbb{R}^2_{\rm h})} dx_3\,.
\end{aligned}
$$

Using the Cauchy-Schwarz inequality gives

$$
\begin{aligned}
J_E^{\sharp,1} &\lesssim \|\nabla_{\rm h} v\|_{L^\infty_{\rm v}(\dot H^{\frac12}(\mathbb{R}^2_{\rm h}))} \|v^3\|_{L^2_{\rm v}(\dot H^{\frac12}(\mathbb{R}^2_{\rm h}))} \|\partial_3 v^\ell\|_{L^2_{\rm v}(\dot H^1(\mathbb{R}^2_{\rm h}))}\\
&\lesssim \|\nabla_{\rm h} v\|_{L^\infty_{\rm v}(\dot H^{\frac12}(\mathbb{R}^2_{\rm h}))} \|v^3\|_{\dot H^{\frac12}(\mathbb{R}^3)} \|\nabla_{\rm h} v^\ell\|_{\dot H^1(\mathbb{R}^3)}\,.
\end{aligned}
$$

Once observed that (see Lemma A.2) $\|\nabla_{\rm h} v\|_{L^\infty_{\rm v}(\dot H^{\frac12}(\mathbb{R}^2_{\rm h}))} \lesssim \|\nabla_{\rm h} v\|_{\dot H^1(\mathbb{R}^3)}$, we infer that

$$
J_E^{\sharp,1} \lesssim \|v^3\|_{\dot H^{\frac12}(\mathbb{R}^3)} \|\nabla_{\rm h} v\|^2_{\dot H^1(\mathbb{R}^3)}\,.
\tag{3.10}
$$

In order to estimate $J_E^{\sharp,2}$, we revisit the proof of Theorem 2.47 in [1] which describes the mapping of paraproducts. Because the support of the Fourier transform of

$$
S^{\rm h}_{k'-1}(\partial_3 v^\ell_{\sharp,E^{-1}}(\cdot,x_3)) \Delta^{\rm h}_{k'} \partial_i v^\ell(\cdot,x_3)
$$

is included in a ring of $\mathbb{R}^2_{\rm h}$ of the type $2^{k'}\widetilde{\mathcal{C}}$, we get, by definition of $T^{\rm h}$ that

$$
J_E^{\sharp,2} = \int_{\mathbb{R}_{\rm v}} \Big(\sum_{k\in\mathbb{Z}} \int_{\mathbb{R}^2_h} \Delta_k^{\rm h} \partial_i v^3(\cdot,x_3) \widetilde\Delta^{\rm h}_k \Big(\sum_{|k'-k|\le N_0} S^{\rm h}_{k'-1}(\partial_3 v^\ell_{\sharp,E^{-1}}(\cdot,x_3)) \Delta^{\rm h}_{k'} \partial_i v^\ell(\cdot,x_3)\Big) dx_{\rm h}\Big) dx_3\,.
$$

Using Cauchy-Schwarz and Bernstein inequalities, we get

$$
\begin{aligned}
J_E^{\sharp,2} \le \int_{\mathbb{R}_{\rm v}} \Big(\sum_{|k-k'|\le N_0} & 2^k \|\Delta_k^{\rm h} v^3(\cdot,x_3)\|_{L^2_{\rm h}}\\
&\times \|S^{\rm h}_{k'-1} \partial_3 v^\ell_{\sharp,E^{-1}}(\cdot,x_3)\|_{L^\infty_{\rm h}} \|\Delta^{\rm h}_{k'} \partial_i v^\ell(\cdot,x_3)\|_{L^2_{\rm h}}\Big) dx_3\,.
\end{aligned}
$$

Using the fact that $|k'-k|\leq N_0$ we get, using the equivalence (A.2) of Sobolev norms,

$$J_E^{\sharp,2} \lesssim \|\nabla_{\rm h} v\|_{L^\infty_{\rm v}(\dot H^{\frac12}(\mathbb{R}^2_{\rm h})}\int_{\mathbb{R}_{\rm v}}\biggl(\sum_{k\in\mathbb{Z}} 2^{\frac k2}\|\Delta_k^{\rm h} v^3(\cdot,x_3)\|_{L^2_{\rm h}} c_k(x_3)$$
$$\times \sum_{|k'-k|\leq N_0} \|S^{\rm h}_{k'-1}\partial_3 v^\ell_{\sharp,E^{-1}}(\cdot,x_3)\|_{L^\infty_{\rm h}}\biggr)dx_3 \quad \text{with} \quad \sum_{k\in\mathbb{Z}} c_k^2(x_3)=1\,.$$

Thanks to Bernstein's inequality and by the equivalence (A.2) of Sobolev norms, we can write

$$\begin{aligned}
\|S^{\rm h}_{k'-1}\partial_3 v^\ell_{\sharp,E^{-1}}(\cdot,x_3)\|_{L^\infty_{\rm h}} & \leq \sum_{E^{-1}\lesssim 2^{k''}\leq 2^{k'-2}} \|\Delta^{\rm h}_{k''}\partial_3 v^\ell_{\sharp,E^{-1}}(\cdot,x_3)\|_{L^\infty_{\rm h}}\\
&\lesssim \sum_{E^{-1}\lesssim 2^{k''}\leq 2^{k'-2}} 2^{k''}\|\Delta^{\rm h}_{k''}\partial_3 v^\ell_{\sharp,E^{-1}}(\cdot,x_3)\|_{L^2_{\rm h}}\\
&\lesssim \sum_{E^{-1}\lesssim 2^{k''}\leq 2^{k'-2}} 2^{k''}\|\Delta^{\rm h}_{k''}\partial_3 v^\ell(\cdot,x_3)\|_{L^2_{\rm h}}\\
&\lesssim \|\nabla_{\rm h}\partial_3 v^\ell(\cdot,x_3)\|_{L^2_{\rm h}} \sum_{E^{-1}\lesssim 2^{k''}\leq 2^{k'-2}} c_{k''}(x_3)
\end{aligned}$$

with $\displaystyle\sum_{k\in\mathbb{Z}} c_k^2(x_3)=1$. Because $|k'-k|\leq N_0$, the Cauchy-Schwarz inequality implies the existence of a constant C such that

$$\forall x_3\in\mathbb{R}_{\rm v}\,,\quad \sum_{E^{-1}\lesssim 2^{k''}\leq 2^{k'-2}} c_{k''}(x_3)\leq C\sqrt{\log(2^kE+e)}\,.$$

Thus we infer that

$$J_E^{\sharp,2} \lesssim \|\nabla_{\rm h} v\|_{L^\infty_{\rm v}(\dot H^{\frac12}(\mathbb{R}^2_{\rm h}))}\int_{\mathbb{R}_{\rm v}} \|\nabla_{\rm h}\partial_3 v^\ell(\cdot,x_3)\|_{L^2_{\rm h}}$$
$$\times\biggl(\sum_{k\in\mathbb{Z}} 2^{\frac k2}\|\Delta_k^{\rm h} v^3(\cdot,x_3)\|_{L^2_{\rm h}} c_k(x_3)\sqrt{\log(2^kE+e)}\biggr)dx_3 \quad \text{with} \quad \sum_{k\in\mathbb{Z}} c_k^2(x_3)=1.$$

Using the Cauchy-Schwarz inequality we get

$$\sum_{k\in\mathbb{Z}} 2^{\frac k2}\|\Delta_k^{\rm h} v^3(\cdot,x_3)\|_{L^2_{\rm h}} c_k(x_3)\sqrt{\log(2^kE+e)} \leq \biggl(\sum_{k\in\mathbb{Z}} 2^k\|\Delta_k^{\rm h} v^3(\cdot,x_3)\|^2_{L^2_{\rm h}}\log(2^kE+e)\biggr)^{\frac12}.$$

Using the Cauchy-Schwarz inequality again we infer that

$$J_E^{\sharp,2} \lesssim \|\nabla_{\rm h} v\|_{L^\infty_{\rm v}(\dot H^{\frac12}(\mathbb{R}^2_{\rm h}))}\|\nabla_{\rm h} v\|_{\dot H^1(\mathbb{R}^3)}\|v^3\|_{\dot H^{\frac12}_{\log_{\rm h},E}}\,.$$

Lemma A.2 claims that

$$\|\nabla_{\rm h} v\|_{L^\infty_{\rm v}(\dot H^{\frac12}(\mathbb{R}^2_{\rm h}))}\lesssim \|\nabla_{\rm h} v\|_{\dot H^1(\mathbb{R}^3)}\,;$$

so we obtain

$$J_E^{\sharp,2} \lesssim \|\nabla_{\rm h} v\|^2_{\dot H^1(\mathbb{R}^3)}\|v^3\|_{\dot H^{\frac12}_{\log_h,E}}\,.$$

Together with (3.8) and (3.10), this concludes the proof of Lemma 2.3. □

Appendix A

Let us recall some elements of Littlewood-Paley theory (see for instance [1] for details). We define frequency truncation operators on $\mathbb{R}^2$,

$$\Delta_k^{\mathrm{h}} a \stackrel{\mathrm{def}}{=} \mathcal{F}^{-1}(\varphi(2^{-k}|\xi_{\mathrm{h}}|)\widehat{a}) \quad \text{and} \quad S_k^{\mathrm{h}} a \stackrel{\mathrm{def}}{=} \mathcal{F}^{-1}(\chi(2^{-k}|\xi_{\mathrm{h}}|)\widehat{a})\,, \tag{A.1}$$

where $\xi = (\xi_{\mathrm{h}}, \xi_3)$ and $\xi_{\mathrm{h}} = (\xi_1, \xi_2)$. We have denoted $\mathcal{F}a$ and $\widehat{a}$ for the Fourier transform of the distribution a, and χ and φ are smooth functions such that

$$\begin{aligned} &\mathrm{Supp}\ \varphi \subset \Big\{\tau \in \mathbb{R} \,/\ \ \frac{3}{4} \leq |\tau| \leq \frac{8}{3}\Big\} \quad \text{and} \quad \forall \tau > 0\,,\ \sum_{j\in\mathbb{Z}} \varphi(2^{-j}\tau) = 1\,,\\ &\mathrm{Supp}\ \chi \subset \Big\{\tau \in \mathbb{R} \,/\ \ |\tau| \leq \frac{4}{3}\Big\} \qquad \text{and} \quad \forall \tau \in \mathbb{R}\,,\ \chi(\tau) + \sum_{j\geq 0} \varphi(2^{-j}\tau) = 1\,. \end{aligned}$$

It is obvious that

$$\|a\|_{\dot H^s_{\mathrm{h}}} \stackrel{\mathrm{def}}{=} \|a\|_{\dot H^s(\mathbb{R}^2)} \stackrel{\mathrm{def}}{=} \big\||\cdot|^s \widehat{a}\big\|_{L^2(\mathbb{R}^2)} \sim \big\|(2^{ks}\|\Delta_k^{\mathrm{h}} a\|_{L^2(\mathbb{R}^2)})\big\|_{\ell^2(\mathbb{Z})}\,. \tag{A.2}$$

We recall the definition of horizontal Besov norms

$$\|a\|_{(\dot B^s_{p,q})_{\mathrm{h}}} \stackrel{\mathrm{def}}{=} \|a\|_{\dot B^s_{p,q}(\mathbb{R}^2)} \stackrel{\mathrm{def}}{=} \big\|(2^{ks}\|\Delta_k^{\mathrm{h}} a\|_{L^p(\mathbb{R}^2)})\big\|_{\ell^q(\mathbb{Z})}\,.$$

We also recall the following anisotropic Bernstein type lemma from [10, 22].

Lemma A.1. *Consider $\mathcal{B}_{\mathrm{h}}$ a ball of $\mathbb{R}^2_{\mathrm{h}}$ and $\mathcal{C}_{\mathrm{h}}$ a ring of $\mathbb{R}^2_{\mathrm{h}}$; fix $1 \leq p_2 \leq p_1 \leq \infty$. Then the following properties hold:*

- If the support of $\widehat{a}$ is included in $2^k\mathcal{B}_{\mathrm{h}}$, then

$$\|\partial^\alpha_{x_{\mathrm{h}}} a\|_{L^{p_1}_{\mathrm{h}}} \lesssim 2^{k(|\alpha|+2(1/p_2-1/p_1))}\|a\|_{L^{p_2}_{\mathrm{h}}}\,.$$

- If the support of $\widehat{a}$ is included in $2^k\mathcal{C}_{\mathrm{h}}$, then

$$\|a\|_{L^{p_1}_{\mathrm{h}}} \lesssim 2^{-k}\|\nabla_{\mathrm{h}} a\|_{L^{p_1}_{\mathrm{h}}}\,.$$

The following lemma is also useful in this anisotropic context.

Lemma A.2. *For any function a in the space $\dot H^{s+\frac12}(\mathbb{R}^3)$ with $1/2 \leq s < 1$, there holds*

$$\|a\|_{L^\infty_{\mathrm{v}}(\dot H^s_{\mathrm{h}})} \leq \sqrt{2}\|a\|_{\dot H^{s+\frac12}(\mathbb{R}^3)}\,.$$

Proof. By density we can assume that v is smooth and compactly supported. Let us write that

$$\frac{d}{dx_3}\int_{\mathbb{R}^2_{\mathrm{h}}} |\xi_{\mathrm{h}}|^{2s}|\widehat{a}(\xi_{\mathrm{h}},x_3)|^2\,d\xi_{\mathrm{h}} = 2\Re e\int_{\mathbb{R}^2_{\mathrm{h}}} |\xi_{\mathrm{h}}|^{s+\frac12}\widehat{a}(\xi_{\mathrm{h}},x_3)|\xi_{\mathrm{h}}|^{s-\frac12}\partial_{x_3}\widehat{a}(\xi_{\mathrm{h}},x_3)\,d\xi_{\mathrm{h}}\,.$$

The Cauchy-Schwarz inequality implies that

$$\Big|\frac{d}{dx_3}\int_{\mathbb{R}^2_{\mathrm{h}}} |\xi_{\mathrm{h}}|^{2s}|\widehat{a}(\xi_{\mathrm{h}},x_3)|^2\,d\xi_{\mathrm{h}}\Big| \leq 2\|a(\cdot,x_3)\|_{\dot H^{s+\frac12}_{\mathrm{h}}}\|\partial_{x_3}a(\cdot,x_3)\|_{\dot H^{s-\frac12}_{\mathrm{h}}}\,.$$

Taking the L^1 norm with respect to x_3 and using again the Cauchy-Schwarz inequality gives, for any x_3 in $\mathbb{R}_{\mathrm{v}}$,

$$\int_{\mathbb{R}^2_{\mathrm{h}}} |\xi_{\mathrm{h}}|^{2s}|\widehat{a}(\xi_{\mathrm{h}},x_3)|^2\,d\xi_{\mathrm{h}} \leq 2\|a\|_{L^2_{\mathrm{v}}(\dot H^{s+\frac12}_{\mathrm{h}})}\|\partial_{x_3}a\|_{L^2_{\mathrm{v}}(\dot H^{s-\frac12}_{\mathrm{h}})} \leq 2\|a\|^2_{\dot H^{s+\frac12}(\mathbb{R}^3)}\,.$$

The lemma follows. □

We recall that

$$\|ab\|_{\dot H^{\frac12}(\mathbb{R}^2)} \lesssim \|a\|_{\dot B^1_{2,1}(\mathbb{R}^2)}\|b\|_{\dot H^{\frac12}(\mathbb{R}^2)}\,.$$

The following law of product in $\mathbb{R}^2$ is also useful.

Lemma A.3. *For any functions $a \in L^\infty \cap \dot{H}^1(\mathbb{R}^2)$ and $b \in \dot{H}^{\frac{1}{2}}(\mathbb{R}^2)$, there holds*

$$\|ab\|_{\dot{H}^{\frac{1}{2}}(\mathbb{R}^2)} \lesssim \big(\|a\|_{L^\infty(\mathbb{R}^2)} + \|a\|_{\dot{H}^1(\mathbb{R}^2)}\big)\|b\|_{\dot{H}^{\frac{1}{2}}(\mathbb{R}^2)}\,.$$

Proof. We recall Bernstein's inequality

$$\|\Delta_k^{\rm h} a\|_{L^\infty(\mathbb{R}^2)} + \|\nabla_{\rm h}\Delta_k^{\rm h} a\|_{L^2(\mathbb{R}^2)} \lesssim 2^k\|\Delta_k^{\rm h} a\|_{L^2(\mathbb{R}^2)}\,.$$

Let us introduce Bony's decomposition (in a simplified version of [3]) writing that

$$ab = T_a^{\rm h} b + \widetilde{T}_b^{\rm h} a \quad \text{with} \quad T_a^{\rm h} b \stackrel{\rm def}{=} \sum_k S_{k-1}^{\rm h} a \Delta_k^{\rm h} b \quad \text{and} \quad \widetilde{T}_b^{\rm h} a \stackrel{\rm def}{=} \sum_k S_{k+2}^{\rm h} b \Delta_k^{\rm h} a\,. \tag{A.3}$$

Theorem 2.47 and Theorem 2.52 of [1] claim that

$$\|T_a^{\rm h} b\|_{\dot{H}^{\frac{1}{2}}(\mathbb{R}^2)} \lesssim \|a\|_{L^\infty(\mathbb{R}^2)}\|b\|_{\dot{H}^{\frac{1}{2}}(\mathbb{R}^2)} \quad \text{and} \quad \|\widetilde{T}_b^{\rm h} a\|_{\dot{H}^{\frac{1}{2}}(\mathbb{R}^2)} \lesssim \|a\|_{\dot{H}^1(\mathbb{R}^2)}\|b\|_{\dot{H}^{\frac{1}{2}}(\mathbb{R}^2)}\,. \tag{A.4}$$

It is clear that (A.3) and (A.4) imply the lemma. □

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(J.-Y. Chemin) Laboratoire Jacques Louis Lions - UMR 7598, Sorbonne Université, Boîte courrier 187, 4 place Jussieu, 75252 Paris Cedex 05, France
E-mail address: chemin@ann.jussieu.fr

(I. Gallagher) DMA, École normale supérieure, CNRS, PSL Research University, 75005 Paris, and UFR de mathématiques, Université Paris-Diderot, Sorbonne Paris-Cité, 75013 Paris, France.
E-mail address: gallagher@math.ens.fr

(P. Zhang) Academy of Mathematics & Systems Science and Hua Loo-Keng Key Laboratory of Mathematics, The Chinese Academy of Sciences, China, and School of Mathematical Sciences, University of Chinese Academy of Sciences, Beijing 100049, China.
E-mail address: zp@amss.ac.cn