Multi-scale analysis of compressible viscous and rotating fluids

Eduard Feireisl* Isabelle Gallagher† David Gerard-Varet‡ Antonín Novotný

Institute of Mathematics of the Academy of Sciences of the Czech Republic
Žitná 25, 115 67 Praha 1, Czech Republic

Institut de Mathématiques UMR 7586, Université Paris Diderot
175, Rue de Chevaleret, 75013 Paris, France

IMATH Université du Sud Toulon-Var
BP 20132, 83957 La Garde, France

Abstract

We study a singular limit for the compressible Navier-Stokes system when the Mach and Rossby numbers are proportional to certain powers of a small parameter ε. If the Rossby number dominates the Mach number, the limit problem is represented by the 2-D incompressible Navier-Stokes system describing the horizontal motion of vertical averages of the velocity field. If they are of the same order then the limit problem turns out to be a linear, 2-D equation with a unique radially symmetric solution. The effect of the centrifugal force is taken into account.

1 Introduction

Rotating fluid systems appear in many applications of fluid mechanics, in particular in models of atmospheric and geophysical flows, see the monograph [3]. Earth's rotation, together with the influence of gravity and the fact that atmospheric Mach number is typically very small, give rise to a large variety of singular limit problems, where some of these characteristic numbers become large or tend to zero, see Klein [13]. We consider a simple situation, where the Rossby number is proportional to a small parameter ε, while the Mach number behaves like ε^m, with $m \geq 1$. Scaling of this type with various choices of m appears, for instance, in meteorological models (cf. [13, Section 1.3]).

We neglect the influence of the temperature and we write the equations of motion in the rotating frame attached to the Earth. Assuming that the rotation axis is parallel to x_3, we set $\mathbf{b} = [0, 0, 1]$, and the associated centrifugal force is denoted $\nabla_x G \approx \nabla_x |x_h|^2$, where we have written $x_h = [x^1, x^2]$, so that the time derivative of the velocity field $\mathbf{u} = \mathbf{u}(t, x)$ is changed into $\partial_t \mathbf{u} + \frac{1}{\varepsilon}(\mathbf{b} \times \varrho \mathbf{u}) - \frac{1}{\varepsilon^2} \varrho \nabla_x G$. Finally we arrive

*The work was supported by Grant 201/09/ 0917 of GA ČR as a part of the general research programme of the Academy of Sciences of the Czech Republic, Institutional Research Plan AV0Z10190503.

†The work was partially supported by the A.N.R grant ANR-08-BLAN-0301-01 "Mathocéan", as well as the Institut Universitaire de France.

‡The work was partially supported by the A.N.R grant ANR-08-JCJC-0104 "RUGO".

at the following scaled *Navier-Stokes system* describing the time evolution of the fluid density $\varrho = \varrho(t,x)$ and the velocity field $\mathbf{u} = \mathbf{u}(t,x)$:

$$\partial_t \varrho + \mathrm{div}_x(\varrho \mathbf{u}) = 0, \tag{1.1}$$

$$\partial_t(\varrho \mathbf{u}) + \mathrm{div}_x(\varrho \mathbf{u} \otimes \mathbf{u}) + \frac{1}{\varepsilon}(\mathbf{b} \times \varrho \mathbf{u}) + \frac{1}{\varepsilon^{2m}} \nabla_x p(\varrho) = \mathrm{div}_x \mathbb{S}(\nabla_x \mathbf{u}) + \frac{1}{\varepsilon^2} \varrho \nabla_x G, \tag{1.2}$$

where p is the pressure, and $\mathbb{S}$ is the viscous stress tensor determined by Newton's rheological law

$$\mathbb{S}(\nabla_x \mathbf{u}) = \mu \Big(\nabla_x \mathbf{u} + \nabla_x^t \mathbf{u} - \frac{2}{3} \mathrm{div}_x \mathbf{u} \mathbb{I} \Big), \ \mu > 0. \tag{1.3}$$

For the sake of simplicity, we have omitted possible influence of the so-called bulk viscosity component in the viscous stress.

We consider a very simple geometry of the underlying physical space $\Omega \subset \mathbb{R}^3$, namely Ω is an infinite slab,

$$\Omega = \mathbb{R}^2 \times (0,1).$$

Moreover, to eliminate entirely the effect of the boundary on the motion, we prescribe the *complete slip* boundary conditions for the velocity field:

$$\mathbf{u} \cdot \mathbf{n}|_{\partial\Omega} = 0, \ [\mathbb{S}\mathbf{n}] \times \mathbf{n}|_{\partial\Omega} = 0, \tag{1.4}$$

where $\mathbf{n}$ denotes the outer normal vector to the boundary. Note that the more common *no-slip* boundary condition

$$\mathbf{u}|_{\partial\Omega} = 0$$

would yield a trivial result in the asymptotic limit, namely $\mathbf{u} \to 0$ for $\varepsilon \to 0$. On the other hand, the so-called *Navier's boundary condition*

$$\mathbf{u} \cdot \mathbf{n}|_{\partial\Omega} = 0, \ \beta \mathbf{u}_{\tan} + [\mathbb{S}\mathbf{n}]_{\tan}|_{\partial\Omega} = 0, \ \beta > 0, \tag{1.5}$$

gives rise to a friction term in the limit system known as Ekman's pumping, see Section 2.5 below.

As is well known (see Ebin [7]), the boundary conditions (1.4) may be conveniently reformulated in terms of geometrical restrictions imposed on the state variables that are *periodic* with respect to the vertical variable x_3. More specifically, we take

$$\Omega = \mathbb{R}^2 \times \mathcal{T}^1, \tag{1.6}$$

where $\mathcal{T}^1 = [-1,1]|_{\{-1,1\}}$ is a one-dimensional torus, on which the fluid density ϱ as well as the horizontal component of the velocity $\mathbf{u}_h = [u^1, u^2]$ are extended to be even in x_3, while the vertical component u^3 is taken odd:

$$\begin{gathered} \varrho(x_1,x_2,-x_3) = \varrho(x_1,x_2,x_3), \ u^i(x_1,x_2,-x_3) = u^i(x_1,x_2,x_3), \ i = 1,2, \\ u^3(x_1,x_2,-x_3) = -u^3(x_1,x_2,x_3). \end{gathered} \tag{1.7}$$

As shown in [3], *incompressible* rotating fluids stabilize to a 2D motion described by the vertical averages of the velocity provided the Rossby number ε is small enough. Besides, the stabilizing effect of rotation has been exploited by many authors, see e.g. Babin, Mahalov and Nicolaenko [1], [2]. On the other hand, compressible fluid flows in the low Mach number regime behave like the incompressible ones, see

Klainerman and Majda [12], Lions and Masmoudi [17], among many others. Thus, at at least for $m \gg 1$, solutions of the scaled system (1.1), (1.2) are first rapidly driven to incompressibility and then stabilize to a purely horizontal motion as $\varepsilon \to 0$. On the other hand, the above mentioned scenario changes completely if $m = 1$. In this case the speed of rotation and incompressibility act on the same scale. Accordingly, the limit behaviour of the fluid is described by a single (linear) equation, see Section 2.5. Note that such a picture is in sharp contrast with [9], where the effect of the centrifugal force is neglected.

However, a rigorous justification of the above programme is hampered by serious mathematical difficulties:

- The main issue in the low Mach number limit, at least in the case of the so-called ill prepared initial data, is the presence of rapidly oscillating acoustic waves, cf. Desjardins and Grenier [5], Desjardins et al. [6], Lions and Masmoudi [17]. Similarly to [5], given the geometry of the spatial domain Ω, we may expect a local decay of the acoustic energy as a result of dispersive effects. Unfortunately, however, the fluid is driven by the centrifugal force that becomes large for $|x_h| \to \infty$. Specifically, we have $G \approx \varepsilon^{-2m}$ on the sphere of the radius ε^{-m}, whereas the speed of sound in the fluid is proportional to ε^{-m}. In other words, the centrifugal force changes effectively propagation of acoustic waves and this effect cannot be neglected, not even on compact subsets of the physical domain.
- The dispersive estimates of Strichartz' type exploited in [3] cannot be used in the present setting as the acoustic waves, represented by the gradient component of the velocity field, remain large for $|x_h| \to \infty$.

Our approach is based on combination of dispersive estimates for acoustic waves with the local method developed in [11]. As already mentioned, we focus on two qualitatively different situations: one where $m \gg 1$, and one where $m = 1$.

- THE CASE $m \gg 1$.

  In order to eliminate the effect of the centrifugal force, we compute the exact rate of the local decay of acoustic energy, here proportional to $\varepsilon^{m/2}$, by adapting the argument of Metcalfe [20] (cf. also D'Ancona and Racke [4], Smith and Sogge [21]). Accordingly, the associated acoustic equation can be localized to balls of radii $\varepsilon^{-\alpha}$ for a certain $\alpha > m/2$, on which $G(x) \approx \varepsilon^{-2\alpha}$. This step requires m to be sufficiently large so that the dispersive effects dominate the action of the centrifugal force. Having established local decay of acoustic waves, we use the method developed in [11], based on cancellation of several quantities in the convective term, similar to the local approach of Lions and Masmoudi [18].

- THE CASE $m = 1$.

  In this case both the high rotation and weak compressibility limits occur at the same scale. We can compute immediately the limiting diagnostic equations, and, similarly to the previous case, careful analysis of cancellations in the convective term allows to identifiy the limit system. In contrast with the situation studied in [9], the limit system is linear as a consequence of strong stratification caused by the centrifugal force.

The paper is organized in the following way. In Section 2, we introduce the concept of finite energy weak solutions to the compressible Navier-Stokes system (1.1 - 1.4) and recall their basic properties. In particular, we present uniform bounds on solutions independent of the scaling parameter $\varepsilon \to 0$. The main results concerning the singular limit of solutions to (1.1 - 1.4) are stated in Section 2. The anisotropic situation, when $m \gg 1$, is analyzed in Section 3 by means of several steps: The easy part concerns the formal identification of the limit system, while, in Section 3.2, the propagation of acoustic waves is studied as well as their local decay in Ω. Finally, the limit systems for $m \gg 1$ is justified in Section 3.3 by means of a careful analysis of the convective term. Finally, the isotropic case, when $m = 1$, is examined in Section 4, where the proofs borrow several ingredients introduced in [11].

2 Preliminaries and statement of the main results

In this section we introduce the main hypotheses, recall some known results concerning existence of solutions to the primitive system as well as the nowadays standard uniform bounds independent of the scaling parameter, and, finally, formulate our main results.

2.1 Main assumptions

Consider a family of solutions ϱ_ε, $\mathbf{u}_\varepsilon$ of the Navier-Stokes system (1.1 - 1.4) in $(0,T) \times \Omega$, emanating from the initial data

$$\varrho_\varepsilon(0,\cdot) = \varrho_{0,\varepsilon},\ \mathbf{u}_\varepsilon(0,\cdot) = \mathbf{u}_{0,\varepsilon}.$$

We assume that the initial data are *ill-prepared*, specifically,

$$\varrho_{0,\varepsilon} = \tilde{\varrho}_\varepsilon + \varepsilon^m r_{0,\varepsilon}, \tag{2.1}$$

where $\tilde{\varrho}_\varepsilon$ is a solution of the associated *static problem*:

$$\nabla_x p(\tilde{\varrho}_\varepsilon) = \varepsilon^{2(m-1)} \tilde{\varrho}_\varepsilon \nabla_x G \text{ in } \Omega.$$

Consequently,

$$P(\tilde{\varrho}_\varepsilon) = \varepsilon^{2(m-1)} G + \text{const}, \text{ where } P(\varrho) = \int_1^\varrho \frac{p'(z)}{z}\ \mathrm{d}z.$$

Furthermore, we suppose that

$$p \in C^1[0,\infty) \cap C^2(0,\infty),\ p'(\varrho) > 0 \text{ for all } \varrho > 0,\ \lim_{\varrho \to \infty} \frac{p'(\varrho)}{\varrho^{\gamma-1}} = c > 0 \tag{2.2}$$

for a certain $\gamma > 1$ specified below, and that

$$\begin{gathered} G \in W^{1,\infty}(\Omega),\ G(x) \geq 0,\ G(x_1, x_2, -x_3) = G(x_1, x_2, x_3), \\ |\nabla_x G(x)| \leq c(1 + |x_h|) \text{ for all } x \in \Omega. \end{gathered} \tag{2.3}$$

Finally, we normalize

$$P(\tilde{\varrho}_\varepsilon) = \varepsilon^{2(m-1)} G, \tag{2.4}$$

noticing that

$$\tilde{\varrho}_\varepsilon(x) \geq 1,\ \tilde{\varrho}_\varepsilon(x) \to 1 \text{ for any } x \in \Omega \text{ as } \varepsilon \to 0 \text{ for } m > 1, \tag{2.5}$$

whereas

$$\tilde{\varrho}_\varepsilon \equiv \tilde{\varrho} \text{ is independent of } \varepsilon \text{ provided } m = 1. \tag{2.6}$$

2.2 Energy inequality

Introducing the *relative entropy*

$$E(\varrho,\tilde{\varrho}) := H(\varrho) - H'(\tilde{\varrho})(\varrho-\tilde{\varrho}) - H(\tilde{\varrho}), \quad H(\varrho) := \varrho \int_1^\varrho \frac{p(z)}{z^2} \mathrm{d}z,$$

we note that $H'(\varrho) = P(\varrho) + \mathrm{const}$; therefore we may assume that

$$\int_\Omega \left(\frac{1}{2}\varrho_\varepsilon |\mathbf{u}_\varepsilon|^2 + \frac{1}{\varepsilon^{2m}} E(\varrho_\varepsilon, \tilde{\varrho}_\varepsilon) \right)(\tau,\cdot)\ \mathrm{d}x + \int_0^\tau \int_\Omega \mathbb{S}(\nabla_x \mathbf{u}_\varepsilon) : \nabla_x \mathbf{u}_\varepsilon \ \mathrm{d}x\ \mathrm{d}\tau' \\ \leq \int_\Omega \left(\frac{1}{2}\varrho_{0,\varepsilon} |\mathbf{u}_{0,\varepsilon}|^2 + \frac{1}{\varepsilon^{2m}} E(\varrho_{0,\varepsilon}, \tilde{\varrho}_\varepsilon) \right)\ \mathrm{d}x \tag{2.7}$$

including implicitly the *mass compatibility condition*

$$\int_\Omega (\varrho_\varepsilon - \tilde{\varrho}_\varepsilon)\ \mathrm{d}x = 0.$$

Now, in order to establish uniform bounds independent of the scaling parameter ε, the initial data specified in (2.1) must be chosen in such a way that the expression on the right-hand side of (2.7) remains bounded uniformly for $\varepsilon \to 0$. Thus, if $\gamma \leq 2$ in (2.2), it is enough to assume that

$$\{r_{0,\varepsilon}\}_{\varepsilon>0} \text{ bounded in } L^2 \cap L^\infty(\Omega),\ \{\sqrt{\tilde{\varrho}_\varepsilon}\mathbf{u}_{0,\varepsilon}\}_{\varepsilon>0} \text{ bounded in } L^2(\Omega; R^3).$$

In general, we suppose that

$$\left\{ \tilde{\varrho}_\varepsilon^{\frac{\gamma-2}{2}} r_{0,\varepsilon} \right\}_{\varepsilon>0} \text{ bounded in } L^2(\Omega),\ \{r_{0,\varepsilon}\}_{\varepsilon>0} \text{ bounded in } L^2\cap L^\infty(\Omega), \\ \{\sqrt{\tilde{\varrho}_\varepsilon}\mathbf{u}_{0,\varepsilon}\}_{\varepsilon>0} \text{ bounded in } L^2(\Omega; R^3). \tag{2.8}$$

2.3 Finite energy weak solutions

We say that ϱ, $\mathbf{u}$ is a finite energy weak solution of the Navier-Stokes system (1.1 - 1.3) in $(0,T)\times\Omega$, supplemented with the initial data (2.1), if:

- the energy inequality

$$\int_\Omega \left(\frac{1}{2}\varrho |\mathbf{u}|^2 + \frac{1}{\varepsilon^{2m}} E(\varrho, \tilde{\varrho}_\varepsilon) \right)(\tau,\cdot)\ \mathrm{d}x + \int_0^\tau \int_\Omega \mathbb{S}(\nabla_x \mathbf{u}) : \nabla_x \mathbf{u} \ \mathrm{d}x\ \mathrm{d}\tau' \\ \leq \int_\Omega \left(\frac{1}{2}\varrho_{0,\varepsilon} |\mathbf{u}_{0,\varepsilon}|^2 + \frac{1}{\varepsilon^{2m}} E(\varrho_{0,\varepsilon}, \tilde{\varrho}_\varepsilon) \right)\ \mathrm{d}x$$

  holds for a.a. $\tau \in (0,T)$;

- equation (1.1) is satisfied in the sense of distributions, specifically,

$$\int_0^T \int_\Omega \Big(\varrho \partial_t \varphi + \varrho \mathbf{u} \cdot \nabla_x \varphi \Big)\ \mathrm{d}x\ \mathrm{d}t = -\int_\Omega \varrho_{0,\varepsilon} \varphi(0,\cdot)\ \mathrm{d}x$$

  for any $\varphi \in C^\infty_c([0,T)\times\Omega)$;

- the pressure p is locally integrable in $[0,T)\times\Omega$, equation (1.2) holds in the sense of distributions:

$$\int_0^T\int_\Omega\Big(\varrho\mathbf{u}\cdot\partial_t\varphi+\varrho\mathbf{u}\otimes\mathbf{u}:\nabla_x\varphi-\frac{1}{\varepsilon}\mathbf{b}\times(\varrho\mathbf{u})\cdot\varphi+\frac{1}{\varepsilon^{2m}}p(\varrho)\mathrm{div}_x\varphi\Big)\ \mathrm{d}x\ \mathrm{d}t \\ =\int_0^T\int_\Omega\Big(\mathbb{S}(\nabla_x\mathbf{u}):\nabla_x\varphi-\frac{1}{\varepsilon^2}\varrho\nabla_x G\cdot\varphi\Big)\ \mathrm{d}x\ \mathrm{d}t-\int_\Omega\varrho_{0,\varepsilon}\mathbf{u}_{0,\varepsilon}\cdot\varphi(0,\cdot)\ \mathrm{d}x \tag{2.9}$$

  for any $\varphi\in C_c^\infty([0,T)\times\Omega;R^3)$.

Under hypothesis (2.2), the *existence* of finite energy weak solutions can be established by the method developed by Lions [16], with the necessary modifications specified in [10] in order to handle the physically relevant range of adiabatic exponents $\gamma>3/2$.

2.4 Uniform bounds

In order to study the asymptotic behavior of solutions, we first establish *uniform bounds* independent of the scaling parameter $\varepsilon\to 0$. As a matter of fact, all of them follow from the energy inequality (2.7). Similarly to [8, Chapter 5], we introduce the *essential* and *residual* component of a function h as

$$h=h_{\rm ess}+h_{\rm res},$$

where

$$h_{\rm ess}(t,x)=h(t,x)\text{ for }(t,x)\text{ such that }\varrho_\varepsilon(t,x)\in(1/2,2),\ \ h_{\rm ess}(t,x)=0\text{ otherwise.}$$

Now, by virtue of (2.4),

$$1\le\tilde\varrho_\varepsilon(x)\le 1+c(r)\varepsilon^{2(m-1-\alpha)}\text{ for all }x\in B_{r/\varepsilon^\alpha},\ 0\le\alpha\le m-1, \tag{2.10}$$

where we have denoted

$$B_R=\{x\in\Omega\ |\ |x_h|\le R\}.$$

It follows directly from energy inequality (2.7) that when $m>1+\alpha$

$$\mathop{\rm ess\ sup}_{t\in(0,T)}\left\|\left[\frac{\varrho_\varepsilon-\tilde\varrho_\varepsilon}{\varepsilon^m}\right]_{\rm ess}\right\|_{L^2(B_{r/\varepsilon^\alpha})}\le c(r) \tag{2.11}$$

$$\mathop{\rm ess\ sup}_{t\in(0,T)}\int_{B_{r/\varepsilon^\alpha}}[\varrho_\varepsilon]^\gamma_{\rm res}\ \mathrm{d}x\le\varepsilon^{2m}c(r) \tag{2.12}$$

$$\mathop{\rm ess\ sup}_{t\in(0,T)}\int_{B_{r/\varepsilon^\alpha}}1_{\rm res}\ \mathrm{d}x\le\varepsilon^{2m}c(r) \tag{2.13}$$

$$\mathop{\rm ess\ sup}_{t\in(0,T)}|\{x\in\Omega\ |\ \varrho_\varepsilon(t,x)\le 1/2\}|\le c\varepsilon^{2m}\le c \tag{2.14}$$

$$\mathop{\rm ess\ sup}_{t\in(0,T)}\|\sqrt{\varrho_\varepsilon}\mathbf{u}_\varepsilon\|_{L^2(\Omega;R^3)}\le c \tag{2.15}$$

and

$$\int_0^T\int_\Omega\mathbb{S}(\nabla_x\mathbf{u}_\varepsilon):\nabla_x\mathbf{u}_\varepsilon\ \mathrm{d}x\ \mathrm{d}t\le c. \tag{2.16}$$

In the case when $m = 1$, the bounds (2.11)-(2.13) should be replaced by

$$\text{ess}\sup_{t\in(0,T)} \left\| \left[\frac{\varrho_\varepsilon - \tilde{\varrho}_\varepsilon}{\varepsilon^m} \right]_{\text{ess}} \right\|_{L^2(B_r)} \leq c(r) \tag{2.17}$$

$$\text{ess}\sup_{t\in(0,T)} \int_{B_r} [\varrho_\varepsilon]^\gamma_{\text{res}} \, \mathrm{d}x \leq \varepsilon^{2m} c(r) \tag{2.18}$$

$$\text{ess}\sup_{t\in(0,T)} \int_{B_r} 1_{\text{res}} \, \mathrm{d}x \leq \varepsilon^{2m} c(r). \tag{2.19}$$

Finally, combining (2.14 - 2.16) with a variant of Korn's inequality (see [8, Theorem 10.17]), we obtain

$$\int_0^T \|\mathbf{u}_\varepsilon\|^2_{W^{1,2}(\Omega;R^3)} \, \mathrm{d}t \leq c. \tag{2.20}$$

All generic constants in the previous estimates are independent of the scaling parameter ε.

2.5 Main results

In what follows, the symbol $\mathbf{H}$ denotes the standard Helmholtz projection onto the space of solenoidal functions in Ω, specifically,

$$\widehat{\mathbf{H}[\mathbf{v}]}(\xi,k) = \mathbf{v}(\xi,k) - \frac{1}{|\xi|^2 + k^2} \Big[\xi \Big(\xi \cdot \widehat{\mathbf{v}}_h(\xi,k) + k\widehat{v}_3(\xi,k) \Big), k \Big(\xi \cdot \widehat{\mathbf{v}}_h(\xi,k) + k\widehat{v}_3(\xi,k) \Big) \Big],$$

$$\mathbf{H}^\perp[\mathbf{v}] = \mathbf{v} - \mathbf{H}[\mathbf{v}],$$

where the symbol $\widehat{\mathbf{v}}(\xi,k)$, $\xi \in R^2, k \in Z$, denotes the Fourier transform of $\mathbf{v} = \mathbf{v}(x_h, x_3)$. Similarly, the Laplace operator Δ is identified through

$$\widehat{\Delta v} \approx -(|\xi|^2 + k^2) \, \widehat{v}(\xi,k).$$

Finally, we introduce the vertical average of a function v as

$$\langle v \rangle (x_h) = \frac{1}{|\mathcal{T}^1|} \int_{\mathcal{T}^1} v(x_h, x_3) \, \mathrm{d}x_3. \tag{2.21}$$

2.5.1 Multiscale limit ($m \gg 1$)

Theorem 1 *Let the pressure p and the potential of the driving force G satisfy hypotheses (2.2), (2.3), with $\gamma > 3/2$. Let ϱ_ε, $\mathbf{u}_\varepsilon$ be a finite energy weak solution of the Navier-Stokes system in $(0,T) \times \Omega$ belonging to the symmetry class (1.7), emanating from the initial data (2.1), (2.8). In addition, suppose that*

$$m > 10$$

and that

$$\mathbf{u}_{0,\varepsilon} \to \mathbf{U}_0 \ \textit{weakly in} \ L^2(\Omega; R^3).$$

Then

$$\text{ess}\sup_{t\in(0,T)} \|\varrho_\varepsilon - 1\|_{(L^2+L^\gamma)(K)} \leq \varepsilon^m c(K) \ \textit{for any compact} \ K \subset \Omega,$$

$$\mathbf{u}_\varepsilon \to \mathbf{U} \ \textit{weakly in } L^2(0,T;W^{1,2}(\Omega;R^3)),$$

where $\mathbf{U} = [\mathbf{U}_h(x_h), 0]$ *is the unique solution to the 2D incompressible Navier-Stokes system*

$$\mathrm{div}_h \mathbf{U}_h = 0, \tag{2.22}$$

$$\partial_t \mathbf{U}_h + \mathrm{div}_h(\mathbf{U}_h \otimes \mathbf{U}_h) + \nabla_h \Pi = \mu \Delta_h \mathbf{U}_h, \tag{2.23}$$

with the initial data

$$\mathbf{U}_h(0,\cdot) = \Big[\mathbf{H}\Big[[\langle \mathbf{U}_0 \rangle_h, 0]\Big]\Big]_h$$

Remark 2.5.1 *A short inspection of the proof of Theorem 1 given in Section 3 below reveals that replacing the complete slip condition (1.4) by the Navier's slip condition (1.5) would produce an extra term* $\beta \mathbf{U}_h$ *on the left-hand side of (2.23) known as Ekman's pumping.*

2.5.2 Stratified limit ($m = 1$)

Theorem 2 *Let the pressure* p *satisfy hypotheses (2.2), with* $\gamma > 3$ *and let* $G(x_h) = |x_h|^2$. *Let* ϱ_ε, $\mathbf{u}_\varepsilon$ *be a finite energy weak solution of the Navier-Stokes system in* $(0,T) \times \Omega$ *belonging to the symmetry class (1.7), emanating from the initial data (2.1), (2.8), where*

$$m = 1.$$

In addition, suppose that

$$r_{0,\varepsilon} \to r_0 \ \textit{weakly in } L^2(\Omega), \ \mathbf{u}_{0,\varepsilon} \to \mathbf{U}_0 \ \textit{weakly in } L^2(\Omega;R^3).$$

Then

$$r_\varepsilon \equiv \frac{\varrho_\varepsilon - \tilde{\varrho}}{\varepsilon} \to r \ \textit{weakly-(*) in } L^\infty(0,T;L^2(K)) \textit{ for any compact } K \subset \Omega,$$

$$\mathbf{u}_\varepsilon \to \mathbf{U} \ \textit{weakly in } L^2(0,T;W^{1,2}(\Omega;R^3)),$$

with

$$r = r(t,x_h) \textit{ radially symmetric }, \quad \mathbf{U} = [\mathbf{U}_h(t,x_h), 0], \tag{2.24}$$

and

$$\nabla_h(P'(\tilde{\varrho})r) + \mathbf{U}_h^\perp = 0. \tag{2.25}$$

Moreover, the function r *satisfies*

$$\partial_t\Big(r - \mathrm{div}_h(\tilde{\varrho}\nabla_h(P'(\tilde{\varrho})r))\Big) + \Delta_h^2\Big(P'(\tilde{\varrho})r\Big) = 0. \tag{2.26}$$

In addition, the initial value $r(0)$ *is the unique radially symmetric function satisfying the integral identity*

$$\int_{R^2}\Big(\tilde{\varrho}\nabla_h\left(P'(\tilde{\varrho})r(0)\right)\cdot\nabla_h\psi \ + \ r(0)\,\psi\Big)\mathrm{d}x_h \ = \ \int_{R^2}\Big(\langle\tilde{\varrho}\mathbf{U}_{0,h}\rangle\cdot\nabla_h^\perp\psi + \langle r_0\rangle\,\psi\Big)\mathrm{d}x_h \tag{2.27}$$

for all radially symmetric $\psi = \psi(x_h) \in C_c^\infty(R^2)$.

Remark 2.5.2 *The initial condition* (2.27) *can be interpreted in polar coordinates as*

$$r(0,s) - \frac{1}{s}\partial_s\Big(s\tilde{\varrho}\partial_s\Big(P'(\tilde{\varrho})r(0,s)\Big)\Big) = \frac{1}{2\pi s}\int_{|x_h|=s}\Big(\mathrm{curl}_h\,\langle\tilde{\varrho}\mathbf{U}_{0,h}\rangle + \langle r_0\rangle\Big)\mathrm{dS}_{x_h}.$$

where $s = |x_h|$*, provided all quantities are sufficiently smooth.*

Remark 2.5.3 *Note that (2.25) implies that* $\nabla_h\tilde{\varrho}\cdot\mathbf{U}_h = 0$*, or, equivalently,*

$$\mathbf{U}_h\cdot x_h = 0,\ \textit{meaning the limit velocity } \mathbf{U}_h \textit{ is tangent to the level sets } \big\{|x_h| = \mathrm{const}\big\},$$

Moreover it follows from (2.24-2.25) that $|\mathbf{U}_h|$ *is constant on* $\big\{|x_h| = \mathrm{const}\big\}$*, and* $\mathrm{div}_h\mathbf{U}_h = 0$.

We have chosen $G(x_h) = |x_h|^2$ for simplicity (and because it corresponds to the physical situation of the centrifugal force). However more general functions of x_h could be treated the same way. It is remarkable that the limit equation (2.26) is linear for $m = 1$, in sharp contrast with the homogeneous case $\tilde{\varrho} = \mathrm{const}$ treated in [9]. This is related to the fact that the limit density $\tilde{\varrho}$ is stratified (non-constant). More precisely, the absence of nonlinearity is related to the smallness of the kernel of the penalized operator, defined by (2.25). This phenomenon had already been identified for fluids with variable rotation axis, see [11]. Much of the analysis that we shall follow to prove Theorem 2 borrows to this last reference.

The rest of the paper is devoted to the proof of Theorems 1 and 2, in Sections 3 and 4 respectively.

3 Anisotropic scaling: Proof of Theorem 1

3.1 Preliminary remarks

We start with some simple observations that follow directly from the uniform bounds (2.11 - 2.20). Clearly, relations (2.10), (2.11), and (2.12) imply that

$$\mathrm{ess}\sup_{t\in(0,T)}\|\varrho_\varepsilon - 1\|_{(L^2+L^\gamma)(K)} \le \varepsilon^m c(K) \text{ for any compact } K\subset\Omega,$$

while (2.20) yields immediately

$$\mathbf{u}_\varepsilon \to \mathbf{U} \text{ weakly in } L^2(0,T;W^{1,2}(\Omega;R^3)),\ \text{at least for a suitable subsequence.}$$

Consequently, letting $\varepsilon\to 0$ in (1.1) yields

$$\mathrm{div}_x\mathbf{U} = 0 \text{ a.a. in } (0,T)\times\Omega. \tag{3.1}$$

Moreover, it follows from equation (1.2) that

$$\mathbf{H}[\mathbf{b}\times\mathbf{U}] = 0,\ \text{meaning, } \mathbf{b}\times\mathbf{U} = \nabla_x\Phi \text{ for a certain potential } \Phi.$$

Consequently, Φ and $\mathbf{U}_h$ are independent of the vertical coordinate x_3, and moreover, $\mathrm{div}_h\mathbf{U}_h = 0$. Since $\mathbf{U}$ is solenoidal, we have $\partial_{x_3}U^3 = 0$; whence, as U^3 has zero vertical mean,

$$U^3 = 0,\ \mathbf{U} = [\mathbf{U}_h(t,x_h),0].$$

Finally, in view of the uniform bounds established in (2.11), (2.20), it is easy to pass to the limit in all terms appearing in the weak formulation of momentum equation (2.9), tested on $\varphi = [\phi(t,x_h),0]$, with $\mathrm{div}_h\phi = 0$, with the exception of the convective term $\mathrm{div}_x(\varrho_\varepsilon\mathbf{u}_\varepsilon\otimes\mathbf{u}_\varepsilon)$ that will be analyzed in the next two sections.

3.2 Acoustic waves

Since $m > 10$, there exists α such that

$$1 + \frac{m}{2} < \alpha < \frac{3}{4}(m-2)\cdot \tag{3.2}$$

This choice of the parameter α will become clear in Section 3.3. Moreover, we introduce a family of cut-off functions χ_ε such that

$$\begin{aligned} &\chi_\varepsilon \in C_c^\infty(R^2),\ 0 \le \chi_\varepsilon \le 1,\ \chi_\varepsilon(x_h) = 1 \text{ for } |x_h| \le \frac{1}{\varepsilon^\alpha}\ , \\ &\chi_\varepsilon(x_h) = 0 \text{ for } |x_h| \ge \tfrac{2}{\varepsilon^\alpha},\ |\nabla_x \chi_\varepsilon(x_h)| \le 2\varepsilon^\alpha \text{ for } x_h \in R^2. \end{aligned} \tag{3.3}$$

As the density becomes constant in the asymptotic limit, the basic idea is to "replace" $\mathbf{u}_\varepsilon \approx \chi_\varepsilon \varrho_\varepsilon \mathbf{u}_\varepsilon$ and write

$$\chi_\varepsilon \varrho_\varepsilon \mathbf{u}_\varepsilon = \mathbf{H}[\chi_\varepsilon \varrho_\varepsilon \mathbf{u}_\varepsilon] + \mathbf{H}^\perp[\chi_\varepsilon \varrho_\varepsilon \mathbf{u}_\varepsilon] = \langle \mathbf{H}[\chi_\varepsilon \varrho_\varepsilon \mathbf{u}_\varepsilon] \rangle + \Big(\mathbf{H}[\chi_\varepsilon \varrho_\varepsilon \mathbf{u}_\varepsilon] - \langle \mathbf{H}[\chi_\varepsilon \varrho_\varepsilon \mathbf{u}_\varepsilon] \rangle \Big) + \nabla_x \Psi_\varepsilon,$$

where $\nabla_x \Psi_\varepsilon = \mathbf{H}^\perp[\chi_\varepsilon \varrho_\varepsilon \mathbf{u}_\varepsilon]$.

In the remaining part of this section, we examine the asymptotic behavior of the *acoustic potential* Ψ_ε. More specifically, we show that $\nabla_x \Psi_\varepsilon$ tends to zero on compact subsets of Ω and therefore becomes negligible in the asymptotic limit $\varepsilon \to 0$.

3.2.1 Acoustic equation

Following Lighthill [14], [15], we rewrite the Navier-Stokes system (1.1), (1.2) in the form:

$$\begin{aligned} &\varepsilon^m \partial_t \left(\frac{\varrho_\varepsilon - \tilde\varrho_\varepsilon}{\varepsilon^m} \right) + \mathrm{div}_x (\varrho_\varepsilon \mathbf{u}_\varepsilon) = 0, \\ &\varepsilon^m \partial_t (\varrho_\varepsilon \mathbf{u}_\varepsilon) + p'(1) \nabla_x \left(\frac{\varrho_\varepsilon - \tilde\varrho_\varepsilon}{\varepsilon^m} \right) = -\frac{1}{\varepsilon^m} \nabla_x \Big(p(\varrho_\varepsilon) - p(\tilde\varrho_\varepsilon) - p'(1)(\varrho_\varepsilon - \tilde\varrho_\varepsilon) \Big) \\ &\qquad + \varepsilon^m \mathrm{div}_x \Big(\mathbb{S}(\nabla_x \mathbf{u}_\varepsilon) - (\varrho_\varepsilon \mathbf{u}_\varepsilon \times \mathbf{u}_\varepsilon) \Big) - \varepsilon^{m-1} \left(\mathbf{b} \times \varrho_\varepsilon \mathbf{u}_\varepsilon \right) + \varepsilon^{2(m-1)} \frac{\varrho_\varepsilon - \tilde\varrho_\varepsilon}{\varepsilon^m} \nabla_x G \end{aligned}$$

where, exactly as in Section 2.3, equations are understood in the sense of distributions in $(0,T) \times \Omega$. Furthermore, introducing new variables

$$S_\varepsilon = \chi_\varepsilon \frac{\varrho_\varepsilon - \tilde\varrho_\varepsilon}{\varepsilon^m},\ \mathbf{m}_\varepsilon = \chi_\varepsilon \varrho_\varepsilon \mathbf{u}_\varepsilon,$$

where χ_ε is the cut-off function specified in (3.3), we arrive at the equation

$$\varepsilon^m \partial_t S_\varepsilon + \mathrm{div}_x \mathbf{m}_\varepsilon = \nabla_x \chi_\varepsilon \cdot (\varrho_\varepsilon \mathbf{u}_\varepsilon), \tag{3.4}$$

while $\mathbf{m}_\varepsilon$ satisfies

$$\begin{aligned} &\varepsilon^m \partial_t \mathbf{m}_\varepsilon + p'(1) \nabla_x S_\varepsilon = -\varepsilon^{m-1} \left(\mathbf{b} \times \mathbf{m}_\varepsilon \right) + \varepsilon^{2(m-1)} \chi_\varepsilon \frac{\varrho_\varepsilon - \tilde\varrho_\varepsilon}{\varepsilon^m} \nabla_x G + p'(1) \nabla_x \chi_\varepsilon \frac{\varrho_\varepsilon - \tilde\varrho_\varepsilon}{\varepsilon^m} \\ &\quad + \varepsilon^m \mathrm{div}_x \Big(\chi_\varepsilon \mathbb{S}(\nabla_x \mathbf{u}_\varepsilon) - \chi_\varepsilon (\varrho_\varepsilon \mathbf{u}_\varepsilon \times \mathbf{u}_\varepsilon) \Big) - \varepsilon^m \Big(\mathbb{S}(\nabla_x \mathbf{u}_\varepsilon) - \varrho_\varepsilon \mathbf{u}_\varepsilon \times \mathbf{u}_\varepsilon \Big) \cdot \nabla_x \chi_\varepsilon \\ &\quad - \frac{1}{\varepsilon^m} \nabla_x \Big[\chi_\varepsilon \Big(p(\varrho_\varepsilon) - p(\tilde\varrho_\varepsilon) - p'(1)(\varrho_\varepsilon - \tilde\varrho_\varepsilon) \Big) \Big] + \frac{1}{\varepsilon^m} \Big(p(\varrho_\varepsilon) - p(\tilde\varrho_\varepsilon) - p'(1)(\varrho_\varepsilon - \tilde\varrho_\varepsilon) \Big) \nabla_x \chi_\varepsilon. \end{aligned} \tag{3.5}$$

3.2.2 Uniform bounds

Our next goal is to deduce uniform bounds on all quantities appearing in the acoustic equation (3.4), (3.5). To begin, it follows from (2.11 - 2.13) that

$$\{S_\varepsilon\}_{\varepsilon>0} \text{ is bounded in } L^\infty(0,T;L^2+L^1(\Omega)). \tag{3.6}$$

As a matter of fact, we have

$$\{S_\varepsilon\}_{\varepsilon>0} \text{ is bounded in } L^\infty(0,T;(L^2+L^1\cap L^\gamma)(\Omega))$$

as the "residual set" is of small measure, cf. (2.13).

Similarly, we deduce from (2.12), (2.15) that

$$\{\mathbf{m}_\varepsilon\}_{\varepsilon>0} \text{ is bounded in } L^\infty(0,T;L^2+L^q(\Omega;R^3)),\ q=\frac{2\gamma}{\gamma+1}\cdot \tag{3.7}$$

Moreover, combining (3.3), hypothesis (2.3) and (3.6) we may infer that

$$\left\{\varepsilon^\alpha \chi_\varepsilon \frac{\varrho_\varepsilon-\tilde\varrho_\varepsilon}{\varepsilon^m}\nabla_x G\right\}_{\varepsilon>0} \text{ is bounded in } L^\infty(0,T;L^2+L^1(\Omega)). \tag{3.8}$$

Finally, by virtue of (2.15), (2.20),

$$\{\mathbb{S}(\nabla_x\mathbf{u}_\varepsilon)\}_{\varepsilon>0} \text{ is bounded in } L^2(0,T;L^2(\Omega;R^{3\times 3})), \tag{3.9}$$

and

$$\{\varrho_\varepsilon\mathbf{u}_\varepsilon\otimes\mathbf{u}_\varepsilon\}_{\varepsilon>0} \text{ is bounded in } L^\infty(0,T;L^1(\Omega;R^{3\times 3})). \tag{3.10}$$

Now let us estimate the pressure perturbation. Writing

$$p(\varrho_\varepsilon)-p(\tilde\varrho_\varepsilon)-p'(1)(\varrho_\varepsilon-\tilde\varrho_\varepsilon)=p(\varrho_\varepsilon)-p(\tilde\varrho_\varepsilon)-p'(\tilde\varrho_\varepsilon)(\varrho_\varepsilon-\tilde\varrho_\varepsilon)+\Big(p'(\tilde\varrho_\varepsilon)-p'(1)\Big)(\varrho_\varepsilon-\tilde\varrho_\varepsilon)$$

we have

$$p(\varrho_\varepsilon)-p(\tilde\varrho_\varepsilon)-p'(\tilde\varrho_\varepsilon)(\varrho_\varepsilon-\tilde\varrho_\varepsilon)=\Big[p(\varrho_\varepsilon)-p(\tilde\varrho_\varepsilon)-p'(\tilde\varrho_\varepsilon)(\varrho_\varepsilon-\tilde\varrho_\varepsilon)\Big]_{\rm ess}+\Big[p(\varrho_\varepsilon)-p(\tilde\varrho_\varepsilon)-p'(\tilde\varrho_\varepsilon)(\varrho_\varepsilon-\tilde\varrho_\varepsilon)\Big]_{\rm res}$$

where, since p is twice continuously differentiable in $(0,\infty)$ and $\tilde\varrho_\varepsilon$ satisfies (2.10),

$$\left|\Big[p(\varrho_\varepsilon)-p(\tilde\varrho_\varepsilon)-p'(\tilde\varrho_\varepsilon)(\varrho_\varepsilon-\tilde\varrho_\varepsilon)\Big]_{\rm ess}(t,x)\right|\le c\,[\varrho_\varepsilon-\tilde\varrho_\varepsilon]^2_{\rm ess}(t,x) \text{ provided } x\in B_{2/\varepsilon^\alpha}.$$

Similarly, by virtue of (2.12), (2.13)

$$\operatorname*{ess\ sup}_{t\in(0,T)}\int_{B_{2/\varepsilon^\alpha}}\left|\Big[p(\varrho_\varepsilon)-p(\tilde\varrho_\varepsilon)-p'(\tilde\varrho_\varepsilon)(\varrho_\varepsilon-\tilde\varrho_\varepsilon)\Big]_{\rm res}\right|\ {\rm d}x\le c\varepsilon^{2m}.$$

Finally, in accordance with (2.10),

$$|p'(\tilde\varrho_\varepsilon(x))-p'(1)|\le c|\tilde\varrho_\varepsilon(x)-1|\le c\varepsilon^{2(m-1-\alpha)} \text{ for } x\in B_{2/\varepsilon^\alpha}.$$

Thus, summing up the previous estimates, we conclude that

$$\operatorname*{ess\ sup}_{t\in(0,T)}\left\|\frac{1}{\varepsilon^{2m}}\Big(p(\varrho_\varepsilon)-p(\tilde\varrho_\varepsilon)-p'(1)(\varrho_\varepsilon-\tilde\varrho_\varepsilon)\Big)\right\|_{(L^1+L^2)(B_{2/\varepsilon^\alpha};R^3)}\le c(1+\varepsilon^{m-2-2\alpha}). \tag{3.11}$$

3.2.3 Spatial regularization

In view of the uniform bounds obtained in the previous section, equations (3.4), (3.5) can be written in the form

$$\varepsilon^m \partial_t S_\varepsilon + \mathrm{div}_x \mathbf{m}_\varepsilon = \varepsilon^\alpha F^1_\varepsilon, \tag{3.12}$$

$$\varepsilon^m \partial_t \mathbf{m}_\varepsilon + p'(1)\nabla_x S_\varepsilon = (\varepsilon^m + \varepsilon^{2(m-1-\alpha)})\mathrm{div}_x \mathbb{F}^2_\varepsilon + (\varepsilon^{m-1} + \varepsilon^\alpha + \varepsilon^{2(m-1-\alpha)})\mathbf{F}^3_\varepsilon, \tag{3.13}$$

with

$$\left\{ \begin{array}{c} \{F^1_\varepsilon\}_{\varepsilon>0} \text{ bounded in } L^\infty(0,T; L^2 + L^1(\Omega)) \\ \\ \{\mathbb{F}^2_\varepsilon\}_{\varepsilon>0} \text{ bounded in } L^2(0,T; L^2 + L^1(\Omega; R^{3\times 3})) \\ \\ \{\mathbf{F}^3_\varepsilon\}_{\varepsilon>0} \text{ bounded in } L^2(0,T; L^2 + L^1(\Omega; R^3)). \end{array} \right\}$$

Equations (3.12), (3.13) are satisfied in the sense of distributions. For future analysis, however, it is more convenient to deal with classical (smooth) solutions. To this end, we introduce

$$v_\delta = \kappa_\delta * v,$$

where $\kappa_\delta = \kappa_\delta(x)$ is a family of regularizing kernels acting in the spatial variable. Accordingly, we may regularize (3.12), (3.13) to obtain

$$\varepsilon^m \partial_t S_{\varepsilon,\delta} + \mathrm{div}_x \mathbf{m}_{\varepsilon,\delta} = \varepsilon^\alpha F^1_{\varepsilon,\delta} \tag{3.14}$$

$$\varepsilon^m \partial_t \mathbf{m}_{\varepsilon,\delta} + p'(1)\nabla_x S_{\varepsilon,\delta} = (\varepsilon^m + \varepsilon^{2(m-1-\alpha)})\mathrm{div}_x \mathbb{F}^2_{\varepsilon,\delta} + (\varepsilon^{m-1} + \varepsilon^\alpha + \varepsilon^{2(m-1-\alpha)})\mathbf{F}^3_{\varepsilon,\delta} \tag{3.15}$$

where

$$\left\{ \begin{array}{c} \|F^1_{\varepsilon,\delta}\|_{L^2(0,T;W^{k,2}(\Omega))} \le c(k,\delta), \\ \\ \|\mathbb{F}^2_{\varepsilon,\delta}\|_{L^2(0,T;W^{k,2}(\Omega;R^{3\times 3}))} \le c(k,\delta), \\ \\ \|\mathbf{F}^3_{\varepsilon,\delta}\|_{L^2(0,T;W^{k,2}(\Omega;R^3)} \le c(k,\delta) \end{array} \right\} \tag{3.16}$$

for any $k = 0, 1, \dots$ uniformly for $\varepsilon \to 0$.

3.2.4 New formulation of the regularized acoustic wave equation

Our aim is to rewrite the regularized acoustic equation (3.14), (3.15) in terms of the acoustic potential. To this end, we decompose

$$\mathbf{m}_{\varepsilon,\delta} = \mathbf{Y}_{\varepsilon,\delta} + \nabla_x \Psi_{\varepsilon,\delta},$$

where $\mathbf{Y}_{\varepsilon,\delta} = \mathbf{H}[\mathbf{m}_{\varepsilon,\delta}]$.

Introducing new unknowns $S_{\varepsilon,\delta}$, $\Psi_{\varepsilon,\delta}$, we may rewrite the acoustic equation (3.14), (3.15) in the form:

$$\varepsilon^m \partial_t S_{\varepsilon,\delta} + \Delta \Psi_{\varepsilon,\delta} = \varepsilon^\alpha F^1_{\varepsilon,\delta}, \tag{3.17}$$

$$\varepsilon^m \partial_t \Psi_{\varepsilon,\delta} + p'(1) S_{\varepsilon,\delta} = (\varepsilon^m + \varepsilon^{2(m-1-\alpha)})\Delta^{-1}\mathrm{div}_x \mathrm{div}_x[\mathbb{F}^2_{\varepsilon,\delta}] + (\varepsilon^{m-1} + \varepsilon^\alpha + \varepsilon^{2(m-1-\alpha)})\Delta^{-1}\mathrm{div}_x \mathbf{F}^3_{\varepsilon,\delta}. \tag{3.18}$$

Our aim is to use dispersive estimates for (3.17), (3.18) to deduce local decay of the acoustic potential. To this end, we present a simple result, the proof of which is a straightforward adaptation of Metcalfe [20, Lemma 4.1] (cf. also D'Ancona and Racke [4, Example 1.2]) :

Lemma 3.1 *Consider $\varphi \in C_c^\infty(R^2)$. Then*

$$\int_{-\infty}^{\infty}\int_\Omega \left|\varphi(x_h)\exp\left(\mathrm{i}\sqrt{-\Delta}t\right)[v]\right|^2 \ \mathrm{d}x\ \mathrm{d}t \leq c(\varphi)\|v\|^2_{L^2(\Omega)}. \tag{3.19}$$

Proof: For a function $w = w(t, x_h, x_3)$, we denote by $\widehat{w}(\tau, \xi, k)$ its Fourier transform in the *space-time* variables, $\tau \in R$, $\xi \in R^2$, $k \in Z$. Accordingly, by virtue of Parseval's identity,

$$\int_{-\infty}^{\infty}\int_\Omega \left|\varphi(x_h)\exp\left(\mathrm{i}\sqrt{-\Delta}t\right)[v]\right|^2 \ \mathrm{d}x\ \mathrm{d}t$$

$$= c\sum_{k\in Z}\int_{-\infty}^{\infty}\int_{R^2}\left|\int_{R^2}\widehat{\varphi}(\xi-\eta)\delta(\tau-\sqrt{|\eta|^2+k^2})\widehat{v}(\eta,k)\ \mathrm{d}\eta\right|^2 \mathrm{d}\xi\ \mathrm{d}\tau$$

$$= c\sum_{k\in Z}\int_{-\infty}^{\infty}\int_{R^2}\left|\int_{\{\tau=\sqrt{|\eta|^2+k^2}\}}\widehat{\varphi}(\xi-\eta)\widehat{v}(\eta,k)\ \mathrm{d}S_\eta\right|^2 \mathrm{d}\xi\ \mathrm{d}\tau.$$

Furthermore, by the Cauchy-Schwartz inequality,

$$\sum_{k\in Z}\int_{-\infty}^{\infty}\int_{R^2}\left|\int_{\{\tau=\sqrt{|\eta|^2+k^2}\}}\widehat{\varphi}(\xi-\eta)\widehat{v}(\eta,k)\ \mathrm{d}S_\eta\right|^2 \mathrm{d}\xi\ \mathrm{d}\tau$$

$$\leq \sum_{k\in Z}\int_{-\infty}^{\infty}\int_{R^2}\left(\int_{\{\tau=\sqrt{|\eta|^2+k^2}\}}|\widehat{\varphi}(\xi-\eta)|\ \mathrm{d}S_\eta\right)\left(\int_{\{\tau=\sqrt{|\eta|^2+k^2}\}}|\widehat{\varphi}(\xi-\eta)||\widehat{v}(\eta,k)|^2\ \mathrm{d}S_\eta\right)\mathrm{d}\xi\ \mathrm{d}\tau$$

$$\leq c(\varphi)\sum_{k\in Z}\int_{R^2}\int_{-\infty}^{\infty}\int_{\{\tau=\sqrt{|\eta|^2+k^2}\}}|\widehat{\varphi}(\xi-\eta)||\widehat{v}(\eta,k)|^2\ \mathrm{d}S_\eta\ \mathrm{d}\tau\ \mathrm{d}\xi$$

$$\leq c(\varphi)\sum_{k\in Z}\int_{R^2}\int_{R^2}|\widehat{\varphi}(\xi-\eta)||\widehat{v}(\eta,k)|^2\ \mathrm{d}\eta\ \mathrm{d}\xi \leq c(\varphi)\|v\|^2_{L^2(\Omega)}.$$

That proves Lemma 3.1.

Q.E.D.

The gradient of the acoustic potential $\Psi_{\varepsilon,\delta}$ can be written by means of Duhamel's formula:

$$\nabla_x\Psi_{\varepsilon,\delta} = \frac{1}{2}\exp\left(\mathrm{i}\sqrt{-\Delta}\frac{t}{\varepsilon^m}\right)\left[\nabla_x\Psi_{0,\varepsilon,\delta}+\mathrm{i}\frac{1}{\sqrt{-\Delta}}\nabla_x[S_{0,\varepsilon,\delta}]\right]$$

$$+\frac{1}{2}\exp\left(-\mathrm{i}\sqrt{-\Delta}\frac{t}{\varepsilon^m}\right)\left[\nabla_x\Psi_{0,\varepsilon,\delta}-\mathrm{i}\frac{1}{\sqrt{-\Delta}}\nabla_x[S_{0,\varepsilon,\delta}]\right]$$

$$+\frac{\varepsilon^{\alpha-m}}{2}\int_0^t\left(\exp\left(\mathrm{i}\sqrt{-\Delta}\frac{t-s}{\varepsilon^m}\right)-\exp\left(-\mathrm{i}\sqrt{-\Delta}\frac{t-s}{\varepsilon^m}\right)\right)\left[\frac{\mathrm{i}}{\sqrt{-\Delta}}\nabla_x F^1_{\varepsilon,\delta}\right]\ \mathrm{d}s$$

$$+\frac{1+\varepsilon^{m-2-2\alpha}}{2}\int_0^t\left(\exp\left(\mathrm{i}\sqrt{-\Delta}\frac{t-s}{\varepsilon^m}\right)+\exp\left(-\mathrm{i}\sqrt{-\Delta}\frac{t-s}{\varepsilon^m}\right)\right)\left[\nabla_x\Delta^{-1}\mathrm{div}_x\mathrm{div}_x\mathbb{F}^2_{\varepsilon,\delta}\right]\ \mathrm{d}s$$

$$+\frac{\varepsilon^{-1}+\varepsilon^{\alpha-m}+\varepsilon^{m-2-2\alpha}}{2}\int_0^t\left(\exp\left(\mathrm{i}\sqrt{-\Delta}\frac{t-s}{\varepsilon^m}\right)+\exp\left(-\mathrm{i}\sqrt{-\Delta}\frac{t-s}{\varepsilon^m}\right)\right)\left[\nabla_x\Delta^{-1}\mathrm{div}_x\mathbf{F}^3_{\varepsilon,\delta}\right]\mathrm{d}s$$

where, for the sake of simplicity, we have set $p'(1) = 1$. Now, in accordance with Lemma 3.1,

$$\int_0^T \int_K \left| \exp\left({\rm i}\sqrt{-\Delta}\frac{t}{\varepsilon^m}\right)[v]\right|^2 {\rm d}x\ {\rm d}t \leq \varepsilon^m \int_0^\infty \int_K \left| \exp\left({\rm i}\sqrt{-\Delta}t\right)[v]\right|^2 {\rm d}x\ {\rm d}t \leq \varepsilon^m c \|v\|^2_{L^2(\Omega)}, \tag{3.20}$$

and, similarly,

$$\begin{aligned}
&\int_0^T \int_K \left| \int_0^t \exp\left({\rm i}\sqrt{-\Delta}\frac{t-s}{\varepsilon^m}\right)[g(s)]\ {\rm d}s\right|^2\ {\rm d}x\ {\rm d}t \\
&\leq T \int_0^T \int_0^T \int_K \left| \exp\left({\rm i}\sqrt{-\Delta}\frac{t-s}{\varepsilon^m}\right)[g(s)]\ \right|^2\ {\rm d}x\ {\rm d}t\ {\rm d}s \\
&\leq cT\varepsilon^m \int_0^T \left\| \exp\left(-{\rm i}\frac{s}{\varepsilon}\right)[g(s)]\right\|^2_{L^2(\Omega)} = \varepsilon^m \|g\|^2_{L^2((0,T)\times\Omega)}
\end{aligned} \tag{3.21}$$

for any compact $K \subset \Omega$. Combining (3.20), (3.21) with the uniform bounds (3.16) and hypotheses (2.1), (2.8) we infer that

$$\int_0^T \|\nabla_x \Psi_{\varepsilon,\delta}\|^2_{L^2(K;R^3)}\ {\rm d}t \leq \varepsilon^{2\beta} c(\delta, K, T) \text{ for any compact } K \subset \Omega \tag{3.22}$$

uniformly for $\varepsilon \to 0$, where $\beta > 1$ provided α satisfies (3.2). Thus the effect of acoustic waves becomes negligible in the limit $\varepsilon \to 0$. Accordingly, the information about the asymptotic behavior is provided by the solenoidal component of the velocity field analyzed in the following section.

3.3 Solenoidal part

In order to control the solenoidal component of the velocity field, we write the momentum equation (3.5) in the form:

$$\begin{aligned}
\varepsilon\partial_t \mathbf{m}_{\varepsilon,\delta} + \mathbf{b}\times\mathbf{m}_{\varepsilon,\delta} &= \varepsilon {\rm div}_x \Big[\chi_\varepsilon \mathbb{S}(\nabla_x \mathbf{u}_\varepsilon) - \chi_\varepsilon(\varrho_\varepsilon \mathbf{u}_\varepsilon \times \mathbf{u}_\varepsilon)\Big]_\delta - \varepsilon\Big[\Big(\mathbb{S}(\nabla_x \mathbf{u}_\varepsilon) - \varrho_\varepsilon \mathbf{u}_\varepsilon \times \mathbf{u}_\varepsilon\Big)\cdot\nabla_x\chi_\varepsilon\Big]_\delta \\
&+\varepsilon^{m-1}\Big[\chi_\varepsilon \frac{\varrho_\varepsilon - \tilde\varrho_\varepsilon}{\varepsilon^m}\nabla_x G\Big]_\delta + \varepsilon^{1-2m}\Big(\nabla_x[\chi_\varepsilon(p(\tilde\varrho_\varepsilon) - p(\varrho_\varepsilon))]_\delta - [\nabla_x\chi_\varepsilon(p(\tilde\varrho_\varepsilon) - p(\varrho_\varepsilon))]_\delta\Big)
\end{aligned}$$

in other words

$$\varepsilon\partial_t \mathbf{m}_{\varepsilon,\delta} + \mathbf{b}\times\mathbf{m}_{\varepsilon,\delta} = (\varepsilon + \varepsilon^{m-1-\alpha})\mathbf{Q}_{\varepsilon,\delta} + \varepsilon^{1-2m}\Big(\nabla_x[\chi_\varepsilon(p(\tilde\varrho_\varepsilon) - p(\varrho_\varepsilon))]_\delta - [\nabla_x\chi_\varepsilon(p(\tilde\varrho_\varepsilon) - p(\varrho_\varepsilon))]_\delta\Big), \tag{3.23}$$

where

$$\{\mathbf{Q}_{\varepsilon,\delta}\}_{\varepsilon>0} \text{ is bounded in } L^2(0,T;W^{k,2}(\Omega;R^3)) \text{ for any fixed } k,\ \delta > 0.$$

3.3.1 Compactness of vertical averages

Taking the vertical average of equation (3.23) we obtain

$$\begin{aligned}
\varepsilon\partial_t \langle \mathbf{m}_{\varepsilon,\delta}\rangle + \mathbf{b}\times\langle\mathbf{m}_{\varepsilon,\delta}\rangle &= (\varepsilon + \varepsilon^{m-1-\alpha})\langle\mathbf{Q}_{\varepsilon,\delta}\rangle \\
&+ \varepsilon^{1-2m}\Big(\nabla_x \langle[\chi_\varepsilon(p(\tilde\varrho_\varepsilon) - p(\varrho_\varepsilon))]_\delta\rangle - \langle[\nabla_x\chi_\varepsilon(p(\tilde\varrho_\varepsilon) - p(\varrho_\varepsilon))]_\delta\rangle\Big).
\end{aligned} \tag{3.24}$$

Recalling

$$\mathbf{m}_{\varepsilon,\delta} = \mathbf{Y}_{\varepsilon,\delta} + \nabla_x \Psi_{\varepsilon,\delta},\ \ \mathbf{Y}_{\varepsilon,\delta} \equiv \mathbf{H}[\mathbf{m}_{\varepsilon,\delta}]$$

we check easily that

$$\mathbf{b} \times \langle \mathbf{Y}_{\varepsilon,\delta} \rangle = \begin{bmatrix} -\left\langle Y^2_{\varepsilon,\delta} \right\rangle \\ \left\langle Y^1_{\varepsilon,\delta} \right\rangle \\ 0 \end{bmatrix}$$

is an exact gradient as $\mathbf{Y}_{\varepsilon,\delta}$ is solenoidal.

Consequently, testing equation (3.24) on $\varphi \in C^\infty_c(\Omega; R^3)$, $\mathrm{div}_x \varphi = 0$ we get

$$\partial_t \int_\Omega \langle \mathbf{m}_{\varepsilon,\delta} \rangle \cdot \varphi \ \mathrm{d}x = \int_\Omega \langle \mathbf{Q}_{\varepsilon,\delta} \rangle \cdot \varphi \ \mathrm{d}x - \frac{1}{\varepsilon} \int_\Omega \langle \mathbf{b} \times \nabla_x \Psi_{\varepsilon,\delta} \rangle \cdot \varphi \ \mathrm{d}x$$

provided ε is small enough so that $\chi_\varepsilon|_{\mathrm{supp}\varphi} \equiv 1$. Thus we may use (3.22) to conclude that

$$\langle \mathbf{Y}_{\varepsilon,\delta} \rangle \to \mathbf{U}_\delta \text{ strongly in } L^2((0,T) \times K; R^2) \text{ for any compact } K \subset \Omega \text{ and any fixed } \delta > 0. \tag{3.25}$$

We note that this step depends essentially on (3.2) that requires the rather strong assumption $m > 10$.

3.3.2 Oscillations

We write any function v in the form

$$v(x) = \langle v \rangle (x_h) + \{v\} (x). \tag{3.26}$$

Since $\{v\} = v - \langle v \rangle$ has zero vertical mean, it can be written in the form

$$\{v\} (x) \equiv \partial_{x_3} I[v] \quad \text{with} \quad \int_{\mathcal{T}^1} I[v](x) \ \mathrm{d}x_3 = 0.$$

Moreover, we define

$$\omega^{i,j}_{\varepsilon,\delta} = \partial_{x_i} Y^j_{\varepsilon,\delta} - \partial_{x_j} Y^i_{\varepsilon,\delta} = \partial_{x_i} m^j_{\varepsilon,\delta} - \partial_{x_j} m^i_{\varepsilon,\delta}.$$

Going back to equation (3.23), we deduce that

$$\varepsilon \partial_t \omega^{1,2}_{\varepsilon,\delta} + \mathrm{div}_h [\mathbf{Y}_{\varepsilon,\delta}]_h = \varepsilon \Big(\partial_{x_1} Q^2_{\varepsilon,\delta} - \partial_{x_2} Q^1_{\varepsilon,\delta} \Big) - \Delta_h \Psi_{\varepsilon,\delta}, \tag{3.27}$$

$$\varepsilon \partial_t \omega^{1,3}_{\varepsilon,\delta} + \partial_{x_3} Y^2_{\varepsilon,\delta} = \varepsilon \Big(\partial_{x_1} Q^3_{\varepsilon,\delta} - \partial_{x_3} Q^1_{\varepsilon,\delta} \Big) - \partial^2_{x_3,x_2} \Psi_{\varepsilon,\delta}, \tag{3.28}$$

and, finally,

$$\varepsilon \partial_t \omega^{2,3}_{\varepsilon,\delta} - \partial_{x_3} Y^1_{\varepsilon,\delta} = \varepsilon \Big(\partial_{x_2} Q^3_{\varepsilon,\delta} - \partial_{x_3} Q^2_{\varepsilon,\delta} \Big) + \partial^2_{x_3,x_1} \Psi_{\varepsilon,\delta} \tag{3.29}$$

for all $t \in (0,T)$, $x \in B_{1/\varepsilon^\alpha}$.

3.3.3 Analysis of the convective term

Following step by step the analysis performed in [11, Section 3] we observe that the only problematic component of the convective term reads

$$\mathbf{H}\left[\int_{\mathcal{T}^1} [\mathrm{div}_x(\mathbf{Y}_{\varepsilon,\delta}\otimes\mathbf{Y}_{\varepsilon,\delta})]_h \ \mathrm{d}x_3\right].$$

- **Step 1:**

  Since $\mathbf{Y}_{\varepsilon,\delta}$ is solenoidal, we have

$$\mathrm{div}_x(\mathbf{Y}_{\varepsilon,\delta}\otimes\mathbf{Y}_{\varepsilon,\delta}) = \frac{1}{2}\nabla_x|\mathbf{Y}_{\varepsilon,\delta}|^2 - \mathbf{Y}_{\varepsilon,\delta}\times(\mathbf{curl}[\mathbf{Y}_{\varepsilon,\delta}]).$$

As the former term is a gradient, we concentrate on the latter.

- **Step 2:**

  Write

$$\begin{aligned}\mathbf{Y}_{\varepsilon,\delta}\times\mathbf{curl}[\mathbf{Y}_{\varepsilon,\delta}] &= \langle\mathbf{Y}_{\varepsilon,\delta}\rangle\times\mathbf{curl}\,\langle\mathbf{Y}_{\varepsilon,\delta}\rangle + \partial_{x_3}\Big(\langle\mathbf{Y}_{\varepsilon,\delta}\rangle\times\mathbf{curl}\ I[\mathbf{Y}_{\varepsilon,\delta}] + I[\mathbf{Y}_{\varepsilon,\delta}]\times\mathbf{curl}\,\langle\mathbf{Y}_{\varepsilon,\delta}\rangle\Big)\\ &\quad + \partial_{x_3}I[\mathbf{Y}_{\varepsilon,\delta}]\times\partial_{x_3}\mathbf{curl}(I[\mathbf{Y}_{\varepsilon,\delta}]),\end{aligned}$$

where the term in the brackets has zero vertical mean, while, in accordance with (3.25), the first term is pre-compact. Finally, we have

$$\begin{aligned}&\Big[\partial_{x_3}I[\mathbf{Y}_{\varepsilon,\delta}]\times\partial_{x_3}\mathbf{curl}(I[\mathbf{Y}_{\varepsilon,\delta}])\Big]^j\\ &= \partial_{x_3}I[Y^i_{\varepsilon,\delta}]\partial_{x_3}\Big(\partial_{x_i}I[Y^j_{\varepsilon,\delta}] - \partial_{x_j}I[Y^i_{\varepsilon,\delta}]\Big) = \partial_{x_3}I[Y^i_{\varepsilon,\delta}]\partial_{x_3}I[\omega^{i,j}_{\varepsilon,\delta}],\ j=1,2,3.\end{aligned}$$

- **Step 3:**

  In accordance with (3.27 - 3.29), we get

$$\varepsilon\partial_t(\partial_{x_3}I[\omega^{1,3}_{\varepsilon,\delta}]) + \partial^2_{x_3}I[Y^2_{\varepsilon,\delta}] = \varepsilon\Big(\partial_{x_1}(Q^3_{\varepsilon,\delta} - \big\langle Q^3_{\varepsilon,\delta}\big\rangle) - \partial_{x_3}Q^1_{\varepsilon,\delta}\Big) - \partial^2_{x_3,x_2}\Psi_{\varepsilon,\delta},$$

and

$$\varepsilon\partial_t(\partial_{x_3}I[\omega^{2,3}_{\varepsilon,\delta}]) - \partial^2_{x_3}I[Y^1_{\varepsilon,\delta}] = \varepsilon\Big(\partial_{x_2}(Q^3_{\varepsilon,\delta} - \big\langle Q^3_{\varepsilon,\delta}\big\rangle) - \partial_{x_3}Q^2_{\varepsilon,\delta}\Big) - \partial^2_{x_3,x_1}\Psi_{\varepsilon,\delta}$$

at least for $x\in B_{1/\varepsilon^\alpha}$. Next, compute

$$\begin{aligned}[\partial_{x_3}I[\mathbf{Y}_{\varepsilon,\delta}]\times\partial_{x_3}\mathbf{curl}(I[\mathbf{Y}_{\varepsilon,\delta}])]^1 &= \partial_{x_3}I[Y^2_{\varepsilon,\delta}]\partial_{x_3}I[\omega^{2,1}_{\varepsilon,\delta}] + \partial_{x_3}I[Y^3_{\varepsilon,\delta}]\partial_{x_3}I[\omega^{3,1}_{\varepsilon,\delta}]\\ &= \partial_{x_3}\Big(\partial_{x_3}I[Y^2_{\varepsilon,\delta}]I[\omega^{2,1}_{\varepsilon,\delta}]\Big) - \partial^2_{x_3}I[Y^2_{\varepsilon,\delta}]I[\omega^{2,1}_{\varepsilon,\delta}] - I[\mathrm{div}_h[\mathbf{Y}_{\varepsilon,\delta}]_h]\partial_{x_3}I[\omega^{3,1}_{\varepsilon,\delta}],\end{aligned}$$

where, furthermore,

$$\partial^2_{x_3} I[Y^2_{\varepsilon,\delta}] I[\omega^{2,1}_{\varepsilon,\delta}] + I[{\rm div}_h [\mathbf{Y}_{\varepsilon,\delta}]_h] \partial_{x_3} I[\omega^{3,1}_{\varepsilon,\delta}] = -\varepsilon \partial_t \Big(\partial_{x_3} I[\omega^{1,3}_{\varepsilon,\delta}] \Big) I[\omega^{2,1}_{\varepsilon,\delta}]$$

$$+\varepsilon \Big(\partial_{x_1} (Q^3_{\varepsilon,\delta} - \langle Q^3 \rangle_{\varepsilon,\delta}) - \partial_{x_3} Q^1_{\varepsilon,\delta} \Big) I[\omega^{2,1}_{\varepsilon,\delta}] - \partial^2_{x_3,x_2} \Psi_{\varepsilon,\delta} I[\omega^{2,1}_{\varepsilon,\delta}] - \varepsilon \partial_t \Big(I[\omega^{1,2}_{\varepsilon,\delta}] \Big) \partial_{x_3} I[\omega^{3,1}_{\varepsilon,\delta}]$$

$$+\varepsilon I \Big[\partial_{x_1} Q^2_{\varepsilon,\delta} - \partial_{x_2} Q^1_{\varepsilon,\delta} \Big] \partial_{x_3} I[\omega^{3,1}_{\varepsilon,\delta}] - I[\Delta_h \Psi_{\varepsilon,\delta}] I[\omega^{3,1}_{\varepsilon,\delta}]$$

$$= \varepsilon \partial_t \Big(\partial_{x_3} I[\omega^{1,3}_{\varepsilon,\delta}] I[\omega^{1,2}_{\varepsilon,\delta}] \Big) + \varepsilon \Big(\partial_{x_1} (Q^3_{\varepsilon,\delta} - \langle Q^3 \rangle_{\varepsilon,\delta}) - \partial_{x_3} Q^1_{\varepsilon,\delta} \Big) I[\omega^{2,1}_{\varepsilon,\delta}] - \partial^2_{x_3,x_2} \Psi_{\varepsilon,\delta} I[\omega^{2,1}_{\varepsilon,\delta}]$$

$$+\varepsilon I \Big[\partial_{x_1} Q^2_{\varepsilon,\delta} - \partial_{x_2} Q^1_{\varepsilon,\delta} \Big] \partial_{x_3} I[\omega^{3,1}_{\varepsilon,\delta}] - I[\Delta_h \Psi_{\varepsilon,\delta}] I[\omega^{3,1}_{\varepsilon,\delta}].$$

for $t \in (0,T)$, $x \in B_{1/\varepsilon^\alpha}$. Treating the second component in a similar fashion, we conclude that

$$\frac{1}{|\mathcal{T}^1|} \int_0^T \int_\Omega [{\rm div}_x (\mathbf{Y}_{\varepsilon,\delta} \otimes \mathbf{Y}_{\varepsilon,\delta})] \cdot \varphi \ {\rm d}x \ {\rm d}t \to \int_0^T \int_{R^2} {\rm div}_x (\mathbf{U}_\delta \otimes \mathbf{U}_\delta) \cdot \phi \ {\rm d}x_h \ {\rm d}t \ \text{as } \varepsilon \to 0 \tag{3.30}$$

for any fixed $\delta > 0$, and for any $\varphi = [\phi(t,x_h), 0]$, $\phi \in C^\infty_c([0,T) \times R^2; R^2)$, ${\rm div}_h \phi = 0$.

3.4 Convergence - conclusion

In order to complete the proof of Theorem 1, write

$$\varrho_\varepsilon \mathbf{u}_\varepsilon \otimes \mathbf{u}_\varepsilon = (\varrho_\varepsilon - 1) \mathbf{u}_\varepsilon \otimes \mathbf{u}_\varepsilon + (\mathbf{u}_{\varepsilon,\delta} - \mathbf{u}_\varepsilon) \otimes \mathbf{u}_{\varepsilon,\delta} + \mathbf{u}_{\varepsilon,\delta} \otimes (\mathbf{u}_{\varepsilon,\delta} - \mathbf{u}_\varepsilon)$$
$$+ [(1-\varrho_\varepsilon)\mathbf{u}_\varepsilon]_\delta \otimes \mathbf{u}_{\varepsilon,\delta} + \mathbf{u}_{\varepsilon,\delta} \otimes [(1-\varrho_\varepsilon)\mathbf{u}_\varepsilon]_\delta \mathbf{m}_{\varepsilon,\delta} \otimes \mathbf{m}_{\varepsilon,\delta}$$

for $t \in (0,T)$ and $x \in K \subset \Omega$, K compact.

Now, it is easy to observe that

$$(\varrho_\varepsilon - 1) \mathbf{u}_\varepsilon \otimes \mathbf{u}_\varepsilon, \ [(1-\varrho_\varepsilon)\mathbf{u}_\varepsilon]_\delta \otimes \mathbf{u}_{\varepsilon,\delta} \to 0 \text{ in } L^1((0,T) \times K) \text{ for } \varepsilon \to 0$$

while

$$\| \mathbf{u}_{\varepsilon,\delta} - \mathbf{u}_\varepsilon \|_{L^2(K;R^3)} \le c\delta \| \mathbf{u}_\varepsilon \|_{W^{1,2}(K;R^3)} \text{ uniformly for } \varepsilon > 0.$$

Consequently, testing the momentum equation (1.2) on a compactly supported solenoidal function, the convective term $\varrho_\varepsilon \mathbf{u}_\varepsilon \otimes \mathbf{u}_\varepsilon$ can be replaced by $\mathbf{m}_{\varepsilon,\delta} \otimes \mathbf{m}_{\varepsilon,\delta}$ handled in detail in Sections 3.2, 3.3. This completes the proof of Theorem 1.

Q.E.D.

Note that, since the limit problem (2.22), (2.23) admits a *unique* solution for any square integrable initial data, there is no need to consider subsequences.

4 Isotropic scaling: Proof of Theorem 2

We consider the situation where the Rossby and Mach numbers have the same scaling. Accordingly, we have $m = 1$, together with the uniform bounds derived in Paragraph 2.4. We recall that the static solution $\tilde\varrho_\varepsilon = \tilde\varrho(|x_h|)$ is now independent of ε and we set

$$r_\varepsilon = \frac{\varrho_\varepsilon - \tilde\varrho}{\varepsilon}\cdot$$

4.1 Preliminary results

In accordance with the uniform bounds established in (2.17-2.19) and (2.14-2.16), we may assume (up to taking subsequences) that

$$\varrho_\varepsilon \to \tilde{\varrho} \ \text{ in } L^\infty(0,T;L^\gamma(K)), \qquad r_\varepsilon \to r \ \text{ weakly-}(^*) \text{ in } L^\infty(0,T;L^2(K)) \tag{4.1}$$

for any compact $K \subset \Omega$, recalling that $\gamma > 3$. Moreover,

$$\mathbf{u}_\varepsilon \to \mathbf{U} \ \text{ weakly in } \ L^2(0,T;W^{1,2}(\Omega;R^3)). \tag{4.2}$$

The main point is to derive the equations satisfied by r and $\mathbf{U}$. Similarly to Section 3.2, we rewrite the Navier-Stokes system in the following form:

$$\begin{cases} \varepsilon\partial_t r_\varepsilon + \mathrm{div}_x(\varrho_\varepsilon \mathbf{u}_\varepsilon) = 0, \\ \varepsilon\partial_t(\varrho_\varepsilon\mathbf{u}_\varepsilon) + \varepsilon\mathrm{div}_x\left(\varrho_\varepsilon\mathbf{u}_\varepsilon\otimes\mathbf{u}_\varepsilon\right) + \nabla_x\left(p'(\tilde{\varrho})r_\varepsilon\right) + \mathbf{b}\times(\varrho_\varepsilon\mathbf{u}_\varepsilon) \\ = -\varepsilon\mathrm{div}_x\mathbb{S}(\nabla_x\mathbf{u}_\varepsilon) - \dfrac{1}{\varepsilon}\nabla_x\left(p(\varrho_\varepsilon) - p(\tilde{\varrho}) - p'(\tilde{\varrho})(\varrho_\varepsilon - \tilde{\varrho})\right) + r_\varepsilon\nabla_x G. \end{cases} \tag{4.3}$$

The bounds (2.14-2.19) allow to pass to the limit in the previous system, in the sense of distributions, to derive the following diagnostic equations:

$$\mathrm{div}_x(\tilde{\varrho}\mathbf{U}) = 0, \quad \nabla_x(p'(\tilde{\varrho})\,r) + \mathbf{b}\times\tilde{\varrho}\mathbf{U} = r\nabla_x G.$$

Moreover, introducing

$$R \equiv P'(\tilde{\varrho})r,$$

one can write the previous equations as

$$\mathrm{div}_x(\tilde{\varrho}\mathbf{U}) = 0, \quad \nabla_x R + \mathbf{b}\times\mathbf{U} = 0.$$

From there, one has easily that

$$R = R(x_h), \quad \mathbf{U} = [\mathbf{U}_h(x_h), 0] \tag{4.4}$$

together with

$$\mathrm{div}_h\left(\tilde{\varrho}\mathbf{U}_h\right) = 0, \quad \nabla_h R + \mathbf{U}_h^\perp = 0. \tag{4.5}$$

Note that in addition to the constraint $\mathrm{div}_h\left(\tilde{\varrho}\mathbf{U}_h\right) = 0$, the last equation implies that $\mathrm{div}_h\mathbf{U}_h = 0$, and $\nabla_x\tilde{\varrho}\cdot\mathbf{U}_h = 0$, therefore $R = R(|x_h|)$ is radially symmetric.

These constraints do not determine the time evolution of r and $\mathbf{U}_h$. Our aim is to show that, in accordance with the conclusion of Theorem 2, the function r satisfies equation (2.26).

The rest of the paper is devoted to the proof of (2.26). To this end, we perform the analysis of the high frequency waves generated by the linear part of system (4.3). Indeed, one can recast (4.3) in the general form

$$\varepsilon\partial_t\mathcal{U}_\varepsilon + L[\mathcal{U}_\varepsilon] = \varepsilon\mathcal{N}_\varepsilon + \varepsilon\mathcal{F}_\varepsilon$$

where $\mathcal{U}_\varepsilon \equiv [r_\varepsilon, \mathbf{V}_\varepsilon]$, with $\mathbf{V}_\varepsilon \equiv \varrho_\varepsilon\mathbf{u}_\varepsilon$, while

$$L[\mathcal{U}_\varepsilon] \equiv [\mathrm{div}_x\mathbf{V}_\varepsilon,\ \tilde{\varrho}\,\nabla_x(P'(\tilde{\varrho})\,r_\varepsilon) + \mathbf{b}\times\mathbf{V}_\varepsilon]$$

stands for the linear part of the equation, and

$$\mathcal{N}_\varepsilon = [0, \mathbf{N}_\varepsilon], \quad \mathbf{N}_\varepsilon = -\mathrm{div}_x \left(\varrho_\varepsilon \mathbf{u}_\varepsilon \otimes \mathbf{u}_\varepsilon \right). \tag{4.6}$$

Furthermore, $\mathcal{F}_\varepsilon$ includes the remaining terms:

$$\mathcal{F}_\varepsilon = (0, \mathbf{F}_\varepsilon), \quad \mathbf{F}_\varepsilon = -\mathrm{div}_x \mathbb{S}(\nabla_x \mathbf{u}_\varepsilon) - \frac{1}{\varepsilon^2} \nabla_x \Big(p(\varrho_\varepsilon) - p(\tilde{\varrho}) - p'(\tilde{\varrho})(\varrho_\varepsilon - \tilde{\varrho}) \Big). \tag{4.7}$$

It can be checked that L defines a skew-symmetric operator with respect to the scalar product

$$(\mathcal{U}, \mathcal{U}') \ \equiv \ \int_\Omega \Big(r r' P'(\tilde{\varrho}) \ + \ \mathbf{V} \cdot \mathbf{V}' \, \tilde{\varrho}^{-1} \Big) \, \mathrm{d}x;$$

whence it generates time oscillations with frequency ε^{-1}, whose nonlinear interaction is not obvious. In fact, the the principal difficulty of this part of the paper is the fact that the spectral properties of the operator L are unclear, and thus, in contrast with the previous part, employing dispersive effects seems difficult if not impossible. We shall therefore rely on local methods and use as much as possible the structure of the equations, to control the asymptotic behavior of the convective term.

4.2 Handling the convective term

The goal of this section is to better understand the behavior of $\mathbf{N}_\varepsilon$, and more precisely of

$$\widetilde{\mathbf{N}}_\varepsilon \equiv \mathrm{div}_h \left(\frac{1}{\tilde{\varrho}} \langle \mathbf{V}_{\varepsilon,h} \otimes \mathbf{V}_{\varepsilon,h} \rangle \right), \quad \text{where} \quad \mathbf{V}_{\varepsilon,h} = \varrho_\varepsilon \mathbf{u}_{\varepsilon,h}. \tag{4.8}$$

Later on, we shall compare this term with the average with respect to x_3 of

$$-\mathbf{N}_{\varepsilon,h} = \mathrm{div}_x \left(\varrho_\varepsilon \mathbf{u}_\varepsilon \otimes \mathbf{u}_{\varepsilon,h} \right),$$

which reads

$$\langle -\mathbf{N}_{\varepsilon,h} \rangle = \mathrm{div}_h \left(\langle \varrho_\varepsilon \mathbf{u}_{\varepsilon,h} \otimes \mathbf{u}_{\varepsilon,h} \rangle \right). \tag{4.9}$$

These are two main steps in deriving the limit equation (2.26). We start by showing that

$$\Big(\tilde{\mathbf{N}}_\varepsilon | \nabla_h^\perp \psi \Big) \to 0, \quad \varepsilon \to 0,$$

for all $\psi \in C_c^\infty((0,T) \times R^2)$ such that $\nabla_h \tilde{\varrho} \cdot \nabla_h^\perp \psi = 0$. Here and hereafter, the symbol $(\cdot|\cdot)$ denotes the standard duality pairing. We follow a general approach initiated by Lions and Masmoudi [17], [18] in the framework of the incompressible limit, and later adapted in [11] to the case of rotating fluids. This approach, used also in the last step of the proof of Theorem 1, is reminiscent of compensated compactness arguments. It relies on various cancellations, obtained by multiple use of the structure of (4.3) which roughly speaking reads

$$\begin{cases} \varepsilon \partial_t r_\varepsilon + \mathrm{div}_x (\varrho_\varepsilon \mathbf{u}_\varepsilon) = 0, \\ \varepsilon \partial_t (\varrho_\varepsilon \mathbf{u}_\varepsilon) + \tilde{\varrho} \, \nabla_x \left(P'(\tilde{\varrho}) r_\varepsilon \right) + \mathbf{b} \times (\varrho_\varepsilon \mathbf{u}_\varepsilon) = O(\varepsilon) \end{cases}$$

and will provide some compactness for appropriate combinations of $r_\varepsilon, \varrho_\varepsilon, \mathbf{u}_\varepsilon$.

Similarly to Section 3.2, the complete treatment of $\widetilde{\mathbf{N}}_\varepsilon$ involves spatial regularizations:

$$\mathbf{u}_{\varepsilon,\delta} = \kappa_\delta * \mathbf{u}_\varepsilon, \quad \mathbf{V}_{\varepsilon,\delta} \ = \ \kappa_\delta * (\varrho_\varepsilon \mathbf{u}_\varepsilon), \quad r_{\varepsilon,\delta} \ = \ \kappa_\delta * r_\varepsilon,$$

where $\kappa_\delta = \kappa_\delta(x)$ is a family of regularizing kernels. Note that the quantities $\mathbf{V}_{\varepsilon,\delta}$ and $r_{\varepsilon,\delta}$ are bounded in $L^\infty(0,T;L^1(K))$ for any compact $K \subset \Omega$, and $\mathbf{u}_{\varepsilon,\delta}$ in $L^2(0,T;W^{1,2}(\Omega;R^3))$, uniformly in δ, ε. We claim the following result.

Proposition 4.1 *The fields $\mathbf{u}_{\varepsilon,\delta}$, $\mathbf{V}_{\varepsilon,\delta}$ and $r_{\varepsilon,\delta}$ satisfy the following properties:*

$$\mathbf{V}_{\varepsilon,\delta} = \varepsilon \mathbf{t}^1_{\varepsilon,\delta} + \mathbf{t}^2_{\varepsilon,\delta}, \quad \textit{and} \quad \mathrm{curl}_x \left(\frac{1}{\tilde{\varrho}} \ \mathbf{V}_{\varepsilon,\delta} \right) = \varepsilon \mathbf{T}^1_{\varepsilon,\delta} + \mathbf{T}^2_{\varepsilon,\delta} \tag{4.10}$$

where

$$\begin{gathered} \|\mathbf{t}^1_{\varepsilon,\delta}\|_{L^2(0,T;W^{k,2}(K;R^3))} + \|\mathbf{T}^1_{\varepsilon,\delta}\|_{L^2(0,T;W^{k,2}(K;R^3))} \ \leq \ c(\delta), \\ \|\mathbf{t}^2_{\varepsilon,\delta}\|_{L^2(0,T;W^{1,2}(K;R^3))} + \|\mathbf{T}^2_{\varepsilon,\delta}\|_{L^2((0,T)\times K;R^3)} \leq c \end{gathered} \tag{4.11}$$

for any compact $K \subset \Omega$ and $k = 0, 1, \ldots$ Moreover,

$$\begin{gathered} \sup_{\varepsilon>0} \|r_\varepsilon - r_{\varepsilon,\delta}\|_{L^\infty(0,T;W^{s,2}(K))} \to 0 \ \textit{as} \ \delta \to 0 \ \ \textit{for all} \ s < 0, \\ \sup_{\varepsilon>0} \|\mathbf{u}_\varepsilon - \mathbf{u}_{\varepsilon,\delta}\|_{L^2(0,T;W^{s,2}(K;R^3))} \to 0 \ \textit{as} \ \delta \to 0 \ \ \textit{for all} \ s < 1, \\ \sup_{\varepsilon>0} \|\mathbf{V}_\varepsilon - \mathbf{V}_{\varepsilon,\delta}\|_{L^2(0,T;W^{s,2}(K;R^3))} \to 0 \ \textit{as} \ \delta \to 0 \ \ \textit{for all} \ s < -1/2 \end{gathered} \tag{4.12}$$

for any compact $K \subset \Omega$. Finally the following approximate wave equation is satisfied:

$$\varepsilon \partial_t r_{\varepsilon,\delta} + \mathrm{div}_x \mathbf{V}_{\varepsilon,\delta} = 0, \tag{4.13}$$

$$\varepsilon \partial_t \mathbf{V}_{\varepsilon,\delta} + \tilde{\varrho} \nabla_x \left(P'(\tilde{\varrho})\, r_{\varepsilon,\delta} \right) + \mathbf{b} \times \mathbf{V}_{\varepsilon,\delta} = \varepsilon \mathbf{F}^1_{\varepsilon,\delta} + \mathbf{F}^2_{\varepsilon,\delta}, \tag{4.14}$$

where the source terms are smooth and satisfy

$$\sup_{\varepsilon>0} \|\mathbf{F}^1_{\varepsilon,\delta}\|_{L^2(0,T;W^{k,2}(K;R^3))} \ \leq \ c(\delta), \tag{4.15}$$

$$\sup_{\varepsilon>0} \left(\|\mathbf{F}^2_{\varepsilon,\delta}\|_{L^2((0,T)\times K;R^3)} \ + \ \left\| \mathrm{curl}_x \big(\frac{1}{\tilde{\varrho}} \mathbf{F}^2_{\varepsilon,\delta} \big) \right\|_{L^2((0,T)\times K;R^3)} \right) \to 0 \ \ \textit{as} \ \delta \to 0 \tag{4.16}$$

for any compact $K \subset \Omega$, $k = 0, 1, \ldots$

We postpone the proof of this proposition to Section 4.3. This regularization process allows to establish the following convergence result.

Proposition 4.2 *The nonlinear quantity*

$$\widetilde{\mathbf{N}}_{\varepsilon,\delta} \ = \ \mathrm{div}_h \left(\frac{1}{\tilde{\varrho}} \left\langle \mathbf{V}_{\varepsilon,\delta,h} \otimes \mathbf{V}_{\varepsilon,\delta,h} \right\rangle \right) \tag{4.17}$$

satisfies

$$\lim_{\delta\to 0} \left(\limsup_{\varepsilon\to 0} \ \left| \left(\widetilde{\mathbf{N}}_{\varepsilon,\delta} | \nabla_h^\perp \psi \right) \right| \right) \ = \ 0,$$

for all $\psi \in C^\infty_c((0,T) \times R^2)$ such that $\nabla_h \tilde{\varrho} \cdot \nabla_h^\perp \psi = 0$.

The rest of this part is devoted to the proof of Proposition 4.2. Actually we shall prove more precisely that

$$\widetilde{\mathbf{N}}_{\varepsilon,\delta} \;=\; F_{\varepsilon,\delta}\,\nabla_h\phi \;+\; g_{\varepsilon,\delta}\nabla_h\tilde{\varrho} \;+\; \mathbf{s}_{\varepsilon,\delta}$$

for explicit functions $\phi = \phi(t,x_h)$, $F_{\varepsilon,\delta} = F_{\varepsilon,\delta}(\tilde{\varrho})$, $g_{\varepsilon,\delta} = g_{\varepsilon,\delta}(t,x_h)$ and a remainder $\mathbf{s}_{\varepsilon,\delta} = \mathbf{s}_{\varepsilon,\delta}(t,x_h)$ satisfying

$$\lim_{\delta\to 0}\left(\limsup_{\varepsilon\to 0}\;|\,(\mathbf{s}_{\varepsilon,\delta}|\varphi)\,|\right) = 0 \;\text{ for all }\; \varphi\in C_c^\infty((0,T)\times\Omega;R^2). \tag{4.18}$$

This will of course imply the result.

The notation $o(1)$ will refer hereafter to any term $s_{\varepsilon,\delta}$ satisfying (4.18). Moreover we drop from now on the lowerscript δ, except if some ambiguity is liable to occur.

Let us set

$$\widetilde{\mathbf{N}}_\varepsilon = \widetilde{\mathbf{N}}^1_\varepsilon + \widetilde{\mathbf{N}}^2_\varepsilon \;\equiv\; \mathrm{div}_h\left(\frac{1}{\tilde{\varrho}}\,\langle \mathbf{V}_{\varepsilon,h}\rangle\otimes\langle \mathbf{V}_{\varepsilon,h}\rangle\right) \;+\; \mathrm{div}_h\left(\frac{1}{\tilde{\varrho}}\,\langle\{\mathbf{V}_{\varepsilon,h}\}\otimes\{\mathbf{V}_{\varepsilon,h}\}\rangle\right),$$

where the notation $\langle\cdot\rangle$, $\{\cdot\}$ has been introduced in (3.26), and treat these parts separately.

- **Treatment of $\widetilde{\mathbf{N}}^1_\varepsilon$:**

We notice that

$$\begin{aligned}
\widetilde{\mathbf{N}}^1_\varepsilon \;&=\; \mathrm{div}_h\left(\langle \mathbf{V}_{\varepsilon,h}\rangle\otimes\frac{1}{\tilde{\varrho}}\,\langle \mathbf{V}_{\varepsilon,h}\rangle\right) \;=\; \frac{1}{\tilde{\varrho}}\,\langle \mathbf{V}_{\varepsilon,h}\rangle\,\mathrm{div}_h(\,\langle \mathbf{V}_{\varepsilon,h}\rangle) \;+\; \langle \mathbf{V}_{\varepsilon,h}\rangle\cdot\nabla_h\left(\frac{1}{\tilde{\varrho}}\,\langle \mathbf{V}_{\varepsilon,h}\rangle\right)\\
&=\; \frac{1}{\tilde{\varrho}}\,\langle \mathbf{V}_{\varepsilon,h}\rangle\,\mathrm{div}_h(\,\langle \mathbf{V}_{\varepsilon,h}\rangle) \;+\; \frac{1}{2}\tilde{\varrho}\,\nabla_h\left(|\frac{1}{\tilde{\varrho}}\,\langle \mathbf{V}_{\varepsilon,h}\rangle\,|^2\right) \;+\; \left(\mathrm{curl}_h\frac{1}{\tilde{\varrho}}\,\langle \mathbf{V}_{\varepsilon,h}\rangle\,\right)\langle \mathbf{V}_{\varepsilon,h}\rangle^\perp.
\end{aligned}$$

On the one hand, averaging (4.13) with respect to x_3, and multiplying by $\langle \mathbf{V}_{\varepsilon,h}\rangle$, we get that

$$\frac{1}{\tilde{\varrho}}\,\langle \mathbf{V}_{\varepsilon,h}\rangle\,\mathrm{div}_h(\langle \mathbf{V}_{\varepsilon,h}\rangle) = -\varepsilon(\partial_t\,\langle r_\varepsilon\rangle)\frac{1}{\tilde{\varrho}}\,\langle \mathbf{V}_{\varepsilon,h}\rangle = -\varepsilon\partial_t\left(\frac{1}{\tilde{\varrho}}\,\langle r_\varepsilon\rangle\,\langle \mathbf{V}_{\varepsilon,h}\rangle\right) \;+\; \frac{1}{\tilde{\varrho}}\,\langle r_\varepsilon\rangle\;\varepsilon\,(\partial_t\,\langle \mathbf{V}_{\varepsilon,h}\rangle)\,,$$

where the first term at the right-hand side is of order $o(1)$. As regards the second term, averaging the horizontal components of (4.14) with respect to x_3 and multiplying by $\langle r_\varepsilon\rangle$, we end up with

$$\begin{aligned}
\frac{1}{\tilde{\varrho}}\,\langle r_\varepsilon\rangle\;\varepsilon\,(\partial_t\,\langle \mathbf{V}_{\varepsilon,h}\rangle) &= -\nabla_h\,(P'(\tilde{\varrho})\;\langle r_\varepsilon\rangle)\,\langle r_\varepsilon\rangle \;-\; \frac{1}{\tilde{\varrho}}\,\langle \mathbf{V}_{\varepsilon,h}\rangle^\perp\,\langle r_\varepsilon\rangle \;+\; \frac{1}{\tilde{\varrho}}\,\langle \varepsilon\mathbf{F}^1_{\varepsilon,h}+\mathbf{F}^2_{\varepsilon,h}\rangle\,\langle r_\varepsilon\rangle\\
&= -\frac{1}{P'(\tilde{\varrho})}\nabla_h\left(\frac{1}{2}\,|P'(\tilde{\varrho})\,\langle r_\varepsilon\rangle|^2\right) \;-\; \frac{1}{\tilde{\varrho}}\,\langle \mathbf{V}_{\varepsilon,h}\rangle^\perp\,\langle r_\varepsilon\rangle \;+\; o(1),
\end{aligned}$$

where the $o(1)$ comes from the properties of the $\mathbf{F}^i_\varepsilon$'s and the fact that $r_\varepsilon = r_{\varepsilon,\delta}$ is uniformly bounded in $L^2_{loc}((0,T)\times\Omega)$ with respect to ε and δ, see (4.1). Thus $\widetilde{\mathbf{N}}^1_\varepsilon$ can be written in the form

$$\widetilde{\mathbf{N}}^1_\varepsilon = \left(\left(\mathrm{curl}_h\frac{1}{\tilde{\varrho}}\,\langle \mathbf{V}_{\varepsilon,h}\rangle\right) - \frac{1}{\tilde{\varrho}}\,\langle r_\varepsilon\rangle\right)\langle \mathbf{V}_{\varepsilon,h}\rangle^\perp \;+\; F(\tilde{\varrho})\,\nabla_h\phi \;+\; o(1).$$

It remains to handle the first term at the r.h.s. Therefore, we average (4.14) with respect to x_3, divide by $\tilde{\varrho}$, take the curl of the horizontal components, and subtract average of (4.13) with respect to x_3. We obtain,

$$\varepsilon\partial_t \left(\left(\mathrm{curl}_h \frac{1}{\tilde{\varrho}} \langle \mathbf{V}_{\varepsilon,h} \rangle \right) - \frac{1}{\tilde{\varrho}} \langle r_\varepsilon \rangle \right) \; = \; \frac{1}{\tilde{\varrho}^2} \langle \mathbf{V}_{\varepsilon,h} \rangle \cdot \nabla_h \tilde{\varrho} + \widetilde{F}_\varepsilon^h \tag{4.19}$$

where

$$\widetilde{F}_\varepsilon^h := \mathrm{curl}_h \frac{1}{\tilde{\varrho}} \left\langle \varepsilon \mathbf{F}^1_{\varepsilon,h} + \mathbf{F}^2_{\varepsilon,h} \right\rangle .$$

According to Proposition 4.1 there is a function $f(\delta)$ going to zero with δ such that, uniformly in ε,

$$\sup_{\varepsilon>0} \left\| \mathrm{curl}_h \frac{1}{\tilde{\varrho}} \mathbf{F}^2_{\varepsilon,h} \right\|_{L^2((0,T)\times K;R^2)} \leq f(\delta). \tag{4.20}$$

Now let us notice that by definition of $\tilde{\varrho}$ one has

$$\nabla_h \tilde{\varrho} \;\; P'(\tilde{\varrho}) = 2x_h, \quad \text{so} \quad \nabla_h \tilde{\varrho} \;\; p'(\tilde{\varrho}) = 2\tilde{\varrho}\, x_h.$$

By Assumption (2.2), this gives in particular that there is a constant C such that for all x_h,

$$|\nabla_h \tilde{\varrho}(x_h)| \geq C|x_h|. \tag{4.21}$$

Now let us consider a smooth function χ_δ defined by

$$\chi_\delta(x_h) := \chi \left(\frac{\nabla_h \tilde{\varrho}(x_h)}{\sqrt{f(\delta)}} \right).$$

where χ is a function of $C_c^\infty(R^2;[0,1])$ such that $\chi(x_h) = 1$ if $|x_h| \leq 1$. Using (4.10) we get

$$\chi_\delta \left(\left(\mathrm{curl}_h \frac{1}{\tilde{\varrho}} \langle \mathbf{V}_{\varepsilon,h} \rangle \right) - \frac{1}{\tilde{\varrho}} \langle r_\varepsilon \rangle \right) \langle \mathbf{V}_{\varepsilon,h} \rangle^\perp = \; \chi_\delta \left(\varepsilon T^1_{\varepsilon,3} + T^2_{\varepsilon,3} - \frac{1}{\tilde{\varrho}} \langle r_\varepsilon \rangle \right) \left\langle \varepsilon \mathbf{t}^1_{\varepsilon,h} + \mathbf{t}^2_{\varepsilon,h} \right\rangle^\perp .$$

Now we can write, by Hölder's inequality and the continuous embedding $W^{1,2} \subset L^6$

$$\begin{aligned} &\left\| \chi_\delta \left(\varepsilon T^1_{\varepsilon,3} + T^2_{\varepsilon,3} - \frac{1}{\tilde{\varrho}} \langle r_\varepsilon \rangle \right) \left\langle \varepsilon \mathbf{t}^1_{\varepsilon,h} + \mathbf{t}^2_{\varepsilon,h} \right\rangle^\perp \right\|_{L^1((0,T)\times K;R^2)} \\ &\leq \|\chi_\delta\|_{L^3(K)} \left\| \varepsilon T^1_{\varepsilon,3} + T^2_{\varepsilon,3} - \frac{1}{\tilde{\varrho}} \langle r_\varepsilon \rangle \right\|_{L^2(0,T;L^2(K))} \|\varepsilon \mathbf{t}^1_{\varepsilon,h} + \mathbf{t}^2_{\varepsilon,h}\|_{L^2(0,T;W^{1,2}(K;R^2))} \end{aligned}$$

Recalling that r_ε is uniformly bounded in ε, δ in $L^\infty(0,T;L^2_{\mathrm{loc}}(\Omega))$, and using (4.11) and (4.21) we infer that

$$\begin{aligned} \left\| \chi_\delta \left(\varepsilon T^1_{\varepsilon,3} + T^2_{\varepsilon,3} - \frac{1}{\tilde{\varrho}} \langle r_\varepsilon \rangle \right) \left\langle \varepsilon \mathbf{t}^1_{\varepsilon,h} + \mathbf{t}^2_{\varepsilon,h} \right\rangle^\perp \right\|_{L^1((0,T)\times K;R^2)} &\leq (\varepsilon c(\delta) + c) \left| \left\{ x_h, \; |\nabla_h \tilde{\varrho}(x_h)| \leq \sqrt{f(\delta)} \right\} \right|^{\frac{1}{3}} \\ &\leq (\varepsilon c(\delta) + c) \left| B(0, C\sqrt{f(\delta)}) \right|^{\frac{1}{3}} \\ &= o(1). \end{aligned}$$

This allows to conclude that

$$\chi_\delta \left(\left(\mathrm{curl}_h \frac{1}{\tilde{\varrho}} \langle \mathbf{V}_{\varepsilon,h} \rangle \right) - \frac{1}{\tilde{\varrho}} \langle r_\varepsilon \rangle \right) \langle \mathbf{V}_{\varepsilon,h} \rangle^\perp = o(1).$$

Then we can write

$$\begin{aligned}
&(1-\chi_\delta) \left(\left(\mathrm{curl}_h \frac{1}{\tilde{\varrho}} \langle \mathbf{V}_{\varepsilon,h} \rangle \right) - \frac{1}{\tilde{\varrho}} \langle r_\varepsilon \rangle \right) \langle \mathbf{V}_{\varepsilon,h} \rangle^\perp \\
&= (1-\chi_\delta) \left(\left(\mathrm{curl}_h \frac{1}{\tilde{\varrho}} \langle \mathbf{V}_{\varepsilon,h} \rangle \right) - \frac{1}{\tilde{\varrho}} \langle r_\varepsilon \rangle \right) \left(\frac{\langle \mathbf{V}_{\varepsilon,h} \rangle^\perp \cdot \nabla_h^\perp \tilde{\varrho}}{|\nabla_h \tilde{\varrho}|^2} \nabla_h^\perp \tilde{\varrho} \,+\, \frac{\langle \mathbf{V}_{\varepsilon,h} \rangle^\perp \cdot \nabla_h \tilde{\varrho}}{|\nabla_h \tilde{\varrho}|^2} \nabla_h \tilde{\varrho} \right) \\
&= (1-\chi_\delta) \left(\left(\mathrm{curl}_h \frac{1}{\tilde{\varrho}} \langle \mathbf{V}_{\varepsilon,h} \rangle \right) - \frac{1}{\tilde{\varrho}} \langle r_\varepsilon \rangle \right) \frac{\langle \mathbf{V}_{\varepsilon,h} \rangle \cdot \nabla_h \tilde{\varrho}}{|\nabla_h \tilde{\varrho}|^2} \nabla_h^\perp \tilde{\varrho} \;+\; g^1 \nabla_h \tilde{\varrho}.
\end{aligned}$$

for

$$g^1 \;\equiv\; (1-\chi_\delta) \left(\left(\mathrm{curl}_h \frac{1}{\tilde{\varrho}} \langle \mathbf{V}_{\varepsilon,h} \rangle \right) - \frac{1}{\tilde{\varrho}} \langle r_\varepsilon \rangle \right) \frac{\langle \mathbf{V}_{\varepsilon,h} \rangle^\perp \cdot \nabla_h \tilde{\varrho}}{|\nabla_h \tilde{\varrho}|^2}.$$

As regards the first term at the right-hand side of the last equality, we use (4.19) to derive

$$\begin{aligned}
&(1-\chi_\delta) \left(\left(\mathrm{curl}_h \frac{1}{\tilde{\varrho}} \langle \mathbf{V}_{\varepsilon,h} \rangle \right) - \frac{1}{\tilde{\varrho}} \langle r_\varepsilon \rangle \right) \frac{\langle \mathbf{V}_{\varepsilon,h} \rangle \cdot \nabla_h \tilde{\varrho}}{|\nabla_h \tilde{\varrho}|^2} \nabla_h^\perp \tilde{\varrho} \\
&= -\varepsilon \partial_t \left(\frac{(1-\chi_\delta)}{2|\nabla_h \tilde{\varrho}|^2} \left(\tilde{\varrho} \left(\mathrm{curl}_h \frac{1}{\tilde{\varrho}} \langle \mathbf{V}_{\varepsilon,h} \rangle \right) - \langle r_\varepsilon \rangle \right)^2 \nabla_h^\perp \tilde{\varrho} \right) + \tilde{\varrho}^2 \left(\varepsilon T^1_{\varepsilon,3} + T^2_{\varepsilon,3} - \frac{1}{\tilde{\varrho}} \langle r_\varepsilon \rangle \right) \widetilde{F}^h_\varepsilon \frac{\nabla_h^\perp \tilde{\varrho}}{|\nabla_h \tilde{\varrho}|^2} (1-\chi_\delta)
\end{aligned}$$

with the notation introduced in (4.10). We get, by (4.11),

$$\begin{aligned}
\left\| \left(\varepsilon T^1_{\varepsilon,3} + T^2_{\varepsilon,3} - \frac{1}{\tilde{\varrho}} \langle r_\varepsilon \rangle \right) \widetilde{F}^h_\varepsilon \frac{\nabla_h^\perp \tilde{\varrho}}{|\nabla_h \tilde{\varrho}|^2} (1-\chi_\delta) \right\|_{L^2(0,T;L^1(K))} &\le \left\| \frac{(1-\chi_\delta)}{|\nabla_h \tilde{\varrho}|} \right\|_{L^\infty(K)} (\varepsilon c(\delta) + c) \, \|\widetilde{F}^h_\varepsilon\|_{L^2((0,T)\times K)} \\
&\le \varepsilon c(\delta) + \frac{c}{\sqrt{f(\delta)}} \|\widetilde{F}^h_\varepsilon\|_{L^2((0,T)\times K)}
\end{aligned}$$

which is $o(1)$ thanks to (4.20). Combining the previous inequalities leads to

$$\widetilde{\mathbf{N}}^1_\varepsilon = g^1 \nabla_h \tilde{\varrho} \;+\; F(\tilde{\varrho}) \nabla_h \phi \;+\; o(1),$$

as expected.

- **Treatment of $\widetilde{\mathbf{N}}^2_\varepsilon$:**

This time, we consider

$$\begin{aligned}
\widetilde{\mathbf{N}}^2_\varepsilon \;&=\; \mathrm{div}_h \left(\frac{1}{\tilde{\varrho}} \langle \{ \mathbf{V}_{\varepsilon,h} \} \otimes \{ \mathbf{V}_{\varepsilon,h} \} \rangle \right) \\
&=\; \left\langle \frac{1}{\tilde{\varrho}} \, \{ \mathbf{V}_{\varepsilon,h} \} \, \mathrm{div}_h \left(\{ \mathbf{V}_{\varepsilon,h} \} \right) \right\rangle \;+\; \frac{1}{2} \tilde{\varrho} \left\langle \nabla_h \left| \frac{1}{\tilde{\varrho}} \{ \mathbf{V}_{\varepsilon,h} \} \right|^2 \right\rangle \;+\; \left\langle \mathrm{curl}_h \Big(\frac{1}{\tilde{\varrho}} \{ \mathbf{V}_{\varepsilon,h} \} \Big) \; \{ \mathbf{V}_{\varepsilon,h} \}^\perp \right\rangle.
\end{aligned}$$

As $\frac{1}{\tilde{\varrho}}\{\mathbf{V}_{\varepsilon,h}\}$ has zero vertical average, we can write

$$\mathrm{curl}_x\frac{1}{\tilde{\varrho}}\{\mathbf{V}_\varepsilon\} = \begin{pmatrix}\partial_3\mathbf{\Omega}_{\varepsilon,h}\\ \omega_\varepsilon\end{pmatrix}, \quad \mathbf{\Omega}_{\varepsilon,h} := \frac{1}{\tilde{\varrho}}\{\mathbf{V}_{\varepsilon,h}\}^\perp - \partial_3^{-1}\nabla_h^\perp\frac{1}{\tilde{\varrho}}\{\mathbf{V}_{\varepsilon,3}\}, \quad \omega_\varepsilon := \mathrm{curl}_h\frac{1}{\tilde{\varrho}}\{\mathbf{V}_{\varepsilon,h}\},$$

where we have set $\partial_3^{-1}a = I(a)$, see (3.26). Applying $\mathbf{curl}_x$ to the momentum equation (4.14) yields

$$\varepsilon\partial_t\mathbf{\Omega}_{\varepsilon,h} = \frac{1}{\tilde{\varrho}}\{\mathbf{V}_{\varepsilon,h}\} + \left\{\partial_3^{-1}(\mathbf{curl}_x\frac{1}{\tilde{\varrho}}(\varepsilon\mathbf{F}_\varepsilon^1+\mathbf{F}_\varepsilon^2))_h\right\}, \tag{4.22}$$

$$\varepsilon\partial_t\omega_\varepsilon = -\mathrm{div}_h\frac{1}{\tilde{\varrho}}\{\mathbf{V}_{\varepsilon,h}\} + \left\{\mathrm{curl}_h\frac{1}{\tilde{\varrho}}(\varepsilon\mathbf{F}_\varepsilon^1+\mathbf{F}_\varepsilon^2)_h\right\}. \tag{4.23}$$

From there, we deduce that

$$\begin{aligned}\left\langle\mathrm{curl}_h\Big(\frac{1}{\tilde{\varrho}}\{\mathbf{V}_{\varepsilon,h}\}\Big)\ \{\mathbf{V}_{\varepsilon,h}\}^\perp\right\rangle = \left\langle\omega_\varepsilon\ \{\mathbf{V}_{\varepsilon,h}\}^\perp\right\rangle &= \tilde{\varrho}\left\langle\omega_\varepsilon\,\varepsilon\partial_t\mathbf{\Omega}_{\varepsilon,h}^\perp\right\rangle + o(1)\\ &= -\tilde{\varrho}\left\langle(\varepsilon\partial_t\omega_\varepsilon)\mathbf{\Omega}_{\varepsilon,h}^\perp\right\rangle + \left\langle\varepsilon\partial_t\left(\tilde{\varrho}\,\omega_\varepsilon\,\mathbf{\Omega}_{\varepsilon,h}^\perp\right)\right\rangle + o(1)\\ &= \tilde{\varrho}\left\langle\mathbf{\Omega}_{\varepsilon,h}^\perp\mathrm{div}_h\frac{1}{\tilde{\varrho}}\{\mathbf{V}_{\varepsilon,h}\}\right\rangle + o(1)\end{aligned}$$

Here, we have used repeatedly equations (4.22), (4.23). The error terms generated by

$$\tilde{\varrho}\partial_3^{-1}\mathbf{curl}_x\frac{1}{\tilde{\varrho}}\left\langle\varepsilon\mathbf{F}_\varepsilon^1+\tilde{\mathbf{F}}_\varepsilon^2\right\rangle$$

are responsible for the $o(1)$ term, as can be verified using as previously (4.10)-(4.11). Therefore

$$\begin{aligned}\widetilde{\mathbf{N}}_\varepsilon^2 &= \left\langle\frac{1}{\tilde{\varrho}}\{\mathbf{V}_{\varepsilon,h}\}\mathrm{div}_h(\{\mathbf{V}_{\varepsilon,h}\})\right\rangle + \tilde{\varrho}\left\langle\mathbf{\Omega}_{\varepsilon,h}^\perp\mathrm{div}_h\frac{1}{\tilde{\varrho}}\{\mathbf{V}_{\varepsilon,h}\}\right\rangle + F(\tilde{\varrho})\nabla_h\psi + o(1)\\ &= \frac{1}{\tilde{\varrho}}\left\langle\left(\{\mathbf{V}_{\varepsilon,h}\}+\tilde{\varrho}\mathbf{\Omega}_{\varepsilon,h}^\perp\right)\mathrm{div}_h(\{\mathbf{V}_{\varepsilon,h}\})\right\rangle + \left\langle\tilde{\varrho}\mathbf{\Omega}_{\varepsilon,h}^\perp\{\mathbf{V}_{\varepsilon,h}\}\cdot\nabla_h\frac{1}{\tilde{\varrho}}\right\rangle + F(\tilde{\varrho})\nabla_h\psi + o(1).\end{aligned}$$

By straightforward manipulations we have

$$\frac{1}{\tilde{\varrho}}\left(\{\mathbf{V}_{\varepsilon,h}\}+\tilde{\varrho}\mathbf{\Omega}_{\varepsilon,h}^\perp\right)\partial_3\{V_{\varepsilon,3}\} = \partial_3\left(\frac{1}{\tilde{\varrho}}\{V_{\varepsilon,3}\}\left(\{\mathbf{V}_{\varepsilon,h}\}+\tilde{\varrho}\mathbf{\Omega}_{\varepsilon,h}^\perp\right)\right) - \frac{1}{2\tilde{\varrho}}\nabla_h\left|\{V_{\varepsilon,3}\}\right|^2 - \left|\{V_{\varepsilon,3}\}\right|^2\nabla_h\frac{1}{\tilde{\varrho}}$$

We can replace the first term by using equation (4.14)

$$\frac{1}{\tilde{\varrho}}\left\langle\left(\{\mathbf{V}_{\varepsilon,h}\}+\tilde{\varrho}\mathbf{\Omega}_{\varepsilon,h}^\perp\right)\mathrm{div}_x\{\mathbf{V}_\varepsilon\}\right\rangle = -\frac{1}{\tilde{\varrho}}\left\langle(\varepsilon\partial_t\{r_\varepsilon\})\left(\{\mathbf{V}_{\varepsilon,h}\}+\tilde{\varrho}\mathbf{\Omega}_{\varepsilon,h}^\perp\right)\right\rangle$$

which leaves us with

$$\begin{aligned}\widetilde{\mathbf{N}}_\varepsilon^2 &= -\frac{1}{\tilde{\varrho}}\left\langle(\varepsilon\partial_t\{r_\varepsilon\})\left(\{\mathbf{V}_{\varepsilon,h}\}+\tilde{\varrho}\mathbf{\Omega}_{\varepsilon,h}^\perp\right)\right\rangle + \left\langle\tilde{\varrho}\mathbf{\Omega}_{\varepsilon,h}^\perp\{\mathbf{V}_{\varepsilon,h}\}\cdot\nabla_h\frac{1}{\tilde{\varrho}}\right\rangle + F(\tilde{\varrho})\nabla_h\psi + g^2\nabla_h\tilde{\varrho} + o(1)\\ &= \frac{1}{\tilde{\varrho}}\left\langle\{r_\varepsilon\}\ \varepsilon\partial_t\left(\{\mathbf{V}_{\varepsilon,h}\}+\tilde{\varrho}\mathbf{\Omega}_{\varepsilon,h}^\perp\right)\right\rangle + \left\langle\tilde{\varrho}\mathbf{\Omega}_{\varepsilon,h}^\perp\{\mathbf{V}_{\varepsilon,h}\}\cdot\nabla_h\frac{1}{\tilde{\varrho}}\right\rangle + F(\tilde{\varrho})\nabla_h\psi + g^2\nabla_h\tilde{\varrho} + o(1).\end{aligned}$$

Now, thanks to (4.14) and (4.22),

$$\frac{1}{\tilde{\varrho}}\left\langle \{r_\varepsilon\}\ \varepsilon\partial_t\left(\{\mathbf{V}_{\varepsilon,h}\}+\tilde{\varrho}\mathbf{\Omega}^{\perp}_{\varepsilon,h}\right)\right\rangle \ = \ -\frac{1}{2P'(\tilde{\varrho})}\nabla_h\left|P'(\tilde{\varrho})\left\{r_\varepsilon\right\}\right|^2+o(1),$$

so that $\widetilde{\mathbf{N}}^2_\varepsilon$ resumes to

$$\widetilde{\mathbf{N}}^2_\varepsilon \ = \ \left\langle \tilde{\varrho}\mathbf{\Omega}^{\perp}_{\varepsilon,h}[\mathbf{V}_{\varepsilon,h}]\cdot\nabla_h\frac{1}{\tilde{\varrho}}\right\rangle \ + \ F(\tilde{\varrho})\nabla_h\psi \ + \ g^2\nabla_h\tilde{\varrho} \ + \ o(1).$$

Finally, we proceed as previously in the case of $\widetilde{\mathbf{N}}^1_\varepsilon$, with the first term at the right-hand side, this time omitting for simplicity the cut-off near $\nabla_h\tilde{\varrho}=0$. We write

$$\begin{aligned}\left(\{\mathbf{V}_{\varepsilon,h}\}\cdot\nabla_h\tilde{\varrho}\right)\mathbf{\Omega}^{\perp}_{\varepsilon,h} = & \ \left(\{\mathbf{V}_{\varepsilon,h}\}\cdot\nabla_h\tilde{\varrho}\right)\left(\frac{\mathbf{\Omega}_{\varepsilon,h}\cdot\nabla_h\tilde{\varrho}}{|\nabla_h\tilde{\varrho}|^2}\nabla^{\perp}_h\tilde{\varrho} \ + \ \frac{\mathbf{\Omega}^{\perp}_{\varepsilon,h}\cdot\nabla_h\tilde{\varrho}}{|\nabla_h\tilde{\varrho}|^2}\nabla_h\tilde{\varrho}\right)\\ = & \ \tilde{\varrho}\left(\varepsilon\partial_t\mathbf{\Omega}_{\varepsilon,h}\cdot\nabla_h\tilde{\varrho}\right)\frac{\mathbf{\Omega}_{\varepsilon,h}\cdot\nabla_h\tilde{\varrho}}{|\nabla_h\tilde{\varrho}|^2}\nabla^{\perp}_h\tilde{\varrho} \ + \ g^3\nabla_h\tilde{\varrho} \ + \ o(1)\end{aligned}$$

where we have used (4.22) in the passage from the second to the third line. This yields

$$\left(\{\mathbf{V}_{\varepsilon,h}\}\cdot\nabla_h\tilde{\varrho}\right)\mathbf{\Omega}^{\perp}_{\varepsilon,h} = g\nabla\tilde{\varrho} \ + \ o(1)$$

and finally we get an expression of the form:

$$\widetilde{\mathbf{N}}^2_\varepsilon \ = \ F(\tilde{\varrho})\nabla_h\psi \ + \ g\nabla_h\tilde{\varrho} \ + \ o(1)$$

as expected. Proposition 4.2 is proved.

Q.E.D.

4.3 Regularization process

In this section we shall prove Proposition 4.1. We start by recalling that

$$\varepsilon\partial_t r_\varepsilon+\mathrm{div}_x(\mathbf{V}_\varepsilon)=0,$$

and

$$\varepsilon\partial_t\mathbf{V}_\varepsilon+\tilde{\varrho}\nabla_x\left(P'(\tilde{\varrho})\,r_\varepsilon\right)+\mathbf{b}\times\mathbf{V}_\varepsilon=\varepsilon\mathbf{N}_\varepsilon+\varepsilon\mathbf{F}_\varepsilon,$$

with notation (4.6) and (4.7). The first step consists in establishing some bounds for $\mathbf{N}_\varepsilon$ and $\mathbf{F}_\varepsilon$. The energy bound clearly implies that $\mathbf{N}_\varepsilon$ is bounded in $L^\infty(0,T;W^{-1,1}(K;R^3))$. As for $\mathbf{F}_\varepsilon$, one has clearly that $\mathrm{div}_x\mathbb{S}(\nabla_x\mathbf{u}_\varepsilon)$ is bounded in $L^2(0,T;W^{-1,2}(K;R^3))$, and $\frac{1}{\varepsilon^2}\nabla_x\left(p(\varrho_\varepsilon)-p(\tilde{\varrho})-p'(\tilde{\varrho})(\varrho_\varepsilon-\tilde{\varrho})\right)$ is bounded in $L^\infty(0,T;W^{-1,1}(K;R^3))$. Therefore in particular

$$\mathbf{N}_\varepsilon+\mathbf{S}_\varepsilon \quad \text{is bounded in} \quad L^2(0,T;W^{-5/2,2}(K;R^3)) \tag{4.24}$$

for any compact $K\subset\Omega$.

Now let us proceed to the regularization. First we notice that

$$\mathbf{V}_{\varepsilon,\delta} = \varepsilon\kappa_\delta * (r_\varepsilon\,\mathbf{u}_\varepsilon)\ +\ \kappa_\delta * (\tilde{\varrho}\mathbf{u}_\varepsilon)\ =:\ \varepsilon\mathbf{t}^1_{\varepsilon,\delta} + \mathbf{t}^2_{\varepsilon,\delta}$$

and

$$\mathbf{curl}_x\left(\frac{1}{\tilde{\varrho}}\ \mathbf{V}_{\varepsilon,\delta}\right) = \varepsilon\mathbf{curl}_x\left(\frac{1}{\tilde{\varrho}}\kappa_\delta * (r_\varepsilon\,\mathbf{u}_\varepsilon)\right)\ +\ \mathbf{curl}_x\left(\frac{1}{\tilde{\varrho}}\kappa_\delta * (\tilde{\varrho}\mathbf{u}_\varepsilon)\right)\ =:\ \varepsilon\mathbf{T}^1_{\varepsilon,\delta} + \mathbf{T}^2_{\varepsilon,\delta}$$

so thanks to the L^2 bound on r_ε and the $W^{1,2}$ bound on $\mathbf{u}_\varepsilon$, we deduce easily that for all k, K

$$\|\mathbf{t}^1_{\varepsilon,\delta}\|_{L^2(0,T;W^{k,2}(K;R^3))} + \|\mathbf{T}^1_{\varepsilon,\delta}\|_{L^2(0,T;W^{k,2}(K;R^3))}\ \leq\ c(\delta),$$

$$\|\mathbf{t}^2_{\varepsilon,\delta}\|_{L^2(0,T;W^{1,2}(K;R^3))} + \|\mathbf{T}^2_{\varepsilon,\delta}\|_{L^2((0,T)\times K;R^3)} \leq c$$

uniformly in ε (and δ for the second bound). This proves (4.11). The uniform bounds derived previously also give directly the convergences (4.12).

Now let us turn to the wave equations. By convolution we get (with obvious notation)

$$\varepsilon\partial_t r_{\varepsilon,\delta} + \mathrm{div}_x\mathbf{V}_{\varepsilon,\delta} = 0, \tag{4.25}$$

and

$$\varepsilon\partial_t\mathbf{V}_{\varepsilon,\delta} +\ \tilde{\varrho}\,\nabla_x\left(P'(\tilde{\varrho})\,r_{\varepsilon,\delta}\right) + \mathbf{b}\times\mathbf{V}_{\varepsilon,\delta} = \varepsilon\mathbf{F}^1_{\varepsilon,\delta} + \mathbf{F}^2_{\varepsilon,\delta} \tag{4.26}$$

with

$$\mathbf{F}^1_{\varepsilon,\delta} = \mathbf{N}_{\varepsilon,\delta} + \mathbf{F}_{\varepsilon,\delta}$$

and

$$\mathbf{F}^2_{\varepsilon,\delta} = \tilde{\varrho}\,\nabla_x\left(P'(\tilde{\varrho})\,r_{\varepsilon,\delta}\right) - \left(\tilde{\varrho}\,\nabla_x\left(P'(\tilde{\varrho})\,r_\varepsilon\right)\right) * \kappa_\delta.$$

Clearly (4.24) implies (4.15), so let us turn to the statement (4.16).

In order to see (4.16), we use [19, Proposition 4.1] (which forces the restriction $\gamma > 3$) on compactness of solutions to (4.25), (4.26), namely,

$$\|r_\varepsilon - r_{\varepsilon,\delta}\|_{L^p([0,T];L^2(K))} \to 0 \text{ as } \delta\to 0 \text{ for any compact } K\subset\Omega \text{ and any } p\geq 1,$$

together with Lemma 3.3 (2) of [19]. Note that, compared with the situation treated in [19, Proposition 4.1], the present system contains an extra term $\mathbf{b}\times\mathbf{V}_{\varepsilon,\delta}$ already known to be compact with respect to the space variable.

The vanishing of $\mathbf{F}^2_{\varepsilon,\delta}$ (uniformly in ε) follows directly. To handle the convergence of $\mathrm{curl}_x\frac{1}{\tilde{\varrho}}\mathbf{F}^2_{\varepsilon,\delta}$, we then notice that

$$\mathbf{curl}_x\mathbf{F}^2_{\varepsilon,\delta} = A(\tilde{\varrho})\nabla r_{\varepsilon,\delta} - (A(\tilde{\varrho})\nabla r_\varepsilon) * \kappa_\delta$$

for some smooth matrix function A. Still using Lemma 3.3 (2) in [19], we obtain the vanishing of $\mathbf{curl}_x\mathbf{F}^2_{\varepsilon,\delta}$, which together with the one of $\mathbf{F}^2_{\varepsilon,\delta}$ completes the proof of Proposition 4.1.

Q.E.D.

4.4 Conclusion

Thanks to Proposition 4.1, we can conclude the proof of Theorem 2. We keep the notation $\mathbf{N}_{\varepsilon,h}$, $\widetilde{\mathbf{N}}_\varepsilon$ and $\widetilde{\mathbf{N}}_{\varepsilon,\delta}$ of Section 4.2, see (4.8), (4.9), (4.17).
Let $\psi = \psi(t, x_h) \in C_c^\infty((0,T) \times R^2)$ such that $\nabla_h \tilde{\varrho} \cdot \nabla_h^\perp \psi = 0$. We write

$$\left| \left(\langle -\mathbf{N}_{\varepsilon,h} \rangle \, | \nabla_h^\perp \psi \right) - \left(\widetilde{\mathbf{N}}_{\varepsilon,\delta} | \nabla_h^\perp \psi \right) \right| = \left| \int_0^T \int_\Omega \left(\varrho_\varepsilon \mathbf{u}_{\varepsilon,h} \otimes \mathbf{u}_{\varepsilon,h} - \frac{1}{\tilde{\varrho}} \mathbf{V}_{\varepsilon,\delta,h} \otimes \mathbf{V}_{\varepsilon,\delta,h} \right) : \nabla_h \otimes \nabla_h^\perp \psi \mathrm{d}x\mathrm{d}t \right|$$

$$\leq \left| \int_0^T \int_\Omega \left((\mathbf{V}_{\varepsilon,h} - \mathbf{V}_{\varepsilon,\delta,h}) \otimes \mathbf{u}_{\varepsilon,h} \right) : (\nabla_h \otimes \nabla_h^\perp \psi) \, \mathrm{d}x\mathrm{d}t \right|$$

$$+ \left| \int_0^T \int_\Omega \left(\mathbf{V}_{\varepsilon,\delta,h} \otimes (\mathbf{u}_{\varepsilon,h} - \frac{\mathbf{V}_{\varepsilon,h}}{\tilde{\varrho}}) \right) : (\nabla_h \otimes \nabla_h^\perp \psi) \, \mathrm{d}x\mathrm{d}t \right|$$

$$+ \left| \int_0^T \int_\Omega \left(\mathbf{V}_{\varepsilon,\delta,h} \otimes \frac{\mathbf{V}_{\varepsilon,h} - \mathbf{V}_{\varepsilon,\delta,h}}{\tilde{\varrho}} \right) : (\nabla_h \otimes \nabla_h^\perp \psi) \mathrm{d}x\mathrm{d}t \right| =: \, I_{\varepsilon,\delta}^1 + I_{\varepsilon,\delta}^2 + I_{\varepsilon,\delta}^3 \, .$$

We have:

$$I_{\varepsilon,\delta}^1 = \left| \int_0^T \int_\Omega \mathbf{V}_{\varepsilon,h} \cdot \left((\nabla_h \otimes \nabla_h^\perp \psi) \mathbf{u}_{\varepsilon,h} - \kappa_\delta * \left((\nabla_h \otimes \nabla_h^\perp \psi) \mathbf{u}_{\varepsilon,h} \right) \right) dx dt \right| = O(\delta)$$

uniformly in ε, using that

$$\| (\nabla_h \otimes \nabla_h^\perp \psi) \mathbf{u}_{\varepsilon,h} - \kappa_\delta * \left((\nabla_h \otimes \nabla_h^\perp \psi) \mathbf{u}_{\varepsilon,h} \right) \|_{L^2((0,T) \times \Omega; R^2)} \; \leq \; C \delta \| \mathbf{u}_{\varepsilon,h} \|_{L^2(0,T; W^{1,2}(K; R^2))}$$

for some compact K containing the support of ψ. Then, noticing that

$$\frac{\mathbf{V}_\varepsilon}{\tilde{\varrho}} = \mathbf{u}_\varepsilon \, + \, \varepsilon \frac{r_\varepsilon}{\tilde{\varrho}} \mathbf{u}_\varepsilon$$

one obtains easily

$$I_{\varepsilon,\delta}^2 \leq c(\delta) \varepsilon.$$

Finally, we remark that

$$\mathbf{V}_\varepsilon - \mathbf{V}_{\varepsilon,\delta} \; = \; (\kappa_\delta * (\tilde{\varrho} \mathbf{u}_\varepsilon) - \tilde{\varrho} \mathbf{u}_\varepsilon) \; + \; \varepsilon \, (r_\varepsilon \mathbf{u}_\varepsilon - \kappa_\delta * (r_\varepsilon \mathbf{u}_\varepsilon))$$

to obtain

$$I_{\varepsilon,\delta}^3 \; \leq \; c \, \delta \; + \; c(\delta) \, \varepsilon.$$

Putting these inequalities altogether yields

$$\limsup_{\delta \to 0} \limsup_{\varepsilon \to 0} \, (I_{\varepsilon,\delta}^1 + I_{\varepsilon,\delta}^2 + I_{\varepsilon,\delta}^3) \; = \; 0.$$

Combining this with Proposition 4.1, we deduce that

$$\lim_{\varepsilon\to 0}\left(\mathbf{N}_{\varepsilon,h}|\nabla_h^\perp\psi\right) \ = \ 0$$

for all $\psi=\psi(t,x_h)\in C_c^\infty((0,T)\times R^2)$ such that $\nabla_h\tilde{\varrho}\cdot\nabla_h^\perp\psi=0$, meaning the function ψ is radially symmetric.

We are now at the point of getting the equation satisfied by $r,\mathbf{u}$, cf. (2.24). The horizontal part of the momentum equation reads

$$\partial_t(\varrho_\varepsilon\mathbf{u}_{\varepsilon,h})+\mathrm{div}_x\left(\varrho_\varepsilon\mathbf{u}_\varepsilon\otimes\mathbf{u}_{\varepsilon,h}\right)+\varepsilon^{-2}\nabla_h p(\varrho_\varepsilon)+\varepsilon^{-1}\varrho_\varepsilon\mathbf{u}_{\varepsilon,h}^\perp=[\mathrm{div}_x\mathbb{S}(\nabla_x\mathbf{u}_\varepsilon)]_h+\frac{\varrho_\varepsilon}{\varepsilon^2}\nabla_h G.$$

We recall that $\nabla_h G=\nabla_h P(\tilde{\varrho})=P'(\tilde{\varrho})\nabla_h\tilde{\varrho}$. We integrate with respect to x_3 the last equation, and apply curl_h. We obtain

$$\begin{aligned}&\partial_t\mathrm{curl}_h\left\langle\varrho_\varepsilon\mathbf{u}_{\varepsilon,h}\right\rangle\ +\ \mathrm{curl}_h\mathrm{div}_h\left\langle\varrho_\varepsilon\mathbf{u}_{\varepsilon,h}\otimes\mathbf{u}_{\varepsilon,h}\right\rangle+\varepsilon^{-1}\mathrm{div}_h\left\langle\varrho_\varepsilon\mathbf{u}_{\varepsilon,h}\right\rangle\\&\qquad=\mathrm{curl}_h\mathrm{div}_h\mathbb{S}_{h,h}(\nabla_h\left\langle\mathbf{u}_{\varepsilon,h}\right\rangle)+\mathrm{curl}_h\left(\frac{P'(\tilde{\varrho})\langle\varrho_\varepsilon\rangle}{\varepsilon^2}\nabla_h\tilde{\varrho}\right),\end{aligned}\tag{4.27}$$

where

$$\mathbb{S}_{h,h}(\nabla_h\mathbf{u}_h)=\mu(\nabla_h\mathbf{u}_h+\nabla_h^\perp\mathbf{u}_h-\frac{2}{3}\mathrm{div}_h(\mathbf{u}_h)\mathbb{I}_{h,h})\text{ with }\mathbb{I}_{h,h}\text{ identity matrix in }R^2.$$

Continuity equation yields $\mathrm{div}_h\left\langle\varrho_\varepsilon\mathbf{u}_{\varepsilon,h}\right\rangle=-\varepsilon\partial_t\langle r_\varepsilon\rangle$; we employ this fact and (4.27), where we use a radially symmetric test function $\psi\in C_c^\infty((0,T)\times R^2)$ to get

$$\begin{aligned}&\Big(\partial_t\big(\mathrm{curl}_h\left\langle\varrho_\varepsilon\mathbf{u}_{\varepsilon,h}\right\rangle-\langle r_\varepsilon\rangle\big)-\mathrm{curl}_h\mathrm{div}_h\mathbb{S}_{h,h}(\nabla_x\langle\mathbf{u}_{\varepsilon,h}\rangle)|\psi\Big)\\&\quad=\ \left(\langle N_{\varepsilon,h}\rangle|\nabla_h^\perp\psi\right)-\frac{1}{\varepsilon^2}\int_0^T\int_{R^2}P'(\tilde{\varrho})\langle\varrho_\varepsilon\rangle\nabla_h\tilde{\varrho}\cdot\nabla_h^\perp\psi\,dx_h\,dt\end{aligned}$$

The first (convective) term at the r.h.s. goes to zero, whereas the second one is identically zero by the properties of ψ. All other quantities converge easily to yield

$$\Big(\partial_t\Big(\mathrm{curl}_h\left\langle\tilde{\varrho}\mathbf{U}_h\right\rangle-\langle r\rangle\Big)-\mu\Delta_h\mathrm{curl}_h\mathbf{U}_h,\psi\Big)=0$$

for any radially symmetric ψ. Using the properties (2.24– 2.25), we arrive at (2.26).

Finally, repeating the same procedure with $\psi\in C_c^\infty([0,T)\times R^2)$, ψ radially symmetric, we obtain

$$-\left((\mathrm{curl}_h(\tilde{\varrho}\mathbf{u}_h)-r)|\partial_t\psi\right)-\mu\left(\Delta_h\mathrm{curl}_h\mathbf{u}_h|\psi\right)=-\left(\mathrm{curl}_h\langle\tilde{\varrho}\mathbf{u}_{0,h}\rangle-\langle r_0\rangle|\psi_{|t=0}\right),\tag{4.28}$$

where $(\mathbf{u}_0,r_0)$ are the weak limits of the family of initial data $\mathbf{u}_{0,\varepsilon},r_{0,\varepsilon}$. This justifies the initial condition stated in (2.27). Moreover, in view of hypothesis (2.8), we have

$$\sqrt{P'(\tilde{\varrho})}\langle r_0\rangle\in L^2(R^2),\quad\sqrt{\tilde{\varrho}}\langle\mathbf{u}_{0,h}\rangle\in L^2(R^2;R^2).\tag{4.29}$$

Under these circumstances, it is easy to show that (2.24)-(2.25)-(4.28) admits a unique solution. Indeed, taking $\psi = P'(\tilde{\varrho})r(0)$ as a test function in (2.27), we check that (4.29) implies

$$\int_{R^2} \left(\tilde{\varrho} \left| \nabla_h \left(P'(\tilde{\varrho}) r(0) \right) \right|^2 \,+\, P'(\tilde{\varrho}) |r(0)|^2 \right) \mathrm{d}x \,<\, +\infty;$$

whence uniqueness follows from standard energy arguments. See [9, section 5.2]. Thus, there is exactly one accumulation point for the sequence $(r_\varepsilon, \mathbf{u}_\varepsilon)$, and the whole sequence converges to it.

References

[1] A. Babin, A. Mahalov, and B. Nicolaenko. Global regularity of 3D rotating Navier-Stokes equations for resonant domains. *Indiana Univ. Math. J.*, 48(3):1133–1176, 1999.

[2] A. Babin, A. Mahalov, and B. Nicolaenko. 3D Navier-Stokes and Euler equations with initial data characterized by uniformly large vorticity. *Indiana Univ. Math. J.*, 50(Special Issue):1–35, 2001. Dedicated to Professors Ciprian Foias and Roger Temam (Bloomington, IN, 2000).

[3] J.-Y. Chemin, B. Desjardins, I. Gallagher, and E. Grenier. *Mathematical geophysics*, volume 32 of *Oxford Lecture Series in Mathematics and its Applications*. The Clarendon Press Oxford University Press, Oxford, 2006.

[4] P. D'Acona and R. Racke. Evolution equations in non-flat waveguides. *arXiv:1010.0817*.

[5] B. Desjardins and E. Grenier. Low Mach number limit of viscous compressible flows in the whole space. *R. Soc. Lond. Proc. Ser. A Math. Phys. Eng. Sci.*, 455(1986):2271–2279, 1999.

[6] B. Desjardins, E. Grenier, P.-L. Lions, and N. Masmoudi. Incompressible limit for solutions of the isentropic Navier-Stokes equations with Dirichlet boundary conditions. *J. Math. Pures Appl.*, **78**:461–471, 1999.

[7] D. B. Ebin. Viscous fluids in a domain with frictionless boundary. *Global Analysis - Analysis on Manifolds, H. Kurke, J. Mecke, H. Triebel, R. Thiele Editors, Teubner-Texte zur Mathematik 57, Teubner, Leipzig*, pages 93–110, 1983.

[8] E. Feireisl and A. Novotný. *Singular limits in thermodynamics of viscous fluids*. Birkhauser, Basel, 2009.

[9] E. Feireisl, I. Gallagher and A. Novotný. A singular limit for compressible rotating fluids. submitted.

[10] E. Feireisl, A. Novotný, and H. Petzeltová. On the existence of globally defined weak solutions to the Navier-Stokes equations of compressible isentropic fluids. *J. Math. Fluid Mech.*, **3**:358–392, 2001.

[11] I. Gallagher and L. Saint-Raymond. Weak convergence results for inhomogeneous rotating fluid equations. *Journal d'Analyse Mathématique*, **99**:1-34, 2006.

[12] S. Klainerman and A. Majda. Singular limits of quasilinear hyperbolic systems with large parameters and the incompressible limit of compressible fluids. *Comm. Pure Appl. Math.*, **34**:481–524, 1981.

[13] R. Klein. Scale-dependent models for atmospheric flows. *Annual Rev. Fluid Mechanics*, **42**:249–274, 2010.

[14] J. Lighthill. On sound generated aerodynamically I. General theory. *Proc. of the Royal Society of London*, **A 211**:564–587, 1952.

[15] J. Lighthill. On sound generated aerodynamically II. General theory. *Proc. of the Royal Society of London*, **A 222**:1–32, 1954.

[16] P.-L. Lions. *Mathematical topics in fluid dynamics, Vol.2, Compressible models.* Oxford Science Publication, Oxford, 1998.

[17] P.-L. Lions and N. Masmoudi. Incompressible limit for a viscous compressible fluid. *J. Math. Pures Appl.*, **77**:585–627, 1998.

[18] P.-L. Lions and N. Masmoudi. Une approche locale de la limite incompressible. *C.R. Acad. Sci. Paris Sér. I Math.*, **329 (5)**:230–240, 1999.

[19] N. Masmoudi. Rigorous derivation of the anelastic approximation. *Journal de Math. Pures et Appl.*, **88 (5)**:387–392, 2007.

[20] J. L. Metcalfe. Global Strichartz estimates for solutions to the wave equation exterior to a convex obstacle. *Trans. Amer. Math. Soc.*, **356(12)**:4839–4855 (electronic), 2004.

[21] H. F. Smith and C. D. Sogge. Global Strichartz estimates for nontrapping perturbations of the Laplacian. *Comm. Partial Differential Equations*, **25(11-12)**:2171–2183, 2000.