RESEARCH ARTICLE

Long-time correlations for a hard-sphere gas at equilibrium

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Abstract
It has been known since Lanford that the dynamics of a hard-sphere gas is described in the low density limit by the Boltzmann equation, at least for short times. The classical strategy of proof fails for longer times, even close to equilibrium. In this paper, we introduce a weak convergence method coupled with a sampling argument to prove that the covariance of the fluctuation field around equilibrium is governed by the linearized Boltzmann equation globally in time (including in diffusive regimes). This method is much more robust and simpler than the one devised in Bodineau et al which was specific to the 2D case.

CONTENTS

1 | INTRODUCTION

The goal of this paper is to study the dynamical fluctuations of a hard sphere gas at equilibrium in the low density limit. The equilibrium is described by a Gibbs measure, which

is a product measure up to the spatial exclusion of the particles, and stationary under the microscopic dynamics.

A major challenge in statistical physics is to understand the long time behavior of the correlations even in an equilibrium regime. Our goal is to prove that the fluctuations are described in the low density limit by the fluctuating Boltzmann equation on long kinetic times. The present paper provides a first step of this program, by characterizing the evolution of the covariance of the fluctuations on such time scales.

Time correlations are expected to evolve deterministically as dictated by the linearized Boltzmann equation. At the mathematical level, such a result can be regarded as a variant of the rigorous validity of the nonlinear Boltzmann equation, which was first obtained for short times in [22] (see also [9, 10, 13–15, 19, 25, 28]). In fact the same method as in [22], combined with a low density expansion of the invariant measure, was applied in [30] to prove the validity of the linearized Boltzmann equation. The result in [30] suffered however from the same time restriction of the nonlinear case, in spite of the fact that the solution to the linearized Boltzmann equation is globally well defined.

This limitation was finally overcome in [3], in the case of a two-dimensional gas of hard disks. The method of [3] used, in particular, that the canonical partition function is uniformly bounded in two space dimensions. For $d \geq 3$ the limit is however more singular, as the accessible volume in phase space is exponentially small. The goal of the present paper is to present a much more robust method, based on weak convergence and on a duality argument, which does not depend on dimension. Our analysis is quantitative and the validity holds for arbitrarily large kinetic times, even slowly diverging. Hence a hydrodynamical limit can be also obtained in the same way as in [3], but we shall not repeat this discussion here.

The weak convergence method discussed in this paper allows actually to construct the limit of higher order moments of the fluctuation field and show their asymptotic factorization according to the Wick rule, providing a central limit theorem and thereby completing the program. This result, which requires a nontrivial combination with the cumulant techniques developed in [5, 6], will be presented in a companion work [4].

1.1 | The hard-sphere model

The microscopic model consists of identical hard spheres of unit mass and of diameter ε.

The motion of N such hard spheres (see Figure 1) is governed by a system of ordinary differential equations, which are set in $\mathbb{D}^N := (\mathbb{T}^d \times \mathbb{R}^d)^N$ where $\mathbb{T}^d$ is the unit d-dimensional periodic box: writing $\mathbf{x}_i^\varepsilon \in \mathbb{T}^d$ for the position of the center of the particle labeled i and $\mathbf{v}_i^\varepsilon \in \mathbb{R}^d$ for its velocity, one has

$$\frac{d\mathbf{x}_i^\varepsilon}{dt} = \mathbf{v}_i^\varepsilon, \quad \frac{d\mathbf{v}_i^\varepsilon}{dt} = 0 \quad \text{as long as } |\mathbf{x}_i^\varepsilon(t) - \mathbf{x}_j^\varepsilon(t)| > \varepsilon \quad \text{for } 1 \leq i \neq j \leq N, \tag{1.1}$$

with specular reflection at collisions:

$$\left. \begin{aligned} \left(\mathbf{v}_i^\varepsilon\right)' &:= \mathbf{v}_i^\varepsilon - \frac{1}{\varepsilon^2} \left(\mathbf{v}_i^\varepsilon - \mathbf{v}_j^\varepsilon\right) \cdot \left(\mathbf{x}_i^\varepsilon - \mathbf{x}_j^\varepsilon\right) \left(\mathbf{x}_i^\varepsilon - \mathbf{x}_j^\varepsilon\right) \\ \left(\mathbf{v}_j^\varepsilon\right)' &:= \mathbf{v}_j^\varepsilon + \frac{1}{\varepsilon^2} \left(\mathbf{v}_i^\varepsilon - \mathbf{v}_j^\varepsilon\right) \cdot \left(\mathbf{x}_i^\varepsilon - \mathbf{x}_j^\varepsilon\right) \left(\mathbf{x}_i^\varepsilon - \mathbf{x}_j^\varepsilon\right) \end{aligned} \right\} \quad \text{if } \left|\mathbf{x}_i^\varepsilon(t) - \mathbf{x}_j^\varepsilon(t)\right| = \varepsilon. \tag{1.2}$$
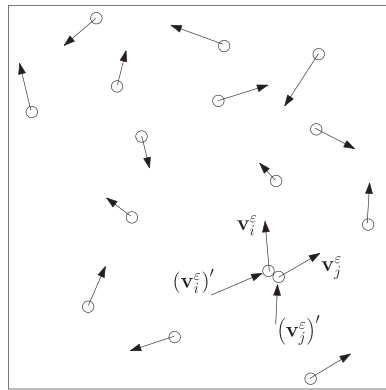
FIGURE 1 Transport and collisions in a hard-sphere gas. The square box represents the d-dimensional torus.

The sign of the scalar product $(\mathbf{v}_i^\varepsilon - \mathbf{v}_j^\varepsilon) \cdot (\mathbf{x}_i^\varepsilon - \mathbf{x}_j^\varepsilon)$ identifies post-collisional $(+)$ and pre-collisional $(-)$ configurations. This flow does not cover all possible situations, as multiple collisions are excluded. But one can show (see [1]) that for almost every admissible initial configuration $(\mathbf{x}_i^{\varepsilon 0}, \mathbf{v}_i^{\varepsilon 0})_{1 \leq i \leq N}$, there are neither multiple collisions, nor accumulations of collision times, so that the dynamics is globally well defined.

We are not interested here in one specific realization of the dynamics, but rather in a statistical description. This is achieved by introducing a measure at time 0, on the phase space we now specify. The collections of N positions and velocities are denoted respectively by $X_N := (x_1, \dots, x_N)$ in $\mathbb{T}^{dN}$ and $V_N := (v_1, \dots, v_N)$ in $\mathbb{R}^{dN}$, we set $Z_N := (X_N, V_N)$ in $(\mathbb{T}^d \times \mathbb{R}^d)^N$, with $Z_N = (z_1, \dots, z_N)$. Our fundamental random variable is the time-zero configuration, consisting of the initial positions and velocities of all the particles of the gas $(\mathbf{z}_i^{\varepsilon 0})_i = (\mathbf{x}_i^{\varepsilon 0}, \mathbf{v}_i^{\varepsilon 0})_i$ in the phase space

$$\mathcal{D}_N^\varepsilon := \left\{ Z_N \in \mathbb{D}^N \,/\, \forall i \neq j, \quad |x_i - x_j| > \varepsilon \right\}.$$

The particle dynamics $\mathbf{Z}_N^\varepsilon = (\mathbf{z}_1^\varepsilon, \dots, \mathbf{z}_N^\varepsilon)$ solution of the hard-sphere flow (1.1)–(1.2) with random initial data $\mathbf{Z}_N^{\varepsilon 0}$, evolves in $\mathcal{D}_N^\varepsilon$ (and it is well defined with probability 1). Actually, to avoid spurious correlations due to a given total number of particles, we shall consider a grand canonical state: the total number of particles is itself a random variable, which we will denote by $\mathcal{N}$. The particle configuration is therefore $\mathbf{Z}_\mathcal{N}^\varepsilon = (\mathbf{z}_1^\varepsilon, \dots, \mathbf{z}_\mathcal{N}^\varepsilon)$, distributed according to the equilibrium measure as follows. The probability density of finding N particles in Z_N is given by

$$\frac{1}{N!} M_N^\varepsilon(Z_N) := \frac{1}{\mathcal{Z}^\varepsilon} \frac{\mu_\varepsilon^N}{N!} \mathbf{1}_{D_N^\varepsilon}(Z_N) M^{\otimes N}(V_N), \qquad \text{for } N = 0, 1, 2, \dots \tag{1.3}$$

with $\mu_\varepsilon > 0$ (typical number of particles) tuned as explained below,

$$M(v) := \frac{1}{(2\pi)^{\frac{d}{2}}} \exp\left(-\frac{|v|^2}{2}\right), \qquad M^{\otimes N}(V_N) := \prod_{i=1}^N M(v_i), \tag{1.4}$$

and the partition function given by

$$\mathcal{Z}^\varepsilon := 1 + \sum_{N \geq 1} \frac{\mu_\varepsilon^N}{N!} \int_{\mathbb{T}^{dN} \times \mathbb{R}^{dN}} \left(\prod_{i \neq j} \mathbf{1}_{|x_i - x_j| > \varepsilon} \right) \left(\prod_{i=1}^N M(v_i) \right) dX_N \, dV_N. \tag{1.5}$$
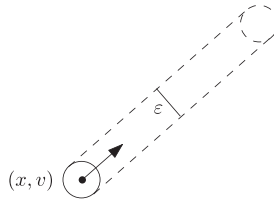
FIGURE 2 The collision cylinder of diameter ε has volume proportional to $\varepsilon^{d-1}|v|\tau$, where τ is the time of free flight.

In the following the probability of an event A with respect to the Gibbs measure (1.3) will be denoted $\mathbb{P}_\varepsilon(A)$, and $\mathbb{E}_\varepsilon$ will be the expected value.

In the low density regime, the density (average total number) of particles is tuned by the parameter $\mu_\varepsilon := \varepsilon^{-(d-1)}$, ensuring that the mean free path between collisions is of order one [16]. With this choice of μ_ε, (1.3)–(1.5) imply indeed that $\mathcal{N}$ (distributed according to a quasi-Poisson process) satisfies

$$\lim_{\varepsilon\to 0} \mathbb{E}_\varepsilon(\mathcal{N})\varepsilon^{d-1} = 1\,,$$

and the fraction of volume occupied by the spheres $\sim \mathbb{E}_\varepsilon(\mathcal{N})\varepsilon^d$ goes to zero. Furthermore, the volume covered by a particle (with velocity of order 1) in a unit of time is $O(\varepsilon^{d-1})$ (see Figure 2). Hence the above scaling relation implies that the typical free flight time between collisions is of order 1, as well as the mean free path.

If the particles are distributed according to the Gibbs measure (1.3)–(1.5), the limit $\varepsilon \to 0$ provides then an ideal gas with velocity distribution M.

1.2 | The linearized Boltzmann equation

Out of equilibrium, if the particles are initially identically distributed according to a smooth, sufficiently decaying function f^0, for example according to a grand canonical density

$$\frac{\mu_\varepsilon^N}{N!}\,\mathbf{1}_{D_N^\varepsilon}(Z_N)\,(f^0)^{\otimes N}(Z_N) \tag{1.6}$$

(generalizing (1.3)), then in the low density regime $\mu_\varepsilon \to \infty$, the average behavior is governed for short times by the Boltzmann equation [22]

$$\begin{cases} \partial_t f + v \cdot \nabla_x f = \displaystyle\int_{\mathbb{R}^d} \int_{\mathbb{S}^{d-1}} (f(t,x,w')f(t,x,v') - f(t,x,w)f(t,x,v)) \\ \qquad\qquad\qquad\qquad\qquad \times ((v-w)\cdot\omega)_+ \, d\omega \, dw\,, \\ f(0,x,v) = f^0(x,v) \end{cases}$$

where the velocities (v', w') are defined by the scattering law

$$v' := v - ((v-w)\cdot\omega)\,\omega\,, \qquad w' := w + ((v-w)\cdot\omega)\,\omega\,. \tag{1.7}$$

At equilibrium, the convergence holds for all times since the Gibbs measure (1.3)–(1.5) is invariant under the microscopic flow (1.1)–(1.2) and M is a stationary solution to the Boltzmann equation. In particular, the empirical density defined by

$$\pi_t^\varepsilon := \frac{1}{\mu_\varepsilon} \sum_{i=1}^{\mathcal{N}} \delta_{\mathbf{z}_i^\varepsilon(t)} \tag{1.8}$$

concentrates on M: for any test function $h : \mathbb{D} \to \mathbb{R}$ and any $\delta > 0, t \in \mathbb{R}$,

$$\mathbb{P}_\varepsilon\left(\left|\pi_t^\varepsilon(h) - \mathbb{E}_\varepsilon\left(\pi_t^\varepsilon(h)\right)\right| > \delta\right) \xrightarrow[\mu_\varepsilon \to \infty]{} 0 . \tag{1.9}$$

It is well-known that the Boltzmann equation dissipates entropy, contrary to the original particle system (1.1)–(1.2) which is time reversible. Thus some information is lost in the low density regime, and describing the fluctuations is a first way to capture part of this lost information. As in the standard central limit theorem, we expect these fluctuations to be of order $1/\sqrt{\mu_\varepsilon}$. We therefore define the fluctuation field ζ^ε by

$$\zeta_t^\varepsilon(h) := \sqrt{\mu_\varepsilon} \left(\pi_t^\varepsilon(h) - \mathbb{E}_\varepsilon\left(\pi_t^\varepsilon(h)\right)\right) \tag{1.10}$$

for any test function h. This process ζ^ε has been studied for short times in [5, 6] and was proved to solve a fluctuating equation. Here we focus on the time correlation

$$\mathrm{Cov}_\varepsilon(t, g_0, h) := \mathbb{E}_\varepsilon\left(\zeta_0^\varepsilon(g_0)\zeta_t^\varepsilon(h)\right) . \tag{1.11}$$

Before stating our main result, let us define the linearized Boltzmann operator

$$\mathcal{L}g := -v \cdot \nabla_x g + \int_{\mathbb{R}^d \times \mathbb{S}^{d-1}} M(w)((v - w) \cdot \omega)_+ \left[g(v') + g(w') - g(v) - g(w)\right] d\omega \, dw$$

which is well-defined in the space L_M^2, denoting for $1 \leq p < \infty$

$$L_M^p := \left\{ g : \mathbb{T}^d \times \mathbb{R}^d \to \mathbb{R}, \ \|g\|_{L_M^p} := \left(\int_{\mathbb{T}^d \times \mathbb{R}^d} |g|^p \, M dx dv\right)^{\frac{1}{p}} < \infty \right\}. \tag{1.12}$$

Theorem 1.1 (Linearized Boltzmann equation). *Consider a system of hard spheres at equilibrium in a d-dimensional periodic box with $d \geq 3$. Let g_0 and h be two functions in L_M^2. Then, in the low density regime $\mu_\varepsilon \to \infty$, the covariance of the fluctuation field $(\zeta_t^\varepsilon)_{t \geq 0}$ defined by (1.11) converges on $\mathbb{R}^+$ to $\int Mg(t)h dx dv$ where $g \in L_t^\infty(L_M^2)$ is the solution of the linearized Boltzmann equation*

$$\partial_t g = \mathcal{L}g , \tag{1.13}$$

with $g_{|t=0} = g_0$.

Remark 1.1. It is classical that there is a unique solution to the linearized Boltzmann equation, which is bounded globally in time in L_M^2 (see e.g., Section 7 in [9]).

The limit is stated for any fixed time $t > 0$, however one can choose $t \in [0, \theta]$ with θ diverging slowly with ε, as $o((\log |\log \varepsilon|)^{1/4})$. As shown in Section 2, in the case of smooth data $h, g_0 \in W^{1,\infty}(\mathbb{D})$ there holds for any $\tau \ll 1 \ll \theta$

$$\sup_{t \in [0,\theta]} \left| \mathrm{Cov}_\varepsilon(t, g_0, h) - \int Mg(t)h \, dx \, dv \right|$$

$$\leq C \|h\|_{W^{1,\infty}} \|g_0\|_{W^{1,\infty}} \left((\theta^3 \tau)^{1/2} + (C\theta)^{2^{\theta/\tau}} \varepsilon^{\frac{1}{8d}} \right).$$

In particular, the hydrodynamical limits hold true leading to the acoustic equations and Stokes-Fourier equations, as explained in [3]. Theorem 1.1 is a consequence of this estimate by a density argument (see Section 2).

Remark 1.2. The same result as Theorem 1.1 was proved in dimension 2 in [3] with a different, more technical and less robust strategy. The proof presented here could be adapted to the two-dimensional case, at the price of slightly more intricate geometric estimates (see Appendix B), but we choose not to deal with this case.

Remark 1.3. Previous work on the (more general) non-equilibrium setting (1.6) has led to construct the Gaussian limiting fluctuation field for short times by using cumulant expansions [5, 6, 25, 26]. For further discussions on the fluctuation theory of the hard sphere gas we refer to these references, as well as to [12, 27, 28]. By combining cumulant techniques and the weak convergence method of the present paper, we can actually derive the fluctuating Boltzmann equation at equilibrium for long times (see the companion paper [4]).

1.3 | Strategy and overview

Let us explain now our strategy in a very informal way, referring to Section 2 below for the technical details. Our goal is to construct the limit of $\mathrm{Cov}_\varepsilon(t, g_0, h)$ as $\mu_\varepsilon \to \infty$. We will therefore compute expectations of observables of the following type:

$$\mu_\varepsilon \mathbb{E}_\varepsilon \left(\pi_0^\varepsilon(g_0) \pi_t^\varepsilon(h) \right). \tag{1.14}$$

A classical way to proceed ([26, 30]) is to introduce the non-equilibrium measure obtained from the invariant measure by perturbing it with the sum $\sum_i g_0(\mathbf{z}_i^{\varepsilon 0})$

$$\mathbb{E}_\varepsilon \left(\left(\sum_{i=1}^{\mathcal{N}} g_0 \left(\mathbf{z}_i^{\varepsilon 0} \right) \right) \pi_t^\varepsilon(h) \right) =: \mathbb{E}_\varepsilon^{g_0} \left(\pi_t^\varepsilon(h) \right).$$

To compute expectations of the empirical measure $\pi_t^\varepsilon(h)$ under this non-equilibrium measure, one transports the non-equilibrium measure along the microscopic dynamics, and then takes its one-dimensional projection $G_1^\varepsilon(t)$:

$$\int G_1^\varepsilon(t, z) h(z) \, dz = \mathbb{E}_\varepsilon^{g_0} \left(\pi_t^\varepsilon(h) \right). \tag{1.15}$$

This leads to consider the whole family of finite-dimensional projections $G_k^\varepsilon(t)$ of the transported measure, namely the so-called correlations functions defined by

$$\int G_k^\varepsilon(t, Z_k) h_k(Z_k) dZ_k = \mathbb{E}_\varepsilon^{g_0}\left(\frac{1}{\mu_\varepsilon^k} \sum_{(i_1,\dots,i_k)} h_k(\mathbf{z}_{i_1}^\varepsilon(t), \dots, \mathbf{z}_{i_k}^\varepsilon(t)) \right), \qquad k = 1, 2, \dots \quad (1.16)$$

for arbitrary test functions h_k, where $(i_1, \dots, i_k)$ are k-tuples of particle labels. In fact, these functions satisfy an infinite hierarchy of coupled linear equations, referred to as the BBGKY hierarchy; see for example, [12] for the particular application to fluctuation fields.

The explicit, iterated (Duhamel) solution to this hierarchy is the basic formula in the proof of Lanford [22], and in most of the mathematical literature on the low density regime. Its main drawback is the above-mentioned time restriction, coming from too many terms in the iterated formula. This is ultimately due to the fact that we are unable to take advantage of cancellations between gain and loss terms, and this is true *even* in our equilibrium setting (as pointed out in [30]).

In this paper, we propose to take advantage in a more systematic way of the invariance of the Gibbs measure, in particular exploiting the symmetry between g_0 and h in formula (1.14). The general idea is to use the Duhamel iteration of the BBGKY hierarchy, which correlates configurations at time 0 with configurations at time t, only when the number of collisions is under control. In other cases, we would like to identify locally a pathological behavior (typically a set $\mathcal{B}$ of trajectories with an anomalously large number of collisions), and prove that the contribution of this bad set $\mathcal{B}$ to the covariance is negligible. To do this we use a *time decoupling* (and the invariance of the Gibbs measure), as in the following Cauchy-Schwarz estimate

$$\left| \mathbb{E}_\varepsilon\left(\zeta_0^\varepsilon(g_0) \zeta_t^\varepsilon(h) \mathbf{1}_\mathcal{B} \right) \right| \leq \mathbb{E}_\varepsilon(\mathbf{1}_\mathcal{B})^{1/4} \mathbb{E}_\varepsilon\left(\left(\zeta_0^\varepsilon(h)^4 \right)^{1/4} \mathbb{E}_\varepsilon\left(\left(\zeta_0^\varepsilon(g_0)^2 \right)^{1/2} \right). \quad (1.17)$$

In order to implement this intuition, we will actually proceed iteratively, by using the Duhamel formula on elementary (small) time intervals. There are then two main ingredients:

(i) suitable *stopping rules* on collision processes. We will use a refined version of the sampling procedure introduced in [2, 3] (and reminiscent of those explained in [11] in a quantum setting). In essence, one checks trajectories locally in each time interval, and stops at $t_{\text{stop}} \in (0, t)$ when a pathological behavior is found; see Section 2.4 for details.

(ii) a *weak convergence* argument relating the Duhamel expansion and some geometric representation of the correlations. The issue here is to introduce geometric constraints on the trajectories of finite subsets of particles, using as integration variables the configurations of particles at time t_{stop}, and characterizing locally the pathological sets. This representation, which will be discussed in Section 2.3, is the key tool to rewrite remainder terms as an *expectation* over symmetric sets of pathological trajectories, allowing an effective time decoupling (see Equation (2.19) below).

The paper is organized as follows. In Section 2 we setup our strategy, introduce several error terms and list the corresponding estimates. Section 3 contains a general bound in a L^2-norm (based on a cluster expansion), which is then used in Sections 4, 5, 6 to control the principal part and the error terms. The required geometric estimates on recollision sets are discussed in Appendix B,

restricting this part for brevity to $d \geq 3$. Appendix A is devoted to the proof of some a priori estimates on the moments of the fluctuation field.

2 | PROOF OF THEOREM 1.1. MAIN STEPS

2.1 | Reduction to smooth mean free data

Let us first prove that, without loss of generality, we can restrict our attention to functions g_0, h satisfying

$$\int M g_0 dz = \int M h dz = 0. \tag{2.1}$$

We start by noticing that there is a constant c_ε such that for all $h \in L_M^2$,

$$\mathbb{E}_\varepsilon\big(\pi_t^\varepsilon(h)\big) = c_\varepsilon \int_\mathbb{D} M(v) h(z) dz. \tag{2.2}$$

Indeed

$$\mathbb{E}_\varepsilon\big(\pi_t^\varepsilon(h)\big) = \frac{1}{\mathcal{Z}^\varepsilon} \sum_{n \geq 1} \frac{\mu_\varepsilon^{n-1}}{(n-1)!} \int_{D_n^\varepsilon} M^{\otimes n}(V_n) h(z_1) \, dZ_n$$

$$= \int dz_1 M(v_1) h(z_1) \left(\frac{1}{\mathcal{Z}^\varepsilon} \sum_{p \geq 0} \frac{\mu_\varepsilon^p}{p!} \int_{D_\varepsilon^p} d\bar{Z}_p \prod_{1 \leq i \leq p} \mathbf{1}_{|x_1 - \bar{x}_i| > \varepsilon} M^{\otimes p}(\bar{V}_p) \right)$$

$$= c_\varepsilon \int_\mathbb{D} M(v) h(z) dz$$

using the translation invariance. Expanding the exclusion condition $\prod_{1 \leq i \leq p} \mathbf{1}_{|x_1 - \bar{x}_i| > \varepsilon}$ actually leads to $c_\varepsilon = 1 + O(\varepsilon)$ but this fact will not be used in the following.

Denoting by $\langle \cdot \rangle$ the average with respect to the probability measure $M dv dx$, and setting $\widehat{g} := g - \langle g \rangle$, we get according to (2.2),

$$\mathbb{E}_\varepsilon\big(\pi_t^\varepsilon(\widehat{g}_0)\big) = \mathbb{E}_\varepsilon\big(\pi_t^\varepsilon(\widehat{h})\big) = 0.$$

Now, shifting g_0 and h by their averages boils down to recording the fluctuation of the total number of particles (in the grand canonical ensemble)

$$\mathrm{Cov}_\varepsilon(t, g_0, h) = \mathrm{Cov}_\varepsilon(t, \widehat{g}_0, \widehat{h}) + \langle g_0 \rangle \mathbb{E}_\varepsilon\big(\zeta^\varepsilon(1)\zeta_t^\varepsilon(\widehat{h})\big) + \langle h \rangle \mathbb{E}_\varepsilon\big(\zeta^\varepsilon(1)\zeta_0^\varepsilon(\widehat{g}_0)\big)$$

$$+ \langle h \rangle \langle g_0 \rangle \mathbb{E}_\varepsilon\big(\zeta^\varepsilon(1)^2\big),$$

where we used the time independent field $\zeta^\varepsilon(1) = \frac{1}{\sqrt{\mu_\varepsilon}}(\mathcal{N} - \mathbb{E}_\varepsilon(\mathcal{N}))$. Using the time invariance of the Gibbs measure, the time evolution of Cov_ε is unchanged

$$\partial_t \mathrm{Cov}_\varepsilon(t, g_0, h) = \partial_t \mathrm{Cov}_\varepsilon(t, \widehat{g}_0, \widehat{h})$$

and the result follows from (1.13) and the fact that for all functions h_1 and h_2 in L_M^2

$$\int M(\mathcal{L}\widehat{h}_1)\widehat{h}_2 \, dx dv = \int M(\mathcal{L}h_1)h_2 \, dx dv \,.$$

It will be also useful in the following to work with functions g_0 and h with additional smoothness (namely assuming g_0 Lipschitz in space, and both functions to be in L^∞ and not only L_M^2). For this we notice that we can introduce sequences of smooth, mean free functions $(g_0^\alpha)_{\alpha>0}$ and $(h^\alpha)_{\alpha>0}$ approximating g_0 and h in L_M^2 as $\alpha \to 0$. By the Cauchy-Schwarz inequality there holds for all mean free functions h_1 and h_2 in L_M^2

$$\text{Cov}_\varepsilon(t, h_1, h_2) = \mathbb{E}_\varepsilon\big(\zeta_0^\varepsilon(h_1)\zeta_t^\varepsilon(h_2)\big)$$

$$\leq \mathbb{E}_\varepsilon\big(\zeta_0^\varepsilon(h_1)^2\big)^{\frac{1}{2}} \mathbb{E}_\varepsilon\big(\zeta_t^\varepsilon(h_2)^2\big)^{\frac{1}{2}} \,,$$

which is bounded uniformly (for small ε) by virtue of the a priori estimate (see [28] or Remark 3.3 below)

$$\forall h \in L_M^2 \,, \quad \mathbb{E}_\varepsilon\big(\zeta_t^\varepsilon(h)^2\big)^{\frac{1}{2}} \leq C\|h\|_{L_M^2} \,, \quad C > 0 \,. \tag{2.3}$$

In particular

$$\big|\text{Cov}_\varepsilon(t, g_0, h) - \text{Cov}_\varepsilon(t, g_0^\alpha, h^\alpha)\big| \longrightarrow 0 \,, \quad \alpha \to 0 \,,$$

uniformly in ε. In the following, we therefore assume that g_0 and h are mean free and smooth.

2.2 | The Duhamel iteration

For any test function $h : \mathbb{D} \to \mathbb{R}$, let us compute

$$\mathbb{E}_\varepsilon\big(\zeta_0^\varepsilon(g_0)\zeta_t^\varepsilon(h)\big) = \frac{1}{\mu_\varepsilon}\mathbb{E}_\varepsilon\left(\left(\sum_{i=1}^{\mathcal{N}} g_0\big(\mathbf{z}_i^\varepsilon(0)\big)\right)\left(\sum_{i=1}^{\mathcal{N}} h\big(\mathbf{z}_i^\varepsilon(t)\big)\right)\right).$$

Thanks to the exchangeability of the particles, this can be written

$$\mathbb{E}_\varepsilon\big(\zeta_0^\varepsilon(g_0)\zeta_t^\varepsilon(h)\big) = \int G_1^\varepsilon(t, z) \, h(z) \, dz \tag{2.4}$$

where G_1^ε is the one-particle correlation function

$$G_1^\varepsilon(t, z_1) := \frac{1}{\mu_\varepsilon} \sum_{p=0}^{\infty} \frac{1}{p!} \int_{\mathbb{D}^p} dz_2 \dots dz_{1+p} \, W_{1+p}^\varepsilon(t, Z_{1+p}) \,,$$

and $W_N^\varepsilon(t)$ is defined as follows. At time zero we set

$$\frac{1}{N!}W_N^{\varepsilon 0}(Z_N) := \frac{1}{\mathcal{Z}^\varepsilon}\frac{\mu_\varepsilon^N}{N!}\mathbf{1}_{\mathcal{D}_N^\varepsilon}(Z_N)M^{\otimes N}(V_N)\sum_{i=1}^N g_0(z_i), \tag{2.5}$$

and $W_N^\varepsilon(t)$ solves the Liouville equation

$$\partial_t W_N^\varepsilon + V_N \cdot \nabla_{X_N}W_N^\varepsilon = 0 \quad \text{on} \quad \mathcal{D}_N^\varepsilon, \tag{2.6}$$

with specular reflection (1.2) on the boundary $|x_i - x_j| = \varepsilon$. We actually extend W_N^ε by zero outside $\mathcal{D}_N^\varepsilon$.

As a consequence, to prove Theorem 1.1 we need to prove that $G_1^\varepsilon(t)$ converges for all times to $Mg(t)$, where g solves the linearized Boltzmann equation (1.13) with initial datum g_0.

Similarly for any test function $h_n : \mathbb{D}^n \to \mathbb{R}$, one defines the n-particle correlation function

$$G_n^\varepsilon(t, Z_n) := \frac{1}{\mu_\varepsilon^n}\sum_{p=0}^\infty \frac{1}{p!}\int_{\mathbb{D}^p} dz_{n+1}\dots dz_{n+p}\, W_{n+p}^\varepsilon(t, Z_{n+p}) \tag{2.7}$$

so that

$$\mathbb{E}_\varepsilon\left(\frac{1}{\mu_\varepsilon^n}\left(\sum_{i=1}^{\mathcal{N}} g_0\big(\mathbf{z}_i^\varepsilon(0)\big)\right)\left(\sum_{(i_1,\dots i_n)} h_n\big(\mathbf{z}_{i_1}^\varepsilon(t),\dots,\mathbf{z}_{i_n}^\varepsilon(t)\big)\right)\right) = \int G_n^\varepsilon(t, Z_n)\, h_n(Z_n)\, dZ_n.$$

Here and below we use the shortened notation for n-tuples

$$\sum_{(i_1,\dots,i_n)} = \sum_{\substack{i_1,\dots,i_n\in\{1,\dots,\mathcal{N}\} \\ i_j\neq i_k,\, j\neq k}}.$$

Remark 2.1. From the explicit structure of the Gibbs measure, the following uniform bound is derived in [30] (see also Lemma 6.1 below)

$$\left|G_n^\varepsilon(0, Z_n)\right| \leq C^n M^{\otimes n}(V_n)\|g_0\|_\infty$$

for some constant $C > 0$ independent of ε. However, one is unable to propagate this initial estimate in time improving the rough a priori bound

$$\left|G_n^\varepsilon(t, Z_n)\right| \leq \mu_\varepsilon C^n M^{\otimes n}(V_n)\|g_0\|_\infty.$$

For this reason, the theorem of Lanford cannot be applied iteratively to reach arbitrary times in this close-to-equilibrium setting; see [30] for details.

Using the Liouville equation (for fixed ε), we obtain that the one-particle correlation function $G_1^\varepsilon(t, x_1, v_1)$ satisfies

$$\partial_t G_1^\varepsilon + v_1 \cdot \nabla_{x_1}G_1^\varepsilon = C_{1,2}^\varepsilon G_2^\varepsilon \tag{2.8}$$

where the collision operator comes from the boundary terms in Green's formula (using the reflection condition to rewrite the gain part in terms of pre-collisional velocities):

$$(C_{1,2}^{\varepsilon} G_2^{\varepsilon})(x_1, v_1) := \int G_2^{\varepsilon}(x_1, v_1', x_1 + \varepsilon\omega, v_2')((v_2 - v_1) \cdot \omega)_+ \, d\omega dv_2$$

$$- \int G_2^{\varepsilon}(x_1, v_1, x_1 + \varepsilon\omega, v_2)((v_2 - v_1) \cdot \omega)_- \, d\omega dv_2 , \tag{2.9}$$

with as in (1.7)

$$v_1' = v_1 - (v_1 - v_2) \cdot \omega \, \omega , \quad v_2' = v_2 + (v_1 - v_2) \cdot \omega \, \omega .$$

Similarly, we have the following evolution equation for the n-particle correlation function:

$$\partial_t G_n^{\varepsilon} + V_n \cdot \nabla_{X_n} G_n^{\varepsilon} = C_{n,n+1}^{\varepsilon} G_{n+1}^{\varepsilon} \quad \text{on} \quad \mathcal{D}_n^{\varepsilon} , \tag{2.10}$$

with specular boundary reflection as in (2.6). This is the well-known BBGKY hierarchy (see [8]), which is the elementary brick in the proof of Lanford's theorem for short times. As $C_{1,2}^{\varepsilon}$ above, $C_{n,n+1}^{\varepsilon}$ describes collisions between one "fresh" particle (labeled $n + 1$) and one given particle $i \in \{1, \dots, n\}$. As in (2.9), this term is decomposed into two parts according to the hemisphere $\pm(v_{n+1} - v_i) \cdot \omega > 0$:

$$C_{n,n+1}^{\varepsilon} G_{n+1}^{\varepsilon} := \sum_{i=1}^{n} C_{n,n+1}^{\varepsilon,i} G_{n+1}^{\varepsilon}$$

with

$$(C_{n,n+1}^{\varepsilon,i} G_{n+1}^{\varepsilon})(Z_n) := \int G_{n+1}^{\varepsilon}(Z_n^{\langle i \rangle}, x_i, v_i', x_i + \varepsilon\omega, v_{n+1}')((v_{n+1} - v_i) \cdot \omega)_+ \, d\omega \, dv_{n+1}$$

$$- \int G_{n+1}^{\varepsilon}(Z_n, x_i + \varepsilon\omega, v_{n+1})((v_{n+1} - v_i) \cdot \omega)_- \, d\omega \, dv_{n+1} ,$$

where (v_i', v_{n+1}') is recovered from (v_i, v_{n+1}) through the scattering laws (1.7), and with the notation

$$Z_n^{\langle i \rangle} := (z_1, \dots, z_{i-1}, z_{i+1}, \dots, z_n) .$$

Note that performing the change of variables $\omega \longmapsto -\omega$ in the pre-collisional term gives rise to

$$(C_{n,n+1}^{\varepsilon,i} G_{n+1}^{\varepsilon})(Z_n) := \int \left(G_{n+1}^{\varepsilon}(Z_n^{\langle i \rangle}, x_i, v_i', x_i + \varepsilon\omega, v_{n+1}') - G_{n+1}^{\varepsilon}(Z_n, x_i - \varepsilon\omega, v_{n+1}) \right)$$

$$\times ((v_{n+1} - v_i) \cdot \omega)_+ \, d\omega \, dv_{n+1} .$$

Since the equation on G_n^{ε} involves G_{n+1}^{ε}, obtaining the convergence of G_1^{ε} requires understanding the behavior of the whole family $(G_n^{\varepsilon})_{n \geq 1}$. A natural first step consists in obtaining uniform

bounds. Denote by S_n^ε the group associated with free transport in $\mathcal{D}_n^\varepsilon$ (with specular reflection on the boundary). Iterating Duhamel's formula

$$G_n^\varepsilon(t) = S_n^\varepsilon(t)G_n^{\varepsilon 0} + \int_0^t S_n^\varepsilon(t-t_1)C_{n,n+1}^\varepsilon G_{n+1}^\varepsilon(t_1)\,dt_1$$

we can express formally the solution $G_n^\varepsilon(t)$ of the hierarchy (2.10) as a sum of operators acting on the initial data:

$$G_n^\varepsilon(t) = \sum_{m \geq 0} Q_{n,n+m}^\varepsilon(t)G_{n+m}^{\varepsilon 0}, \tag{2.11}$$

where we have defined for $t > 0$

$$Q_{n,n+m}^\varepsilon(t)G_{n+m}^{\varepsilon 0} := \int_0^t \int_0^{t_1} \cdots \int_0^{t_{m-1}} S_n^\varepsilon(t-t_1)C_{n,n+1}^\varepsilon S_{n+1}^\varepsilon(t_1-t_2)C_{n+1,n+2}^\varepsilon$$
$$\ldots S_{n+m}^\varepsilon(t_m)G_{n+m}^{\varepsilon 0}\,dt_m\ldots dt_1 \tag{2.12}$$

and $Q_{n,n}^\varepsilon(t)G_n^{\varepsilon 0} := S_n^\varepsilon(t)G_n^{\varepsilon 0}$.

Let us sketch how an a priori L^∞ bound can be derived from the series expansion (2.11). We say that a belongs to the set of (ordered, signed) collision trees $\mathcal{A}_{n,m}^\pm$ if $a = (a_i, s_i)_{1 \leq i \leq m}$ with labels $a_i \in \{1, \ldots, n+i-1\}$ describing which particle collides with particle $n+i$, and with signs $s_i \in \{-, +\}$ specifying the collision hemispheres. Each elementary integral appearing in the operator $Q_{n,n+m}^\varepsilon$ thus corresponds to a collision tree in $\mathcal{A}_{n,m}^\pm$ with m branching points, involving a simplex in time ($t_1 > t_2 > \ldots > t_m$). If we replace, for simplicity, the cross-section factors by a bounded function (cutting off high energies), we immediately get that the integrals are bounded, for each fixed collision tree $a \in \mathcal{A}_{n,m}^\pm$, by

$$\|G_{m+n}^{\varepsilon 0}\|_\infty \frac{(Ct)^m}{m!} \leq \|g_0\|_{L^\infty} C_0^{n+m} \frac{(Ct)^m}{m!}$$

(see Remark 2.1). Since $|\mathcal{A}_{n,m}^\pm| = 2^m(m+n-1)!/(n-1)!$, summing over all trees gives rise to a bound $C^{n+m}t^m\|g_0\|_{L^\infty}$. The series expansion is therefore uniformly absolutely convergent only for short times. In the presence of the true cross-section factor, the result remains valid (with a slightly different value of the convergence radius), though the proof requires some extra care [21, 22].

2.3 | Pseudo-trajectories and symmetric expectations

In Lanford's strategy described above, the number of collisions m is not under control a priori and this is the reason for the short time of validity of the result in [30]. To extend the convergence for long times, it is therefore crucial to control the number of collisions. The idea is to introduce a sampling, apply Lanford's strategy on elementary time intervals, and discard terms corresponding locally to too many collisions. For this, we shall use a geometric interpretation of the expansion (2.11)-(2.12), which we introduce next (see e.g., [6]).

2.3.1 | Pseudo-trajectories

For all parameters $(t_i, \omega_i, v_{n+i})_{i=1,\dots,m}$ with $t_i > t_{i+1}$ and all collision trees $a \in \mathcal{A}_{n,m}^{\pm}$, one constructs *pseudo-trajectories* on $[0, t]$

$$\Psi_{n,m}^{\varepsilon} = \Psi_{n,m}^{\varepsilon}\big(Z_n, (a_i, s_i, t_i, \omega_i, v_{n+i})_{i=1,\dots,m}\big)$$

iteratively on $i = 1, 2, \dots, m$ as follows (denoting by $Z_{n+i}^{\varepsilon}(\tau)$ the coordinates of the pseudo-particles at time $\tau \leq t_i$, and setting $t_0 = t$):

- starting from Z_n at time t,
- transporting all existing particles backward on (t_i, t_{i-1}) (on $\mathcal{D}_{n+i-1}^{\varepsilon}$ with specular reflection on the boundary),
- adding a new particle labeled $n + i$ at time t_i, at position $x_{a_i}^{\varepsilon}(t_i) + \varepsilon s_i \omega_i$ and with velocity v_{n+i},
- applying the scattering rule (1.7) if $s_i > 0$.

We discard non admissible parameters for which this procedure is ill-defined; in particular we exclude values of ω_i corresponding to an overlap of particles (two spheres at distance strictly smaller than ε) as well as those such that $\omega_i \cdot (v_{n+i} - v_{a_i}^{\varepsilon}(t_i^+)) \leq 0$. In the following we denote by $\mathcal{G}_m^{\varepsilon}(a, Z_n)$ the set of admissible parameters.

Definition 2.2. We call <u>creation</u> the addition of a new particle $n + i$ at time t_i (with scattering or without).

We call <u>recollision</u> a collision between pre-existing particles corresponding to a configuration in $\partial \mathcal{D}_{n+i-1}^{\varepsilon}$ for some time in (t_i, t_{i-1}). In particular, a recollision does not involve a fresh (just created) particle in the backward pseudo-trajectory.

With these notations, one gets the following geometric representation of the correlation function G_n^{ε}:

$$G_n^{\varepsilon}(t, Z_n) = \sum_{m \geq 0} \sum_{a \in \mathcal{A}_{n,m}^{\pm}} \int_{\mathcal{G}_m^{\varepsilon}(a, Z_n)} dT_m d\Omega_m dV_{n+1, n+m}$$

$$\times \left(\prod_{i=1}^{m} s_i \big((v_{n+i} - v_{a_i}^{\varepsilon}(t_i^+)) \cdot \omega_i \big)_+ \right) G_{n+m}^{\varepsilon 0} \big(Z_{n+m}^{\varepsilon}(0) \big),$$

where $(T_m, \Omega_m, V_{n+1, n+m}) := (t_i, \omega_i, v_{n+i})_{1 \leq i \leq m}$.

In the following we concentrate on the case $n = 1$ since as explained above, it is the key to studying the covariance of the fluctuation field: our goal is indeed to study

$$\int dz_1 G_1^{\varepsilon}(t, z_1) h(z_1) = \sum_{m \geq 0} I_m$$

where

$$I_m := \sum_{m \geq 0} \int dz_1 h(z_1) \Big(Q_{1,1+m}^{\varepsilon}(t) G_{1+m}^{\varepsilon 0} \Big)(z_1).$$

Since we would like to control the collision process independently of the structure of the initial data, it is useful to define a "dual" operator $Q_{1,1+m}^{\varepsilon*}(t)$

$$\int dz_1 h(z_1) \Big(Q_{1,1+m}^{\varepsilon}(t) G_{1+m}^{\varepsilon 0} \Big)(z_1) = \int dZ_{1+m} \Big(Q_{1,1+m}^{\varepsilon*}(t) h(z_1) \Big) G_{1+m}^{\varepsilon 0}(Z_{1+m}). \tag{2.13}$$

By this procedure, pathological behaviors may be identified directly at the level of the test function $Q_{1,1+m}^{\varepsilon*}(t)h(z_1)$. This corresponds to changing integration variables, in such a way that the standard Duhamel iterated formula on the left hand side assumes a more practical (and more symmetric) geometric representation, in terms of trajectories evolving forward in time.

2.3.2 | The duality argument in the absence of recollisions

Let us assume momentarily that there is no recollision in the pseudo-dynamics. Denoting by $Q_{1,1+m}^{\varepsilon 0}$ the restriction of $Q_{1,1+m}^{\varepsilon}$ to pseudo-trajectories without recollision, and recalling the series expansion (2.11), we therefore focus in this paragraph on

$$I^0 := \sum_{m \geq 0} I_m^0 := \sum_{m \geq 0} \int dz_1 h(z_1) Q_{1,1+m}^{\varepsilon 0}(t) G_{1+m}^{\varepsilon 0}.$$

Let us fix the integer $m \geq 0$. Expanding the collision operators leads to

$$I_m^0 = \sum_{a \in \mathcal{A}_{1,m}^{\pm}} \int_{\mathcal{P}_a} dz_1 h(z_1) dT_m d\Omega_m dV_{2,m+1} \left(\prod_{i=1}^{m} s_i \big((v_{1+i} - v_{a_i}^{\varepsilon}(t_i^+)) \cdot \omega_i \big)_+ \right)$$
$$\times G_{1+m}^{\varepsilon 0} \big(Z_{1+m}^{\varepsilon}(0) \big),$$

where $\mathcal{P}_a$ is the subset of $\mathbb{D} \times ([0,t] \times \mathbb{S}^{d-1} \times \mathbb{R}^d)^m$ such that for $z_1, (t_i, \omega_i, v_{1+i})_{1 \leq i \leq m}$ in $\mathcal{P}_a$, the associate pseudo-trajectory is well-defined and satisfies the requirements that as time goes from t to 0, there are exactly m creations according to the collision tree a, and no recollision. Recall that a tree a encodes both the labels of the colliding particles (namely $1 + i$ and a_i) and the signs s_i prescribing at each creation if there is scattering or not.

Given a tree $a \in \mathcal{A}_{1,m}^{\pm}$, consider the change of variables, of range $\mathcal{R}_a$:

$$\big(z_1, (t_i, \omega_i, v_{1+i})_{1 \leq i \leq m} \big) \in \mathcal{P}_a \longmapsto Z_{1+m}^{\varepsilon}(0) \in \mathcal{R}_a. \tag{2.14}$$

Definition 2.3. We call <u>forward flow</u> the reconstruction of the dynamics on $[0,t]$ starting from the configuration $Z_{1+m}^{\varepsilon}(0)$.

In the forward dynamics, two particles are said to <u>encounter</u> if they find themselves at distance ε leading to a creation or a recollision in the corresponding backward pseudo-dynamics.

In the case without recollision, if we start from some $Z_{1+m} \in \mathcal{R}_a$ at time 0, we can reconstruct the forward dynamics on $[0,t]$ by removing at each encounter the particle with highest index, and possibly scattering the other colliding particle according to the sequence $(s_{m-i+1})_i$. The collision

parameters $(t_i, \omega_i, v_{1+i})_{1 \leq i \leq m}$ are thus uniquely defined. This proves that the change of variables (2.14) is injective. Note that the knowledge of the sequence (a_i) is not useful in this construction.

From the encounter condition

$$x_{i+1}^\varepsilon(t_i) - x_{a_i}^\varepsilon(t_i) = x_{i+1}^\varepsilon(t_{i+1}) - x_{a_i}^\varepsilon(t_{i+1}) + (t_i - t_{i+1})(v_{i+1}^\varepsilon(t_{i+1}^+) - v_{a_i}^\varepsilon(t_{i+1}^+)) = \varepsilon s_i \omega_i$$

we deduce that

$$dx_{i+1} dv_{i+1} = \frac{1}{\mu_\varepsilon} \Big(\big(v_{1+i} - v_{a_i}^\varepsilon(t_i^+) \big) \cdot \omega_i \Big)_+ dt_i d\omega_i dv_{i+1} .$$

Thus the Jacobian of the change of variable (2.14) can be computed recursively

$$\frac{1}{\mu_\varepsilon^m} \prod_{i=1}^m \Big(\big(v_{1+i} - v_{a_i}^\varepsilon(t_i^+) \big) \cdot \omega_i \Big)_+ .$$

Denoting by $z_1^\varepsilon(t, Z_{1+m})$ the configuration of particle 1 at time t starting from $Z_{1+m} \in \mathcal{R}_a$ at time 0, one can therefore write

$$I_m^0 = \sum_{a \in \mathcal{A}_{1,m}^\pm} \mu_\varepsilon^m \int_{\mathcal{R}_a} dZ_{1+m} G_{1+m}^{\varepsilon 0}(Z_{1+m}) h\big(z_1^\varepsilon(t, Z_{1+m})\big) \prod_{i=1}^m s_i .$$

Note that the restriction to $\mathcal{R}_a$ implies that Z_{1+m} is configured in such a way that the encounters will take place in a prescribed order (first $1 + m$ with a_m, then m with a_{m-1}, etc.). This is related to the symmetry breaking in the iterated Duhamel formula. Usually this symmetry breaking is not an issue since we work with L^∞ estimates on correlation functions, and therefore L^1 bounds on test functions. But here we intend to work with different estimates (see (1.17)), and it is important to keep the symmetry as much as possible. Using the exchangeability of the initial distribution, we therefore symmetrize over the labels of particles and set

$$\Phi_{m+1}^0(Z_{m+1}) := \frac{\mu_\varepsilon^m}{(m+1)!} \sum_{\sigma \in \mathfrak{S}_{m+1}} \sum_{a \in \mathcal{A}_{1,m}^\pm} h\Big(z_{\sigma(1)}^\varepsilon(t, Z_\sigma) \Big) \mathbf{1}_{\{Z_\sigma \in \mathcal{R}_a\}} \prod_{i=1}^m s_i \qquad (2.15)$$

where $\mathfrak{S}_{m+1}$ denotes the permutations of $\{1, \ldots, m+1\}$, and

$$Z_\sigma = (z_{\sigma(1)}, \ldots, z_{\sigma(m+1)}) .$$

Remark 2.4. Note that the change of variables

$$\big(\sigma, z_1, (t_i, \omega_i, v_{1+i})_{1 \leq i \leq m}\big) \longmapsto Z_{1+m}^\varepsilon(0)$$

is almost injective: it suffices to prescribe two sequences of m signs to fix the label of the particle to be removed at each encounter, and the possible scattering, to reconstruct the pseudo-dynamics. We therefore expect the different terms of the sum with respect to permutations σ in (2.15) to have essentially disjoint supports, and therefore the L^2 norm of the symmetrized function $\Phi_{m+1}^0(Z_{m+1})$

to be smaller by a factor $\sqrt{(m+1)!}$ than the original function

$$\mu_\varepsilon^m \sum_{a \in \mathcal{A}_{1,m}^\pm} h\big(z_1^\varepsilon(t, Z_{1+m})\big) \mathbf{1}_{\{Z_{1+m} \in \mathcal{R}_a\}} \prod_{i=1}^m s_i.$$

By definition, Φ_{m+1}^0 encodes m independent constraints of size t/μ_ε (the size of the cylinder spanned by each particle between two collisions) corresponding to the creations in the pseudo-dynamics on $[0, t]$, so we expect

$$\int |\Phi_{m+1}^0(Z_{m+1})| M^{\otimes(m+1)}(V_{m+1}) dZ_{m+1} \leq \|h\|_{L^\infty(\mathbb{D})} C(Ct)^m$$

for some $C > 0$. In order to estimate

$$I_m^0 = \int dZ_{m+1} G_{m+1}^{\varepsilon 0}(Z_{m+1}) \Phi_{m+1}^0(Z_{m+1}), \tag{2.16}$$

the key idea is now to use the Cauchy-Schwarz inequality to decouple the initial fluctuation from the dynamics on $[0, t]$: indeed, setting

$$\mathbb{E}_\varepsilon(\Phi_{m+1}^0) = \mathbb{E}_\varepsilon\left(\frac{1}{\mu_\varepsilon^{m+1}}\left(\sum_{(i_1, \dots i_{m+1})} \Phi_{m+1}^0\left(\mathbf{z}_{i_1}^\varepsilon, \dots, \mathbf{z}_{i_{m+1}}^\varepsilon\right)\right)\right) \tag{2.17}$$

and introducing the centered variable

$$\hat{\Phi}_{m+1}^0\left(\mathbf{Z}_{\mathcal{N}}^\varepsilon\right) := \frac{1}{\mu_\varepsilon^{m+1}} \sum_{(i_1, \dots i_{m+1})} \Phi_{m+1}^0\left(\mathbf{z}_{i_1}^\varepsilon, \dots, \mathbf{z}_{i_{m+1}}^\varepsilon\right) - \mathbb{E}_\varepsilon(\Phi_{m+1}^0), \tag{2.18}$$

we have

$$\sum_{m \geq 0} \mathbb{E}_\varepsilon\left(\hat{\Phi}_{m+1}^0\left(\mathbf{Z}_{\mathcal{N}}^\varepsilon\right) \sum_{i=1}^{\mathcal{N}} g_0\left(z_i^\varepsilon\right)\right) := \sum_{m \geq 0} \mathbb{E}_\varepsilon\left(\mu_\varepsilon^{\frac{1}{2}} \hat{\Phi}_{m+1}^0 \zeta_0^\varepsilon(g_0)\right)$$

$$\leq \mathbb{E}_\varepsilon\big((\zeta_0^\varepsilon(g_0))^2\big)^{1/2} \sum_{m \geq 0} \mathbb{E}_\varepsilon\left(\mu_\varepsilon\big(\hat{\Phi}_{m+1}^0\big)^2\right)^{1/2}, \tag{2.19}$$

and I^0 differs from the above quantity by a small error coming from the subtraction of the average (which will be shown to be negligible).

One important step in this paper will be the estimate of the last expectation in (2.19). It requires to expand the square and to control the cross products using the clustering structure of $\hat{\Phi}_{m+1}^0(Z_{m+1})\hat{\Phi}_{m+1}^0(Z'_{m+1})$. This will be achieved in Proposition 3.1.

At this stage, the weak convergence method does not seem to be much better than the Lanford's method, since we expect an estimate of the form

$$|I_m^0| \leq C(Ct)^m,$$

which diverges as $m \to \infty$ despite the fact that it does not even take into account pseudo-dynamics involving recollisions, for which the change of variables (2.14) is not injective.

However, since the duality argument "decouples" the dynamics and the initial distribution, it will be easier to introduce additional constraints on the dynamics. Typically we will require that

- the total number m of collisions remains under control (much smaller than $|\log \varepsilon|$);
- the number of recollisions per particle is bounded, in order to control the defect of injectivity in (2.14).

Hence our strategy is to apply the above explained argument at a suitably defined stopping time, as introduced in the following section.

2.4 | Sampling

As in [3], we introduce a pruning procedure to control the number of terms in the expansion (2.11) as well as the occurrence of recollisions. We shall rely on the geometric interpretation of this expansion: to have a convergent series expansion on a long time $(0, \theta)$ with $\theta \gg 1$, we shall stop the (backward) iteration whenever one of the two following conditions is fulfilled:

- super-exponential branching: on the time interval $(\theta - k\tau, \theta - (k-1)\tau)$, with $\tau \ll 1$ to be tuned, the number n_k of created particles is larger than 2^k;
- recollision: on $(\theta - (k-1)\tau - r\delta, \theta - (k-1)\tau - (r-1)\delta)$ with $\delta \ll \tau$ to be tuned, there is at least one recollision.

Note that this sampling is more involved than in [3] since we essentially stop the iteration as soon as there is one recollision in the pseudo-dynamics: this will be used to apply the duality argument. Note also that both conditions (controlled growth and absence of recollision) have to be dealt with simultaneously: it is indeed hopeless to control the number of recollisions if the number of collisions can be much larger than $|\log \varepsilon|$.

The principal part of the expansion will correspond to all pseudo-trajectories for which the number of created particles on each time step $(\theta - k\tau, \theta - (k-1)\tau)$, for $1 \leq k \leq \theta/\tau$, is smaller than 2^k, and for which there is no recollision. Recalling that $Q_{n,n+m}^{\varepsilon 0}$ denotes the restriction of $Q_{n,n+m}^{\varepsilon}$ to pseudo-trajectories without recollision, and setting $K := \theta/\tau$ and $N_k = 1 + \cdots + n_k$, we thus define the main part of the expansion as

$$G_1^{\varepsilon,\mathrm{main}}(\theta) := \sum_{(n_k \leq 2^k)_{k \leq K}} Q_{1,N_1}^{\varepsilon 0}(\tau) \dots Q_{N_{K-1},N_K}^{\varepsilon 0}(\tau) G_{N_K}^{\varepsilon 0}. \tag{2.20}$$

In order to prove that $G_1^{\varepsilon} - G_1^{\varepsilon,\mathrm{main}}$ is small, we will use the duality argument discussed in Section 2.3.2, which requires an a priori control on the number of recollisions allowed in the dynamics.

This means that we do not work with arbitrary realizations of the hard-sphere dynamics: we rather condition the measure to avoid atypical configurations, defined as follows. Given an integer $\gamma \in \mathbb{N}$, we call *microscopic cluster of size γ* a set $\mathcal{G}$ of γ particle configurations in $\mathbb{T}^d \times \mathbb{R}^d$ such

that $(z, z') \in \mathcal{G} \times \mathcal{G}$ if and only if there are $z_1 = z, z_2, \dots, z_\ell = z'$ in $\mathcal{G}$ such that

$$|x_i - x_{i+1}| \leq 3\sqrt{\gamma} \mathbb{V} \delta, \quad \forall 1 \leq i \leq \ell - 1,$$

where $\mathbb{V} \in \mathbb{R}^+$ is related to an energy truncation. To fix ideas, we choose from now on

$$\varepsilon \ll \tau \ll 1 \ll \theta \ll (\log|\log\varepsilon|)^{1/4} \quad \text{and} \quad \gamma = 4d, \quad \mathbb{V} = |\log\varepsilon|, \quad \delta = \varepsilon^{1-\frac{1}{2d}}. \tag{2.21}$$

Definition 2.5. Given $\gamma \in \mathbb{N}$, we define the set Y_N^ε as the set of initial configurations $\mathbf{Z}_N^{\varepsilon 0} \in \mathcal{D}_N^\varepsilon$ such that for any integer $1 \leq k \leq \theta/\tau$ and any integer $r \in [0, \tau/\delta]$, the configuration at time $\theta - (k-1)\tau - r\delta$ satisfies

$$\forall 1 \leq j \leq N, \quad |v_j| \leq \mathbb{V},$$

and any microscopic cluster of particles is of size at most γ.

Thus the main contribution to the Duhamel expansion will be given by the restriction to configurations in Y_N^ε. For this reason, we introduce the tilted measures

$$\widetilde{W}_N^\varepsilon = W_N^\varepsilon \, \mathbf{1}_{Y_N^\varepsilon} \tag{2.22}$$

and the corresponding correlation functions $(\widetilde{G}_n^\varepsilon)_{n \geq 1}$ defined as in (2.7).

Remark 2.6. For the measure supported on Y_N^ε, it is easy to see that on the time interval $(\theta - (k-1)\tau - r\delta, \theta - (k-1)\tau - (r-1)\delta)$, two particles from different clusters will not be able to recollide. Indeed the total energy of each microscopic cluster is at most $\gamma \mathbb{V}^2/2$ so that the variation of the relative distance between two particles from different clusters is at most $2\sqrt{\gamma} \mathbb{V} \delta$, which prevents any collision.

Now recall that $K = \theta/\tau$ and $N_k = 1 + \cdots + n_k$ (where n_k is the number of created particles on the interval $(\theta - k\tau, \theta - (k-1)\tau)$ in the backward dynamics), and let us set $R := \tau/\delta$. Defining

$$Q_{n,n+m}^{\text{rec}} := Q_{n,n+m}^\varepsilon - Q_{n,n+m}^{\varepsilon 0}$$

the restriction of $Q_{n,n+m}^\varepsilon$ to pseudo-trajectories which have at least one recollision, we can write the following decomposition of $\widetilde{G}_1^\varepsilon$:

$$\widetilde{G}_1^\varepsilon(\theta) = G_1^{\varepsilon,\text{main}}(\theta) - G_1^{\varepsilon,\text{clust}}(\theta) + G_1^{\varepsilon,\text{exp}}(\theta) + G_1^{\varepsilon,\text{rec}}(\theta) \tag{2.23}$$

with

$$G_1^{\varepsilon,\text{clust}}(\theta) := \sum_{(n_k \leq 2^k)_{k \leq K}} Q_{1,N_1}^{\varepsilon 0}(\tau) \dots Q_{N_{K-1}, N_K}^{\varepsilon 0}(\tau) (G_{N_K}^{\varepsilon 0} - \widetilde{G}_{N_K}^{\varepsilon 0}).$$

The term

$$G_1^{\varepsilon,\exp}(\theta) := \sum_{k=1}^{K} \sum_{(n_j \le 2^j)_{j \le k-1}} \sum_{n_k > 2^k} Q_{1,N_1}^{\varepsilon 0}(\tau) \dots Q_{N_{k-1},N_k}^{\varepsilon 0}(\tau) \widetilde{G}_{N_k}^{\varepsilon}(\theta - k\tau)$$

is the error encoding super-exponential trees. The term $G_1^{\varepsilon,\mathrm{rec}}(\theta)$ encodes the occurrence of a rec-ollision. We denote by $n_k^{\mathrm{rec}} \ge 0$ the number of particles added on the time step $(\theta - (k-1)\tau - r\delta, \theta - (k-1)\tau - (r-1)\delta)$ (on which by definition there is a recollision), and by $n_k^0 := n_k - n_k^{\mathrm{rec}}$ the number of particles added on the time step $(\theta - (k-1)\tau - (r-1)\delta, \theta - (k-1)\tau)$ (on which by definition there is no recollision). We then define

$$G_1^{\varepsilon,\mathrm{rec}}(\theta) := \sum_{k=1}^{K} \sum_{(n_j \le 2^j)_{j \le k-1}} \sum_{r=1}^{R} \sum_{n_k \ge 0} \sum_{n_k^0 + n_k^{\mathrm{rec}} = n_k} Q_{1,N_1}^{\varepsilon 0}(\tau) \dots Q_{N_{k-2},N_{k-1}}^{\varepsilon 0}(\tau)$$

$$\circ Q_{N_{k-1},N_{k-1}+n_k^0}^{\varepsilon 0}((r-1)\delta) Q_{N_{k-1}+n_k^0,N_{k-1}+n_k^0+n_k^{\mathrm{rec}}}^{\mathrm{rec}}(\delta) \widetilde{G}_{N_k}^{\varepsilon}(\theta - (k-1)\tau - r\delta).$$

2.5 | Analysis of the remainder terms

Recall that our aim is to compute the integral in (2.4). By definition,

$$\mathbb{E}_{\varepsilon}\Big(\zeta_0^{\varepsilon}(g_0)\zeta_{\theta}^{\varepsilon}(h)\Big) = \int G_1^{\varepsilon,\mathrm{main}}(\theta) h(z)\, dz$$

$$- \int G_1^{\varepsilon,\mathrm{clust}}(\theta) h(z)\, dz + \mathbb{E}_{\varepsilon}\Big(\mathbf{1}_{{}^c\mathrm{Y}_{\mathcal{N}}^{\varepsilon}}\zeta_0^{\varepsilon}(g_0)\zeta_{\theta}^{\varepsilon}(h)\Big) \qquad (2.24)$$

$$+ \int G_1^{\varepsilon,\exp}(\theta) h(z)\, dz + \int G_1^{\varepsilon,\mathrm{rec}}(\theta) h(z)\, dz.$$

The first two remainder terms consist essentially in measuring the cost of the constraint on $\mathrm{Y}_{\mathcal{N}}^{\varepsilon}$. They are easily shown to be small thanks to the invariant measure: the following proposition is proved in Section 6.1.

Proposition 2.7 (Cost of the conditioning). *With the previous choices (2.21) of parameters, the conditioning is negligible in the sense that*

$$\mathbb{P}_{\varepsilon}\big({}^c\mathrm{Y}_{\mathcal{N}}^{\varepsilon}\big) \le \theta\, \varepsilon^d. \qquad (2.25)$$

In particular,

$$\left| \int dz_1 G_1^{\varepsilon,\mathrm{clust}}(\theta, z_1) h(z_1) \right| \le C \|h\|_{L^{\infty}(\mathbb{D})} \|g_0\|_{L^{\infty}(\mathbb{D})} (C\theta)^{2\theta/\tau} (\theta\varepsilon)^{1/2},$$

$$\left| \mathbb{E}_{\varepsilon}\big(\mathbf{1}_{{}^c\mathrm{Y}_{\mathcal{N}}^{\varepsilon}}\zeta_0^{\varepsilon}(g_0)\zeta_{\theta}^{\varepsilon}(h)\big) \right| \le C \|h\|_{L^{\infty}(\mathbb{D})} \|g_0\|_{L^2_M}(\theta\varepsilon^d)^{1/4}. \qquad (2.26)$$

Furthermore, there holds

$$\left| \mathbb{E}_\varepsilon \left(\zeta_0^\varepsilon(g_0) \mathbf{1}_{Y_{\mathcal{N}}^\varepsilon} \right) \right| \le C \|g_0\|_{L_M^2} \, (\theta \varepsilon^d)^{1/2} \,. \tag{2.27}$$

It remains to study $G_1^{\varepsilon,\exp}(\theta)$ and $G_1^{\varepsilon,\mathrm{rec}}(\theta)$. For these two terms we use the a priori L^2 control on fluctuations, and thus rework the duality argument of Paragraph 2.3.2. The following proposition is proved in Section 4 thanks to the quasi-orthogonality estimates of Section 3 and the clustering estimates of Section 4, the extra smallness coming from the assumption that the tree becomes superexponential on a short time interval of size τ.

Proposition 2.8 (Superexponential trees). *If θ, τ are chosen such that*

$$\lim_{\mu_\varepsilon \to \infty} \theta^3 \tau = 0 \,, \tag{2.28}$$

then

$$\left| \int dz_1 G_1^{\varepsilon,\exp}(\theta, z_1) h(z_1) \right| \le C \|h\|_{L^\infty(\mathbb{D})} \|g_0\|_{L_M^2} (\theta^3 \tau)^{1/2} \,.$$

The possibility of recollisions makes the analysis of $G_1^{\varepsilon,\mathrm{rec}}$ more intricate: it is however possible to revisit the arguments of Section 4, to gain smallness thanks to the presence of a recollision on a time interval of size δ. The following proposition is proved in Section 5.

Proposition 2.9 (Recollisions). *Under the previous scaling conditions,*

$$\left| \int dz_1 G_1^{\varepsilon,\mathrm{rec}}(\theta, z_1) h(z_1) \right| \le C \|h\|_{L^\infty(\mathbb{D})} \|g_0\|_{L_M^2} (C\theta)^{2\theta/\tau} \varepsilon^{\frac{1}{8d}} \,.$$

To conclude the proof of the main theorem, it remains to study the convergence of the principal part.

Proposition 2.10 (Principal part). *Under the previous scaling assumptions, there holds*

$$\left| \int G_1^{\varepsilon,\mathrm{main}}(\theta, z) \, h(z) \, dz - \int M(v) \, g(\theta, z) \, h(z) \, dz \right|$$

$$\le C \|h\|_{L^\infty(\mathbb{D})} \|g_0\|_{L^\infty(\mathbb{D})} \left(\varepsilon^{1/2} (C\theta)^{2\theta/\tau} + \theta \tau \right) + C \|h\|_{L^\infty(\mathbb{D})} \|\nabla_x g_0\|_{L^\infty(\mathbb{D})} (C\theta)^{2\theta/\tau} \varepsilon \,,$$

where g is the solution of the linearized Boltzmann equation (1.13) with initial datum g_0.

The proof of this proposition is the content of Section 6.2.

Collecting this together with the decomposition (2.24) and the previous propositions, Theorem 1.1 is proved, provided that the scaling assumptions are compatible. The convergence holds globally in time, that is, for any finite θ and even for very slowly diverging $\theta = o((\log|\log \varepsilon|)^{1/4})$. Choosing for instance

$$\tau = (\theta^2 \log|\log \varepsilon|)^{-1/2} \,, \tag{2.29}$$

we check that

$$\lim_{\mu_\varepsilon \to 0} \frac{\theta}{\tau \log|\log\varepsilon|} = 0 \qquad \text{and} \quad \lim_{\mu_\varepsilon \to \infty} \theta^3 \tau = 0, \tag{2.30}$$

so that (2.28) is satisfied and all remainders converge to 0. $\qquad\square$

3 | QUASI-ORTHOGONALITY ESTIMATES

To control the remainders associated with super exponential branching $G_1^{\varepsilon,\exp}(\theta)$ and recollisions $G_1^{\varepsilon,\text{rec}}(\theta)$, we shall follow the strategy presented in Section 2.3.2 using a duality argument. More precisely, in order to use the L^2 estimate (2.3) on the initial fluctuation field $\zeta_0^\varepsilon(g_0)$, we need to establish L^2 estimates on the associate test functions Φ_{N_k}, see (2.17)–(2.19). We prove here a general statement which will be applied to the superexponential case in Section 4, and to the case of recollisions in Section 5.

In the following we denote for $i < j$

$$Z_{i,j} := (z_i, z_{i+1}, \dots, z_j).$$

Proposition 3.1. *Let Φ_N be a symmetric function of N variables satisfying*

$$\sup_{x_N \in \mathbb{T}^d} \int |\Phi_N(Z_N)| M^{\otimes N}(V_N) \, dX_{N-1} dV_N \le C^N \rho_0 \tag{3.1}$$

$$\sup_{x_{2N-\ell} \in \mathbb{T}^d} \int |\Phi_N(Z_N)\Phi_N(Z_\ell, Z_{N+1,2N-\ell})| M^{\otimes(2N-\ell)}(V_{2N-\ell}) \, dX_{2N-\ell-1} dV_{2N-\ell} \tag{3.2}$$

$$\le C^N \frac{\mu_\varepsilon^{\ell-1}}{N^\ell} \rho_\ell, \qquad \ell = 1, \dots, N,$$

for some $C, \rho_0, \rho_\ell > 0$. Define the mean $\mathbb{E}_\varepsilon(\Phi_N)$ and the centered variable $\mathring{\Phi}_N$ as in (2.17)-(2.18). Then there is a constant $\tilde{C} > 0$ such that

$$|\mathbb{E}_\varepsilon(\Phi_N)| \le \tilde{C}^N \rho_0 \tag{3.3}$$

and

$$\mathbb{E}_\varepsilon\big(\mu_\varepsilon \mathring{\Phi}_N^2\big) \le \tilde{C}^N \sum_{\ell=1}^N \rho_\ell + O\big(\tilde{C}^N \rho_0^2 \varepsilon\big). \tag{3.4}$$

Remark 3.2. Properties (3.1) and (3.2) will come from the fact that the Φ_{N_k} are sums of elementary functions of size $\mu_\varepsilon^{N_k-1}$ in L^∞, supported on dynamical clusters. These clusters can be represented by minimally connected graphs with N_k vertices, where each edge has a cost in L^1 of the order of $O(\theta/\mu_\varepsilon)$. In order to compute the L^1 norm of tensor products, we will then extract minimally connected graphs from the union of two such trees, which provides independent variables of integration. Additional smallness (encoded in the constants ρ_0, ρ_ℓ) will come from the conditions that there are recollisions, or that many creations of particles are localized in a small time interval (see Sections 4 and 5).

Proof. We start by computing the expectation

$$
\begin{aligned}
\mathbb{E}_\varepsilon(\Phi_N) &= \frac{1}{\mu_\varepsilon^N} \mathbb{E}_\varepsilon \left(\sum_{(i_1,\dots,i_N)} \Phi_N \left(\mathbf{z}_{i_1}^\varepsilon, \dots, \mathbf{z}_{i_N}^\varepsilon \right) \right) \\
&= \frac{1}{\mathcal{Z}^\varepsilon} \sum_{p \geq 0} \int dZ_{N+p} \frac{\mu_\varepsilon^p}{p!} \mathbf{1}_{D_{N+p}^\varepsilon}(Z_{N+p}) M^{\otimes(N+p)}(V_{N+p}) \Phi_N(Z_N) .
\end{aligned}
\tag{3.5}
$$

This expression will be estimated by expanding the exclusion condition on $Z_{N+p} = (Z_N, \bar{Z}_p)$ using classical cluster techniques. We will consider Z_N as a block represented by one vertex, and $(\bar{z}_i)_{1 \leq i \leq p}$ as p separate vertices. We denote by $d(y, y^*)$ the minimum relative distance (in position) between elements $y, y^* \in \{Z_N, \bar{z}_1, \dots \bar{z}_p\}$. We then have

$$
\begin{aligned}
\mathbf{1}_{D_{N+p}^\varepsilon}(Z_{N+p}) &= \mathbf{1}_{D_N^\varepsilon}(Z_N) \prod_{\substack{y,y^* \in \{Z_N, \bar{z}_1, \dots \bar{z}_p\} \\ y \neq y^*}} \mathbf{1}_{d(y,y^*)>\varepsilon} \\
&= \mathbf{1}_{D_N^\varepsilon}(Z_N) \sum_{\sigma_0 \subset \{1,\dots,p\}} \mathbf{1}_{D_{|\sigma_0|}^\varepsilon}(\bar{Z}_{\sigma_0}) \varphi(Z_N, \bar{Z}_{\sigma_0^c})
\end{aligned}
$$

where σ_0 is a (possibly empty) part of $\{1, \dots, p\}$, σ_0^c is its complement, and where the cumulants φ are defined as follows

$$
\varphi(Z_N, \bar{Z}_\sigma) := \sum_{G \in \mathcal{C}_{1+|\sigma|}} \prod_{(y,y^*) \in E(G)} (-\mathbf{1}_{d(y,y^*)\leq \varepsilon}),
\tag{3.6}
$$

denoting by $\mathcal{C}_n$ the set of connected graphs with n vertices, and by $E(G)$ the set of edges of such a graph G. By exchangeability of the background particles, we therefore obtain

$$
\begin{aligned}
\mathbb{E}_\varepsilon(\Phi_N) &= \frac{1}{\mathcal{Z}^\varepsilon} \left(\sum_{p_0 \geq 0} \frac{\mu_\varepsilon^{p_0}}{p_0!} \int M^{\otimes p_0} \mathbf{1}_{D_{p_0}^\varepsilon}(\bar{Z}_{p_0}) d\bar{Z}_{p_0} \right) \\
&\quad \times \sum_{p_1 \geq 0} \frac{\mu_\varepsilon^{p_1}}{p_1!} \int M^{\otimes(N+p_1)} \varphi(Z_N, \bar{Z}_{p_1}) \mathbf{1}_{D_N^\varepsilon}(Z_N) \Phi_N(Z_N) dZ_N d\bar{Z}_{p_1} \\
&= \sum_{p_1 \geq 0} \frac{\mu_\varepsilon^{p_1}}{p_1!} \int M^{\otimes(N+p_1)} \varphi(Z_N, \bar{Z}_{p_1}) \mathbf{1}_{D_N^\varepsilon}(Z_N) \Phi_N(Z_N) dZ_N d\bar{Z}_{p_1} ,
\end{aligned}
\tag{3.7}
$$

where in the last step we used the definition of the grand canonical partition function $\mathcal{Z}^\varepsilon$.

A powerful tool to sum cluster expansions of exclusion processes is the tree inequality due to Penrose ([23], see also [20]) estimating sums over connected graphs in terms of sums over minimally connected graphs. It states that the cumulants defined by (3.6) satisfy

$$
|\varphi(Z_N, \bar{Z}_{p_1})| \leq \sum_{T \in \mathcal{T}_{1+p_1}} \prod_{(y,y^*) \in E(T)} \mathbf{1}_{d(y,y^*)\leq \varepsilon} ,
\tag{3.8}
$$

where $\mathcal{T}_{1+p_1}$ is the set of minimally connected graphs with $1 + p_1$ vertices.

The product of indicator functions in (3.8) is a sequence of p_1 constraints, confining the space coordinates to balls of size ε centered at the positions $X_N, \bar{x}_1, \dots \bar{x}_{p_1}$. We rewrite it as a constraint on the positions $x_N, \bar{x}_1, \dots \bar{x}_{p_1}$ (recalling that X_N is considered as a block, meaning that the relative

positions inside it are fixed). Integrating the indicator function with respect to $\bar{X}_{p_1}$ provides a factor $C^{p_1} N^{d_1} \varepsilon^{d p_1}$ where d_1 is the degree of the vertex X_N in T. Then, using (3.1) to integrate with respect to X_{N-1}, V_N provides a factor $C^N \rho_0$.

It is classical (see for instance [6, Lemma 2.4.1] for a proof) that the number of minimally connected graphs with specified vertex degrees $d_1, \ldots, d_{1+p_1}$ is given by

$$(p_1 - 1)! / \prod_{i=1}^{1+p_1} (d_i - 1)! . \tag{3.9}$$

Therefore, combining (3.7) and (3.8), we conclude that there exists $C' > 0$ such that

$$|\mathbb{E}_\varepsilon(\Phi_N)| \le C^N \rho_0 \sum_{p_1 \ge 0} \left[(C' \varepsilon^d \mu_\varepsilon)^{p_1} \sum_{d_1, \ldots, d_{p_1+1} \ge 1} \frac{N^{d_1}}{\prod_{i=1}^{p_1+1} (d_i - 1)!} \right], \tag{3.10}$$

from which (3.3) follows by taking ε small enough and using the fact that using the series expansion of the exponential

$$\sum_{d_1, \ldots, d_{p_1+1} \ge 1} \frac{N^{d_1}}{\prod_{i=1}^{p_1+1} (d_i - 1)!} = \sum_{d_1 \ge 1} \frac{N^{d_1}}{(d_1 - 1)!} \sum_{d_2 \ge 1} \frac{1}{(d_2 - 1)!} \cdots \sum_{d_{p_1+1} \ge 1} \frac{1}{(d_{p_1+1} - 1)!}$$

$$= N e^N e^{p_1} .$$

In order to establish (3.4), we note that

$$\mathbb{E}_\varepsilon \left(\mu_\varepsilon \hat{\Phi}_N^2 \right) = \frac{1}{\mu_\varepsilon^{2N-1}} \mathbb{E}_\varepsilon \left[\left(\sum_{(i_1, \ldots, i_N)} \Phi_N \left(\mathbf{z}_{i_1}^\varepsilon, \ldots, \mathbf{z}_{i_N}^\varepsilon \right) \right)^2 \right] - \mu_\varepsilon (\mathbb{E}_\varepsilon(\Phi_N))^2 \tag{3.11}$$

and first expand the square

$$\mathbb{E}_\varepsilon \left(\left(\sum_{(i_1, \ldots, i_N)} \Phi_N \left(\mathbf{z}_{i_1}^\varepsilon, \ldots, \mathbf{z}_{i_N}^\varepsilon \right) \right)^2 \right)$$

$$= \mathbb{E}_\varepsilon \left(\sum_{(i_1, \ldots, i_N)} \Phi_N \left(\mathbf{z}_{i_1}^\varepsilon, \ldots, \mathbf{z}_{i_N}^\varepsilon \right) \sum_{(i_1', \ldots, i_N')} \Phi_N \left(\mathbf{z}_{i_1'}^\varepsilon, \ldots, \mathbf{z}_{i_N'}^\varepsilon \right) \right).$$

There are two configurations of (different) particles labeled by $(i_1, \ldots, i_N)$ and $(i_1', \ldots, i_N')$, with a certain number ℓ of particles in common, $\ell = 0, 1, \ldots, N$. Using the symmetry of the function Φ_N, we can choose $i_1 = i_1', i_2 = i_2', \ldots, i_\ell = i_\ell'$ as the common indices and we find that

$$\mathbb{E}_\varepsilon \left(\left(\sum_{(i_1, \ldots, i_N)} \Phi_N \left(\mathbf{z}_{i_1}^\varepsilon, \ldots, \mathbf{z}_{i_N}^\varepsilon \right) \right)^2 \right) = \sum_{\ell=0}^{N} \binom{N}{\ell}^2 \ell!$$

$$\times \mathbb{E}_\varepsilon \left(\sum_{(i_k)_{k \in \{1, \ldots, 2N-\ell\}}} \Phi_N \left(\mathbf{z}_{i_1}^\varepsilon, \ldots, \mathbf{z}_{i_N}^\varepsilon \right) \Phi_N \left(\mathbf{z}_{i_1}^\varepsilon, \ldots, \mathbf{z}_{i_\ell}^\varepsilon, \mathbf{z}_{i_{N+1}}^\varepsilon, \ldots, \mathbf{z}_{i_{2N-\ell}}^\varepsilon \right) \right), \tag{3.12}$$
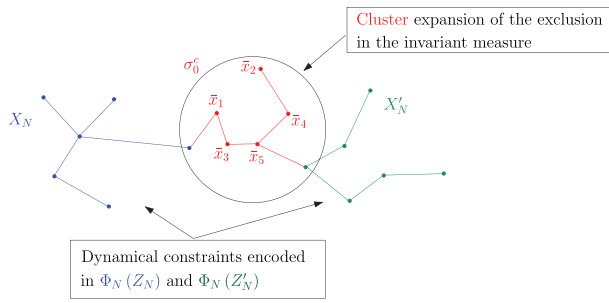
FIGURE 3 Cluster expansion of the exclusion when Z_N and Z'_N are disjoint. The red graph is a minimally connected graph on X_N (blue graph), X'_N (green graph) each seen as one vertex and $\sigma_0^c = \{1, 2, 3, 4, 5\}$. This graph encodes 6 constraints, independent from the dynamical constraints encoded in $\Phi_N(Z_N)$ and $\Phi_N(Z'_N)$; these dynamical constraints on X_N and X'_N are represented by two (minimally connected) graphs with N vertices, which corresponds more or less to the situations treated in Sections 4 and 5 (see Remark 3.2).

where the combinatorial factor $\binom{N}{\ell}^2$ comes from all possible choices for sets A and A' in $\{1, \dots N\}$, with $|A| = |A'| = \ell$, corresponding to the positions of the common indices in both N-uplets. The factor $\ell!$ is due to all possible bijections between A and A', corresponding to the permutations of the repeated indices.

Next we treat separately the cases $\ell = 0$ and $\ell \neq 0$.

Step 1. The case when all indices are different $\ell = 0$. Let us compute

$$
\frac{1}{\mu_\varepsilon^{2N-1}} \mathbb{E}_\varepsilon \left(\sum_{(i_1, \dots, i_{2N})} \Phi_N\left(\mathbf{z}_{i_1}^\varepsilon, \dots, \mathbf{z}_{i_N}^\varepsilon\right) \Phi_N\left(\mathbf{z}_{i_{N+1}}^\varepsilon, \dots, \mathbf{z}_{i_{2N}}^\varepsilon\right) \right)
$$

$$
= \frac{\mu_\varepsilon}{\mathcal{Z}^\varepsilon} \sum_{p \geq 0} \int dZ_{2N+p} \frac{\mu_\varepsilon^p}{p!} \mathbf{1}_{D_{2N+p}^\varepsilon}(Z_{2N+p}) M^{\otimes(2N+p)}(V_{2N+p}) \Phi_N(Z_N) \Phi_N(Z_{N+1,2N}).
$$
(3.13)

We can proceed as in the proof of (3.3) by expanding the exclusion condition on $Z_{2N+p} = (Z_N, Z'_N, \bar{Z}_p)$ (see the red part in Figure 3) and considering Z_N and Z'_N as blocks represented each by one vertex. We then have

$$
\mathbf{1}_{D_{2N+p}^\varepsilon}(Z_{2N+p}) = \mathbf{1}_{D_N^\varepsilon}(Z_N) \mathbf{1}_{D_N^\varepsilon}(Z'_N) \sum_{\sigma_0 \subset \{1, \dots, p\}} \mathbf{1}_{D_{|\sigma_0|}^\varepsilon}(\bar{Z}_{\sigma_0}) \Bigg[\varphi\left(Z_N, Z'_N, \bar{Z}_{\sigma_0^c}\right)
$$

$$
+ \sum_{\substack{\sigma \cup \sigma' = \sigma_0^c \\ \sigma \cap \sigma' = \emptyset}} \varphi(Z_N, \bar{Z}_\sigma) \varphi(Z'_N, \bar{Z}_{\sigma'}) \Bigg]
$$

where $\sigma_0, \sigma, \sigma'$ are (possibly empty) parts of $\{1, \dots, p\}$, and where we use (3.6) and

$$
\varphi(Z_N, Z'_N, \bar{Z}_\sigma) := \sum_{G \in \mathcal{C}_{2+|\sigma|}} \prod_{(y, y^*) \in E(G)} \left(-\mathbf{1}_{d(y, y^*) \leq \varepsilon}\right).
$$

By exchangeability of the background particles, we therefore obtain (as in (3.7))

$$\frac{\mu_\varepsilon}{\mathcal{Z}^\varepsilon} \sum_{p \geq 0} \frac{\mu_\varepsilon^p}{p!} \int M^{\otimes(2N+p)} \mathbf{1}_{D_{2N+p}^\varepsilon}(Z_N, Z_N', \bar{Z}_p) \Phi_N(Z_N) \Phi_N(Z_N') dZ_N dZ_N' d\bar{Z}_p$$

$$= \sum_{p_1 \geq 0} \frac{\mu_\varepsilon^{p_1+1}}{p_1!} \int M^{\otimes(2N+p_1)} \varphi(Z_N, Z_N', \bar{Z}_{p_1}) \mathbf{1}_{D_N^\varepsilon}(Z_N) \mathbf{1}_{D_N^\varepsilon}(Z_N')$$

$$\times \Phi_N(Z_N) \Phi_N(Z_N') dZ_N dZ_N' d\bar{Z}_{p_1}$$

$$+ \mu_\varepsilon \left(\sum_{p_1 \geq 0} \frac{\mu_\varepsilon^{p_1}}{p_1!} \int M^{\otimes(N+p_1)} \varphi(Z_N, \bar{Z}_{p_1}) \mathbf{1}_{D_N^\varepsilon}(Z_N) \Phi_N(Z_N) dZ_N d\bar{Z}_{p_1} \right)^2. \tag{3.14}$$

The last term is equal to $\mu_\varepsilon(\mathbb{E}_\varepsilon(\Phi_N))^2$ by (3.7), therefore it cancels out in the computation of (3.11).

The second line in (3.14) is treated as before. By the tree inequality

$$|\varphi(Z_N, Z_N', \bar{Z}_{p_1})| \leq \sum_{T \in \mathcal{T}_{2+p_1}} \prod_{(y,y^*) \in E(T)} \mathbf{1}_{d(y,y^*) \leq \varepsilon},$$

we reduce to $p_1 + 1$ constraints confining the space coordinates to balls of size ε centered at the positions $X_N, X_N', \bar{x}_1, \dots \bar{x}_{p_1}$, which we can rewrite as a constraint on the positions $x_N, x_N', \bar{x}_1, \dots \bar{x}_{p_1}$ (recalling that X_N and X_N' are considered as blocks, meaning that the relative positions inside each one of these blocks are fixed). Integrating the indicator function with respect to the variables $\bar{X}_{p_1}, x_N, x_N'$ provides a factor $N^{d_1+d_2} \varepsilon^{d(p_1+1)}$ where d_1 and d_2 are the degrees of the vertices X_N and X_N' in T. Then, using (3.1) to integrate with respect to $X_{N-1}, X_{N-1}', V_N, V_N'$ provides a factor $(C^N \rho_0)^2$. We conclude that the second line in (3.14) is bounded by

$$(C^N \rho_0)^2 \sum_{p_1 \geq 0} \left[(C' \varepsilon^d \mu_\varepsilon)^{p_1+1} \sum_{d_1, \dots, d_{p_1+2} \geq 1} \frac{N^{d_1+d_2}}{\prod_{i=1}^{p_1+2} (d_i - 1)!} \right] = O(\tilde{C}^N \rho_0^2 \varepsilon) \tag{3.15}$$

and it follows that

$$\frac{1}{\mu_\varepsilon^{2N-1}} \mathbb{E}_\varepsilon \left(\sum_{(i_1, \dots, i_{2N})} \Phi_N \left(\mathbf{z}_{i_1}^\varepsilon, \dots, \mathbf{z}_{i_N}^\varepsilon \right) \Phi_N \left(\mathbf{z}_{i_{N+1}}^\varepsilon, \dots, \mathbf{z}_{i_{2N}}^\varepsilon \right) \right)$$

$$= \mu_\varepsilon(\mathbb{E}_\varepsilon(\Phi_N))^2 + O(\tilde{C}^N \rho_0^2 \varepsilon). \tag{3.16}$$

Step 2. The case when some indices are repeated. For $\ell \in [1, N]$ given, we consider

$$\frac{1}{\mu_\varepsilon^{2N-1}} \mathbb{E}_\varepsilon \left(\sum_{(i_k)_{k \in \{1, \dots 2N-\ell\}}} \Phi_N \left(\mathbf{z}_{i_1}^\varepsilon, \dots, \mathbf{z}_{i_N}^\varepsilon \right) \Phi_N \left(\mathbf{z}_{i_1}^\varepsilon, \dots, \mathbf{z}_{i_\ell}^\varepsilon, \mathbf{z}_{i_{N+1}}^\varepsilon, \dots, \mathbf{z}_{i_{2N-\ell}}^\varepsilon \right) \right)$$

$$= \frac{\mu_\varepsilon^{1-\ell}}{\mathcal{Z}^\varepsilon} \sum_{p \geq 0} \frac{\mu_\varepsilon^p}{p!} \int \mathbf{1}_{D_{2N+p-\ell}^\varepsilon}(Z_{2N+p-\ell}) M^{\otimes(2N+p-\ell)}(V_{2N+p-\ell})$$

$$\times \Phi_N(Z_N) \Phi_N(Z_N') dZ_{2N+p-\ell}$$

denoting $Z_N = (Z_\ell, Z_{\ell+1,N})$, $Z_N' = (Z_\ell, Z_{N+1,2N-\ell})$ and $\bar{Z}_p = Z_{2N-\ell+1,2N-\ell+p}$.

This expression is of the same form as (3.5), but $\Phi_N(Z_N)$ is now replaced by $\Phi_N(Z_N)\Phi_N(Z_N')$ which is a function of $2N - \ell$ particle variables. It can be therefore estimated in exactly the same
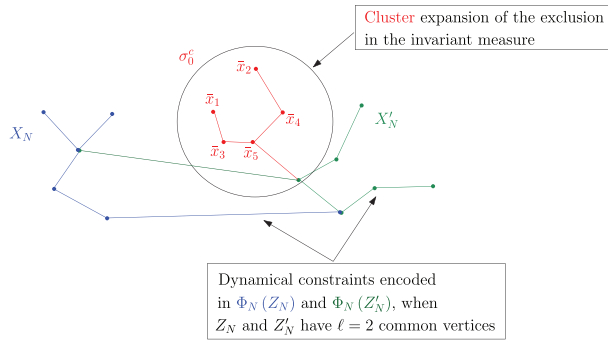
FIGURE 4 Cluster expansion of the exclusion when Z_N and Z'_N have ℓ common elements. The constraints imposed by $\Phi_N(Z_N)$ and $\Phi_N(Z'_N)$ are no longer independent, and the $2(N-1)$ dynamical constraints will be replaced by $2N - \ell - 1$ "independent" dynamical constraints (by extracting from the blue and green graphs a minimally connected graph). In this case $X_{2N-\ell}$ is considered as one vertex for the exclusion expansion.

way, by considering $Z_{2N-\ell}$ as one block since the dynamical constraints will provide a cluster structure on $Z_{2N-\ell}$. Note that taking $\ell = 0$ in this computation does not reduce to the case treated in Step 1, since X_N and X'_N were considered there as two vertices, meaning that one additional clustering constraint came from the exclusion (one more red edge in Figure 3 than in Figure 4). The role of the cluster estimate (3.1) is now played by (3.2) and this leads to (see (3.10))

$$\frac{1}{\mu_\varepsilon^{2N-1}} \mathbb{E}_\varepsilon \left(\sum_{(i_k)_{k \in \{1, \dots 2N-\ell\}}} \Phi_N\left(\mathbf{z}_{i_1}^\varepsilon, \dots, \mathbf{z}_{i_N}^\varepsilon\right) \Phi_N\left(\mathbf{z}_{i_1}^\varepsilon, \dots, \mathbf{z}_{i_\ell}^\varepsilon, \mathbf{z}_{i_{N+1}}^\varepsilon, \dots, \mathbf{z}_{i_{2N-\ell}}^\varepsilon\right) \right) \tag{3.17}$$

$$\leq C^N \frac{\rho_\ell}{N^\ell} \sum_{p_1 \geq 0} \left[(C'\varepsilon^d \mu_\varepsilon)^{p_1} \sum_{d_1, \dots, d_{p_1+1} \geq 1} \frac{(2N-\ell)^{d_1}}{\prod_{i=1}^{p_1+1}(d_i - 1)!} \right] \leq \tilde{C}^N \rho_\ell N^{-\ell} .$$

Combining (3.11), (3.12), (3.16) and (3.17) we conclude that

$$\mathbb{E}_\varepsilon\left(\mu_\varepsilon \hat{\Phi}_N^2\right) \leq \mu_\varepsilon(\mathbb{E}_\varepsilon(\Phi_N))^2 + O(\tilde{C}^N \rho_0^2 \varepsilon) + \sum_{\ell=0}^N \binom{N}{\ell}^2 \ell! \, \tilde{C}^N \rho_\ell N^{-\ell} - \mu_\varepsilon(\mathbb{E}_\varepsilon(\Phi_N))^2 \tag{3.18}$$

and, remarking that $\ell! \leq N^\ell$, this leads to (3.4) by enlarging the constant $\tilde{C}$. $\qquad \square$

Remark 3.3. For $N = 1$ and $\Phi_1 = h \in L_M^2$, one has $\mathbb{E}_\varepsilon(\mu_\varepsilon \hat{\Phi}_1^2) = \mathbb{E}_\varepsilon(\zeta^\varepsilon(h)^2)$. A simple corollary of the above proof leads then to (2.3).

4 | CLUSTERING ESTIMATES

In this section we prove Proposition 2.8. We consider

$$\int G_1^{\varepsilon, \exp}(\theta) h(z) \, dz$$

$$= \sum_{k=1}^K \sum_{(n_j \leq 2^j)_{j \leq k-1}} \sum_{n_k \geq 2^k} \int dz \, h(z) \, Q_{1,N_1}^{\varepsilon 0}(\tau) \dots Q_{N_{K-1},N_K}^{\varepsilon 0}(\tau) \, \tilde{G}_{N_K}^{\tilde{\varepsilon}}(\theta - k\tau),$$

corresponding to pseudo-trajectories satisfying the following constraints:

(i) there are n_j particles added on the time intervals $(\theta - j\tau, \theta - (j-1)\tau)$ for $j \leq k$,
(ii) there is no recollision on $(t_{\text{stop}}, \theta)$.

Each term of the sum will be estimated by using Proposition 3.1. Introducing the notation $t_{\text{stop}} := \theta - k\tau$, we set

$$I_{\mathbf{n}_k} := \int h(z_1) Q_{1,N_1}^{\varepsilon 0}(\tau) \dots Q_{N_{k-1},N_k}^{\varepsilon 0}(\tau) \widetilde{G}_{N_k}^{\varepsilon}(t_{\text{stop}}) dz_1 \tag{4.1}$$

where $1 \leq k \leq K$ is fixed, as well as the set $\mathbf{n}_k = (n_j)_{1 \leq j \leq k}$ of integers. Given a collision tree $a \in \mathcal{A}_{1,N_k-1}^{\pm}$, we will use, as explained in (2.14), the injectivity of the change of variables

$$\left(z_1, (t_i, \omega_i, v_{1+i})_{1 \leq i \leq N_k-1}\right) \longmapsto Z_{N_k}^{\varepsilon}(0) \in \mathcal{R}_{a,\mathbf{n}_k}, \tag{4.2}$$

where the configurations in $\mathcal{R}_{a,\mathbf{n}_k}$ are obtained by pseudo-trajectories satisfying (i)(ii) when the addition of new particles is prescribed by the collision tree a.

We can thus write

$$I_{\mathbf{n}_k} = \int \Phi_{N_k}(Z_{N_k}) \widetilde{G}_{N_k}^{\varepsilon}(t_{\text{stop}}, Z_{N_k}) dZ_{N_k},$$

with

$$\Phi_{N_k}(Z_{N_k}) := \frac{\mu_{\varepsilon}^{N_k-1}}{N_k!} \sum_{\sigma \in \mathfrak{S}_{N_k}} \sum_{a \in \mathcal{A}_{1,N_k-1}^{\pm}} h\left(z_{\sigma(1)}^{\varepsilon}(\theta, Z_{\sigma})\right) \mathbf{1}_{\{Z_{\sigma} \in \mathcal{R}_{a,\mathbf{n}_k}\}} \prod_{i=1}^{N_k-1} s_i. \tag{4.3}$$

Using same the notation as (2.18), we set

$$\hat{\Phi}_N\left(\mathbf{Z}_N^{\varepsilon}\right) := \frac{1}{\mu_{\varepsilon}^N} \sum_{(i_1,\dots i_N)} \Phi_N\left(\mathbf{z}_{i_1}^{\varepsilon}, \dots, \mathbf{z}_{i_N}^{\varepsilon}\right) - \mathbb{E}_{\varepsilon}(\Phi_N), \tag{4.4}$$

so that $I_{\mathbf{n}_k}$ becomes

$$I_{\mathbf{n}_k} = \mathbb{E}_{\varepsilon}\left(\mu_{\varepsilon}^{1/2} \, \hat{\Phi}_{N_k}\left(\mathbf{Z}_{N_k}^{\varepsilon}(t_{\text{stop}})\right) \zeta_0^{\varepsilon}(g_0) \, \mathbf{1}_{Y_{\mathcal{N}}^{\varepsilon}}\right) + \mu_{\varepsilon}^{1/2} \, \mathbb{E}_{\varepsilon}\left(\Phi_{N_k}\right) \mathbb{E}_{\varepsilon}\left(\zeta_0^{\varepsilon}(g_0)\mathbf{1}_{Y_{\mathcal{N}}^{\varepsilon}}\right), \tag{4.5}$$

where the indicator function on $Y_{\mathcal{N}}^{\varepsilon}$ stands for the restriction on the microscopic cluster sizes and on the velocities (recall Definition 2.5). Applying the Cauchy-Schwarz inequality, as in (2.19), leads to the following upper bound

$$|I_{\mathbf{n}_k}| \leq \mathbb{E}_{\varepsilon}\left((\zeta_0^{\varepsilon}(g_0))^2\right)^{1/2} \mathbb{E}_{\varepsilon}\left(\mu_{\varepsilon}\left(\hat{\Phi}_{N_k}\left(\mathbf{Z}_{N_k}^{\varepsilon}(t_{\text{stop}})\right)\right)^2\right)^{1/2}$$

$$+ \mu_{\varepsilon}^{1/2} \left|\mathbb{E}_{\varepsilon}\left(\Phi_{N_k}\right)\mathbb{E}_{\varepsilon}\left(\zeta_0^{\varepsilon}(g_0)\mathbf{1}_{Y_{\mathcal{N}}^{\varepsilon}}\right)\right| \tag{4.6}$$

which can be estimated by Proposition 3.1. To do this, we are going to check, in Lemmas 4.1 and 4.2 stated below, that Φ_{N_k} satisfies the assumptions (3.1) and (3.2) of Proposition 3.1. The last term involving the expectation will be negligible thanks to estimate (2.27) of Proposition 2.7.

Lemma 4.1. *There exists $C > 0$ such that*

$$\sup_{x_{N_k} \in \mathbb{T}^d} \int \left| \Phi_{N_k}(Z_{N_k}) \right| M^{\otimes N_k}(V_{N_k}) \, dX_{N_k-1} dV_{N_k} \leq C^{N_k} \|h\|_{L^\infty(\mathbb{D})} \theta^{N_{k-1}-1} \tau^{n_k} . \tag{4.7}$$

Lemma 4.2. *There exists $C > 0$ such that, for any $\ell = 1, \ldots, N_k$,*

$$\sup_{x_{2N_k-\ell} \in \mathbb{T}^d} \int \left| \Phi_{N_k}(Z_{N_k}) \Phi_{N_k}(Z_\ell, Z_{N_k+1, 2N_k-\ell}) \right| M^{\otimes(2N_k-\ell)}(V_{2N_k-\ell}) \, dX_{2N_k-\ell-1} dV_{2N_k-\ell}$$

$$\tag{4.8}$$

$$\leq C^{N_k} \mu_\varepsilon^{\ell-1} N_k^{-\ell} \|h\|_{L^\infty(\mathbb{D})}^2 \, \theta^{2N_k-\ell-1-n_k} \tau^{n_k} .$$

Assuming those lemmas are true, let us complete the estimate of $I_{\mathbf{n}_k}$. Applying the quasi-orthogonality estimates of Proposition 3.1 to Φ_{N_k}, we obtain the bounds

$$\mathbb{E}_\varepsilon(\Phi_{N_k}) \leq C^{N_k} \|h\|_{L^\infty(\mathbb{D})} \theta^{N_{k-1}-1} \tau^{n_k}$$

and

$$\mathbb{E}_\varepsilon\left(\mu_\varepsilon\left(\hat{\Phi}_{N_k}\left(\mathbf{Z}_{N_k}^\varepsilon(t_{\text{stop}}) \right)^2 \right) \right) \leq \|h\|_{L^\infty(\mathbb{D})}^2 \left(\sum_{\ell=1}^{N_k} C^{N_k} \theta^{2N_k-\ell-1-n_k} \tau^{n_k} + \varepsilon \theta^{2(N_{k-1}-1)} \tau^{2n_k} \right).$$

As noted in Remark 2.4, thanks to the symmetrization and the quasi-orthogonality of the supports in (4.3), we gain a factor $N_k!$.

Starting from (4.6), and using (2.3) to control $\mathbb{E}_\varepsilon((\zeta_0^\varepsilon(g_0))^2)^{1/2}$ and (2.27) of Proposition 2.7 to control $\mathbb{E}_\varepsilon(\zeta_0^\varepsilon(g_0)\mathbf{1}_{Y_{\mathcal{N}}^\varepsilon})$, we finally get

$$|I_{\mathbf{n}_k}| \leq C^{N_k} \|g_0\|_{L_M^2} \|h\|_{L^\infty(\mathbb{D})} \left(\left(\sum_{\ell=1}^{N_k} \theta^{2N_k-\ell-1-n_k} \tau^{n_k} + \varepsilon \theta^{2(N_{k-1}-1)} \tau^{2n_k} \right)^{1/2} \right.$$

$$\tag{4.9}$$

$$\left. + \theta^{N_{k-1}-\frac{1}{2}} \tau^{n_k} \, \mu_\varepsilon^{1/2} \varepsilon^{d/2} \right)$$

$$\leq \|g_0\|_{L_M^2} \|h\|_{L^\infty(\mathbb{D})} (C\theta)^{N_{k-1}+n_k/2} \, \tau^{n_k/2}$$

since $\varepsilon \ll \tau \ll 1 \leq \theta$.

To complete Proposition 2.8, we will show that the contribution of the superexponential trees is negligible. For superexponential trees, then $N_{k-1} \leq 2^k \leq n_k$. This leads to

$$|I_{\mathbf{n}_k}| \leq \|h\|_{L^\infty(\mathbb{D})} \|g_0\|_{L_M^2} \, (C\theta)^{N_{k-1}+n_k/2} \, \tau^{n_k/2} \leq \|h\|_{L^\infty(\mathbb{D})} \, \|g_0\|_{L_M^2} \, (C\theta^3\tau)^{n_k/2}. \tag{4.10}$$

The parameters θ, τ satisfy (2.28) so we can sum over $(n_j)_{j \leq k}$ and the series is controlled by

$$\left| \int dz_1 G_1^{\varepsilon, \exp}(\theta, z_1) h(z_1) \right| \leq \|h\|_{L^\infty(\mathbb{D})} \|g_0\|_{L_M^2} \sum_{k=1}^K 2^{k^2} (C\theta^3\tau)^{2^{k-1}} . \tag{4.11}$$
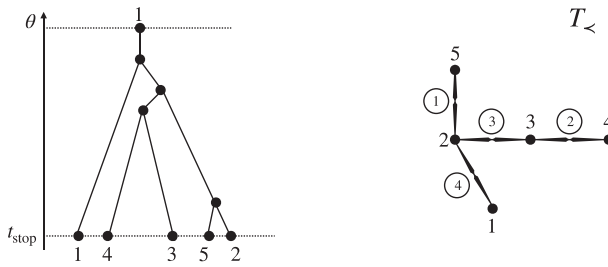
The proof of Proposition 2.8 is complete. □

FIGURE 5 In the figure on the left, an example of a pseudo-trajectory with $N_k = 5$. The associate graph $T_\prec$ is depicted on the right. For instance there holds $G_2 = \{(2,5)(3,4)\}$ and $(q_3, \bar{q}_3) = (2,3)$.

Before proving Lemmas 4.1 and 4.2, let us introduce some notation. For any positive integer N, we shall denote as previously by $\mathcal{T}_N$ the set of trees (minimally connected graphs) with N vertices. We further denote by $\mathcal{T}_N^\prec$ the set of ordered trees. A tree $T_\prec \in \mathcal{T}_N^\prec$ is represented by an ordered sequence of edges $(q_i, \bar{q}_i)_{1 \leq i \leq N-1}$.

Proof of Lemma 4.1. For each configuration Z_{N_k}, there exist at most 4^{N_k-1} different (σ, a) such that $Z_\sigma \in \mathcal{R}_{a, \mathbf{n}_k}$. Indeed at each encounter between two particles in the forward flow, the particle which disappears has to be chosen, as well as a possible scattering. To fix these discrepancies, we introduce two sets of signs $\bar{s}_i$ and s_i which determine respectively which particle should be removed (say $\bar{s}_i = +$ if the particle with largest index remains, $\bar{s}_i = -$ if it disappears) and whether there is scattering ($s_i = +$) or not ($s_i = -$). Note that the signs $(s_i)_{1 \leq i \leq N_k-1}$ are encoded in the tree a while $(\bar{s}_i)_{1 \leq i \leq N_k-1}$ are known if σ is given. If we prescribe the set $\mathbf{S}_{N_k-1} := (s_i, \bar{s}_i)_{1 \leq i \leq N_k-1}$, then the mapping

$$\left(a, \sigma, z_1, (t_i, \omega_i, v_{1+i})_{1 \leq i \leq N_k-1}\right) \longmapsto Z_\sigma^\varepsilon(t_{\text{stop}})$$

restricted to pseudo-trajectories compatible with $\mathbf{S}_{N_k-1}$, is injective. Recalling (4.3) for the definition of Φ_{N_k}, this leads to

$$\left|\Phi_{N_k}(Z_{N_k})\right| \leq \|h\|_{L^\infty(\mathbb{D})} \frac{\mu_\varepsilon^{N_k-1}}{N_k!} \sum_{\mathbf{S}_{N_k-1}} \mathbf{1}_{\{Z_{N_k} \in \mathcal{R}_{\mathbf{S}_{N_k-1}}\}}, \tag{4.12}$$

where $\mathcal{R}_{\mathbf{S}_{N_k-1}}$ is the set of configurations such that the forward flow compatible with $\mathbf{S}_{N_k-1}$ exists, and with the constraints respecting the sampling (we drop the dependence of the sets on $\mathbf{n}_k$, not to overburden notation).

We are now going to evaluate the cost of the constraint $Z_{N_k} \in \mathcal{R}_{\mathbf{S}_{N_k-1}}$ for a given $\mathbf{S}_{N_k-1}$. For this it is convenient to record the encounters in the forward dynamics in an ordered tree $T_\prec = (q_i, \bar{q}_i)_{1 \leq i \leq N_k-1}$: the first encounter, in the forward flow starting at configuration Z_{N_k} at time 0, is between particles q_1 and $\bar{q}_1$ at time $\tau_1 \in (t_{\text{stop}}, \theta)$, and the last encounter is between q_{N_k-1} and $\bar{q}_{N_k-1}$ at time $\tau_{N_k-1} \in (\tau_{N_k-2}, \theta)$. An example is depicted in Figure 5. Notice that compared with the definition of (backward) pseudo-trajectories, since we follow the trajectories forward in time we choose an increasing order in the collision times (namely $\tau_i = t_{N_k-i}$). This leads to

$$\left|\Phi_{N_k}(Z_{N_k})\right| \leq \|h\|_{L^\infty(\mathbb{D})} \frac{\mu_\varepsilon^{N_k-1}}{N_k!} \sum_{\mathbf{S}_{N_k-1}} \sum_{T_\prec \in \mathcal{T}_{N_k}^\prec} \mathbf{1}_{\{Z_{N_k} \in \mathcal{R}_{T_\prec, \mathbf{S}_{N_k-1}}\}}, \tag{4.13}$$

where $\mathcal{R}_{T_\prec, \mathbf{S}_{N_k-1}}$ is the set of configurations such that the forward flow compatible with the couple $(T_\prec, \mathbf{S}_{N_k-1})$ exists, and with the constraints respecting the sampling. Actually note that for a fixed $\mathbf{S}_{N_k-1}$, the above sum over ordered trees corresponds to a partition, meaning that for any given Z_{N_k}, at most one term is non zero.

Given such an admissible tree $T_\prec$ let us define the relative positions at time t_{stop}

$$\hat{x}_i := x_{q_i} - x_{\bar{q}_i}.$$

Given the relative positions $(\hat{x}_s)_{s<i}$ and the velocities V_{N_k}, we fix a forward flow with encounters at times $\tau_1 < \cdots < \tau_{i-1} < \theta$. By construction, q_i and $\bar{q}_i$ belong to two forward pseudo-trajectories that have not interacted yet. In other words, q_i and $\bar{q}_i$ do not belong to the same connected component in the graph $G_{i-1} := (q_j, \bar{q}_j)_{1 \le j \le i-1}$.

Inside each connected component, relative positions are fixed by the previous constraints, and one degree of freedom remains. Therefore we are going to vary $\hat{x}_i$ so that an encounter at time $\tau_i \in (\tau_{i-1}, \theta)$ occurs between q_i and $\bar{q}_i$ (moving rigidly the corresponding connected components). This encounter condition defines a set $\mathcal{B}_{T_\prec, i}(\hat{x}_1, \ldots, \hat{x}_{i-1}, V_{N_k})$. $\qquad\square$

Definition 4.3. We say that the sets $(\mathcal{B}_{T_\prec, i})_{i \le N_k-1}$ are <u>sequentially independent</u> if for all i the set $\mathcal{B}_{T_\prec, i}$ is defined by constraints depending only on $\hat{x}_1, \ldots, \hat{x}_{i-1}, V_{N_k}$.

The particles q_i and $\bar{q}_i$ move in straight lines, therefore the measure of this set can be estimated by

$$|\mathcal{B}_{T_\prec, i}| \le \frac{C}{\mu_\varepsilon} |v_{q_i}^\varepsilon(\tau_{i-1}^+) - v_{\bar{q}_i}^\varepsilon(\tau_{i-1}^+)| \int \mathbf{1}_{\tau_i \ge \tau_{i-1}} d\tau_i$$

and there holds

$$\sum_{q_i, \bar{q}_i} |\mathcal{B}_{T_\prec, i}| \le \frac{C}{\mu_\varepsilon} \left(V_{N_k}^2 + N_k\right) N_k \int \mathbf{1}_{\tau_i \ge \tau_{i-1}} d\tau_i. \tag{4.14}$$

Hence by Fubini's theorem

$$\sum_{T_\prec \in \mathcal{T}_{N_k}^\prec} \int d\hat{X}_{N_k-1} \prod_{i=1}^{N_k-1} \mathbf{1}_{\mathcal{B}_{T_\prec, i}} \le \sum_{T_\prec \in \mathcal{T}_{N_k}^\prec} \int d\hat{x}_1 \mathbf{1}_{\mathcal{B}_{T_\prec, 1}} \int d\hat{x}_2 \cdots \int d\hat{x}_{N_k-1} \mathbf{1}_{\mathcal{B}_{T_\prec, N_k-1}}$$

$$\le \left(\frac{C}{\mu_\varepsilon}\right)^{N_k-1} \left(V_{N_k}^2 + N_k\right)^{N_k-1} N_k^{N_k-1} \int_{t_{\text{stop}}}^\theta d\tau_1 \cdots \int_{\tau_{N_k-2}}^\theta d\tau_{N_k-1} \mathbf{1}_{\mathbf{n}_k} \tag{4.15}$$

where $\mathbf{1}_{\mathbf{n}_k}$ is the constraint on times respecting the sampling in (4.1). Retaining only the information that n_k times are in the interval $(t_{\text{stop}}, t_{\text{stop}} + \tau)$ and the other $N_{k-1} - 1$ times are in $(t_{\text{stop}} + \tau, \theta)$, we get by integrating over these ordered times an upper bound of the form

$$\frac{\tau^{n_k}}{n_k!} \frac{\theta^{N_{k-1}-1}}{(N_{k-1}-1)!} \le \frac{2^{N_k-1}}{(N_k-1)!} \tau^{n_k} \theta^{N_{k-1}-1}. \tag{4.16}$$

Up to a factor C^{N_k}, the factorial $(N_k - 1)!$ compensates the factor $N_k^{N_k}$ in (4.15). Furthermore, for any K, N and dimension $D > 0$

$$\sup_{V \in \mathbb{R}^D} \left\{ \exp\left(-\frac{1}{8}|V|^2\right) (|V|^2 + K)^N \right\} \leq C^N e^K N^N. \tag{4.17}$$

After integrating the velocities with respect to the measure $M^{\otimes N_k}$, we deduce from the previous inequality that the term $(V_{N_k}^2 + N_k)^{N_k}$ leads to another factor of order $N_k^{N_k}$ which is compensated, up to a factor C^{N_k}, by the $N_k!$ in (4.12). Combining all these estimates, we deduce that $\int d\hat{X}_{N_k-1} dV_{N_k} |\Phi_{N_k}| M^{\otimes N_k}$ can be bounded from above uniformly with respect to the one remaining parameter which takes into account the translation invariance of the system: for clarity, we have decided arbitrarily that the remaining degree of freedom is indexed by the variable x_{N_k}. This completes the proof of Lemma 4.1.

Proof of Lemma 4.2. The proof is similar to the one of the previous lemma, however, we have to analyze now the dynamical constraints associated with two configurations $Z_{N_k} = (Z_\ell, Z_{\ell+1, N_k})$ and $Z'_{N_k} = (Z_\ell, Z_{N_k+1, 2N_k-\ell})$ sharing ℓ particles. The parameters $\mathbf{S}_{N_k-1} = (s_i, \bar{s}_i)_{1 \leq i \leq N_k-1}$ and $\mathbf{S}'_{N_k-1} = (s'_i, \bar{s}'_i)_{1 \leq i \leq N_k-1}$ coding the encounters are fixed for each configuration. By analogy with formula (4.12), we get

$$\left| \Phi_{N_k}(Z_{N_k}) \Phi_{N_k}(Z_\ell, Z_{N_k+1, 2N_k-\ell}) \right|$$
$$\leq \|h\|_{L^\infty(\mathbb{D})}^2 \left(\frac{\mu_\varepsilon^{N_k-1}}{N_k!} \right)^2 \sum_{\substack{\mathbf{s}_{N_k-1} \\ \mathbf{s}'_{N_k-1}}} \mathbf{1}_{\{Z_{N_k} \in \mathcal{R}_{\mathbf{s}_{N_k-1}}\}} \mathbf{1}_{\{Z'_{N_k} \in \mathcal{R}_{\mathbf{s}'_{N_k-1}}\}}. \tag{4.18}$$

We consider the forward flows of each set of particles Z_{N_k} and Z'_{N_k} starting at time t_{stop}. Both dynamics evolve independently and each one of them should have exactly $N_k - 1$ encounters to be compatible with an ordered tree as the ones used in the proof of Lemma 4.1. As the configurations Z_{N_k} and Z'_{N_k} share ℓ particles in common, strong correlations are imposed in order to produce a total of $2(N_k - 1)$ encounters. For our purpose, it is enough to relax these constraints and to record only $2N_k - \ell - 1$ (sequentially independent) "clustering encounters" which will be indexed by an ordered graph $T''_\prec$ with $2N_k - \ell - 1$ edges, as well as relative positions $(\hat{x}_i)_{1 \leq i \leq 2N_k-\ell-1}$ at time t_{stop}.

The ordered graph $T''_\prec$ is constructed as follows. As in the proof of Lemma 4.1, we denote by $T_\prec$ the ordered tree corresponding to the forward flow of Z_{N_k}, and by $(\tau_i)_{1 \leq i \leq N_k-1}$ and $(\hat{x}_i)_{1 \leq i \leq N_k-1}$ the corresponding encounter times and relative positions. The first $N_k - 1$ edges $(q_i, \bar{q}_i)_{1 \leq i \leq N_k-1}$ of the graph $T''_\prec$ are the edges of the ordered tree $T_\prec$, so that $T_\prec$ is fully embedded in $T''_\prec$ (this prescribes the constraints on the particles Z_{N_k}). The last $N_k - \ell$ edges in $T''_\prec$ will record the additional constraints on the remaining particles $Z_{N_k+1, 2N_k-\ell}$ which are involved in the dynamics of Z'_{N_k} (see Figure 6).

The edges $(q_i, \bar{q}_i)_{N_k \leq i \leq 2N_k-\ell}$ are added as follows, keeping only the *clustering encounters* in the forward dynamics of Z'_{N_k}, that is, the encounters associated with edges which are not creating cycles in the graph:

- the first clustering encounter is the first encounter in the forward flow of Z'_{N_k} involving at least one particle with label in $[N_k + 1, 2N_k - \ell]$. We denote by $(q_{N_k}, \bar{q}_{N_k})$ the labels of the colliding particles and by τ_{N_k} the corresponding time. We also define the ordered graph
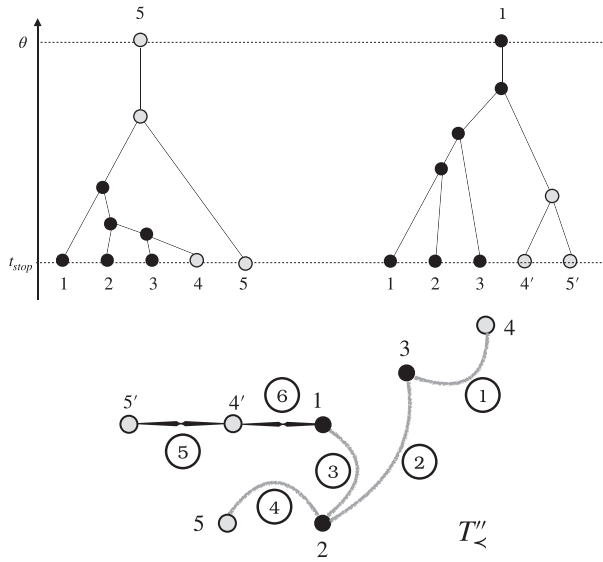
FIGURE 6 In the figure on the left, an example of two pseudo-trajectories sharing $\ell = 3$ particles with $N_k = 5$. The graph $T_<$ associated with the left pseudo-trajectory starting from Z_5 is depicted by the bended grey edges ordered according to the encounter times. The complete tree $T''_<$ is built starting from $T_<$ to which two additional straight edges (numbered 5 and 6) have been added to connect $4'$ and $5'$.

$G_{N_k} = (q_j, \bar{q}_j)_{1 \leq j \leq N_k}$. Note that on Figure 6, the graph G_5 is made of two components $\{1, 2, 3, 4, 5\}$ and $\{4', 5'\}$.

- for $N_k + 1 \leq i \leq 2N_k - \ell - 1$, the i-th clustering encounter is the first encounter (after τ_{i-1}) in the forward flow of Z'_{N_k} involving two particles which are not in the same connected component of the graph G_{i-1}. By construction at least one of these particles belongs to $Z_{N_k+1,2N_k-\ell}$. We denote by $(q_i, \bar{q}_i)$ the labels of the colliding particles and by τ_i the corresponding time. We also define the ordered graph $G_i = (q_j, \bar{q}_j)_{1 \leq j \leq i}$.

By this procedure, we end up with a tree $T''_< := (q_i, \bar{q}_i)_{1 \leq i \leq 2N_k - \ell - 1}$ with no cycles (nor multiple edges). We define as above the relative positions $\hat{x}_i := x_{q_i} - x_{\bar{q}_i}$.

Note that the sequence of times $(\tau_i)_{1 \leq i \leq 2N_k - \ell - 1}$ is only partially ordered. Indeed the times $\tau_1 < \cdots < \tau_{N_k-1}$ associated with Z_{N_k} are ordered, and the same applies to the times $\tau_{N_k} < \cdots < \tau_{2N_k-\ell-1}$ associated with the clustering encounters in Z'_{N_k} but they are not mutually ordered. Nevertheless, this is not a problem since the only important point is that the sets $(\mathcal{B}_{T''_<,i})_{1 \leq i \leq 2N_k - \ell - 1}$, defined as in the proof of Lemma 4.1, only depend on $\hat{x}_1, \ldots, \hat{x}_{i-1}, V_{2N_k-\ell}$. When $i \geq N_k$, this is less obvious than in the previous case since in the construction of $T''_<$ some encounters (those in the forward flow of Z'_{N_k} leading to cycles) have been left out, so one needs to check that the corresponding trajectories before time τ_i can be reconstructed knowing only $\hat{x}_1, \ldots, \hat{x}_{i-1}, V_{2N_k-\ell}$.

By construction, for $i \geq N_k$, the two particles $(q_i, \bar{q}_i)$ which encounter at time τ_i belong to two different connected components $C_{i-1}(q_i)$ and $C_{i-1}(\bar{q}_i)$ of the dynamical graph G_{i-1}. The trajectory of q_i in the pseudo-trajectory of Z'_{N_k} up to time τ_i depends only

- on the relative positions $(\hat{x}_j)_{(q_j, \bar{q}_j) \in C_{i-1}(q_i)}$ at t_{stop}
- and on any root of $C_{i-1}(q_i)$, for instance the position x_{q_i} of q_i at t_{stop}.

The same holds for the trajectory of $\bar{q}_i$. We can therefore write the colliding condition by moving rigidly the two connected components $C_{i-1}(q_i)$ and $C_{i-1}(\bar{q}_i)$, which provides as previously a condition on $\hat{x}_i$.

From this point, we can proceed exactly as in the previous lemma and the sets $\mathcal{B}_{T''_{\prec},i}$ satisfy the same estimates as before:

$$
\sum_{T''_{\prec}} \int d\hat{X}_{2N_k-\ell-1} \prod_{i=1}^{2N_k-\ell-1} \mathbf{1}_{\mathcal{B}_{T''_{\prec},i}}
$$

$$
\leq \left(\frac{C}{\mu_\varepsilon}\right)^{2N_k-\ell-1} \left(V_{N_k}^2 + N_k\right)^{N_k-1} N_k^{N_k-1} \left((V'_{N_k})^2 + N_k\right)^{N_k-\ell} N_k^{N_k-\ell} \tag{4.19}
$$

$$
\times \int_{t_{\text{stop}}}^{\theta} d\tau_1 \dots \int_{\tau_{N_k-2}}^{\theta} d\tau_{N_k-1} \mathbf{1}_{\mathbf{n}_k} \times \int_{t_{\text{stop}}}^{\theta} d\tau_{N_k} \dots \int_{\tau_{2N_k-\ell-2}}^{\theta} d\tau_{2N_k-\ell-1} .
$$

Notice that the first $N_k - 1$ ordered time integrals correspond to the constraints in the tree $T_{\prec}$ and are estimated from above by $\frac{2^{N_k-1}}{(N_k-1)!}\tau^{n_k}\theta^{N_k-1-1}$ as in (4.16). The sampling in (4.1) is omitted for the remaining times which are simply constrained to satisfy the ordering conditions $\tau_{N_k} < \cdots < \tau_{2N_k-\ell-1} \leq \theta$, so that

$$
\int_{t_{\text{stop}}}^{\theta} d\tau_1 \cdots \int_{\tau_{N_k-2}}^{\theta} d\tau_{N_k-1} \mathbf{1}_{\mathbf{n}_k} \times \int_{t_{\text{stop}}}^{\theta} d\tau_{N_k} \cdots \int_{\tau_{2N_k-\ell-2}}^{\theta} d\tau_{2N_k-\ell-1}
$$

$$
\leq \frac{2^{N_k-1}}{(N_k-1)!}\tau^{n_k}\theta^{N_k-1-1} \times \frac{\theta^{N_k-\ell}}{(N_k-\ell)!} \leq \frac{C^{N_k}}{(N_k-\ell)!(N_k-1)!}\tau^{n_k}\theta^{2N_k-\ell-1-n_k} .
$$

Plugging this estimate in (4.19), we deduce that

$$
\sum_{T''_{\prec}} \int d\hat{X}_{2N_k-\ell-1} \prod_{i=1}^{2N_k-\ell-1} \mathbf{1}_{\mathcal{B}_{T''_{\prec},i}}
$$

$$
\leq \left(\frac{C}{\mu_\varepsilon}\right)^{2N_k-\ell-1} \tau^{n_k}\theta^{2N_k-\ell-1-n_k} \left(V_{N_k}^2 + N_k\right)^{N_k-1} \left((V'_{N_k})^2 + N_k\right)^{N_k} N_k^{-\ell} .
$$

We conclude as in the proof of Lemma 4.1 by integrating with respect to velocities $V_{2N_k-\ell}$, and by using the prefactor $(N_k!)^{-2}$ from (4.18) to compensate, up to a factor C^{N_k}, the divergence $N_k^{2N_k}$ coming from (4.17). Lemma 4.2 is proved. $\qquad\square$

5 | THE COST OF NON-CLUSTERING CONSTRAINTS

In this section we prove Proposition 2.9 showing that, compared to the previous section, the presence of a recollision produces extra smallness as μ_ε goes to infinity. Let us recall the setup: the term $G_1^{\varepsilon,\text{rec}}(\theta)$ encodes pseudo-trajectories with no recollision and sub-exponential growth in the first $k - 1$ time intervals of length τ; and the first recollision in the smaller time

interval $(t_{\text{stop}}, t_{\text{stop}} + \delta)$ with $t_{\text{stop}} := \theta - (k-1)\tau - r\delta$. Recall that

$$
G_1^{\varepsilon,\text{rec}}(\theta) := \sum_{k=1}^{K} \sum_{(n_j \leq 2^j)_{j \leq k-1}} \sum_{r=1}^{R} \sum_{n_k \geq 0} \sum_{n_k^0 + n_k^{\text{rec}} = n_k} Q_{1,N_1}^{\varepsilon 0}(\tau) \ldots Q_{N_{k-2},N_{k-1}}^{\varepsilon 0}(\tau)
$$
$$
\circ Q_{N_{k-1}, N_{k-1}+n_k^0}^{\varepsilon 0}((r-1)\delta) Q_{N_{k-1}+n_k^0, N_{k-1}+n_k^0+n_k^{\text{rec}}}^{\text{rec}}(\delta) \widetilde{G}_{N_k}^{\varepsilon}(t_{\text{stop}})
$$

(5.1)

corresponds to pseudo-trajectories such that

(i) the number of new particles added respectively on the three time intervals $(\theta - j\tau, \theta - (j-1)\tau)$, $(\theta - (k-1)\tau - (r-1)\delta, \theta - (k-1)\tau)$ and $(t_{\text{stop}}, t_{\text{stop}} + \delta)$ are $n_j \leq 2^j$, n_k^0 and n_k^{rec},
(ii) there is no recollision on the interval $(t_{\text{stop}} + \delta, \theta)$ and there is at least one on $(t_{\text{stop}}, t_{\text{stop}} + \delta)$.

Furthermore because of the conditioning, we also know that at t_{stop}, each velocity $|v_i|$ $(1 \leq i \leq N_k)$ is less than $\mathbb{V}$, and the configuration has no microscopic cluster of more than γ particles.

We set $\mathbf{n}_k := ((n_j)_{1 \leq j \leq k-1}, n_k^0, n_k^{\text{rec}}, r)$ and

$$
I_{r, \mathbf{n}_k}^{\text{rec}} := \int h(z_1) Q_{1,N_1}^{\varepsilon 0}(\tau) \ldots
$$
$$
\ldots Q_{N_{k-1}, N_{k-1}+n_k^0}^{\varepsilon 0}((r-1)\delta) Q_{N_{k-1}+n_k^0, N_{k-1}+n_k^0+n_k^{\text{rec}}}^{\text{rec}}(\delta) \widetilde{G}_{N_k}^{\varepsilon}(t_{\text{stop}}).
$$

(5.2)

The idea is therefore to combine the argument of the previous section, with a geometric estimate on the strong constraint characterizing the recollision event, which will bring a small factor in ε.

As above we shall use a duality argument in order to write an expression of the type

$$
I_{r, \mathbf{n}_k}^{\text{rec}} = \int \Phi_{N_k}^{\text{rec}}(Z_{N_k}) \widetilde{G}_{N_k}^{\varepsilon}(t_{\text{stop}}, Z_{N_k}) dZ_{N_k}.
$$

Recall however that recollisions have been defined for pseudo-trajectories, which by construction (see Section 2.3) correspond to following the flow of (pseudo)-particles backwards in time. On the other hand the duality argument requires studying the flow forward in time, and this produces two difficulties. First, defining this forward flow uniquely is not possible if the number of recollisions for each particle is not known, so a new parameter needs to be introduced to track this number. Second, we shall need to understand the effect on the forward flow of the presence of a recollision in the (backward) pseudo-trajectories: it will be responsible for the presence of a cycle in the (forward) trees we shall construct.

Let us start by writing $I_{r, \mathbf{n}_k}^{\text{rec}}$ in dual form. The presence of recollisions requires introducing additional parameters to recover the injectivity of the change of variables (2.14). On the time interval $(t_{\text{stop}} + \delta, \theta)$, the situation is the same as in the previous section since there are no recollisions by definition. On $(t_{\text{stop}}, t_{\text{stop}} + \delta)$ however, the construction of the forward dynamics starting from a configuration Z_{N_k} is more intricate since there is at least one recollision. The important fact is that the number of recollisions is under control. We have seen that particles from different microscopic clusters cannot collide on $(t_{\text{stop}}, t_{\text{stop}} + \delta)$ (see Remark 2.6). Therefore, each particle may interact at most with $\gamma - 1$ particles on this small interval. Furthermore, there cannot be any recollision due to periodicity as $\mathbb{V}\delta \ll 1$. Since the total number of collisions for a system of γ hard spheres in the whole space is finite (see Theorem 1.3 in [7] or [18]) say at most $\mathbb{K}_\gamma$, each particle in a pseudo-trajectory cannot have more than $\mathcal{K}_\gamma = \sum_{\ell=2}^{\gamma} \mathbb{K}_\ell$ recollisions during the short amount
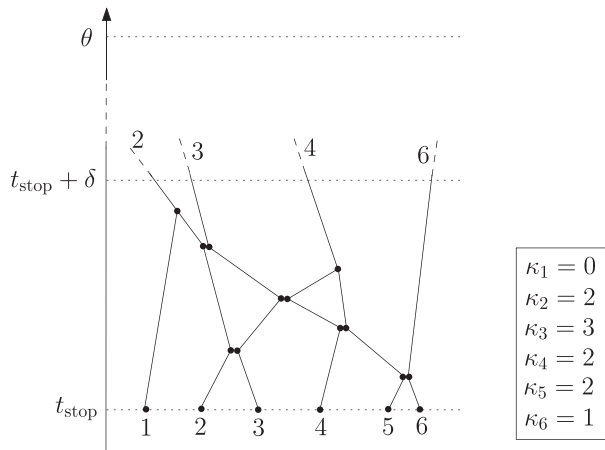
FIGURE 7   Example of pseudo-trajectory with recollisions in the time interval $(t_{\text{stop}}, t_{\text{stop}} + \delta)$. The associated forward flow is determined thanks to the recollision indices at time t_{stop}, listed on the right.

of time δ. This crude upper bound on the number of recollisions takes into account the fact that the number of particles in a cluster may vary due the creation of new particles. We then associate with each particle i an index κ_i (less than $\mathcal{K}_\gamma$) which is zero at time θ and increased by one each time the particle undergoes a recollision in the backward pseudo-dynamics. We denote by $\mathbf{K}_{N_k}$ the set of recollision indices $(\kappa_i)_{1 \leq i \leq N_k}$ at time t_{stop}.

Given a collision tree $a \in \mathcal{A}^{\pm}_{1, N_k-1}$, this new set of parameters enables us to recover the lost injectivity, by applying the following rule to reconstruct the forward dynamics (see Figure 7). At each encounter between two particles,

- if the two particles have a positive index, then it corresponds to a recollision in the backward pseudo-dynamics, and the recollision index of each particle has to be decreased by one in the forward flow,
- if one of the particles has zero index, then it corresponds to a creation in the backward pseudo-dynamics. In the forward flow, a particle must disappear: its label, and the possible scattering of the other colliding particle are prescribed by the collision tree a.

Note that the disappearing particle should have zero index, else the trajectory is not admissible.

Finally let us define, for each a and each $\mathbf{K}_{N_k}$ in $\{0, \dots, \mathcal{K}_\gamma\}^{N_k}$, the set $\mathcal{R}^{\text{rec}}_{\mathbf{K}_{N_k}, a, \mathbf{n}_k}$ of configurations compatible with pseudo-trajectories satisfying (i)(ii) and such that

(iii) the addition of new particles is prescribed by the collision tree a and recollisions between particles are compatible with $\mathbf{K}_{N_k}$.

Then the change of variables, as in (2.14),

$$\left(z_1, (t_i, \omega_i, v_{1+i})_{1 \leq i \leq N_k-1} \right) \longmapsto \left(Z^\varepsilon_{N_k}(t_{\text{stop}}), \mathbf{K}_{N_k} \right)$$

of range

$$\left\{ (Z_{N_k}, \mathbf{K}_{N_k}) \in \mathcal{D}^\varepsilon_{N_k} \times \{0, \dots, \mathcal{K}_\gamma\}^{N_k}, \quad Z_{N_k} \in \mathcal{R}^{\text{rec}}_{\mathbf{K}_{N_k}, a, \mathbf{n}_k} \right\}$$

is injective (of course not surjective).

So we can now write

$$I_{r,\mathbf{n}_k}^{\mathrm{rec}} = \int \Phi_{N_k}^{\mathrm{rec}}(Z_{N_k})\widetilde{G}_{N_k}^{\varepsilon}(t_{\mathrm{stop}},Z_{N_k})dZ_{N_k}$$

where

$$\Phi_{N_k}^{\mathrm{rec}}(Z_{N_k}) := \frac{\mu_{\varepsilon}^{N_k-1}}{N_k!} \sum_{\sigma\in\mathfrak{S}_{N_k}} \sum_{a\in\mathcal{A}_{1,N_k-1}^{\pm}} \sum_{\mathbf{K}_{N_k}} h(z_{\sigma(1)}^{\varepsilon}(\theta))\mathbf{1}_{\{Z_{\sigma}\in\mathcal{R}_{\mathbf{K}_{N_k},a,\mathbf{n}_k}^{\mathrm{rec}}\}} \prod_{i=1}^{N_k-1} s_i . \tag{5.3}$$

Note that as in (2.15), we have enforced the symmetry of the particles which was lost in the Duhamel formulation (see Remark 2.4). Proceeding as in (4.4), we define $\hat{\Phi}_{N_k}^{\mathrm{rec}}$ by subtracting the mean and rewrite $I_{r,\mathbf{n}_k}^{\mathrm{rec}}$ as an expectation

$$I_{r,\mathbf{n}_k}^{\mathrm{rec}} = \mathbb{E}_{\varepsilon}\left(\mu_{\varepsilon}^{1/2} \hat{\Phi}_{N_k}^{\mathrm{rec}}\left(\mathbf{Z}_{\mathcal{N}}^{\varepsilon}(t_{\mathrm{stop}})\right) \zeta_0^{\varepsilon}(g_0) \mathbf{1}_{Y_{\mathcal{N}}^{\varepsilon}}\right) + \mu_{\varepsilon}^{1/2} \mathbb{E}_{\varepsilon}\left(\Phi_{N_k}^{\mathrm{rec}}\right)\mathbb{E}_{\varepsilon}\left(\zeta_0^{\varepsilon}(g_0)\mathbf{1}_{Y_{\mathcal{N}}^{\varepsilon}}\right).$$

Following (4.6), a Cauchy-Schwarz inequality implies

$$|I_{r,\mathbf{n}_k}^{\mathrm{rec}}| \le \mathbb{E}_{\varepsilon}\left((\zeta_0^{\varepsilon}(g_0))^2\right)^{1/2} \mathbb{E}_{\varepsilon}\left(\mu_{\varepsilon}\left(\hat{\Phi}_{N_k}^{\mathrm{rec}}\left(\mathbf{Z}_{\mathcal{N}}^{\varepsilon}(t_{\mathrm{stop}})\right)\right)^2\right)^{1/2}$$
$$+ \mu_{\varepsilon}^{1/2} \mathbb{E}_{\varepsilon}\left(\Phi_{N_k}^{\mathrm{rec}}\right)\mathbb{E}_{\varepsilon}\left(\zeta_0^{\varepsilon}(g_0)\mathbf{1}_{Y_{\mathcal{N}}^{\varepsilon}}\right).$$

As in (4.9), this can be estimated by Proposition 3.1 and using (2.27), once we check that $\Phi_{N_k}^{\mathrm{rec}}$ satisfies the assumptions (3.1) and (3.2) of Proposition 3.1. This is the purpose of the following two lemmas.

Lemma 5.1. *There exists $C > 0$ such that for $d \ge 3$,*

$$\sup_{x_{N_k}\in\mathbb{T}^d} \int \left|\Phi_{N_k}^{\mathrm{rec}}(Z_{N_k})\right| M^{\otimes N_k}(V_{N_k}) dX_{N_k-1}dV_{N_k} \tag{5.4}$$

$$\le C^{N_k}\|h\|_{L^{\infty}(\mathbb{D})}\delta^{\max\left(1,n_k^{\mathrm{rec}}\right)}\tau^{\binom{n_k^0-1}{+}}(\mathbb{V}\theta)^{2d+4}\theta^{N_k-1-1}\varepsilon|\log\varepsilon| .$$

Lemma 5.2. *There exists $C > 0$ such that, for any $\ell = 1,\dots,N_k$ and for $d \ge 3$,*

$$\sup_{x_{2N_k-\ell}\in\mathbb{T}^d} \int \left|\Phi_{N_k}^{\mathrm{rec}}(Z_{N_k})\Phi_{N_k}^{\mathrm{rec}}(Z_{\ell},Z_{N_k+1,2N_k-\ell})\right|$$

$$\times M^{\otimes(2N_k-\ell)}(V_{2N_k-\ell}) dX_{2N_k-\ell-1}dV_{2N_k-\ell} \tag{5.5}$$

$$\le C^{N_k}\mu_{\varepsilon}^{\ell-1}N_k^{-\ell}\|h\|_{L^{\infty}(\mathbb{D})}^2 \delta^{\max(1,n_k^{\mathrm{rec}})}\tau^{(n_k^0-1)_+}(\mathbb{V}\theta)^{2d+4}\theta^{2N_k-\ell-1-n_k}\varepsilon|\log\varepsilon| .$$

Assuming these lemmas are true, let us conclude the proof of Proposition 2.9. Thanks to Proposition 3.1 and using (2.27), there holds with the scaling choices from (2.21)

$$|I^{\rm rec}_{r,\mathbf{n}_k}| \leq C^{N_k} \|h\|_{L^\infty(\mathbb{D})} \|g_0\|_{L^2_M}$$

$$\left[\varepsilon^{\frac{1}{2}} |\log \varepsilon| \left(\sum_{\ell=1}^{N_k} \theta^{2N_k - \ell - 1 - n_k} \right)^{1/2} (\mathbb{V}\theta)^{d+2} \delta^{\frac{1}{2} \max(1, n_k^{\rm rec})} \tau^{\frac{1}{2}(n_k^0 - 1)_+} \right.$$

$$\left. + \varepsilon |\log \varepsilon| (\mathbb{V}\theta)^{2d+4} \theta^{N_{k-1}} \delta^{\max(1, n_k^{\rm rec})} \tau^{(n_k^0 - 1)_+} (\mu_\varepsilon \varepsilon^d)^{1/2} \right].$$

Using the choices (2.21) on the parameters we get

$$|I^{\rm rec}_{r,\mathbf{n}_k}| \leq \varepsilon^{\frac{1}{2}} |\log \varepsilon| \|h\|_{L^\infty(\mathbb{D})} \|g_0\|_{L^2_M} (C\theta)^{N_{k-1} + n_k/2} (\mathbb{V}\theta)^{d+2} \delta^{\frac{1}{2}\max(1, n_k^{\rm rec})} \tau^{\frac{1}{2}(n_k^0 - 1)_+}. \tag{5.6}$$

Finally we are in position to sum over all parameters (recall (5.1) and (5.2)). We find after summation over n_k^0 and $n_k^{\rm rec}$, then $r \leq \tau/\delta$ (which corresponds to the cutting of each interval of size τ into $R = \tau/\delta$ pieces) and finally $(n_j)_{j<k}$ and k,

$$\left| \int dz_1 G_1^{\varepsilon, \rm rec}(\theta) h(z_1) \right| \leq \frac{\tau}{\delta} (\mathbb{V}\theta)^{2d+4} \left(\sum_{k=1}^K 2^{k^2} \right) (C\theta)^{2^K} \delta^{\frac{1}{2}} \varepsilon^{\frac{1}{2}} |\log \varepsilon| \|h\|_{L^\infty(\mathbb{D})} \|g_0\|_{L^2_M}$$

$$\leq (C\theta)^{2^K} \varepsilon^{1/8d} \|h\|_{L^\infty(\mathbb{D})} \|g_0\|_{L^2_M},$$

as $\delta = \varepsilon^{1 - \frac{1}{2d}}$ by (2.21). This ends the proof of Proposition 2.9. □

Proof of Lemma. 5.1 We shall follow the method of the previous section, by introducing the set of signs $\mathbf{S}_{N_k - 1} = (s_i, \bar{s}_i)_{1 \leq i \leq N_k - 1}$, with $(s_i, \bar{s}_i)$ characterizing the i-th creation (namely whether there is scattering or not, and which particle remains). Then if $\mathbf{S}_{N_k - 1}, \mathbf{K}_{N_k}$ are prescribed, the mapping

$$\left(a, \sigma, z_1, (t_i, \omega_i, v_{1+i})_{1 \leq i \leq N_k - 1} \right) \longmapsto \left(Z_\sigma^\varepsilon(t_{\rm stop}) \right) \tag{5.7}$$

is injective and we infer that

$$\left| \Phi^{\rm rec}_{N_k}(Z_{N_k}) \right| \leq \|h\|_{L^\infty(\mathbb{D})} \frac{\mu_\varepsilon^{N_k - 1}}{N_k!} \sum_{\mathbf{K}_{N_k}, \mathbf{S}_{N_k - 1}} \mathbf{1}_{\{Z_{N_k} \in \mathcal{R}^{\rm rec}_{\mathbf{K}_{N_k}, \mathbf{S}_{N_k - 1}}\}}.$$

We have defined $\mathcal{R}^{\rm rec}_{\mathbf{K}_{N_k}, \mathbf{S}_{N_k - 1}}$ as the set of configurations such that the forward flow compatible with $\mathbf{K}_{N_k}, \mathbf{S}_{N_k - 1}$ exists, and with the constraints and conditionings listed in (i)(ii) and (iii), p 34–35.

Now let us fix $\mathbf{K}_{N_k}, \mathbf{S}_{N_k - 1}$, and evaluate the cost of the constraint that $Z_{N_k} \in \mathcal{R}^{\rm rec}_{\mathbf{K}_{N_k}, \mathbf{S}_{N_k - 1}}$. For this we start by splitting the sum according to ordered trees $T_< = (q_i, \bar{q}_i)_{1 \leq i \leq N_k - 1}$ encoding the "clustering encounters" as in the previous section: the first encounter in the forward flow is necessarily clustering, say between particles q_1 and $\bar{q}_1$ at time $\tau_1 \in (t_{\rm stop}, t_{\rm stop} + \delta)$. Clustering encounters are then defined recursively: the i-th clustering encounter is the first encounter after time τ_{i-1} involving two particles which are not in the same connected component of the collision graph $G_{i-1} = (q_j, \bar{q}_j)_{j \leq i-1}$. We then denote by $(q_i, \bar{q}_i)$ the colliding particles and by τ_i the corresponding colliding time. The last clustering encounter is between $q_{N_k - 1}$ and $\bar{q}_{N_k - 1}$ at
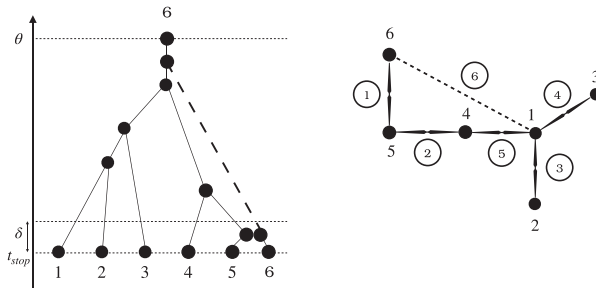
FIGURE 8   In the pseudo-trajectory (with $N_k = 6$) represented on the left figure, a recollision occurs between 5,6 in the time interval $[t_{\text{stop}}, t_{\text{stop}} + \delta]$. Therefore the graph encoding all encounters is not minimal, and it has at least one cycle (or multiple edge). The time ordering of the clustering and non- clustering encounters is represented by the circled numbers and the edges are added dynamically following the forward dynamics, that is, starting from t_{stop}. As a consequence, the recollision between 5,6 in the backward pseudo-dynamics becomes the first clustering encounter in the forward dynamics and the non-clustering encounter is identified with the dashed edge (1,6) occurring close to time θ.

time $\tau_{N_k - 1} \in (\tau_{N_k - 2}, \theta)$. By construction (recall (i) and (ii) above) we know that there are at least $\max(1, n_k^{\text{rec}})$ clustering encounters in the interval $(t_{\text{stop}}, t_{\text{stop}} + \delta)$, and at least $n_k^{\text{rec}} + n_k^0$ clustering encounters in the interval $(t_{\text{stop}}, t_{\text{stop}} + \tau)$. This leads to

$$\left| \Phi_{N_k}^{\text{rec}}(Z_{N_k}) \right| \leq \|h\|_{L^\infty(\mathbb{D})} \frac{\mu_\varepsilon^{N_k - 1}}{N_k!} \sum_{\mathbf{K}_{N_k}, \mathbf{S}_{N_k - 1}} \sum_{T_\prec \in \mathcal{T}_{N_k}^\prec} \mathbf{1}_{\{Z_{N_k} \in \mathcal{R}_{T_\prec, \mathbf{K}_{N_k}, \mathbf{S}_{N_k - 1}}^{\text{rec}}\}}, \tag{5.8}$$

where $\mathcal{R}_{T_\prec, \mathbf{K}_{N_k}, \mathbf{S}_{N_k - 1}}^{\text{rec}}$ is the set of configurations such that the forward flow compatible with $T_\prec, \mathbf{K}_{N_k}, \mathbf{S}_{N_k - 1}$ exists, and again with the above constraints and conditioning.

Notice that, since the pseudo-trajectories involve recollisions, the clustering encounters of the forward dynamics do not coincide in general with the creations in the backward dynamics (see Figure 8). Furthermore, the construction of $T_\prec$ is such to exclude cycles, so that the graph is minimally connected. On the other hand the graph encoding all encounters has more than $(N_k - 1)$ edges, which means that there will be at least one non-clustering encounter in the forward dynamics (see Figure 8): it will be taken into account to gain some smallness.

To begin, we proceed exactly as in the proof of Lemma 4.1. Given an admissible tree $T_\prec$, the relative positions $(\hat{x}_s)_{s < i}$ and the velocities V_{N_k}, we can vary $\hat{x}_i$ so that an encounter at time $\tau_i \in (\tau_{i-1}, \theta)$ occurs between q_i and $\bar{q}_i$ and thus define the set $\mathcal{B}_{T_\prec, i}(\hat{x}_1, \ldots, \hat{x}_{i-1}, V_{N_k})$ of measure

$$\left| \mathcal{B}_{T_\prec, 1} \right| \leq \frac{C}{\mu_\varepsilon} |v_{q_1} - v_{\bar{q}_1}| \delta \tag{5.9}$$

and for $i > 1$

$$\left| \mathcal{B}_{T_\prec, i} \right| \leq \frac{C}{\mu_\varepsilon} \left| v_{q_i}^\varepsilon(\tau_{i-1}^+) - v_{\bar{q}_i}^\varepsilon(\tau_{i-1}^+) \right| \int \mathbf{1}_{\tau_i \geq \tau_{i-1}} d\tau_i, \tag{5.10}$$

recalling the sampling (i)–(ii). The point now is to use the fact that the existence of a non-clustering encounter (which would produce a first cycle in the graph encoding all encounters) strengthens one of these conditions. Given an ordered tree $T_\prec$, the occurrence of a cycle is thus parametrized by an edge $(q, \bar{q})$ and an index c such that $\tau_{\text{cyc}} \in [\tau_c, \tau_{c+1}]$. Then, proceeding as in (4.15)

we see that

$$
\sum_{T_\prec \in \mathcal{T}_{N_k}^\prec} \int d\hat{X}_{N_k-1} \mathbf{1}_{\text{cycle}} \prod_{i=1}^{N_k-1} \mathbf{1}_{B_{T_\prec,i}}
$$

$$
\leq \sum_{T_\prec \in \mathcal{T}_{N_k}^\prec} \sum_{q,\bar{q}} \sum_{c \leq N_k} \int d\hat{x}_1 \mathbf{1}_{B_{T_\prec,1}} \int d\hat{x}_2 \cdots \int d\hat{x}_{N_k-1} \mathbf{1}_{B_{T_\prec,N_k-1}} \mathbf{1}_{\text{cycle defined by } (q,\bar{q}),c} \cdot \tag{5.11}
$$

As shown in Appendix B, the cycle imposes strong geometric constraints on the history of these particles, which produce the estimate

$$
\sum_{T_\prec \in \mathcal{T}_{N_k}^\prec} \int M^{\otimes N_k}(V_{N_k}) \, dV_{N_k} \int d\hat{X}_{N_k-1} \mathbf{1}_{\text{cycle}} \prod_{i=1}^{N_k-1} \mathbf{1}_{B_{T_\prec,i}}
$$

$$
\leq \left(\frac{C}{\mu_\varepsilon}\right)^{N_k-1} (\mathbb{V}\theta)^{2d+4} N_k^3 C^{N_k} N_k^{N_k} N_k^{N_k-1} \int_{t_{\text{stop}}}^{t_{\text{stop}}+\delta} d\tau_1 \cdots \int_{\tau_{N_k-2}}^\theta d\tau_{N_k-1} \varepsilon |\log \varepsilon| \, \mathbf{1}_{\mathbf{n}_k}, \tag{5.12}
$$

recalling that $\mathbf{1}_{\mathbf{n}_k}$ is the constraint on times respecting the sampling in formula (5.1), with possibly 3 time integrals missing due to the iterated use of Propositions B.2 and B.3. Integrating over the simplex in time, we finally obtain (5.4). Lemma 5.1 is proved. $\qquad\square$

Proof of Lemma 5.2. The proof combines arguments from the proofs of Lemmas 4.2 and 5.1. Our starting point is the estimate

$$
\left|\Phi_{N_k}^{\text{rec}}(Z_{N_k})\right| \leq \|h\|_{L^\infty(\mathbb{D})} \frac{\mu_\varepsilon^{N_k-1}}{N_k!} \sum_{\mathbf{K}_{N_k},\mathbf{S}_{N_k-1}} \mathbf{1}_{\{Z_{N_k} \in \mathcal{R}_{\mathbf{K}_{N_k},\mathbf{S}_{N_k-1}}^{\text{rec}}\}} \cdot \tag{5.13}
$$

Let us fix two families $(\mathbf{K}_{N_k}, \mathbf{S}_{N_k-1})$ and $(\mathbf{K}'_{N_k}, \mathbf{S}'_{N_k-1})$ and consider a configuration $Z_{2N_k-\ell}$ such that $Z_{N_k} \in \mathcal{R}_{\mathbf{K}_{N_k},\mathbf{S}_{N_k-1}}^{\text{rec}}$ and $Z'_{N_k} = (Z_\ell, Z_{N_k+1,2N_k-\ell}) \in \mathcal{R}_{\mathbf{K}'_{N_k},\mathbf{S}'_{N_k-1}}^{\text{rec}}$.

We consider the forward flows of each set of particles Z_{N_k} and Z'_{N_k} starting at time t_{stop}. Both dynamics evolve independently and each one of them should have at least one non-clustering encounter. As in the proof of Lemma 5.1 we denote by $T_\prec$ the ordered tree corresponding to the clustering encounters of Z_{N_k}, and by $(\tau_i)_{1 \leq i \leq N_k-1}$ and $(\hat{x}_i)_{1 \leq i \leq N_k-1}$ the corresponding times and relative positions. Recall that the non-clustering encounter on the dynamics of Z_{N_k} strengthens one of the clustering constraint.

Starting from this ordered minimally connected tree $T_\prec$ with N_k vertices, we construct an ordered minimally connected tree with $2N_k - \ell$ vertices by the same procedure as in the proof of Lemma 4.2. The edges $(q_i, \bar{q}_i)_{N_k \leq i \leq 2N_k-\ell}$ are added by keeping only the "clustering encounters" in the forward dynamics of Z'_{N_k}:

- the first clustering encounter is the first encounter in the forward flow of Z'_{N_k} involving at least one particle with label in $[N_k + 1, 2N_k - \ell]$. We denote by $(q_{N_k}, \bar{q}_{N_k})$ the labels of the colliding particles and by τ_{N_k} the corresponding time. We also define the ordered graph $G_{N_k} = (q_j, \bar{q}_j)_{1 \leq j \leq N_k}$,

- for $N_k + 1 \leq i \leq 2N_k - \ell - 1$, the i-th clustering encounter is the first encounter (after τ_{i-1}) in the forward flow of Z'_{N_k} involving two particles which are not in the same connected component of the graph G_{i-1}. We denote by $(q_i, \bar{q}_i)$ the labels of the colliding particles and by τ_i the corresponding time. We also define the ordered graph $G_i = (q_j, \bar{q}_j)_{1 \leq j \leq i}$.

By this procedure we end up with a tree $T''_{\prec} := (q_i, \bar{q}_i)_{1 \leq i \leq 2N_k - \ell - 1}$ with no cycles (nor multiple edges). We define as above the relative positions $\hat{x}_i := x_{q_i} - x_{\bar{q}_i}$.

Necessary conditions to have $Z_{N_k} \in \mathcal{R}^{\text{rec}}_{\mathbf{K}_{N_k}, \mathbf{S}_{N_k - 1}}$ and $Z'_{N_k} \in \mathcal{R}^{\text{rec}}_{\mathbf{K}'_{N_k}, \mathbf{S}'_{N_k - 1}}$ can be expressed recursively in terms of the collision sets $(\mathcal{B}_{T''_{\prec}, i})_{1 \leq i \leq 2N_k - \ell - 1}$:

- the sets $\mathcal{B}_{T''_{\prec}, i}$ only depend on $\hat{x}_1, \ldots, \hat{x}_{i-1}, V_{2N_k - \ell}$ for any $i \leq 2N_k - \ell - 1$ (see Lemma 4.2),
- one set of $(\mathcal{B}_{T''_{\prec}, i})_{1 \leq i \leq N_k - 1}$ has some extra smallness due to the existence of a non-clustering encounter in the dynamics of Z_{N_k} (see Lemma 5.1).

We therefore end up with the estimate

$$
\sum_{T''_{\prec}} \int d\hat{X}_{2N_k - \ell - 1} dV_{2N_k - \ell} M^{\otimes(2N_k - \ell)} \prod_{i=1}^{2N_k - \ell - 1} \mathbf{1}_{\mathcal{B}_{T''_{\prec}, i}}
$$

$$
\leq \left(\frac{C}{\mu_\varepsilon} \right)^{2N_k - \ell - 1} (\mathbb{V}\theta)^{d+1} \delta^{\max(1, n_k^{\text{rec}})} \tau^{(n_k^0 - 1)_+} \theta^{2N_k - \ell - 1 - n_k} \varepsilon |\log \varepsilon| (N_k)^{2N_k - \ell}.
$$

(5.14)

Summing over all possible $(\mathbf{K}_{N_k}, \mathbf{S}_{N_k - 1})$ and $(\mathbf{K}'_{N_k}, \mathbf{S}'_{N_k - 1})$, we obtain the expected estimate. Lemma 5.2 is proved. □

6 | CONCLUSION OF THE PROOF: CONVERGENCE RESULTS

6.1 | The cost of the conditioning

This section is devoted to the proof of Proposition 2.7.

To prove (2.25), we evaluate the occurrence of a microscopic cluster of size larger than γ under the equilibrium measure. This can be estimated by considering the event that $\gamma + 1$ particles are located in a ball of diameter $3\gamma^{3/2} \mathbb{V}\delta$.

$$
\mathbb{P}_\varepsilon(\text{there is a cluster larger than } \gamma \text{ at time } 0)
$$

$$
\leq \mathbb{E}_\varepsilon \left(\sum_{(i_1, \ldots, i_{\gamma+1})} \mathbf{1}_{\{i_1, \ldots, i_{\gamma+1} \text{ are in a cluster}\}} \right) \leq C \mu_\varepsilon^{\gamma+1} \left(\gamma^{3/2} \mathbb{V}\delta \right)^{d\gamma}.
$$

In the set ${}^c\mathrm{Y}^\varepsilon_\mathcal{N}$ a cluster should appear (at least) at one of the θ/δ time steps. In a similar way, the occurrence of a large velocity is given by

$$
\mathbb{P}_\varepsilon(\text{there is a velocity larger than } \mathbb{V} \text{ at time } 0)
$$

$$
\leq \mathbb{E}_\varepsilon \left(\sum_{i_1} \mathbf{1}_{\{i_1 \text{ has a velocity larger than } \mathbb{V}\}} \right) \leq C |\mathbb{V}|^{d-2} \mu_\varepsilon \exp \left(-\frac{1}{2} |\mathbb{V}|^2 \right),
$$

which is much smaller with our choice $\mathbb{V} = |\log \varepsilon|$.

Thus we get by a union bound

$$\mathbb{P}_\varepsilon\left({}^c Y^\varepsilon_{\mathcal{N}}\right) \le C_\gamma \theta \mathbb{V}^{d\gamma} \mu_\varepsilon^{\gamma+1} \delta^{d\gamma-1}. \tag{6.1}$$

Finally (2.25) follows from the choice of parameters (2.21).

Let us now note that the measure restricted to $Y^\varepsilon_{\mathcal{N}}$ can be decomposed as

$$\mathbb{E}_\varepsilon\left(\zeta_0^\varepsilon(g_0)\mathbf{1}_{Y^\varepsilon_{\mathcal{N}}}\right) = \mathbb{E}_\varepsilon\left(\zeta_0^\varepsilon(g_0)\right) - \mathbb{E}_\varepsilon\left(\zeta_0^\varepsilon(g_0)\mathbf{1}_{{}^c Y^\varepsilon_{\mathcal{N}}}\right) = -\mathbb{E}_\varepsilon\left(\zeta_0^\varepsilon(g_0)\mathbf{1}_{{}^c Y^\varepsilon_{\mathcal{N}}}\right),$$

where we used that $\mathbb{E}_\varepsilon(\zeta_0^\varepsilon(g_0)) = 0$. Applying the Cauchy-Schwarz inequality, we get by (2.3), (6.1) and the choice of parameters (2.21) that

$$\left|\mathbb{E}_\varepsilon\left(\zeta_0^\varepsilon(g_0)\mathbf{1}_{Y^\varepsilon_{\mathcal{N}}}\right)\right| \le \mathbb{E}_\varepsilon\left(\zeta_0^\varepsilon(g_0)^2\right)^{1/2} \mathbb{P}_\varepsilon\left({}^c Y^\varepsilon_{\mathcal{N}}\right)^{1/2} \le C\|g_0\|_{L^2_M} \left(\theta\varepsilon^d\right)^{1/2}. \tag{6.2}$$

This completes (2.27).

Next we use the Hölder inequality to get

$$\left|\mathbb{E}_\varepsilon\left(\mathbf{1}_{{}^c Y^\varepsilon_{\mathcal{N}}}\zeta_0^\varepsilon(g_0)\zeta_\theta^\varepsilon(h)\right)\right| \le \mathbb{P}_\varepsilon\left({}^c Y^\varepsilon_{\mathcal{N}}\right)^{1/4} \mathbb{E}_\varepsilon\left(\zeta_0^\varepsilon(g_0)^2\right)^{1/2} \mathbb{E}_\varepsilon\left(\zeta_0^\varepsilon(h)^4\right)^{1/4}.$$

Recall that h is in L^∞. Combining (6.1) with the bounds in Proposition A.1 on the moments of the fluctuation field, we get

$$\left|\mathbb{E}_\varepsilon\left(\mathbf{1}_{{}^c Y^\varepsilon_{\mathcal{N}}}\zeta_0^\varepsilon(g_0)\zeta_\theta^\varepsilon(h)\right)\right| \le C \|h\|_{L^\infty(\mathbb{D})}\|g_0\|_{L^2_M}(\theta\varepsilon^d)^{1/4}.$$

We turn now to proving the estimate on $\int dz_1 G_1^{\varepsilon,\text{clust}}(\theta, z_1)h(z_1)$. Proceeding as in (4.4)-(4.5), we get

$$\int dz_1 G_1^{\varepsilon,\text{clust}}(\theta, z_1)h(z_1) = \sum_{\mathbf{n}_k} \mathbb{E}_\varepsilon\left(\mu_\varepsilon^{1/2}\,\hat{\Phi}_{N_K}\left(\mathbf{Z}^\varepsilon_{\mathcal{N}}(0)\right)\zeta_0^\varepsilon(g_0)\,\mathbf{1}_{{}^c Y^\varepsilon_{\mathcal{N}}}\right)$$

$$+ \sum_{\mathbf{n}_k} \mu_\varepsilon^{1/2}\,\mathbb{E}_\varepsilon\left(\Phi_{N_K}\right)\mathbb{E}_\varepsilon\left(\zeta_0^\varepsilon(g_0)\mathbf{1}_{{}^c Y^\varepsilon_{\mathcal{N}}}\right),$$

where Φ_{N_K} is conditioned on the sampling $\mathbf{n}_k$ with sub-exponential trees and no recollisions in $(0, \theta)$. Applying the Hölder inequality to bound the first term and (6.2) leads to

$$\left|\int dz_1 G_1^{\varepsilon,\text{clust}}(\theta, z_1)h(z_1)\right|$$

$$\le \mathbb{P}_\varepsilon\left({}^c Y^\varepsilon_{\mathcal{N}}\right)^{1/4} \mathbb{E}_\varepsilon\left(\zeta_0^\varepsilon(g_0)^4\right)^{1/4} \sum_{\mathbf{n}_k} \mathbb{E}_\varepsilon\left(\mu_\varepsilon\left(\hat{\Phi}_{N_K}\left(\mathbf{Z}^\varepsilon_{\mathcal{N}}(0)\right)\right)^2\right)^{1/2}$$

$$+ C\|g_0\|_{L^2_M}\left(\theta\varepsilon^d\right)^{1/2} \sum_{\mathbf{n}_k} \mu_\varepsilon^{1/2}\,\mathbb{E}_\varepsilon\left(\Phi_{N_K}\right).$$

Since g_0 belongs to L^∞, the moments of the fluctuation field can be bounded by Proposition A.1. Thus the previous term is estimated as in (4.9) and we find thanks to (6.1)

$$
\left| \int dz_1 G_1^{\varepsilon,\text{clust}}(\theta, z_1) h(z_1) \right| \leq C(\theta \varepsilon^d)^{1/4} \|g_0\|_{L^\infty(\mathbb{D})} \|h\|_{L^\infty(\mathbb{D})} \sum_{(n_k \leq 2^k)_{k \leq K}} (C\theta)^{N_K}
$$

$$
+ C(\theta \varepsilon^d)^{1/2} \|g_0\|_{L^2_M} \|h\|_{L^\infty(\mathbb{D})} \mu_\varepsilon^{1/2} 2^{K^2} (C\theta)^{2^{K+1}}
$$

$$
\leq C \, \|h\|_{L^\infty(\mathbb{D})} \|g_0\|_{L^\infty(\mathbb{D})} (\theta \varepsilon)^{1/2} 2^{K^2} (C\theta)^{2^{K+1}} .
$$

Using the scaling $K = \theta/\tau$, this concludes the proof of Proposition 2.7.

6.2 │ Convergence of the principal part

In this section, we prove Proposition 2.10. This is based on classical arguments relying on L^∞ estimates. We shall refer to the literature for details.

To begin, the limit initial data is identified thanks to the following classical lemma: we refer to [13, 24, 30]. We set

$$
G_n^0(Z_n) := M^{\otimes n}(V_n) \sum_{i=1}^{n} g_0(z_i) , \qquad n \geq 1 . \tag{6.3}
$$

Lemma 6.1. *There exists a positive constant C such that, for any $n \in \mathbb{N}$,*

$$
\left| (G_n^{\varepsilon 0} - G_n^0)(Z_n) \mathbf{1}_{D_n^\varepsilon}(X_n) \right| \leq C^n M^{\otimes n}(V_n) \varepsilon \|g_0\|_\infty ,
$$

when ε is small enough.

Next we define formally a limit hierarchy, and identify its solution with the solution of the linearized Boltzmann equation (1.13). To do so we introduce Boltzmann pseudo-trajectories $\Psi_{1,m}$ on $(0, \theta)$, constructed as follows. For all z_1, all parameters $(t_i, \omega_i, v_{n+i})_{i=1,\dots,m}$ with $t_i > t_{i+1}$ and all collision trees $a \in \mathcal{A}_{1,m}^\pm$ (denoting by $Z_{m+1}(\tau)$ the coordinates of the particles at time $\tau \leq t_m$)

- start from z_1 at time t and, by iteration on $i = 1, 2, \dots, m$,
- transport all existing particles backward on (t_i, t_{i-1}) (on $\mathbb{D}^i$),
- add a new particle labeled $i + 1$ at time t_i, at position $x_{a_i}(t_i)$ and with velocity v_{1+i},
- apply the scattering rule (1.7) if $s_i > 0$.

We then set

$$
G_1(\theta) := \sum_{m \geq 0} Q_{1,m+1}(\theta) G_{m+1}^0 , \qquad n \geq 1 \tag{6.4}
$$

where $Q_{1,m+1}$ is the Boltzmann hierarchy operator

$$Q_{1,m+1}(\theta)G_{m+1}^0 := \sum_{a \in \mathcal{A}_{1,m}^{\pm}} \int dT_m d\Omega_m dV_{2,1+m}$$

$$\times \left(\prod_{i=1}^m s_i \big((v_{1+i} - v_{a_i}(t_i^+)) \cdot \omega_i \big)_+ \right) G_{m+1}^0(Z_{m+1}(0)) .$$

Similarly we define

$$G_n(\theta) := \sum_{m \geq 0} Q_{n,n+m}(\theta)G_{n+m}^0 ,$$

where

$$Q_{n,n+m}(\theta)G_{n+m}^0 := \sum_{a \in \mathcal{A}_{n,m}^{\pm}} \int dT_m d\Omega_m dV_{n+1,n+m}$$

$$\times \left(\prod_{i=1}^m s_i \big((v_{n+i} - v_{a_i}(t_i^+)) \cdot \omega_i \big)_+ \right) G_{n+m}^{\varepsilon 0}(Z_{n+m}(0)) .$$

The following result is due to [30] (see also Section 1.1.3 in [3]). It identifies G_1 to the solution $Mg(\theta)$ of the linearized Boltzmann equation (1.13) with data g_0. We recall (see for instance [16, 17]) that there is a unique solution Mg to (1.13) as soon as g_0 is bounded, which remains bounded for all positive times.

Lemma 6.2. *The solution $G_1(\theta)$ of (6.4) with initial data (6.3) is equal to the solution $Mg(\theta)$ of the linearized Boltzmann equation (1.13) with data g_0. Furthermore, the n-particle correlation function $G_n(t, Z_n)$ is given by the following explicit expression for any $n \geq 1$*

$$\forall t \geq 0, \qquad G_n(t, Z_n) := M^{\otimes n}(V_n) \sum_{i=1}^n g(t, z_i) . \tag{6.5}$$

To prove Proposition 2.10, it now remains to prove that

$$\lim_{\varepsilon \to 0} \int G_1^{\varepsilon,\text{main}}(\theta, z) h(z) \, dz = \int G_1(\theta, z) h(z) \, dz , \qquad \forall \theta \in \mathbb{R}^+ .$$

Following the decomposition (2.23) of $G_1^{\varepsilon}(\theta)$, we write $G_1(\theta)$ as

$$G_1(\theta) = G_1^{\text{main}}(\theta) + G_1^{\text{exp}}(\theta) ,$$

where the main part is given by

$$G_1^{\text{main}}(\theta) := \sum_{(n_k \leq 2^k)_{k \leq K}} Q_{1,N_1}(\tau) \dots Q_{N_{K-1},N_K}(\tau)G_{N_K}^0 ,$$

and the superexponential part by

$$G_1^{\exp}(\theta) := \sum_{k=1}^{K} \sum_{(n_j \le 2^j)_{j \le k-1}} \sum_{n_k > 2^k} Q_{1,N_1}(\tau) \dots Q_{N_{K-1},N_K}(\tau) G_{N_K}(\theta - k\tau).$$

This remainder term is controlled as in [2, 3] using the explicit form (6.5) of the correlation functions G_{N_K}. Since the solution $g(t)$ of the linearized Boltzmann equation (1.13) remains in L^∞ for all positive times, the correlation functions G_{N_K} are also in L^∞ at any time. Therefore

$$\left| \int G_1^{\exp}(\theta, z) h(z) \, dz \right| \le C \|g_0\|_{L^\infty(\mathbb{D})} \|h\|_{L^\infty(\mathbb{D})} \sum_{k \ge 1} 2^{k^2} (C\theta\tau)^{2^k}$$

$$\le C \|g_0\|_{L^\infty(\mathbb{D})} \|h\|_{L^\infty(\mathbb{D})} \theta\tau.$$

Recalling the principal part

$$G_1^{\varepsilon,\mathrm{main}}(\theta) = \sum_{(n_k \le 2^k)_{k \le K}} Q_{1,N_1}^{\varepsilon 0}(\tau) \dots Q_{N_{K-1},N_K}^{\varepsilon 0}(\tau) \, G_{N_K}^{\varepsilon 0},$$

we notice that the differences in this formula with respect to $G_1^{\mathrm{main}}(\theta)$ are due to:

(1) the initial data $G_{N_K}^{\varepsilon 0}$ vs. $G_{N_K}^0$;
(2) the fact that pseudo-trajectories $\Psi_{1,m}^\varepsilon$ are constrained to the set of parameters avoiding recollisions, and also to the set $\mathcal{G}_m^\varepsilon(a, Z_1)$;
(3) the fact that (at creations) particles in $\Psi_{1,m}^\varepsilon$ collide at distance ε while in $\Psi_{1,m}^\varepsilon$ they collide at distance 0.

These errors are controlled as in [22]. First, we borrow an argument from [25] and define $\Psi_{1,m}^E$ an auxiliary pseudo-trajectory defined exactly as $\Psi_{1,m}$, with the only difference that particle $i + 1$ is created at position $x_{a_i}(t_i) + \varepsilon s_i \omega_i$ (this is sometimes called the Boltzmann-Enskog pseudo-trajectory). Correspondingly, we can define $Q_{1,m+1}^E$ exactly as $Q_{1,m+1}$, with $\Psi_{1,m}$ replaced by $\Psi_{1,m}^E$. By definition, $\Psi_{1,m}^E$ and $\Psi_{1,m}$ have identical velocities and the positions cannot differ more than $m\varepsilon$. In particular at time zero we have that the Euclidean norm of the difference $|Z_{N_k}^E(0) - Z_{N_k}^0(0)|$ is bounded by

$$|Z_{N_k}^E(0) - Z_{N_k}^0(0)| \le N_k^{\frac{3}{2}} \varepsilon. \tag{6.6}$$

Next, we can simplify the integral in $G_1^{\varepsilon,\mathrm{main}}$ by removing the constraint in point 2) above. Let $\mathcal{O}^\varepsilon$ be the complement of the set of parameters in 2). Clearly the pseudo-particles in $\Psi_{1,m}^E$ can overlap (they can reach distance strictly smaller than ε). However in absence of recollisions and overlaps, the auxiliary pseudo-trajectory coincides with the BBGKY pseudo-trajectory $\Psi_{1,m}^E = \Psi_{1,m}^\varepsilon$. We can therefore replace $\Psi_{1,m}^\varepsilon$ by $\Psi_{1,m}^E$ in the geometric representation for $G_1^{\varepsilon,\mathrm{main}}$. The contribution of $\mathcal{O}^\varepsilon$ to $Q_{1,1+m}^E$ is bounded by a quantitative version of Lanford's argument: following [25], one can show that there is a constant $\alpha \in (0, 1)$ (which can actually be chosen arbitrarily close to 1) such

that

$$\left| \sum_{a \in \mathcal{A}_{1,m}^{\pm}} \int_{\mathcal{O}^{\varepsilon}} dz \, dT_m d\Omega_m dV_{2,1+m} \, h(z) \, G_{1+m}^0 \big(Z_{1+m}^E(0) \big) \right.$$

$$\left. \times \prod_{i=1}^{m} s_i \big((v_{1+i} - v_{a_i}^E(t_i^+)) \cdot \omega_i \big)_+ \right| \leq \|h\|_{L^{\infty}(\mathbb{D})} \|g_0\|_{L^{\infty}(\mathbb{D})} \varepsilon^{\alpha} (C\theta)^m .$$

Using this after Lemma 6.1, and controlling the error (6.6) thanks to the Lipschitz norm of g_0, we conclude that

$$\left| \int \Big(G_1^{\varepsilon,\mathrm{main}}(\theta) - G_1^{\mathrm{main}}(\theta) \Big) h(z) \, dz \right|$$

$$\leq \|h\|_{L^{\infty}(\mathbb{D})} \Big(\varepsilon^{\alpha} \|g_0\|_{L^{\infty}(\mathbb{D})} + \varepsilon \|\nabla_x g_0\|_{L_M^{\infty}} \Big) \sum_{\substack{n_k \leq 2^k \\ k \leq K}} (C\theta)^{N_K+1}$$

which leads to Proposition 2.10 since α may be chosen arbitrarily close to 1. □

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APPENDIX A: L^p A PRIORI ESTIMATES

For the sake of completeness, we state below some estimates on the moments of the fluctuation field under the equilibrium measure. These bounds follow from a standard cluster expansion approach (see e.g., [29]).

Recall that the functional spaces L_M^p was introduced in (1.12).

Proposition A.1. *Let h be a function in L_M^p. For ε small enough, the moments of the fluctuation field are bounded:*

$$\left| \mathbb{E}_\varepsilon\left(\left(\zeta_0^\varepsilon(h)\right)^p \right) \right| \leq C_p \|h\|_{L_M^p}^p, \qquad 1 \leq p < \infty, \tag{A.1}$$

where the positive constant C_p depends only on p.

Proof. The proof is based on the same algebraic manipulations used in Section 3 to estimate mean and variance of test functions; we repeat them here for moments of arbitrary order of ζ_0^ε. After expanding the product of fluctuation fields in (A.1), we organize the sums by grouping particles with common indices. Then we proceed to compute the expectations under the Gibbs measure, by using a cluster expansion of the exclusion condition. This leads to the exact decomposition (A.5) below, which implies (A.1) thanks to the tree inequality.

Since the time is always zero in this proof, let us drop the indices 0 from now on. By definition of ζ^ε and π^ε and abbreviating $|\sigma| = s$, $|\sigma'| = s'$, $|\lambda_j| = \ell_j$ etc., we have that

$$\mathbb{E}_\varepsilon\left((\zeta^\varepsilon(h))^p\right) = \mu_\varepsilon^{p/2} \sum_{\substack{\sigma \cup \sigma' = \{1,\ldots,p\} \\ \sigma \cap \sigma' = \emptyset}} (-\mathbb{E}_\varepsilon(\pi^\varepsilon(h)))^{s'} \mathbb{E}_\varepsilon\left((\pi^\varepsilon(h))^s\right)$$

$$= \mu_\varepsilon^{p/2} \sum_{\substack{\sigma \cup \sigma' = \{1,\ldots,p\} \\ \sigma \cap \sigma' = \emptyset}} \mu_\varepsilon^{-s}(-\mathbb{E}_\varepsilon(\pi^\varepsilon(h)))^{s'} \mathbb{E}_\varepsilon\left(\sum_{\lambda \in \mathcal{P}_\sigma} \sum_{(i_1,\ldots,i_\ell)} \prod_{j=1}^{\ell} h^{\ell_j}\left(\mathbf{z}_{i_j}^\varepsilon\right)\right),$$

where $\mathcal{P}_\sigma$ indicates the set of partitions of σ. In the second line we have arranged the s sums over particles encoded in $(\pi^\varepsilon(h))^s$ according to the repeated indices: there are $\ell \in [1, s]$ different particles and the partition $\lambda = \{\lambda_1, \ldots, \lambda_\ell\}$ specifies how many test functions are assigned to each particle.

In terms of the correlation functions $(G_k^{\varepsilon,\text{eq}})_{k \geq 1}$ of the equilibrium measure (1.3), whose definition (1.16) we recall:

$$\int G_k^{\varepsilon,\text{eq}}(Z_k) h_k(Z_k) dZ_k = \mathbb{E}_\varepsilon\left(\frac{1}{\mu_\varepsilon^k} \sum_{(i_1,\ldots,i_k)} h_k(\mathbf{z}_{i_1}^\varepsilon(t), \ldots, \mathbf{z}_{i_k}^\varepsilon(t))\right),$$

the previous formula reads

$$\mu_\varepsilon^{p/2} \sum_{\substack{\sigma \cup \sigma' = \{1,\ldots,p\} \\ \sigma \cap \sigma' = \emptyset}} \sum_{\lambda \in \mathcal{P}_\sigma} \mu_\varepsilon^{-s+\ell} \left(-\int d\tilde{z}\, h(\tilde{z})\, G_1^{\varepsilon,\text{eq}}(\tilde{z})\right)^{s'} \int dZ_\ell \prod_{j=1}^{\ell} h^{\ell_j}(z_j) G_\ell^{\varepsilon,\text{eq}}(Z_\ell)$$

$$= \mu_\varepsilon^{p/2} \sum_{\substack{\sigma \cup \sigma'' = \{1,\ldots,p\} \\ \sigma \cap \sigma'' = \emptyset}} \sum_{\lambda \in \mathcal{P}_\sigma^*} \sum_{\sigma' \subset \sigma''} (-1)^{s'} \mu_\varepsilon^{-s+\ell} \left(\int d\tilde{z}\, h(\tilde{z})\, G_1^{\varepsilon,\text{eq}}(\tilde{z})\right)^{s'} \quad (\text{A.2})$$

$$\times \int d\tilde{Z}_{s''-s'}\, dZ_\ell\, h^{\otimes(s''-s')}(\tilde{Z}_{s''-s'}) \bigotimes_{j=1}^{\ell} h^{\ell_j}(Z_\ell)\, G_{\ell+s''-s'}^{\varepsilon,\text{eq}}(Z_\ell, \tilde{Z}_{s''-s'})$$

denoting by $\mathcal{P}_\sigma^*$ the set of partitions of σ without singletons. The equality comes from the renaming of variables

$$\sigma' \cup \{\lambda_i \;;\; |\lambda_i| = 1\} \to \sigma''$$

$$s' + \#\{\lambda_i \;;\; |\lambda_i| = 1\} \to s''$$

$$\lambda \setminus \{\lambda_i \;;\; |\lambda_i| = 1\} \to \lambda$$

$$\ell - \#\{\lambda_i \;;\; |\lambda_i| = 1\} \to \ell$$

In (A.2), we adopt the convention of using the symbol $\tilde{z}$ for variables which correspond to a single test function h, and z for variables which correspond to a power of test functions h^{ℓ_j}.

We use now the cluster expansion of the correlation functions, see for example, Equation (8.1.18) in [6]:

$$
G_k^{\varepsilon,\mathrm{eq}}(Z_k) = \sum_{\rho \in \mathcal{P}_{\{1,\dots,k\}}} \prod_{i=1}^{r} g_{r_i}^{\varepsilon,\mathrm{eq}}(Z_{\rho_i}) ,
$$

$$
g_r^{\varepsilon,\mathrm{eq}}(Z_r) = M^{\otimes r}(V_r) \sum_{p \geq 0} \frac{\mu_\varepsilon^p}{p!} \int M^{\otimes p}(\bar{V}_p) \varphi(z_1,\dots,z_r,\bar{Z}_p) d\bar{Z}_p , \qquad r \geq 1
$$

(A.3)

where the cumulant φ, defined similarly to (3.6), is supported on connected graphs on $r + p$ points. Note that at leading order correlation functions are tensor products of $g_1^{\varepsilon,\mathrm{eq}} = G_1^{\varepsilon,\mathrm{eq}}$.

We recall that the tree inequality (3.8) implies the estimate

$$
\left| g_r^{\varepsilon,\mathrm{eq}}(Z_r) \right| \leq M^{\otimes r}(V_r) \sum_{p \geq 0} \sum_{T \in \mathcal{T}_{r+p}} \frac{\mu_\varepsilon^p}{p!} \int d\bar{Z}_p \, M^{\otimes p}(\bar{V}_p) \prod_{(y,y^*) \in E(T)} \mathbf{1}_{d(y,y^*) \leq \varepsilon} ,
$$

(A.4)

denoting by $d(y,y^*)$ the minimum relative distance (in position) between elements y, y^* in $\{z_1,\dots,z_r,\bar{z}_1,\dots\bar{z}_p\}$. This bound will be useful below.

Inserting the expansion (A.3) inside (A.2), we will show that the terms with singletons disappear. First we notice that, for each partition ρ, the last line in (A.2) factorizes into $r = |\rho|$ independent integrals:

$$
\mathbb{E}_\varepsilon\left((\zeta^\varepsilon(h))^p \right)
$$

$$
= \mu_\varepsilon^{p/2} \sum_{\substack{\sigma \cup \sigma'' = \{1,\dots,p\} \\ \sigma \cap \sigma'' = \emptyset}} \sum_{\lambda \in \mathcal{P}_\sigma^*} \sum_{\sigma' \subset \sigma''} \sum_{\rho \in \mathcal{P}_{\lambda \cup (\sigma'' \setminus \sigma')}} (-1)^{s'} \mu_\varepsilon^{-s+\ell} \left(\int d\tilde{z}\, h(\tilde{z})\, g_1^{\varepsilon,\mathrm{eq}}(\tilde{z}) \right)^{s'}
$$

$$
\times \prod_{i=1}^{r} \int d\tilde{Z}_{r_i^2}\, dZ_{r_i^1}\, h^{\otimes r_i^2}\left(\tilde{Z}_{r_i^2} \right) \bigotimes_{\lambda_j \in \rho_i^1} h^{\ell_j}\left(Z_{r_i^1} \right) g_{r_i}^{\varepsilon,\mathrm{eq}}\left(Z_{r_i^1}, \tilde{Z}_{r_i^2} \right),
$$

where $r_i = |\rho_i| = r_i^1 + r_i^2$, $r_i^1 = |\rho_i^1|$, $r_i^2 = |\rho_i^2|$ with $\rho_i^1 = \rho_i \cap \lambda$ and $\rho_i^2 = (\rho_i^1)^c$ (the complement of ρ_i^1 in ρ_i). For $r_i^1 = 0$, $r_i^2 = 1$ the last line reduces to a singleton factor $\int h\, g_1^{\varepsilon,\mathrm{eq}}$. This suggests to rename again the summation variables as follows:

$$
\sigma' \cup \{\rho_i \; ; \; r_i^1 = 0, \, r_i^2 = 1\} \rightarrow \sigma^{\mathrm{sing}}
$$

$$
\{1,\dots,p\} \setminus \sigma^{\mathrm{sing}} \rightarrow \sigma^* \qquad\qquad ,
$$

$$
\{\rho_i \; ; \; r_i^1 > 0 \text{ or } r_i^2 > 1\} \rightarrow \rho^*
$$

with cardinalities s^{sing}, s^*, r^* respectively. In this way, σ, λ and ρ^* determine a nested partition of σ^*: we first choose $\sigma \subset \sigma^*$ and $\lambda \in \mathcal{P}_\sigma^*$; and secondly we take a partition ρ^* of $\lambda \cup (\sigma^* \setminus \sigma)$. This partition is characterized by the fact that ρ_i^* contains at least two indices in $\{1,\dots,p\}$. We will indicate by $\lambda \hookrightarrow^* \rho^*$ the sum over such nested partitions (the $*$ reminding us of the constraint

excluding singletons). By Fubini we get that

$$
\mathbb{E}_\varepsilon\left((\zeta^\varepsilon(h))^p\right)
$$

$$
= \mu_\varepsilon^{p/2} \sum_{\substack{\sigma^*\cup\sigma^{\mathrm{sing}}=\{1,\dots,p\}\\ \sigma^*\cap\sigma^{\mathrm{sing}}=\emptyset}} \sum_{\sigma\subset\sigma^*} \sum_{\substack{\lambda\in P_\sigma^*\\ \lambda\hookrightarrow^*\rho^*}} \sum_{\sigma'\subset\sigma^{\mathrm{sing}}} (-1)^{s'} \mu_\varepsilon^{-s+\ell}\left(\int d\tilde{z}\, h(\tilde{z})\, g_1^{\varepsilon,\mathrm{eq}}(\tilde{z})\right)^{s^{\mathrm{sing}}}
$$

$$
\times \prod_{i=1}^{r^*}\int d\tilde{Z}_{r_i^{*2}}\, dZ_{r_i^{*1}}\, h^{\otimes r_i^{*2}}\left(\tilde{Z}_{r_i^{*2}}\right) \bigotimes_{\lambda_j\in\rho_i^{*1}} h^{\ell_j}\left(Z_{r_i^{*1}}\right) g_{r_i^*}^{\varepsilon,\mathrm{eq}}\left(Z_{r_i^{*1}},\tilde{Z}_{r_i^{*2}}\right),
$$

$$
= \mu_\varepsilon^{p/2} \sum_{\substack{\sigma^*\cup\sigma^{\mathrm{sing}}=\{1,\dots,p\}\\ \sigma^*\cap\sigma^{\mathrm{sing}}=\emptyset}} \sum_{\substack{\lambda\in P_{\sigma^*}\\ \lambda\hookrightarrow^*\rho^*}} \sum_{\sigma'\subset\sigma^{\mathrm{sing}}} (-1)^{s'} \mu_\varepsilon^{-s^*+\ell}\left(\int d\tilde{z}\, h(\tilde{z})\, g_1^{\varepsilon,\mathrm{eq}}(\tilde{z})\right)^{s^{\mathrm{sing}}}
$$

$$
\times \prod_{i=1}^{r^*}\int dZ_{r_i^*} \bigotimes_{\lambda_j\in\rho_i^*} h^{\ell_j}\left(Z_{r_i^*}\right) g_{r_i^*}^{\varepsilon,\mathrm{eq}}\left(Z_{r_i^*}\right),
$$

where in the second equality we eliminated the sum over σ by removing the constraint on λ. Now the sum over $\sigma' \subset \sigma^{\mathrm{sing}}$ can be performed first. Since $\sum_{\sigma'\subset\sigma^{\mathrm{sing}}}(-1)^{s'}=\delta_{\sigma^{\mathrm{sing}},\emptyset}$, we conclude that

$$
\mathbb{E}_\varepsilon\left((\zeta^\varepsilon(h))^p\right) = \mu_\varepsilon^{-p/2} \sum_{\lambda\in\mathcal{P}_{\{1,\dots,p\}}} \sum_{\rho:\lambda\hookrightarrow^*\rho} \mu_\varepsilon^\ell \prod_{i=1}^r \int dZ_{r_i} \bigotimes_{\lambda_j\in\rho_i} h^{\ell_j}\left(Z_{r_i}\right) g_{r_i}^{\varepsilon,\mathrm{eq}}\left(Z_{r_i}\right) \tag{A.5}
$$

where the sums run over nested partitions of $\{1,\dots,p\}$ with ρ_i containing at least two indices in $\{1,\dots,p\}$.

Observe that (A.5) is factorized in r integrals, each of which can be bounded by Hölder's inequality:

$$
\left|\int dZ_{r_i} \bigotimes_{\lambda_j\in\rho_i} h^{\ell_j}\left(Z_{r_i}\right) g_{r_i}^{\varepsilon,\mathrm{eq}}\left(Z_{r_i}\right)\right| \le \prod_{\lambda_j\in\rho_i}\left(\int dZ_{r_i}\, |h|^{\sum_{\lambda_j}\ell_j}\left(z_{r_i}\right)\, |g_{r_i}^{\varepsilon,\mathrm{eq}}|\left(Z_{r_i}\right)\right)^{\ell_j/\sum_{\lambda_j}\ell_j}.
$$

Moreover by (A.4) and Fubini,

$$
\int dZ_{r_i}\, |h|^{\sum_{\lambda_j\in\rho_i}\ell_j}\left(z_{r_i}\right)\, |g_{r_i}^{\varepsilon,\mathrm{eq}}|\left(Z_{r_i}\right) \le \int dz_{r_i}\, |h|^{\sum_{\lambda_j}\ell_j}\left(z_{r_i}\right) M(v_i)
$$

$$
\times \sum_{p\ge 0}\sum_{T\in\mathcal{T}_{r_i+p}} \frac{\mu_\varepsilon^p}{p!}\int dZ_{r_i-1}d\bar{Z}_p M^{\otimes r_i-1+p}\left(V_{r_i-1},\bar{V}_p\right) \prod_{(y,y^*)\in E(T)} \mathbf{1}_{d(y,y^*)\le\varepsilon}
$$

and the last line of the previous formula is estimated exactly as in Section 3 by $r_i!C^{r_i}\varepsilon^{d(r_i-1)}$ for some $C>0$, uniformly in z_{r_i} for ε small enough. On the other hand, the term in the first line produces a factor

$$
\|h\|_{L_M^{\sum\ell_j}} \le C_p'\|h\|_{L_M^p}
$$

for some pure constant C_p' depending only on p (and not on the partition). From this we deduce that

$$\left| \int dZ_{r_i} \bigotimes_{\lambda_j \in \rho_i} h^{\ell_j}(Z_{r_i}) \, g_{r_i}^{\varepsilon,\mathrm{eq}}(Z_{r_i}) \right| \leq r_i! C^{r_i} \varepsilon^{d(r_i-1)} \left(C_p' \|h\|_{L_M^p} \right)^{\sum_{\lambda_j \in \rho_i} \ell_j}.$$

Hence

$$\left| \mathbb{E}_\varepsilon \left((\zeta^\varepsilon(g))^p \right) \right| \leq \mu_\varepsilon^{-p/2} \left(C_p' \|h\|_{L_M^p} \right)^p \sum_{\substack{\lambda \in \mathcal{P}_{\{1,\ldots,p\}} \\ \rho:\lambda \hookrightarrow^* \rho}} \mu_\varepsilon^\ell \prod_{i=1}^r r_i! C^{r_i} \varepsilon^{d(\ell-r)}.$$

Bounding roughly $\mu_\varepsilon^\ell \varepsilon^{d(\ell-r)} \leq \mu_\varepsilon^\ell \mu_\varepsilon^{-\ell+r} \leq \mu_\varepsilon^{p/2}$, we arrive to the estimate (A.1). $\qquad\square$

APPENDIX B: GEOMETRIC ESTIMATES

In this section, we complete the proof of Lemmas 5.1 and 5.2 and show how the presence of a cycle due to a non-clustering encounter leads to an additional constraint, producing extra smallness and leading to (5.12) and (5.14).

We first give some technical definitions to identify this additional constraint, then state the geometric estimates and finally deduce (5.12) from (5.11) (the argument is the same for (5.14)).

We recall that in deriving (5.11) an ordered tree $T_\prec$ has been constructed, following the forward dynamics. The definitions that follow relate to some particular edges in the tree. In a forward trajectory, encounters are of two types: with annihilation of a particle (corresponding to a creation in the backward pseudo-trajectory) or without (corresponding to a recollision in the backward pseudo-trajectory). Moreover in the first case, the surviving particle can be deflected or not. By deflection we mean here that the particle undergoes a non-zero variation of velocity.

Definition B.1. We call <u>parent</u> p of a group of particles $(q_k)_k$ at time τ the p-th edge with the largest p such that one of the particles $(q_k)_k$ is deflected at $\tau_p \leq \tau$. If such a parent does not exist, then we set $\tau_p := t_{\mathrm{stop}}$.

We define the <u>connector</u> k of two particles $(q, \bar{q})$ the index of the first edge realizing a connected path between q and $\bar{q}$.

The <u>tutor</u> j of two particles $(q, \bar{q})$ at time τ is the largest j with $\tau_j \leq \tau$ such that j is either the parent at time τ or the connector of $(q, \bar{q})$.

Recall that the construction of the admissible tree $T_\prec$ in the proof of Lemma 5.1 is such to exclude cycles, so that the graph $T_\prec$ is minimally connected, whereas the graph encoding all encounters has more than $N_k - 1$ edges. Consider the first encounter in the forward dynamics creating a cycle (or multiple edge) in the graph encoding all encounters. Let $(q, \bar{q})$ be the edge realizing the cycle and τ_{cyc} the corresponding time. The cycle will impose a constraint on the tutor j of $(q, \bar{q})$ at τ_{cyc}, and integrating on the relative position $\hat{x}_j$ as in the proof of Lemma 4.1 will produce the required additional smallness. The following proposition quantifies this smallness, which is different depending on whether the tutor is the parent or not.

Proposition B.2. *Assume that $d \geq 3$. Let q and $\bar{q}$ be the labels of the two particles involved in the first cycle, at time τ_{cyc}, and let j (resp p, k) be the tutor (resp. parent, connector) of $(q, \bar{q})$ at time τ_{cyc},*

as defined in Definition B.1. Then, denoting by $w_q, w_{\bar{q}}, w_{q_j}, w_{\bar{q}_j}$ the velocities of $q, \bar{q}, q_j, \bar{q}_j$ at τ_{j-1}^+, one has if the tutor is a parent p

$$
\int \mathbf{1}_{\text{Cycle with tutor } j = p} \, \mathbf{1}_{B_{T_<,j}} \, d\hat{x}_j \le \frac{C}{\mu_\varepsilon} (\mathbb{V}\theta)^d
$$
$$
\times \left(\frac{\mathbb{V}\varepsilon |\log \varepsilon| \mathbf{1}_{q \ne \bar{q}_j}}{|w_q - w_{\bar{q}_j}|} + \frac{\mathbb{V}\varepsilon |\log \varepsilon| \mathbf{1}_{\bar{q} \ne \bar{q}_j}}{|w_{\bar{q}} - w_{\bar{q}_j}|} + \frac{\mathbb{V}\theta}{\mu_\varepsilon} \right),
\tag{B.1}
$$

where $\bar{q}_j$ is the label of the colliding particle, and if the tutor is a connector k but not a parent

$$
\int \mathbf{1}_{\text{Cycle with tutor } j = k > p} \, \mathbf{1}_{B_{T_<,j}} \, d\hat{x}_j \le \frac{C}{\mu_\varepsilon} (\mathbb{V}\theta)^{d+1}
$$
$$
\times \left[\sum_\zeta \mathbf{1}_{\sin(w_q - w_{\bar{q}}, \zeta) \le \varepsilon} + (\mathbb{V}\theta)^d \min \left(1, \frac{\varepsilon \mathbf{1}_{(q,\bar{q}) \ne (q_j, \bar{q}_j)}}{\sin\left(w_q - w_{\bar{q}}, w_{q_j} - w_{\bar{q}_j} \right)} \right) \right]
\tag{B.2}
$$

where the sum runs over $\zeta \in \mathbb{Z}^d \setminus \{0\}$ contained in the ball of radius $\mathbb{V}\theta$.

The above proposition uses the tutor to gain some smallness from the strong geometric constraint. However, the estimates in (B.1)-(B.2) lead to singularities in the relative velocities. Those singularities have to be integrated out either by using available parents (if any) or by using the Gaussian measure of the velocity distribution at time t_{stop}. The following proposition summarizes the different possibilities.

Proposition B.3.

(i) Let $q \ne \bar{q}$ be two particles of velocities $w_q, w_{\bar{q}}$ with parent ℓ. Let $\zeta \in \mathbb{Z}^d \setminus \{0\}$. Then one has that

$$
\int \left(\frac{\mathbb{V}\varepsilon |\log \varepsilon|}{|w_q - w_{\bar{q}_j}|} + \mathbf{1}_{\sin(w_q - w_{\bar{q}_j}, \zeta) \le \varepsilon} \right) \mathbf{1}_{B_{T_<,\ell}} \, d\hat{x}_\ell \le \frac{C}{\mu_\varepsilon} \mathbb{V}\varepsilon |\log \varepsilon| \left(\delta \mathbf{1}_{\ell=1} + \theta \mathbf{1}_{\ell \ne 1} \right).
\tag{B.3}
$$

(ii) Let $q, \bar{q}, q_j, \bar{q}_j$ be particles with velocities $w_q, w_{\bar{q}}, w_{q_j}, w_{\bar{q}_j}$ and parent ℓ (say deflecting q), such that (q, q_j) and $(\bar{q}, \bar{q}_j)$ belong to different connected components of the dynamical graph.

$$
\int \min \left(1, \frac{\varepsilon \mathbf{1}_{\{q,\bar{q}\} \ne \{q_j, \bar{q}_j\}}}{\sin\left(w_q - w_{\bar{q}}, w_{q_j} - w_{\bar{q}_j} \right)} \right) \mathbf{1}_{B_{T_<,\ell}} d\hat{x}_\ell \le \frac{C}{\mu_\varepsilon} \mathbb{V}\varepsilon |\log \varepsilon| \left(\delta \mathbf{1}_{\ell=1} + \theta \mathbf{1}_{\ell \ne 1} \right)
$$
$$
\times \left(1 + \frac{\theta \mathbb{V} \mathbf{1}_{(q,q_j) \text{ encounter at } \tau_\ell}}{|u_q + u_{q_j} - (w_{\bar{q}_j} + w_{\bar{q}})|} + \frac{\theta \mathbb{V} \mathbf{1}_{q=q_j} \mathbf{1}_{\bar{q} \ne \bar{q}_j}}{|w_{\bar{q}} - w_{\bar{q}_j}|} \right),
\tag{B.4}
$$

denoting by u the pre-collisional velocities.

(iii) Let $q, \bar{q}, q_j, \bar{q}_j$ be particles with velocities $w_q, w_{\bar{q}}, w_{q_j}, w_{\bar{q}_j}$ such that $(q), (q_j)$ and $(\bar{q}, \bar{q}_j)$ belong to different connected components of the dynamical graph. Let ℓ be the first parent of $q, \bar{q}, q_j, \bar{q}_j$

deflecting only one particle of the group.

$$\int \frac{\mathbb{V}\varepsilon|\log\varepsilon|}{|w_q + w_{q_j} - (w_{\bar{q}_j} + w_{\bar{q}})|} \mathbf{1}_{B_{T_<,\ell}} \, d\hat{x}_\ell \le \frac{C}{\mu_\varepsilon} \mathbb{V}\varepsilon|\log\varepsilon| \left(\delta\mathbf{1}_{\ell=1} + \theta\mathbf{1}_{\ell\neq1}\right). \qquad (B.5)$$

(iv) For $q \neq \bar{q}, \zeta \in \mathbb{Z}^d \setminus \{0\}$

$$\int M(w_q)M(w_{q_j})M(w_{\bar{q}})M(w_{\bar{q}_j})\left(\frac{\mathbb{V}\varepsilon|\log\varepsilon|}{|w_q - w_{\bar{q}}|} + \frac{\mathbb{V}\varepsilon|\log\varepsilon|}{|w_q + w_{q_j} - w_{\bar{q}} - w_{\bar{q}_j}|}\right.$$

$$\left. +\mathbf{1}_{\sin(w_q - w_{\bar{q}},\zeta)\le\varepsilon} + \min\left(1, \frac{\varepsilon\mathbf{1}_{(q,\bar{q})\neq(q_j,\bar{q}_j)}}{\sin\left(w_q - w_{\bar{q}}, w_{q_j} - w_{\bar{q}_j}\right)}\right)\right)dw_q dw_{q_j} dw_{\bar{q}} dw_{\bar{q}_j} \qquad (B.6)$$

$$\le C\,\mathbb{V}\varepsilon|\log\varepsilon|.$$

Notice that if the parent lies in the time interval $(t_{\text{stop}}, t_{\text{stop}} + \delta)$, then the estimates are strengthened by a factor δ.

Propositions B.2 and B.3 are proved below. We now explain how to apply these local propositions iteratively, by using the time ordering prescribed by the dynamical graph, so that the singularities are progressively reduced leading finally to an upper bound of order $\varepsilon|\log\varepsilon|$. We recall Inequality (5.11):

$$\sum_{T_< \in \mathcal{T}_{N_k}^<} \int d\hat{X}_{N_k-1}\mathbf{1}_{\text{cycle}} \prod_{i=1}^{N_k-1} \mathbf{1}_{B_{T_<,i}}$$

$$\le \sum_{T_< \in \mathcal{T}_{N_k}^<} \sum_{q,\bar{q}} \sum_{c \le N_k} \int d\hat{x}_1 \mathbf{1}_{B_{T_<,1}} \int d\hat{x}_2 \cdots \int d\hat{x}_{N_k-1}\mathbf{1}_{B_{T_<,N_k-1}}\mathbf{1}_{\text{cycle defined by }(q,\bar{q}), c}.$$

Now we integrate the constraints iteratively with the additional integrals of Propositions B.2 and B.3 which act on local parameters thanks to the time ordering. More precisely we proceed as follows.

- We bound the inner integrals one by one up to the step j given by the tutor of the cycle. At this step, we apply Proposition B.2.
- We continue by estimating the integrals at steps $j - 1, j - 2, \ldots$ up to the step ℓ (if any) defined as the parent in Proposition B.3, items (i) or (ii) (depending on the term to be treated in (B.1)-(B.2)). At step ℓ, we apply (B.3) or (B.4) respectively. Notice that we are left with singularities involving the groups of particles in the right hand side of (B.4).
- We continue by estimating the integrals at steps $\ell - 1, \ell - 2, \ldots$ until we possibly find a parent of the latter group of particles, as defined in Proposition B.3, items (i) or (iii) (respectively for the third and the second term in (B.4)). We then apply (B.3) or (B.5) respectively.
- We continue by estimating the integrals up to step 1. If we have not found enough parents, we may be left with singularities as in the right-hand side of (B.1)–(B.2) or (B.4). By integrating the velocities with respect to the measure $M^{\otimes N_k}$, such singularities are dealt with by (B.6) and (4.17).

This proves (5.12).

Now let us prove Propositions B.2 and B.3. The cycle at time τ_{cyc} is triggered by a tutor j involving an edge $(q_j, \bar{q}_j)$. Notice that the tutor can involve the particles $(q, \bar{q})$ themselves. Below, we are going to distinguish the different cases in the definition of tutor (either $j = p$ or $j = k > p$) as well as a series of subcases to integrate the singularities.

Proof of Proposition B.2. **Case 1a**: $j = p$ **is the parent of** $q, \bar{q}$.

This is the case of a direct, periodic cycle, in which the last deflection of $q, \bar{q}$ in the forward dynamics before τ_{cyc} involves both particles q and $\bar{q}$ at time τ_j (in this case $q = q_j$ and $\bar{q} = \bar{q}_j$). In addition to the condition $\hat{x}_j \in B_{T_{\prec},j}$ which encodes the encounter (recall that $\hat{x}_j := x_{q_j} - x_{\bar{q}_j} = x_q - x_{\bar{q}}$), we obtain the following condition for a cycle

$$\varepsilon \omega_j + (v_q - v_{\bar{q}})(\tau_{\text{cyc}} - \tau_j) = \varepsilon \omega_{\text{cyc}} + \zeta \text{ with } \zeta \in \mathbb{Z}^d \setminus \{0\}, \omega_{\text{cyc}} \in \mathbb{S}^{d-1},$$

$$\text{and } v_q - v_{\bar{q}} = w_q - w_{\bar{q}} - 2(w_q - w_{\bar{q}}) \cdot \omega_j \omega_j$$

where, by definition, $v_q, v_{\bar{q}}$ are the velocities at time τ_j^+, $w_q, w_{\bar{q}}$ are the velocities at time τ_{j-1}^+, and ω_j is the impact parameter at the encounter. We deduce from the first relation that $v_q - v_{\bar{q}}$ has to be in a small cone K_ζ of opening ε, which implies by the second relation that ω_j has to be in a small cone S_ζ of opening ε. Note that the additional parameter $\zeta \in \mathbb{Z}^d \setminus \{0\}$ takes into account the periodic structure of the domain $\mathbb{T}^d$. Since the velocities are bounded by $\mathbb{V}$, it will be enough to consider the parameter ζ in the box $[-\mathbb{V}\theta, \mathbb{V}\theta]^d$.

Using the local change of variables $\hat{x}_j \mapsto (\varepsilon \omega_j, \tau_j)$, it follows that

$$\int \mathbf{1}_{\text{Cycle with } j = p, (q, \bar{q}) = (q_j, \bar{q}_j)} \mathbf{1}_{B_{T_{\prec},j}} d\hat{x}_j \leq C \varepsilon^{d-1} \theta \sum_\zeta \int \mathbf{1}_{\omega_j \in S_\zeta} ((w_q - w_{\bar{q}}) \cdot \omega_j)_+ d\omega_j$$

$$\leq C \varepsilon^{2(d-1)} (\theta \mathbb{V})^{d+1}$$

since there are at most $(\theta \mathbb{V})^d$ possibilities for the ζ's.

Case 1b: $j = p$ **is the parent of** $q = q_j$ **and** $\bar{q} \neq \bar{q}_j$.

In this case (see Figure B1), a third particle is involved in the process, as q is deflected by an encounter with $\bar{q}_j \neq \bar{q}$ at time τ_j (which implies necessarily that $j \geq 2$). By definition of the cycle, the connector k of $(q, \bar{q})$ is such that $\tau_k < \tau_{\text{cyc}}$. Then by definition of the tutor, one has $j \geq k$ so that at time τ_{j-1}^+

- either q and $\bar{q}$ are already in the same connected component;
- or $\bar{q}_j$ and $\bar{q}$ are already in the same connected component.

The encounter at τ_j is encoded by the condition $\hat{x}_j = x_q - x_{\bar{q}_j} \in B_{T_{\prec},j}$, and we can strengthen this condition thanks to the cycle. The new condition will be written as a new constraint between $\hat{x}_j := x_{q_j} - x_{\bar{q}_j} = x_q - x_{\bar{q}_j}, \tau_j$ and ω_j, which is different in both cases but can be estimated in the same way.

Case 1b.a) – If q and $\bar{q}$ are already in the same connected component, then we write that in addition to the condition $\hat{x}_j = x_q - x_{\bar{q}_j} \in B_{T_{\prec},j}$,

$$\left(x_q(\tau_{j-1}) - x_{\bar{q}}(\tau_{j-1})\right) + (w_q - w_{\bar{q}})(\tau_j - \tau_{j-1}) + (v_q - w_{\bar{q}})(\tau_{\text{cyc}} - \tau_j) = \varepsilon \omega_{\text{cyc}} + \zeta,$$

$$\text{with} \quad \zeta \in \mathbb{Z}^d \quad \text{and} \quad v_q = w_q - (w_q - w_{\bar{q}_j}) \cdot \omega_j \omega_j$$
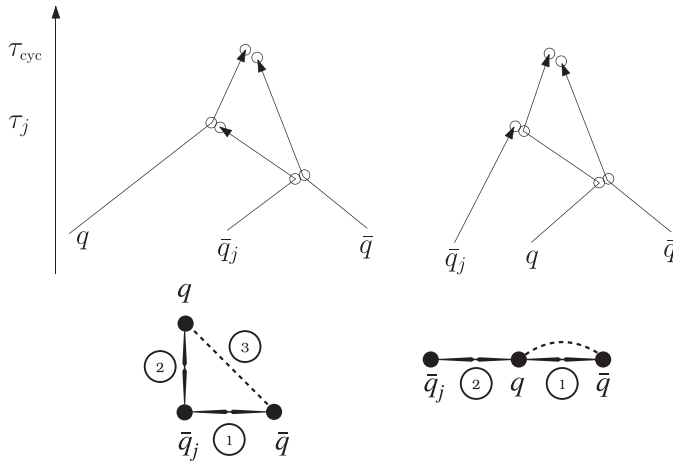
(B.7)

FIGURE B1 Two simple situations corresponding to Case 1b. The non- clustering encounter between $(q, \bar{q})$ is triggered by a previous deflection between $q = q_j$ and $\bar{q}_j$ which is not equal to $\bar{q}$. The parameter $\hat{x}_j$ has to be be tuned so that the encounter between $q, \bar{q}_j$ leads as well to the recollision. The corresponding graphs with a cycle are depicted below.

where we recall that v_q is the velocity at time τ_j^+ and $w_q, w_{\bar{q}_j}$ and $w_{\bar{q}} = v_{\bar{q}}$ are the velocities of $q, \bar{q}_j$ and $\bar{q}$ at time τ_{j-1}^+ (and therefore at time τ_j^-). Since q and $\bar{q}$ are already in the same connected component, their relative distance at τ_{j-1}^+ and the velocities are fixed by $\hat{x}_1, \dots \hat{x}_{j-1}$ and the velocities at t_{stop}. Condition (B.7) therefore expresses a new constraint between $\hat{x}_j, \tau_j$ and ω_j.

Case 1b.b) – If $\bar{q}$ and $\bar{q}_j$ are already in the same connected component, then their relative distance at τ_{j-1}^+ and the velocities are fixed by $\hat{x}_1, \dots \hat{x}_{j-1}$ and the velocities at t_{stop}. Then, in addition to the condition $\hat{x}_j = x_q - x_{\bar{q}_j} \in B_{T_{<,j}}$ which encodes the encounter at τ_j, we obtain

$$(x_{\bar{q}_j}(\tau_{j-1}) - x_{\bar{q}}(\tau_{j-1})) + (w_{\bar{q}_j} - w_{\bar{q}})(\tau_j - \tau_{j-1}) + (v_q - w_{\bar{q}})(\tau_{\text{cyc}} - \tau_j)$$

$$= \varepsilon \omega_{\text{cyc}} - \varepsilon \omega_j + \zeta, \tag{B.8}$$

$$\text{with} \quad \zeta \in \mathbb{Z}^d \quad \text{and} \quad v_q = w_q - (w_q - w_{\bar{q}_j}) \cdot \omega_j \, \omega_j$$

with the same notations as in (B.7) for the pre-collisional and post-collisional velocities.

The first equations in (B.7)(B.8) restate

$$v_q - w_{\bar{q}} = \frac{1}{\delta\tau_{\text{cyc}}} \left(\omega_{\text{cyc}} - \omega_j \mathbf{1}_{\text{Case 1b.b)}} + \delta x_\perp + \delta\tau_j w_{\text{rel}} \right), \quad \delta\tau_{\text{cyc}} := \frac{\tau_{\text{cyc}} - \tau_j}{\varepsilon} \tag{B.9}$$

where

- for (B.7) the relative velocity is $w_{\text{rel}} := w_q - w_{\bar{q}}$ and

$$\delta x := \frac{1}{\varepsilon} \left(x_q(\tau_{j-1}) - x_{\bar{q}}(\tau_{j-1}) - \zeta \right) =: \delta x_\perp + \tau_* w_{\text{rel}}$$

$$\delta x_\perp \perp w_{\text{rel}} \quad \text{and} \quad \delta\tau_j := \frac{1}{\varepsilon}(\tau_j - \tau_{j-1} + \tau_*).$$

In this case there is no term ω_j;

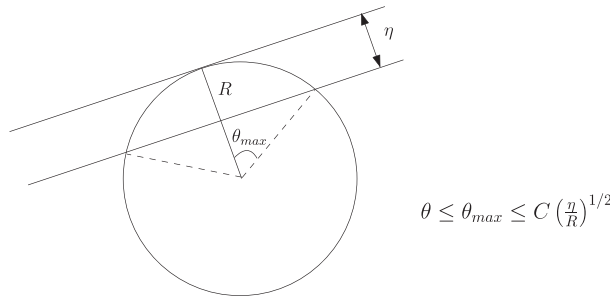$$\theta \leq \theta_{max} \leq C\left(\frac{\eta}{R}\right)^{1/2}$$

FIGURE B2 Intersection of a cylinder and a sphere. The maximal solid angle is obtained in the case when the cylinder is tangent to the sphere. It is always less than $C_d \min(1, (\eta/R)^{(d-1)/2})$.

- for (B.8) the relative velocity is $w_{rel} := w_{\bar{q}_j} - w_{\bar{q}}$,

$$\delta x := \frac{1}{\varepsilon}\left(x_{\bar{q}_j}(\tau_{j-1}) - x_{\bar{q}}(\tau_{j-1}) - \zeta\right) =: \delta x_\perp + \tau_* w_{rel}$$

$$\delta x_\perp \perp w_{rel} \quad \text{and} \quad \delta\tau_j := \frac{1}{\varepsilon}(\tau_j - \tau_{j-1} + \tau_*).$$

Note that, by definition

$$|w_{rel}\tau_*| \leq |\delta x| \leq \frac{C}{\varepsilon} \quad \Rightarrow \quad |w_{rel}\delta\tau_j| \leq \frac{C\mathbb{V}\theta}{\varepsilon}.$$

The recollision will be easier to achieve if δx is small, however this cannot happen (for a large amount of time) if the relative velocity w_{rel} is large enough, as the particles will drift far apart when τ_j changes. In the following, we will integrate over the time τ_j recalling that $d\hat{x}_j = \varepsilon^{d-1}((w_q - w_{\bar{q}_j}) \cdot \omega_j)_+ d\omega_j d\tau_j$.

Subcase (i): Suppose that $|w_{rel}\delta\tau_j| \geq 4$. We get from (B.9) that

$$(v_q - w_{\bar{q}})\delta\tau_{cyc} = \omega_{cyc} - \omega_j \mathbf{1}_{\text{Case 1b.b)}} + \delta x_\perp + w_{rel}\delta\tau_j.$$

Thus the triangular inequality implies

$$\frac{1}{2\delta\tau_{cyc}}|w_{rel}\delta\tau_j| \leq |v_q - w_{\bar{q}}|,$$

from which we deduce

$$\frac{1}{\delta\tau_{cyc}} \leq \frac{4\mathbb{V}}{|w_{rel}\delta\tau_j|}.$$

By (B.9), $v_q - w_{\bar{q}}$ belongs to a cylinder $\mathcal{R}$ of main axis $\delta x_\perp + w_{rel}\delta\tau_j$ and of width $4\mathbb{V}/|w_{rel}\delta\tau_j|$.

Then, v_q has to be both in the sphere of diameter $[w_q, w_{\bar{q}_j}]$ (by the second equation in (B.7)) and in the cylinder $w_{\bar{q}} + \mathcal{R}$ (by (B.9)). This imposes a strong constraint on the deflection angle ω_j in (B.7)(B.8), which has to belong to a union of at most two spherical caps. The maximal solid angle is obtained in the case when the cylinder is tangent to the sphere (see Figure B2). It is always less than $C_d \min(1, (\eta/R)^{(d-1)/2})$ denoting by η the width of the cylinder, and by $R = \frac{1}{2}|w_q - w_{\bar{q}_j}|$ the radius of the sphere.
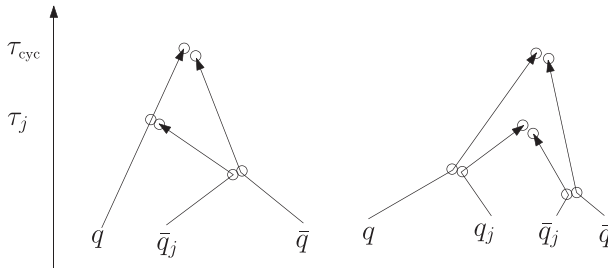
FIGURE B3 Two simple situations corresponding to Case 2. In the picture on the left, only three particles are involved and $\bar{q}_j$ is annihilated in the encounter with q at time τ_j ($j \geq 2$). The picture on the right depicts another possible situation, where 4 particles are involved as $q, \bar{q}$ are both different from $q_j, \bar{q}_j$ ($j \geq 3$).
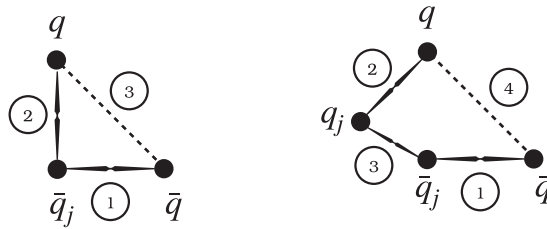

FIGURE B4 Graphs associated to the pseudo-trajectories in Figure B3. Non-clustering encounters are represented by dashed lines.

Thus ω_j has to belong to a union of spherical caps S_ζ, of solid angle less than

$$\int \mathbf{1}_{\omega_j \in S_\zeta} d\omega_j \leq C \left(\frac{\mathbb{V}}{|\delta \tau_j w_{\text{rel}}||w_q - w_{\bar{q}_j}|} \right)^{(d-1)/2}.$$

Note that we can always replace the power $(d-1)/2$ by 1 since we know that the left-hand side is bounded by $|\mathbb{S}^{d-1}|$. Therefore

$$\int \mathbf{1}_{\omega_j \in S_\zeta} ((w_q - w_{\bar{q}_j}) \cdot \omega_j)_+ d\omega_j d\tau_j \leq C\mathbb{V} \int \min\left(\left(\frac{\varepsilon}{|\tau_j w_{\text{rel}}|} \right), 1 \right) d\tau_j.$$

<u>Subcase (ii)</u>: if $|w_{\text{rel}} \delta \tau_j| < 4$, we have a strong constraint on $\delta \tau_j$ and we do not need any additional constraint on ω_j.

Thus, it follows that

$$\int \mathbf{1}_{\text{j parent of } q \text{ with } \bar{q} \neq \bar{q}_j} \mathbf{1}_{B_{T_{<,j}}} d\hat{x}_j \leq \frac{C}{\mu_\varepsilon} (\mathbb{V}\theta)^d \frac{\varepsilon |\log \varepsilon| \mathbb{V}}{|w_{\text{rel}}|},$$

which concludes the proof of (B.1).

Case 2a: $j = k > p$ **is the connector of** $(q, \bar{q}) \neq (q_j, \bar{q}_j)$ **but not its parent**.

The situation when q (or $\bar{q}$) is deflected at time τ_j has already been dealt within Case 1. We will therefore assume that q and $\bar{q}$ are not deflected at time τ_j (see Figures B3-B4 for situations when this can happen).

In this case, the velocities $v_q = w_q$ and $v_{\bar{q}} = w_{\bar{q}}$ are constant on $[\tau_{j-1}^+, \tau_{\mathrm{cyc}}]$. Moreover by definition of the tutor, we know that $\bar{q}$ and $\bar{q}_j$ (resp. q and q_j) belong to the same connected component of the graph $T_\prec$ at time τ_{j-1}^+. As the corresponding connected components move rigidly with respect to $\hat{x}_j$, we notice that the relative distances $x_q(\tau_{j-1}) - x_{q_j}(\tau_{j-1})$ and $x_{\bar{q}}(\tau_{j-1}) - x_{\bar{q}_j}(\tau_{j-1})$ are independent of $\hat{x}_j$. The dynamical constraints state

$$
\begin{aligned}
x_q(\tau_{j-1}) - x_{\bar{q}}(\tau_{j-1}) + (\tau_{\mathrm{cyc}} - \tau_{j-1})(w_q - w_{\bar{q}}) &= \varepsilon\,\omega_{\mathrm{cyc}} + \zeta_{\mathrm{cyc}} \\
x_{q_j}(\tau_{j-1}) - x_{\bar{q}_j}(\tau_{j-1}) + (\tau_j - \tau_{j-1})(w_{q_j} - w_{\bar{q}_j}) &= \varepsilon\,\omega_j + \zeta_j\,.
\end{aligned}
\tag{B.10}
$$

Writing

$$
x_q(\tau_{j-1}) - x_{\bar{q}}(\tau_{j-1}) = \hat{x}_j + \left(x_q(\tau_{j-1}) - x_{q_j}(\tau_{j-1})\right) + \left(x_{\bar{q}_j}(\tau_{j-1}) - x_{\bar{q}}(\tau_{j-1})\right),
$$

it follows that $\hat{x}_j$ has to be in the intersection $\mathcal{R}$ of two cylinders of axis $w_q - w_{\bar{q}}$ and $w_{q_j} - w_{\bar{q}_j}$, and width ε. The volume of this intersection is at most

$$
|\mathcal{R}| \le \frac{1}{\mu_\varepsilon} \min(\mathbb{V}\theta, \frac{\varepsilon}{\sin\Omega})
\tag{B.11}
$$

where Ω is the angle between $w_q - w_{\bar{q}}$ and $w_{q_j} - w_{\bar{q}_j}$. This proves (B.2).

Case 2b: $j = k > p$ **is the connector of** $(q, \bar{q}) = (q_j, \bar{q}_j)$ **but not its parent.**

Since q and $\bar{q}$ are not deflected at τ_j, we obtain a contradiction: one of the particles has to be annihilated and they cannot encounter again. However, in the companion paper [4], when looking at higher moments of the fluctuation field, we will have to consider overlaps, that is, encounters where the two particles survive without being deflected. For the sake of completeness, we therefore deal also with this case. From (B.10) we deduce that

$$
x_q(\tau_{j-1}) - x_{\bar{q}}(\tau_{j-1}) + (\tau_{\mathrm{cyc}} - \tau_{j-1})(w_q - w_{\bar{q}}) = \varepsilon\,\omega_{\mathrm{cyc}} + \zeta_{\mathrm{cyc}}
$$

$$
x_q(\tau_{j-1}) - x_{\bar{q}}(\tau_{j-1}) + (\tau_j - \tau_{j-1})(w_q - w_{\bar{q}}) = \varepsilon\,\omega_j + \zeta_j\,,
$$

so that

$$
(\tau_j - \tau_{\mathrm{cyc}})(w_q - w_{\bar{q}}) = (\zeta_j - \zeta_{\mathrm{cyc}}) + O(\varepsilon)\,.
$$

In other words, the sinus between $w_q - w_{\bar{q}}$ and $\zeta_j - \zeta_{\mathrm{cyc}}$ has to be less than ε, which proves (B.2). This concludes the proof of Proposition B.2. $\qquad\square$

Proof of Proposition B.3. **Integration of the first singularity, proof of** (B.3).

Let us start by dealing with the singularity $1/|w_q - w_{\bar{q}}|$, which we want to integrate by using the parent variables. Denoting by $u_q, u_{\bar{q}}$ the velocities at time τ_ℓ^-, we distinguish again between two subcases.

<u>Subcase (i):</u> $q = q_\ell$ and $\bar{q} = \bar{q}_\ell$, then $|w_q - w_{\bar{q}}| = |u_q - u_{\bar{q}}|$ and there holds

$$
\int \frac{\mathbf{1}_{B_{T_\prec,\ell}}}{|w_q - w_{\bar{q}}|} d\hat{x}_\ell \le \frac{C}{\mu_\varepsilon} \int \frac{|(u_q - u_{\bar{q}}) \cdot \omega_\ell|}{|u_q - u_{\bar{q}}|} d\omega_\ell d\tau_\ell \le \frac{C}{\mu_\varepsilon}(\delta\mathbf{1}_{\ell=1} + \theta\mathbf{1}_{\ell\neq 1})\,.
$$

<u>Subcase (ii):</u> $q = q_\ell$ and $\bar{q} \neq \bar{q}_\ell$, then $\bar{q}$ is not deflected at τ_ℓ. We therefore have

$$
w_q = u_q - (u_q - u_{\bar{q}_\ell}) \cdot \omega_\ell\, \omega_\ell
$$

and

$$\int \frac{\mathbf{1}_{B_{T_{\prec},\ell}}}{|w_q - w_{\bar{q}}|} d\hat{x}_\ell \leq \frac{C}{\mu_\varepsilon} \int \frac{1}{|u_q - u_{\bar{q}} - (u_q - u_{\bar{q}_\ell}) \cdot \omega_\ell \, \omega_\ell|} \big|(u_q - u_{\bar{q}_\ell}) \cdot \omega_\ell\big| d\omega_\ell \, d\tau_\ell \,.$$

Denoting $a := u_q - u_{\bar{q}_\ell}$ and $b := u_q - u_{\bar{q}}$, we therefore have to study the integral

$$\int \frac{1}{|b - (a \cdot \omega)\, \omega|} |a \cdot \omega| d\omega \,.$$

The denominator in the integrand vanishes at

$$\omega_0 := \frac{b}{|b|}, \quad \text{if} \quad (b \cdot a) = |b|^2 \,.$$

Consider an infinitesimal variation η around ω_0. Since $\omega \in \mathbb{S}^{d-1}$, η is orthogonal to ω_0. The first increment of the denominator at ω_0 is

$$|(a \cdot \eta)\omega_0 + (a \cdot \omega_0)\eta| \geq |(a \cdot \omega_0)\eta| \geq |b||\eta| \,.$$

We therefore find that

$$\frac{|a \cdot \omega|}{|b - (a \cdot \omega)\, \omega|} \leq C \frac{|b|}{|\eta||b|} \,.$$

Locally the measure $d\omega$ looks like $|\eta|^{d-2} d\eta$, from which we deduce that

$$\int \frac{1}{|b - (a \cdot \omega)\, \omega|} |a \cdot \omega| d\omega \leq C\mathbb{V}$$

since $d \geq 3$. Integrating with respect to τ_ℓ (and for $\ell = 1$ considering the constraint that $\tau_1 \in [t_{\text{stop}}, t_{\text{stop}} + \delta]$) leads to (B.3) in this case.

The term with small sine in (B.3) is bounded by

$$\frac{1}{\mu_\varepsilon} \int \mathbf{1}_{\sin(w_q - w_{\bar{q}}, \zeta) \leq \varepsilon} \mathbf{1}_{B_{T_{\prec},\ell}} |u_{q_\ell} - u_{\bar{q}_\ell}| d\omega_\ell \, d\tau_\ell \leq \frac{C\mathbb{V}\theta\varepsilon}{\mu_\varepsilon} \big(\delta \mathbf{1}_{\ell=1} + \theta \mathbf{1}_{\ell \neq 1}\big) \,,$$

which concludes the proof (B.3). □

Integration of the second singularity, proof of (B.4)-(B.5).

We want to integrate the singularity

$$\left| \frac{1}{\sin \Omega} \right| = \frac{|w_q - w_{\bar{q}}| \, |w_{q_j} - w_{\bar{q}_j}|}{|(w_q - w_{\bar{q}}) \wedge (w_{q_j} - w_{\bar{q}_j})|} \,, \tag{B.12}$$

by using the parent variable of $(q, \bar{q}, q_j, \bar{q}_j)$. Without loss of generality, we can assume that particle q has a deflection at the encounter ℓ. This singularity is very degenerate as the denominator is equal to 0 as soon as the vectors are aligned. Thus the integration with respect to the first parent may not be enough to control fully the divergence. Nevertheless, the integration will lead to a less singular function of the type $v \mapsto 1/|v|$ which can then be integrated by using an additional parent as in (B.3) (already proved) or (B.5) (proved below).
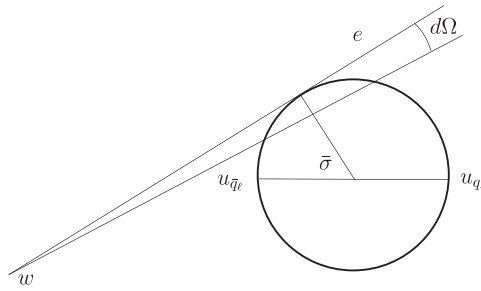
FIGURE B5   The geometry leading to the degeneracy at $\sigma_\ell = \bar\sigma$ of the singular integral (B.13) is depicted.

Subcase (i): Suppose first that $q \neq q_j$ and that q_j is not deflected at time τ_ℓ. The encounter at time τ_ℓ involves particle $q = q_\ell$ and a new particle $\bar q_\ell$. Denoting by σ_ℓ the deflection parameter of the encounter at time τ_ℓ, and by $u_{q_\ell}, u_{\bar q_\ell}$ the velocities at time τ_ℓ^-, we must have

$$w_{q_\ell} = \frac{1}{2}\left(u_{q_\ell} + u_{\bar q_\ell}\right) \pm \frac{1}{2}\left|u_{q_\ell} - u_{\bar q_\ell}\right|\sigma_\ell \, .$$

In (B.12), the velocities $w_{q_j} - w_{\bar q_j}$ and $w_{\bar q}$ are frozen at time τ_ℓ. By definition, given a vector $w := w_{\bar q}$ and a unit vector $e := \frac{w_{q_j} - w_{\bar q_j}}{|w_{q_j} - w_{\bar q_j}|}$, the integral

$$\int \min\left(1, \frac{\varepsilon|w_{q_\ell} - w|}{|(w_{q_\ell} - w) \wedge e|}\right)\mathbf{1}_{B_{T_<,\ell}} d\hat x_\ell$$

$$= \frac{1}{\mu_\varepsilon} \int \min\left(1, \frac{\varepsilon|w_{q_\ell} - w|}{|(w_{q_\ell} - w) \wedge e|}\right)\mathbf{1}_{B_{T_<,\ell}}|u_{q_\ell} - u_{\bar q_\ell}| d\sigma_\ell d\tau_\ell \tag{B.13}$$

has a singularity when $(w_{q_\ell} - w) \wedge e = 0$. Singularities are isolated as soon as $u_{q_\ell} - u_{\bar q_\ell} \neq 0$, and are in general of order 1, but they become degenerate when the line $w + \mathbb{R}e$ is tangent to the sphere of diameter $[u_{q_\ell}, u_{\bar q_\ell}]$ at w_{q_ℓ} (see Figure B5).

Consider now an infinitesimal variation η around a singular value $\sigma_\ell = \bar\sigma$. Since $\sigma_\ell \in \mathbb{S}^{d-1}$, η is orthogonal to $\bar\sigma$. At leading order one has

$$\sin\Omega \sim \frac{|u_{q_\ell} - u_{\bar q_\ell}||\eta \wedge e|}{|w_{q_\ell} - w|} \, .$$

Therefore, if $d \geq 3$ (and even though e is in the plane orthogonal to $\bar\sigma$),

$$\frac{1}{\mu_\varepsilon} \int \min\left(1, \frac{\varepsilon}{\sin\Omega}\right)\mathbf{1}_{B_{T_<,\ell}}|u_{q_\ell} - u_{\bar q_\ell}| d\sigma_\ell d\tau_\ell \leq \frac{C\mathbb{V}\varepsilon|\log\varepsilon|}{\mu_\varepsilon}\left(\delta\mathbf{1}_{\ell=1} + \theta\mathbf{1}_{\ell\neq1}\right), \tag{B.14}$$

which gives (B.4) in the case $q \neq q_j$ and q_j is not deflected.

Subcase (ii): Suppose then that $q \neq q_j$ (as on the right picture in Figure B3) and that q and q_j encounter at time τ_ℓ. In this case, the velocities w_q and w_{q_j} change simultaneously in (B.12). In order to decouple them, we rewrite the denominator by adding $w_q - w_{\bar q}$ and use the upper bound $|w_{q_j} - w_{\bar q_j}| \leq \mathbb{V}$

$$\frac{1}{\sin\Omega} = \frac{|w_q - w_{\bar{q}}||w_{q_j} - w_{\bar{q}_j}|}{|(w_q - w_{\bar{q}}) \wedge (w_q + w_{q_j} - w_{\bar{q}_j} - w_{\bar{q}})|}$$

$$\leq \frac{\mathbb{V}}{|w_q + w_{q_j} - (w_{\bar{q}_j} + w_{\bar{q}})|} \frac{|w_q - w_{\bar{q}}|}{|(w_q - w_{\bar{q}}) \wedge e|},$$

where the vector $e = \frac{w_q + w_{q_j} - w_{\bar{q}_j} - w_{\bar{q}}}{|w_q + w_{q_j} - (w_{\bar{q}_j} + w_{\bar{q}})|}$ is unchanged by the encounter between q, q_j as the momentum is conserved. With the previous notation to describe the encounter at time τ_ℓ, the particles are indexed by $q_\ell = q$, $\bar{q}_\ell = q_j$, the pre-collisional velocities by $u_{q_\ell}, u_{\bar{q}_\ell}$ and the post-collisional velocities by $w_{q_\ell} = w_q$ and $w_{\bar{q}_\ell} = w_{q_j}$. In particular the momentum conservation reads $w_q + w_{q_j} = u_{q_\ell} + u_{\bar{q}_\ell}$.

Thus the term $\frac{|w_q - w_{\bar{q}}|}{|(w_q - w_{\bar{q}}) \wedge e|} = \frac{|w_{q_\ell} - w_{\bar{q}}|}{|(w_{q_\ell} - w_{\bar{q}}) \wedge e|}$ can be integrated as in (B.14)

$$\frac{1}{\mu_\varepsilon} \int \min\left(1, \frac{\varepsilon}{\sin\Omega}\right) \mathbf{1}_{B_{T_<,\ell}} |u_{q_\ell} - u_{\bar{q}_\ell}| d\sigma_\ell d\tau_\ell \leq \frac{C\mathbb{V}\theta\varepsilon|\log\varepsilon|}{\mu_\varepsilon} \frac{\theta\mathbb{V}(\delta\mathbf{1}_{\ell=1} + \theta\mathbf{1}_{\ell\neq1})}{|u_{q_\ell} + u_{\bar{q}_\ell} - (w_{\bar{q}_j} + w_{\bar{q}})|},$$

which leads to (B.4) in this case.

Subcase (iii): Suppose now that $q = q_j$ and $\bar{q} \neq \bar{q}_j$ (left picture in Figure B3). If the parent acts on q, then one has to integrate over the variable $w_q = w_{q_j}$ which appears twice now. To decouple the different occurrences of w_q, we proceed as in the previous step and add $w_{\bar{q}} - w_q$ in the denominator. Then

$$\frac{1}{\sin\Omega} = \frac{|w_q - w_{\bar{q}}||w_q - w_{\bar{q}_j}|}{|(w_q - w_{\bar{q}}) \wedge (w_q - w_{\bar{q}_j})|}$$

$$\leq \frac{|w_q - w_{\bar{q}}|\mathbb{V}}{|(w_q - w_{\bar{q}}) \wedge (w_{\bar{q}} - w_{\bar{q}_j})|} \leq \frac{\mathbb{V}}{|w_{\bar{q}} - w_{\bar{q}_j}|} \frac{|w_q - w|}{|(w_q - w) \wedge e|},$$

with $e = \frac{w_{\bar{q}} - w_{\bar{q}_j}}{|w_{\bar{q}} - w_{\bar{q}_j}|}$. We stress the fact that, by construction, the particle $\bar{q}_\ell$ colliding with q is different from the previous particles. Thus we can repeat the above steps but we will be left with a singularity $1/|w_{\bar{q}} - w_{\bar{q}_j}|$

$$\frac{1}{\mu_\varepsilon} \int \min\left(1, \frac{\varepsilon}{\sin\Omega}\right) \mathbf{1}_{B_{T_<,\ell}} |u_{q_\ell} - u_{\bar{q}_\ell}| d\sigma_\ell d\tau_\ell \leq \frac{C\mathbb{V}\theta\varepsilon|\log\varepsilon|}{\mu_\varepsilon} \frac{\mathbb{V}}{|w_{\bar{q}} - w_{\bar{q}_j}|}.$$

This concludes the proof of (B.4).

Proof of (B.5): We discuss now the singularity arising from the previous subcase (ii). This is a singularity of the form $1/|w_q + w_{q_j} - (w_{\bar{q}_j} + w_{\bar{q}})|$ and respecting the assumptions of item (iii) in Proposition B.3. If the parent ℓ of the group $(q, \bar{q}_j, \bar{q}_j, \bar{q})$ exists and acts on a single particle, then we proceed as in the proof of (B.3), subcase (ii). If instead the first parent of the group $(q, \bar{q}_j, \bar{q}_j, \bar{q})$ involves simultaneously $\bar{q}_j$ and $\bar{q} \neq \bar{q}_j$, then the momentum $w_{\bar{q}_j} + w_{\bar{q}}$ is constant and the parent cannot be used: one then looks for the next available parent (if any) deflecting only one particle, and proceeds always integrating the singularity as in (B.3). This leads to (B.5).

The case of no parent is dealt with by (B.6), which is straightforward. This concludes the proof of Proposition B.3.