# A SINGULAR LIMIT FOR COMPRESSIBLE ROTATING FLUIDS

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Abstract. We consider a singular limit problem for the Navier-Stokes system of a rotating compressible fluid, where the Rossby and Mach numbers tend simultaneously to zero. The limit problem is identified as the 2-D Navier-Stokes system in the "horizontal" variables containing an extra term that accounts for compressibility in the original system.

## 1. Introduction and main result

### 1.1. Setting.

Consider a scaled Navier-Stokes system in the form

$$\partial_t \varrho + \mathrm{div}_x(\varrho \mathbf{u}) = 0, \tag{1.1}$$

$$\partial_t(\varrho \mathbf{u}) + \mathrm{div}_x(\varrho \mathbf{u} \otimes \mathbf{u}) + \frac{1}{\varepsilon}(\mathbf{g} \times \varrho \mathbf{u}) + \frac{1}{\varepsilon^2} \nabla_x p(\varrho) = \mathrm{div}_x \mathbb{S}(\nabla_x \mathbf{u}), \tag{1.2}$$

with the viscous stress tensor

$$\mathbb{S}(\nabla_x \mathbf{u}) = \mu \Big( \nabla_x \mathbf{u} + \nabla_x^t \mathbf{u} - \frac{2}{3} \mathrm{div}_x \mathbf{u} \mathbb{I} \Big), \ \mu > 0, \tag{1.3}$$

and

$$\mathbf{g} = [0, 0, 1].$$

Here $\varrho = \varrho(t, x) \geq 0$ denotes the density and $\mathbf{u}(t, x) = [u_1, u_2, u_3](t, x)$ denotes the velocity of the fluid. Problem (1.1 - 1.2) arises in meteorological applications, modeling rotating compressible fluids with the rotation axis determined by $\mathbf{g}$ and the Rossby and Mach number proportional to a small parameter $\varepsilon$.

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We consider a very simple geometry of the underlying physical space, namely an infinite slab $\Omega$ bounded above and below by two parallel planes,

$$\Omega = \mathbb{R}^2 \times (0,1). \tag{1.4}$$

The velocity $\mathbf{u}$ satisfies the complete slip boundary conditions (known also as Navier boundary conditions),

$$\mathbf{u}\cdot\mathbf{n} = u_3|_{\partial\Omega} = 0,\ \ [\mathbb{S}\mathbf{n}]\times\mathbf{n}|_{\partial\Omega} = [S_{2,3}, -S_{1,3}, 0]|_{\partial\Omega} = 0. \tag{1.5}$$

For the initial data

$$\varrho(0,\cdot) = \varrho_{0,\varepsilon},\ \ \mathbf{u}(0,\cdot) = \mathbf{u}_{0,\varepsilon}, \tag{1.6}$$

our goal is to study the asymptotic behavior of the corresponding solutions $\varrho_\varepsilon$, $\mathbf{u}_\varepsilon$ for $\varepsilon \to 0$. We focus on the interplay between the Coriolis force, here proportional to a singular parameter $1/\varepsilon$, and the acoustic waves created in the low Mach number regime. In particular, we neglect:

- stratification due to the presence of gravitation, here assumed in equilibrium with the hydrostatic pressure; the action of the centrifugal force is also neglected, compared to the gravitational force;
- the effect of a boundary layer (Ekman layer), here eliminated by the choice of the complete slip boundary conditions (see for instance Lopes Filho et al. [11] and the references therein).

We consider *ill-prepared* initial data, specifically,

$$\left\{ \begin{array}{c} \varrho_{0,\varepsilon} = \overline{\varrho} + \varepsilon r_{0,\varepsilon},\ \text{with } \{r_{0,\varepsilon}\}_{\varepsilon>0} \text{ bounded in } L^2\cap L^\infty(\Omega), \\ \\ \text{for some positive constant } \overline{\varrho}, \\ \\ \{\mathbf{u}_{0,\varepsilon}\}_{\varepsilon>0} \text{ bounded in } L^2\cap L^\infty(\Omega;\mathbb{R}^3). \end{array} \right\} \tag{1.7}$$

Because of the prominent role of the "vertical" direction $\mathbf{g}$ in the problem, we introduce the "horizontal" component $\mathbf{v}_h = [v_1, v_2, 0]$ of a vector field $\mathbf{v}$, together with the corresponding differential operators $\nabla_h$, $\mathrm{div}_h$, and, notably, $\mathrm{curl}_h$, which is represented by the *scalar* field

$$\mathrm{curl}_h[\mathbf{v}] = \partial_{x_1} v_2 - \partial_{x_2} v_1.$$

Let $\varrho_\varepsilon$, $\mathbf{u}_\varepsilon$ be a solution of problem (1.1-1.6). Introducing a new quantity

$$r_\varepsilon = \frac{\varrho_\varepsilon - \overline{\varrho}}{\varepsilon}$$

which satisfies

$$\partial_t r_\varepsilon + \frac{1}{\varepsilon}\mathrm{div}_x(\overline{\varrho}\mathbf{u}) + \mathrm{div}_x(r_\varepsilon \mathbf{u}) = 0,$$

we easily check that if

$$r_\varepsilon \to r, \ \mathbf{u}_\varepsilon \to \mathbf{U} \text{ in some sense,}$$

then, at least formally, the limits satisfy a diagnostic equation

$$\mathbf{g} \times \mathbf{U} + \frac{p'(\overline{\varrho})}{\overline{\varrho}}\nabla_x r = 0, \tag{1.8}$$

which in turn implies that

$$r = r(x_1, x_2), \ \mathbf{U} = [\mathbf{U}_h, 0], \ \mathbf{U}_h = \mathbf{U}_h(x_1, x_2). \tag{1.9}$$

Moreover, as we shall see below, $\mathrm{div}_x \mathbf{U} = \mathrm{div}_h \mathbf{U}_h = 0$, and denoting $\nabla_h^\perp r$ the vector $(\partial_{x_2} r, -\partial_{x_1} r)$,

$$\partial_t \Big( \Delta_h r - \frac{1}{p'(\overline{\varrho})} r \Big) + \nabla_h^\perp r \cdot \nabla_h(\Delta_h r) = \frac{\mu}{\overline{\varrho}} \Delta_h^2 r. \tag{1.10}$$

Thus $r$ may be interpreted as a *stream function* associated to the vector field $\mathbf{U}_h$, therefore (1.10) can be viewed as a 2D Navier-Stokes system describing the motion of an incompressible fluid in the horizontal plane $\mathbb{R}^2$, supplemented with an extra term $(1/p'(\overline{\varrho}))\partial_t r$. Equation (1.10) is well-known to physicists in the theory of quasi-geostrophic flows, see Zeitlin [13, Chapters 1,2].

The main goal of the present paper is to provide a rigorous justification of the target system (1.10) in the framework of weak solutions to the primitive equations (1.1), (1.2). In the coming paragraphs, we introduce the weak solutions to both systems, recall their basic properties, and state our main result. In Section 2, we derive the necessary uniform bounds on the family of solutions $\{\varrho_\varepsilon, \mathbf{u}_\varepsilon\}_{\varepsilon>0}$, and pass formally to the limit when $\varepsilon \to 0$. In Section 3, the associated wave equation describing propagation of the acoustic waves in the low Mach number regime is introduced. Using the celebrated RAGE theorem, we show that the acoustic energy tends to zero, at least locally in space. The proof of the main result is completed in Section 4.

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1.2. **Existence of weak solutions, and first uniform bounds.** To begin, we point out that system (1.1 - 1.3), endowed with the boundary conditions (1.5) can be recast as a purely periodic problem with respect to the vertical coordinate $x_3$ provided $\varrho$, $u_1$, $u_2$ were extended as even functions in the $x_3-$variable defined on

$$\Omega = \mathbb{R}^2 \times \mathcal{T}^1,\ \mathcal{T}^1 \equiv [-1,1]|_{\{-1,1\}},$$

while $u_3$ is extended to be odd in $x_3$ on the same set. Throughout the whole text, we therefore tacitly assume that $\varrho$ and $\mathbf{u}$ belong to these symmetry classes. Accordingly, the same convention is adopted for the initial data.

We shall say that functions $\varrho$, $\mathbf{u}$ represent a *weak solution* to problem (1.1 - 1.6) in $(0,T)\times\Omega$ if:

- $\varrho \geq 0$, $(\varrho - \overline{\varrho}) \in L^\infty(0,T;(L^\gamma + L^2)(\Omega))$ for a certain $\gamma > 3/2$, $\mathbf{u} \in L^2(0,T;W^{1,2}(\Omega;\mathbb{R}^3))$;
- equation of continuity (1.1) is satisfied in the sense of renormalized solutions, namely

(1.11)
$$\int_0^T \int_\Omega \Big( (\varrho + b(\varrho))\partial_t\varphi + (\varrho + b(\varrho))\mathbf{u}\cdot\nabla_x\varphi + (b(\varrho) - b'(\varrho)\varrho)\mathrm{div}_x\mathbf{u}\varphi \Big)\ \mathrm{d}x\ \mathrm{d}t$$
$$= - \int_\Omega \Big( \varrho_{0,\varepsilon} + b(\varrho_{0,\varepsilon}) \Big)\varphi(0,\cdot)\ \mathrm{d}x$$

for any $b \in C^\infty[0,\infty)$, $b' \in C^\infty_c[0,\infty)$, and any test function $\varphi \in C^\infty_c([0,T)\times\Omega)$;

- $p = p(\varrho) \in L^1((0,T)\times\Omega)$, momentum equation (1.2) is replaced by a family of integral identities

(1.12)
$$\int_0^T \int_\Omega \Big( \varrho\mathbf{u}\cdot\partial_t\varphi + \varrho(\mathbf{u}\otimes\mathbf{u}) : \nabla_x\varphi + \frac{1}{\varepsilon}(\mathbf{g}\times\varrho\mathbf{u})\cdot\varphi + \frac{1}{\varepsilon^2}p(\varrho)\mathrm{div}_x\varphi \Big)\ \mathrm{d}x\ \mathrm{d}t$$
$$= \int_0^T \int_\Omega \mathbb{S}(\nabla_x\mathbf{u}) : \nabla_x\varphi\ \mathrm{d}x\ \mathrm{d}t - \int_\Omega \varrho_{0,\varepsilon}\mathbf{u}_{0,\varepsilon}\cdot\varphi(0,\cdot)\ \mathrm{d}x$$

for any $\varphi \in C^\infty_c([0,T)\times\Omega;\mathbb{R}^3)$;

- the *energy inequality*

$$\int_\Omega \left( \frac{1}{2}\varrho|\mathbf{u}|^2 + \frac{1}{\varepsilon^2}E(\varrho,\overline{\varrho}) \right)(\tau,\cdot)\ \mathrm{d}x + \int_0^\tau \int_\Omega \mathbb{S}(\nabla_x\mathbf{u}) : \nabla_x\mathbf{u}\ \mathrm{d}x\ \mathrm{d}t \tag{1.13}$$
$$\leq \int_\Omega \left( \frac{1}{2}\varrho_{0,\varepsilon}|\mathbf{u}_{0,\varepsilon}|^2 + \frac{1}{\varepsilon^2}E(\varrho_{0,\varepsilon},\overline{\varrho}) \right)\ \mathrm{d}x$$

holds for a.a. $\tau \in (0,T)$, where

$$E(\varrho,\overline{\varrho}) = H(\varrho) - H'(\overline{\varrho})(\varrho - \overline{\varrho}) - H(\overline{\varrho}),$$

with

$$H(\varrho) = \varrho\int_1^\varrho \frac{p(z)}{z^2}\ \mathrm{d}z.$$

Note that, by virtue of hypothesis (1.7), the quantity on the right-hand side of (1.13) is bounded uniformly for $\varepsilon \to 0$.

*Existence* of global-in-time weak solutions to problem (1.1 - 1.6) can be established by the method developed by P.-L. Lions [10], with the necessary modifications introduced in [8] in order to accommodate a larger class of physically relevant pressure-density state equations, specifically,

(1.14)

$$p \in C^1[0,\infty),\ p(0)=0,\ p'(\varrho)>0 \text{ for } \ \varrho>0,\ \lim_{\varrho\to\infty}\frac{p'(\varrho)}{\varrho^{\gamma-1}} = p_\infty > 0,$$

for a certain $\gamma > 3/2$.

*Uniform bounds* can be obtained in a rather standard way from the energy inequality (1.13). We shall present the bounds in the next section.

1.3. **Main result.** The main result of the present paper is stated as follows.

**Theorem 1.1.** *Assume that the pressure $p$ satisfies (1.14).*

*Let $\{\varrho_\varepsilon, \mathbf{u}_\varepsilon\}_{\varepsilon>0}$ be a family of weak solutions to problem (1.1 - 1.6) in $(0,T)\times\Omega$, where $\Omega$ is specified through (1.4), with the initial data satisfying (1.7), where*

$$r_{0,\varepsilon} \to r_0 \text{ weakly in } L^2(\Omega),\ \mathbf{u}_{0,\varepsilon} \to \mathbf{U}_0 \text{ weakly in } L^2(\Omega;\mathbb{R}^3).$$

*Then after taking a subsequence, the following results hold*

$$r_\varepsilon \equiv \frac{\varrho_\varepsilon - \overline{\varrho}}{\varepsilon} \to r \text{ weakly-(*) in } L^\infty(0,T;L^2(\Omega)+L^\gamma(\Omega)),$$

$$\mathbf{u}_\varepsilon \to \mathbf{U} \text{ weakly in } L^2(0,T;W^{1,2}(\Omega;\mathbb{R}^3)),$$

$$\text{and } \mathbf{u}_\varepsilon \to \mathbf{U} \text{ strongly in } L^2_{\rm loc}((0,T)\times\Omega;\mathbb{R}^3),$$

*where $r$ and $\mathbf{U}$ satisfy (1.8), $\mathrm{div}_x\mathbf{U}=0$, and, moreover, the stream function $r$ solves equation (1.10) in the sense of distributions (see (4.3) below for a weak formulation), supplemented with the initial datum*

$$r(0,\cdot) = \tilde{r}, \tag{1.15}$$

*where $\tilde{r} \in W^{1,2}(\mathbb{R}^2)$ is the unique solution of*

$$-\Delta_h \tilde{r} + \frac{1}{p'(\overline{\varrho})}\tilde{r} = \overline{\varrho}\int_0^1 \mathrm{curl}_h \mathbf{U}_{0,h}\ \mathrm{d}x_3 + \int_0^1 r_0 \mathrm{d}x_3.$$

*If, in addition, $\mathrm{curl_h}\mathbf{U}_{0,h} \in L^2(\Omega)$, then the solution $r$ of (1.10) is uniquely determined by (1.15) in the space $\{r \in \mathcal{D}'(\mathbb{R}^2),\ \nabla_h r \in L^\infty(0,\infty;W^{1,2}(\mathbb{R}^2))\cap L^2(0,\infty;\dot{H}^2(\mathbb{R}^2))\}$ and the convergence holds for the whole sequence of solutions.*

The remaining part of the paper is devoted to the proof of Theorem 1.1. The crucial point of the proof is, of course, the strong (a.e. pointwise) convergence of the velocity field that enables us to carry out the limit in the convective term. Here, the desired pointwise convergence will follow from the celebrated RAGE theorem, together with the fact that the wave propagator in the associated *acoustic equation* commutes with the Fourier transform in both the horizontal variables $(x_1, x_2)$ and the vertical variable $x_3$.

1.4. **Related results.** This work is a contribution to a general research direction consisting in studying singular limits in PDEs arising in fluid mechanics. Without giving an extensive bibliography, one should refer for the first works in this line to Klainerman and Majda [9] and Ukai [12] for the incompressible limit (actually [12] is probably the first work in which dispersive estimates were established in order to prove strong convergence in the whole space), followed by Desjardins et al. [5] and [6]. In the context of rotating fluids one should mention the important work of Babin, Mahalov and Nicolaenko [1], as well as the book [3] and references therein; one also refers to [7] for a survey. Few studies combine both rotation and compressible effects. We refer to Bresch, Desjardins and Gérard-Varet [2] for an analysis in a cylinder, where the well prepared case is studied precisely; the ill prepared case is also addressed but only a conditional result is proved.

## 2. Uniform bounds

We start reviewing rather standard uniform bounds that follow directly from the energy inequality (1.13). To this end, it is convenient to introduce a decomposition

$$h = [h]_{\rm ess} + [h]_{\rm res}, \text{ where } [h]_{\rm ess} = \psi(\varrho_\varepsilon) h,$$

$$\psi \in C^\infty_c(0,\infty), \ 0 \leq \psi \leq 1, \ \psi \equiv 1 \text{ in a neighborhood of } \overline{\varrho}$$

for any function $h$ defined on $(0,T) \times \Omega$. It is understood that the *essential* part $[h]_{\rm ess}$ is the crucial quantity that determines the asymptotic behavior of the system while the *residual* component $[h]_{\rm res}$ "disappears" in the limit $\varepsilon \to 0$.

As already pointed out, our choice of the initial data (1.7) guarantees that the right-hand side of energy inequality (1.13) remains bounded for $\varepsilon \to 0$. After a straightforward manipulation, we deduce the following estimates:

$$\{\sqrt{\varrho_\varepsilon}\mathbf{u}_\varepsilon\}_{\varepsilon>0} \text{ bounded in } L^\infty(0,T;L^2(\Omega;\mathbb{R}^3)), \tag{2.1}$$

$$\{[r_\varepsilon]_{\rm ess}\}_{\varepsilon>0} \text{ bounded in } L^\infty(0,T;L^2(\Omega)), \tag{2.2}$$

$$\operatorname*{ess\,sup}_{t\in(0,T)} \|[\varrho_\varepsilon]_{\rm res}\|^\gamma_{L^\gamma(\Omega)} \le \varepsilon^2 c, \tag{2.3}$$

$$\operatorname*{ess\,sup}_{t\in(0,T)} \|[1]_{\rm res}\|_{L^1(\Omega)} \le \varepsilon^2 c, \tag{2.4}$$

and

$$\left\{\nabla_x \mathbf{u}_\varepsilon + \nabla_x^t \mathbf{u}_\varepsilon - \frac{2}{3}{\rm div}_x \mathbf{u}_\varepsilon \mathbb{I}\right\}_{\varepsilon>0} \quad \text{bounded in } L^2((0,T)\times\Omega;\mathbb{R}^{3\times 3}),$$

which together with the standard Korn's inequality implies that

$$\{\nabla_x \mathbf{u}_\varepsilon\}_{\varepsilon>0} \text{ is bounded in } L^2((0,T)\times\Omega;\mathbb{R}^{3\times 3}). \tag{2.5}$$

In addition, it is easy to observe that (2.2), (2.3) yield

$$\varrho_\varepsilon \to \overline{\varrho} \text{ in } L^\infty(0,T;L^\gamma + L^2(\Omega)). \tag{2.6}$$

Now let us decompose

$$\overline{\varrho}\int_\Omega |\mathbf{u}_\varepsilon|^2 {\rm d}x = \int_\Omega (\overline{\varrho}-\varrho_\varepsilon)|\mathbf{u}_\varepsilon|^2 {\rm d}x + \int_\Omega \varrho_\varepsilon |\mathbf{u}_\varepsilon|^2 {\rm d}x.$$

By (2.1) the second term on the right-hand side is bounded in $L^\infty(0,T)$ so let us consider the first one. Writing due to (2.6)

$$\varrho_\varepsilon - \overline{\varrho} = \varrho_\varepsilon^{(1)} + \varrho_\varepsilon^{(2)}$$

with

$$\varrho_\varepsilon^{(1)} \to 0 \text{ in } L^\infty(0,T;L^\gamma(\Omega)) \text{ and } \varrho_\varepsilon^{(2)} \to 0 \text{ in } L^\infty(0,T;L^2(\Omega)) \tag{2.7}$$

we have by Hölder's inequality

$$\left|\int_\Omega (\overline{\varrho}-\varrho_\varepsilon)|\mathbf{u}_\varepsilon|^2 {\rm d}x\right| \le \left(\int_\Omega |\varrho_\varepsilon^{(1)}|^\gamma {\rm d}x\right)^{\frac{1}{\gamma}} \left(\int_\Omega |\mathbf{u}_\varepsilon|^{2\gamma'} {\rm d}x\right)^{\frac{1}{\gamma'}} + \left(\int_\Omega |\varrho_\varepsilon^{(2)}|^2 {\rm d}x\right)^{\frac{1}{2}} \left(\int_\Omega |\mathbf{u}_\varepsilon|^4 {\rm d}x\right)^{\frac{1}{2}}$$

with $1/\gamma + 1/\gamma' = 1$. By (2.5) and Sobolev embeddings we know that $\mathbf{u}_\varepsilon$ is bounded in $L^\infty(0,T;L^6(\Omega))$ and Hölder's inequality again allows to write

$$\left(\int_\Omega |\mathbf{u}_\varepsilon|^{2\gamma'} {\rm d}x\right)^{\frac{1}{\gamma'}} \le \left(\int_\Omega |\mathbf{u}_\varepsilon|^2 {\rm d}x\right)^{\frac{3}{2\gamma'}-\frac{1}{2}} \left(\int_\Omega |\mathbf{u}_\varepsilon|^6 {\rm d}x\right)^{\frac{1}{2}-\frac{3}{6\gamma'}}$$

and

$$\left(\int_\Omega |\mathbf{u}_\varepsilon|^4 {\rm d}x\right)^{\frac{1}{2}} \le \left(\int_\Omega |\mathbf{u}_\varepsilon|^2 {\rm d}x\right)^{\frac{1}{4}} \left(\int_\Omega |\mathbf{u}_\varepsilon|^6 {\rm d}x\right)^{\frac{1}{4}}.$$

Young's inequality allows to conclude, thanks to (2.7). Finally

$$\{\mathbf{u}_\varepsilon\}_{\varepsilon>0} \text{ bounded in } L^2(0,T;W^{1,2}(\Omega;\mathbb{R}^3)). \tag{2.8}$$

In accordance with (2.2) we may assume that

$$[r_\varepsilon]_{\rm ess} \to r \text{ weakly-}(*) \text{ in } L^\infty(0,T;L^2(\Omega)). \tag{2.9}$$

Due to (2.3), (2.4),

$$\int_\Omega \left[\frac{|\varrho_\varepsilon - \overline{\varrho}|^\gamma}{\varepsilon^\gamma}\right]_{\rm res} {\rm d}x \leq c\left(\int_\Omega \left[\frac{\varrho_\varepsilon^\gamma}{\varepsilon^\gamma}\right]_{\rm res} {\rm d}x + \int_\Omega \left[\frac{1}{\varepsilon^\gamma}\right]_{\rm res} {\rm d}x\right) \leq c\varepsilon^{2-\gamma};$$

whence

$$[r_\varepsilon]_{\rm res} \to 0 \text{ in } L^\infty(0,T;L^q(\Omega)) \text{ for any } 1 \leq q < \min\{\gamma, 2\}. \tag{2.10}$$

Moreover, by virtue of (2.8),

$$\mathbf{u}_\varepsilon \to \mathbf{U} \text{ weakly in } L^2(0,T;W^{1,2}(\Omega;\mathbb{R}^3)), \tag{2.11}$$

passing to suitable subsequences as the case may be.

Letting $\varepsilon \to 0$ in the weak formulation of the continuity equation (1.11), with $b \equiv 0$, we obtain

$${\rm div}_x \mathbf{U} = 0 \text{ a.a. in } (0,T)\times\Omega. \tag{2.12}$$

Finally, multiplying momentum balance (1.12) by $\varepsilon$, we recover (1.8)

$$\overline{\varrho}\left[\begin{array}{c} -U_2 \\ U_1 \end{array}\right] = p'(\overline{\varrho})\nabla_h r, \quad \partial_3 r = 0, \tag{2.13}$$

in particular, $r = r(x_1,x_2)$ is independent of the vertical variable, and

$$\mathbf{U}_h = \mathbf{U}_h(x_1,x_2),\ {\rm div}_h \mathbf{U}_h = 0, \tag{2.14}$$

which, together with (2.12), implies $U_3$ is independent of $x_3$. However, as $\mathbf{U}$ satisfies the complete-slip boundary conditions (1.5) on $\partial\Omega$, we may infer that

$$U_3 \equiv 0. \tag{2.15}$$

## 3. Propagation of acoustic waves

Assume from now on, to simplify notation, that $p'(\overline{\varrho}) = 1$. System (1.11), (1.12) can be written in the form

$$\varepsilon\partial_t r_\varepsilon + {\rm div}_x \mathbf{V}_\varepsilon = 0, \tag{3.1}$$

$$\varepsilon\partial_t \mathbf{V}_\varepsilon + (\mathbf{g}\times\mathbf{V}_\varepsilon + \nabla_x r_\varepsilon) = \varepsilon\mathbf{f}_\varepsilon, \tag{3.2}$$

where we have set

$$r_\varepsilon = \frac{\varrho_\varepsilon - \overline{\varrho}}{\varepsilon},\ \mathbf{V}_\varepsilon = \varrho_\varepsilon\mathbf{u}_\varepsilon,$$

and

$$\mathbf{f}_\varepsilon = {\rm div}_x\mathbb{S}(\nabla_x\mathbf{u}_\varepsilon) - {\rm div}_x(\varrho_\varepsilon\mathbf{u}_\varepsilon\otimes\mathbf{u}_\varepsilon) - \frac{1}{\varepsilon^2}\nabla_x\Big(p(\varrho_\varepsilon) - p'(\overline{\varrho})(\varrho_\varepsilon - \overline{\varrho}) - p(\overline{\varrho})\Big).$$

More precisely, system (3.1), (3.2) should be understood in the weak sense:

$$\int_0^T \int_\Omega \Big( \varepsilon r_\varepsilon \partial_t \varphi + \mathbf{V}_\varepsilon \cdot \nabla_x \varphi \Big) \ \mathrm{d}x \ \mathrm{d}t = - \varepsilon \int_\Omega r_{0,\varepsilon} \varphi(0, \cdot) \ \mathrm{d}x \tag{3.3}$$

for any $\varphi \in C^\infty_c([0,T) \times \overline{\Omega})$,

(3.4)

$$\int_0^T \int_\Omega \Big( \varepsilon \mathbf{V}_\varepsilon \cdot \partial_t \varphi - (\mathbf{g} \times \mathbf{V}_\varepsilon) \cdot \varphi + r_\varepsilon \mathrm{div}_x \varphi \Big) \ \mathrm{d}x \ \mathrm{d}t = - \varepsilon \int_0^T < \mathbf{f}_\varepsilon, \varphi > \ \mathrm{d}t$$

$$- \varepsilon \int_\Omega \varrho_{0,\varepsilon} \mathbf{u}_{0,\varepsilon} \cdot \varphi(0, \cdot) \ \mathrm{d}x,$$

for any test function $\varphi \in C^\infty_c([0,T) \times \overline{\Omega}; R^3)$, $\varphi \cdot \mathbf{n}|_{\partial\Omega} = 0$, where

$$- < \mathbf{f}_\varepsilon, \varphi >= \int_\Omega \Big( \mathbb{S}(\nabla_x \mathbf{u}_\varepsilon) : \nabla_x \varphi - (\varrho_\varepsilon \mathbf{u}_\varepsilon \otimes \mathbf{u}_\varepsilon) : \nabla_x \varphi$$

$$- \frac{1}{\varepsilon^2} \Big( p(\varrho_\varepsilon) - p'(\overline{\varrho})(\varrho_\varepsilon - \overline{\varrho}) - p(\overline{\varrho}) \Big) \mathrm{div}_x \varphi \Big) \ \mathrm{d}x.$$

It follows from the uniform bounds established in (2.1 - 2.8) that

$$< \mathbf{f}_\varepsilon, \varphi >= \int_\Omega \Big( \mathbb{F}^1_\varepsilon : \nabla_x \varphi + \mathbb{F}^2_\varepsilon : \nabla_x \varphi \Big) \ \mathrm{d}x, \tag{3.5}$$

with

$$\mathbb{F}^1_\varepsilon = \varrho_\varepsilon \mathbf{u}_\varepsilon \otimes \mathbf{u}_\varepsilon + \frac{1}{\varepsilon^2} \Big( p(\varrho_\varepsilon) - p'(\overline{\varrho})(\varrho_\varepsilon - \overline{\varrho}) - p(\overline{\varrho}) \Big) \mathbb{I},$$

$$\mathbb{F}^2_\varepsilon = - \mathbb{S} \Big( \nabla_x \mathbf{u}_\varepsilon \Big),$$

$$\{ \mathbb{F}^1_\varepsilon \}_{\varepsilon > 0} \text{ bounded in } L^\infty(0,T; L^1(\Omega; R^{3 \times 3})), \tag{3.6}$$

$$\{ \mathbb{F}^2_\varepsilon \}_{\varepsilon > 0} \text{ bounded in } L^2(0,T; L^2(\Omega; R^{3 \times 3})). \tag{3.7}$$

3.1. **Point spectrum of the acoustic propagator.** Consider an operator $\mathcal{B}$ defined, formally, in $L^2(\Omega) \times L^2(\Omega; R^3)$,

$$\mathcal{B} \left[ \begin{array}{c} r \\ \mathbf{V} \end{array} \right] \equiv \left[ \begin{array}{c} \mathrm{div}_x \mathbf{V} \\ \mathbf{g} \times \mathbf{V} + \nabla_x r \end{array} \right].$$

As a matter of fact, it is more convenient to work in the frequency space, meaning, we associate to a function $v$ its Fourier transform $\tilde{v}$

$$\tilde{v} = \tilde{v}(\xi_h, k), \ \xi_h \equiv (\xi_1, \xi_2) \in R^2, \ k \in Z,$$

where

$$\tilde{v}(\xi_h, k) = \frac{1}{\sqrt{2}} \int_{-1}^1 \int_{R^2} \exp \Big( - \mathrm{i}(\xi_h \cdot x_h) \Big) v(x_h, x_3) \ \mathrm{d}x_h \exp(-\mathrm{i}k x_3) \ \mathrm{d}x_3.$$

We investigate the point spectrum of $\mathcal{B}$, meaning, we look for solutions of the eigenvalue problem

$$\mathrm{div}_x \mathbf{V} = \lambda r, \ \nabla_x r + \mathbf{g} \times \mathbf{V} = \lambda \mathbf{V}, \tag{3.8}$$

or, in the Fourier variables,

$$\mathrm{i}\Big( \sum_{j=1}^{2} \xi_j \tilde{V}_j + k \tilde{V}_3 \Big) - \lambda \tilde{r} = 0, \ \mathrm{i}[\xi_1, \xi_2, k] \tilde{r} - [\tilde{V}_2, -\tilde{V}_1, 0] - \lambda \tilde{\mathbf{V}} = 0.$$

After a bit tedious but straightforward manipulation, we obtain

$$\lambda^2 = -\mu, \ \mu = \frac{1 + |\xi|^2 + k^2 \pm \sqrt{(1 + |\xi|^2 + k^2)^2 - 4k^2}}{2}; \tag{3.9}$$

whence the only eigenvalue is $\lambda = 0$, for which $k = 0$, and consequently, the space of eigenvectors coincides with the null-space of $\mathcal{B}$,

$$\mathrm{Ker}(\mathcal{B}) = \Big\{ [r, \mathbf{V}] \ \Big| \ r = r(x_1, x_2), \tag{3.10}$$

$$\mathbf{V} = [V_1(x_1, x_2), V_2(x_1, x_2), V_3(x_1, x_2)], \ \mathrm{div}_h \mathbf{V}_h = 0, \ \nabla_h r = [V_2, -V_1] \Big\}.$$

3.2. **RAGE theorem.** Our goal is to show that the component of the field $[r_\varepsilon, \mathbf{V}_\varepsilon]$, orthogonal to the null space $\mathrm{Ker}(\mathcal{B})$ decays to zero on any compact subset of $\Omega$. To this end, we use the celebrated RAGE theorem in the following form (see Cycon et al. [4, Theorem 5.8]):

**Theorem 3.1.** *Let $H$ be a Hilbert space, $A : \mathcal{D}(A) \subset H \to H$ a self-adjoint operator, $C : H \to H$ a compact operator, and $P_{\mathrm{cont}}$ the orthogonal projection onto $H_{\mathrm{cont}}$, where*

$$H = H_{\mathrm{cont}} \oplus \mathrm{closure}_H \Big\{ \mathrm{span}\{ w \in H \mid w \textit{ an eigenvector of } A \} \Big\}.$$

*Then*

$$\left\| \frac{1}{\tau} \int_0^\tau \exp(-\mathrm{i}tA) C P_{\mathrm{cont}} \exp(\mathrm{i}tA) \ \mathrm{d}t \right\|_{\mathcal{L}(H)} \to 0 \textit{ for } \tau \to \infty.$$

In addition to the hypotheses of Theorem 3.1, suppose that $C$ is non-negative and self-adjoint in $H$. Thus we may write

$$\frac{1}{T} \int_0^T \left\langle \exp\left(-\mathrm{i}\frac{t}{\varepsilon} A\right) C \exp\left(\mathrm{i}\frac{t}{\varepsilon} A\right) P_{\mathrm{cont}} X, Y \right\rangle_H \mathrm{d}t \le h(\varepsilon) \|X\|_H \|Y\|_H,$$

where $h(\varepsilon) \to 0$ as $\varepsilon \to 0$. Taking $Y = P_{\mathrm{cont}} X$ we deduce

$$\frac{1}{T} \int_0^T \left\| \sqrt{C} \exp\left(\mathrm{i}\frac{t}{\varepsilon} A\right) P_{\mathrm{cont}} X \right\|_H^2 \mathrm{d}t \le h(\varepsilon) \|X\|_H^2. \tag{3.11}$$

Similarly, for $X \in L^2(0,T;H)$, we have

$$\frac{1}{T^2}\left\|\sqrt{C}P_{\rm cont}\int_0^t \exp\left({\rm i}\frac{t-s}{\varepsilon}A\right)X(s)\ {\rm d}s\right\|^2_{L^2(0,T;H)} \tag{3.12}$$

$$\leq \frac{1}{T}\int_0^T\int_0^T\left\|\sqrt{C}\exp\left({\rm i}\frac{t-s}{\varepsilon}A\right)P_{\rm cont}X(s)\right\|^2_H\ {\rm d}t\ {\rm d}s$$

$$\leq h(\varepsilon)\int_0^T\left\|\exp\left(-{\rm i}\frac{s}{\varepsilon}A\right)X(s)\right\|^2_H\ {\rm d}s = h(\varepsilon)\int_0^T\|X(s)\|^2\ {\rm d}s.$$

3.3. **Application of RAGE theorem.** For a fixed $M>0$, we introduce a Hilbert space

$$H = H_M \equiv \{[r,\mathbf{V}]\mid \tilde{r}(\xi_h,k)=0, \tilde{\mathbf{V}}(\xi_h,k)=0 \text{ whenever } |\xi_h|+|k|>M\}.$$

Let

$$P_M : L^2(\Omega)\times L^2(\Omega;\mathbb{R}^3)\to H_M$$

denote the associated orthogonal projection onto $H_M$.

Our goal is to apply RAGE theorem to the operators

$$A = {\rm i}\mathcal{B},\ C[v] = P_M[\chi v],\ \chi\in C^\infty_c(\Omega),\ 0\leq\chi\leq 1,$$

considered on the Hilbert space $H_M$. Indeed $C$ is clearly self-adjoint and non-negative, and its compactness is a consequence of the Rellich-Kondrachov theorem (noticing that the range of $C$ is included in the space of $H^1$, rapidly decaying functions).

Going back to system (3.3), (3.4), we obtain that

$$\varepsilon\frac{\rm d}{{\rm d}t}\left[\begin{array}{c} r_{\varepsilon,M}\\ \mathbf{V}_{\varepsilon,M}\end{array}\right] + \mathcal{B}\left[\begin{array}{c} r_{\varepsilon,M}\\ \mathbf{V}_{\varepsilon,M}\end{array}\right] = \varepsilon\left[\begin{array}{c} 0\\ \mathbf{f}_{\varepsilon,M}\end{array}\right], \tag{3.13}$$

where

$$[r_{\varepsilon,M},\mathbf{V}_{\varepsilon,M}] = P_M[r_\varepsilon,\mathbf{V}_\varepsilon],$$

and

$$\left[\begin{array}{c} 0\\ \mathbf{f}_{\varepsilon,M}\end{array}\right]\in H^*_M\approx H_M,$$

$$\left\langle\left[\begin{array}{c} 0\\ \mathbf{f}_{\varepsilon,M}\end{array}\right],\left[\begin{array}{c} s\\ \mathbf{w}\end{array}\right]\right\rangle_{H_M} = -\int_\Omega\left(\mathbb{F}^1_\varepsilon : \nabla_x\mathbf{w} + \mathbb{F}^2_\varepsilon : \nabla_x\mathbf{w}\right)\ {\rm d}x$$

whenever $(s,w)\in H_M$. Since

$$\|\mathbf{w}\|_{W^{m,\infty}\cap W^{m,2}(\Omega;\mathbb{R}^3)}\leq c(m)\|\mathbf{w}\|_{W^{m+2,2}(\Omega;\mathbb{R}^3)}\leq cM^{m+2}\|\mathbf{w}\|_{L^2(\Omega;\mathbb{R}^3)},$$

we may use the uniform bounds (3.6), (3.7) in order to conclude that

$$\left\|\left[\begin{array}{c} 0\\ \mathbf{f}_{\varepsilon,M}\end{array}\right]\right\|_{L^2(0,T;H_M)}\leq c(M) \tag{3.14}$$

uniformly for $\varepsilon\to 0$.

Writing solutions to (3.13) by means of Duhamel's formula we get

$$\begin{bmatrix} r_{\varepsilon,M} \\ \mathbf{V}_{\varepsilon,M} \end{bmatrix} = \exp(\mathrm{i}A\frac{t}{\varepsilon}) \begin{bmatrix} r_{\varepsilon,M}(0) \\ \mathbf{V}_{\varepsilon,M}(0) \end{bmatrix} + \int_0^t \exp\left(\mathrm{i}\frac{t-s}{\varepsilon}A\right) \begin{bmatrix} 0 \\ \mathbf{f}_{\varepsilon,M} \end{bmatrix} \, \mathrm{d}s; \tag{3.15}$$

whence a direct application of (3.11), (3.12), recalling that the only point spectrum is reduced to 0, yields

$$Q^{\perp} \begin{bmatrix} r_{\varepsilon,M} \\ \mathbf{V}_{\varepsilon,M} \end{bmatrix} \to 0 \text{ in } L^2((0,T)\times K;\mathbb{R}^4)) \text{ as } \varepsilon \to 0, \tag{3.16}$$

for any compact $K \subset \overline{\Omega}$ and any fixed $M$, where we have denoted

$$Q : L^2(\Omega) \times L^2(\Omega;\mathbb{R}^3) \to \mathrm{Ker}(\mathcal{B})$$

the orthogonal projection onto the null space of $\mathcal{B}$. Indeed observe that

$$\left\| \sqrt{C} Q^{\perp} \begin{bmatrix} r_{\varepsilon,M} \\ \mathbf{V}_{\varepsilon,M} \end{bmatrix} \right\|_{H_M}^2 = \left\langle C Q^{\perp} \begin{bmatrix} r_{\varepsilon,M} \\ \mathbf{V}_{\varepsilon,M} \end{bmatrix}, Q^{\perp} \begin{bmatrix} r_{\varepsilon,M} \\ \mathbf{V}_{\varepsilon,M} \end{bmatrix} \right\rangle_{H_M}$$

$$= \int_{\Omega} \chi \left| Q^{\perp} \begin{bmatrix} r_{\varepsilon,M} \\ \mathbf{V}_{\varepsilon,M} \end{bmatrix} \right|^2 \, \mathrm{d}x,$$

where we have used the fact that $P_M$ and $Q$ commute, and the time integration is made possible by use of the Lebesgue dominated convergence theorem.

A direct inspection of (3.15), using also (3.14), shows that the sequence $\partial_t Q \begin{bmatrix} r_{\varepsilon,M} \\ \mathbf{V}_{\varepsilon,M} \end{bmatrix}$ is bounded in $L^2(0,T;H_M)$. Finally, since $H_M$ is compactly imbedded in $L^2(K;\mathbb{R}^3)$ for any fixed $M$ and any compact $K \subset \Omega$, Ascoli-Arzela's theorem yields, in particular,

$$Q \begin{bmatrix} r_{\varepsilon,M} \\ \mathbf{V}_{\varepsilon,M} \end{bmatrix} \to \begin{bmatrix} r_M \\ \overline{\varrho}\mathbf{U}_M \end{bmatrix} \text{ in } L^2((0,T)\times K;\mathbb{R}^4)) \text{ as } \varepsilon \to 0, \tag{3.17}$$

where $r$ and $\mathbf{U}$ are the asymptotic limits identified through (2.9 - 2.13).

3.4. **Strong convergence of the velocity fields.** Relations (3.16), (3.17), together with (2.8 - 2.11), may be used to obtain the desired conclusion

$$\mathbf{u}_\varepsilon \to \mathbf{U} \text{ in } L^2((0,T)\times K;\mathbb{R}^3) \text{ for any compact } K \subset \Omega. \tag{3.18}$$

In fact, since

$$\overline{\varrho} P_M[\mathbf{u}_\varepsilon] = \varepsilon P_M\Big[\frac{\overline{\varrho} - \varrho_\varepsilon}{\varepsilon}\mathbf{u}_\varepsilon\Big] + P_M[\varrho_\varepsilon \mathbf{u}_\varepsilon],$$

we obtain, by virtue of (2.9), (2.10), (3.16), (3.17),

$$P_M[\mathbf{u}_\varepsilon] \to P_M[\mathbf{U}] \text{ in } L^2((0,T)\times K;\mathbb{R}^3)$$

for any fixed $M$, which, together with (2.11) yields (3.18). Indeed we have

$$\mathbf{u}_\varepsilon = P_M[\mathbf{u}_\varepsilon] + [\mathbb{I} - P_M][\mathbf{u}_\varepsilon],$$

where

$$\int_\Omega \Big| [\mathbb{I} - P_M][\mathbf{u}_\varepsilon] \Big|^2 \mathrm{d}x = \underbrace{\sum \int}_{|\xi_h|+|k|\geq M} |\widetilde{\mathbf{u}_\varepsilon}|^2 \, \mathrm{d}\xi_h$$

$$= \underbrace{\sum \int}_{|\xi_h|+|k|\geq M} \frac{(|\xi_h|+|k|)^2}{(|\xi_h|+|k|)^2} |\widetilde{\mathbf{u}_\varepsilon}|^2 \, \mathrm{d}\xi_h \leq \frac{c}{M^2} \|\mathbf{u}_\varepsilon\|^2_{W^{1,2}(\Omega;\mathbb{R}^3)}.$$

## 4. The limit system

4.1. **Identifying the limit system.** With the convergence established in (2.9 - 2.11), and (3.18), it is not difficult to pass to the limit in the weak formulation (1.11), (1.12). To this end, we take

$$\varphi \equiv [\nabla_h^\perp \psi, 0], \;\; \psi \in C_c^\infty([0,T)\times\Omega)$$

as a test function in momentum equation (1.12) to obtain

$$\int_0^T \int_\Omega \Big( \varrho_\varepsilon \mathbf{u}_\varepsilon \cdot \partial_t \varphi + \varrho_\varepsilon \mathbf{u}_\varepsilon \otimes \mathbf{u}_\varepsilon : \nabla_x \varphi - \frac{1}{\varepsilon} \varrho_\varepsilon [\mathbf{u}_\varepsilon]_h \cdot \nabla_x \psi \Big) \, \mathrm{d}x \, \mathrm{d}t \tag{4.1}$$

$$= -\int_\Omega \varrho_{0,\varepsilon} \mathbf{u}_{0,\varepsilon} \cdot \varphi(0,\cdot) \, \mathrm{d}x + \int_0^T \int_\Omega \mathbb{S}(\nabla_x \mathbf{u}_\varepsilon) : \nabla_x \varphi \, \mathrm{d}x \, \mathrm{d}t.$$

Moreover, (3.3) yields

$$\int_0^T \int_\Omega \Big( r_\varepsilon \partial_t \psi + \frac{1}{\varepsilon} \varrho_\varepsilon [\mathbf{u}_\varepsilon]_h \cdot \nabla_x \psi \Big) \, \mathrm{d}x = -\int_\Omega r_{0,\varepsilon} \psi(0,\cdot) \, \mathrm{d}x. \tag{4.2}$$

Letting $\varepsilon \to 0$ in (4.1), (4.2) we may infer that

$$\int_0^T \int_\Omega \Big( \overline{\varrho} \mathbf{U}_h \cdot \partial_t \nabla_h^\perp \psi + \overline{\varrho} [\mathbf{U}_h \otimes \mathbf{U}_h] : \nabla_x (\nabla_h^\perp \psi) + r \partial_t \psi \Big) \, \mathrm{d}x$$

$$= -\int_\Omega \Big( \overline{\varrho} \mathbf{U}_{0,h} \cdot \nabla_h^\perp \psi(0,\cdot) + r_0 \psi(0,\cdot) \Big) \, \mathrm{d}x$$

$$+ \int_0^T \int_\Omega \mu \nabla_h \mathbf{U}_h : \nabla(\nabla_h^\perp \psi) \, \mathrm{d}x \, \mathrm{d}t.$$

Moreover, as the limit functions are independent of $x_3$, we get,

$$(4.3)\quad \int_0^T \int_{\mathbb{R}^2} \Big( \overline{\varrho}\mathbf{U}_h \cdot \partial_t \nabla_h^\perp \psi + \overline{\varrho}[\mathbf{U}_h \otimes \mathbf{U}_h] : \nabla_h(\nabla_h^\perp \psi) + r\partial_t \psi \Big) \, \mathrm{d}x_h \, \mathrm{d}t$$

$$= - \int_{\mathbb{R}^2} \left( \overline{\varrho} \left( \int_0^1 \mathbf{U}_{0,h} \, \mathrm{d}x_3 \right) \cdot \nabla_h^\perp \psi(0,\cdot) + \left( \int_0^1 r_0 \, \mathrm{d}x_3 \right) \psi(0,\cdot) \right) \, \mathrm{d}x_h$$

$$+ \int_0^T \int_{\mathbb{R}^2} \mu \nabla_h \mathbf{U}_h : \nabla_h(\nabla_h^\perp \psi) \, \mathrm{d}x_h \, \mathrm{d}t$$

for all $\psi \in C_c^\infty([0,T) \times \mathbb{R}^2)$.

Finally, by virtue of (1.8), $\mathbf{U}_h = \nabla_h^\perp r$, and (4.3) coincides with a weak formulation of (1.10), (1.15). We have completed the proof of the convergence result, up to a subsequence, of Theorem 1.1.

4.2. **Uniqueness for the limit system.** In this final section we shall prove that the limit system has a unique solution provided the initial data are more regular. In order to do so we shall simply write an energy-type estimate on the difference of two solutions, called $r_1$ and $r_2$, associated with two initial data $\tilde{r}_1$ and $\tilde{r}_2$. This will provide a stability estimate, whose immediate consequence will be a uniqueness result. Notice that the diagnostic equation (1.8) implies that $\tilde{r}$ should be taken in $W^{1,2}(\mathbb{R}^2)$.

The limit system writes

$$\partial_t(\Delta_h r - r) + \nabla_h^\perp r \cdot \nabla_h(\Delta_h r) = \frac{\mu}{\overline{\varrho}} \Delta_h^2 r$$

recalling that for simplicity we have chosen $p'(\overline{\varrho}) = 1$. Multiplying (formally) this equation by $\Delta_h r$ and integrating over $\mathbb{R}^2$ yields

$$\frac{d}{dt}\Big( \|\Delta_h r\|_{L^2}^2 + \|\nabla_h r\|_{L^2}^2 \Big) + \frac{\mu}{\overline{\varrho}} \|\nabla_h \Delta_h r\|_{L^2}^2 = 0,$$

whence the estimate

$$\|\Delta_h r(t)\|_{L^2}^2 + \|\nabla_h r(t)\|_{L^2}^2 + \frac{2\mu}{\overline{\varrho}} \int_0^t \|\nabla_h \Delta_h r(t')\|_{L^2}^2 \, dt' = \|\Delta_h \tilde{r}\|_{L^2}^2 + \|\nabla_h \tilde{r}\|_{L^2}^2.$$

Now suppose $r_1$ and $r_2$ are two solutions as described above, and define $\delta := r_1 - r_2$. Then of course $\delta$ satisfies

$$\partial_t(\Delta_h \delta - \delta) + \nabla_h^\perp \delta \cdot \nabla_h(\Delta_h r_2) + \nabla_h^\perp r_1 \cdot \nabla_h(\Delta_h \delta) = \frac{\mu}{\overline{\varrho}} \Delta_h^2 \delta$$

with initial data $\delta^0 = \tilde{r}_1 - \tilde{r}_2$. Writing a similar energy estimate to the one above yields formally

$$\frac{d}{dt}\Big(\|\Delta_h\delta\|_{L^2}^2 + \|\nabla_h\delta\|_{L^2}^2\Big) + \frac{2\mu}{\overline{\varrho}}\|\nabla_h\Delta_h\delta\|_{L^2}^2 = -\int_{\mathbb{R}^2} \nabla_h^\perp\delta \cdot \nabla_h(\Delta_h r_1)\Delta_h\delta\, dx.$$

Then we simply write, by Hölder's inequality followed by Gagliardo-Nirenberg's inequality

$$\Big|\int_{\mathbb{R}^2} \nabla_h^\perp\delta \cdot \nabla_h(\Delta_h r_1)\Delta_h\delta\, dx\Big| \leq \|\nabla_h^\perp\delta\|_{L^4}\|\nabla_h\Delta_h r_1\|_{L^2}\|\Delta_h\delta\|_{L^4}$$
$$\leq C\|\nabla_h\delta\|_{L^2}^{\frac12}\|\Delta_h\delta\|_{L^2}^{\frac12}\|\nabla_h\Delta_h r_1\|_{L^2}\|\Delta_h\delta\|_{L^2}^{\frac12}\|\nabla_h\Delta_h\delta\|_{L^2}^{\frac12}.$$

This implies that

$$\Big|\int_{\mathbb{R}^2} \nabla_h^\perp\delta \cdot \nabla_h(\Delta_h r_1)\Delta_h\delta\, dx\Big| \leq \frac{\mu}{\overline{\varrho}}\|\nabla_h\Delta_h\delta\|_{L^2}^2 + \|\nabla_h\delta\|_{L^2}^2 + C\sqrt{\frac{\overline{\varrho}}{\mu}}\|\Delta_h\delta\|_{L^2}^2\|\nabla_h\Delta_h r_1\|_{L^2}^2.$$

Finally Gronwall's inequality allows to obtain

$$\|\Delta_h\delta(t)\|_{L^2}^2 + \|\nabla_h\delta(t)\|_{L^2}^2 + \frac{\mu}{\overline{\varrho}}\int_0^t \|\nabla_h\Delta_h\delta(t')\|_{L^2}^2\, dt'$$
$$\leq \Big(\|\Delta_h\delta^0\|_{L^2}^2 + \|\nabla_h\delta^0\|_{L^2}^2\Big)\exp\Big(C\sqrt{\frac{\overline{\varrho}}{\mu}}\int_0^t \|\nabla_h\Delta_h r_1(t')\|_{L^2}^2\, dt' + Ct\Big).$$

This allows to conclude to stability, hence uniqueness for the limit system (leaving the usual regularization procedure to make the above arguments rigorous to the reader) provided the initial datum enjoys the extra regularity stated in Theorem 1.1.

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