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LONG-TIME DERIVATION AT EQUILIBRIUM OF THE FLUCTUATING BOLTZMANN EQUATION

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Abstract. We study a hard sphere gas at equilibrium, and prove that in the low density limit, the fluctuations converge to a Gaussian process governed by the fluctuating Boltzmann equation. This result holds for arbitrarily long times. The method of proof builds upon the weak convergence method introduced in the companion paper [8] which is improved by considering clusters of pseudo-trajectories as in [7].

1. Introduction

In this paper, we prove that dynamical fluctuations in the empirical measure of a hard sphere gas at equilibrium are governed, in the low density limit (Boltzmann-Grad limit), by the fluctuating Boltzmann equation, and this for arbitrarily long kinetic times. In particular, we show that the limiting process is Gaussian. The fluctuating equation is a stochastic equation given by a linearized Boltzmann collision operator, forced by a Gaussian noise, white in space and time, whose structure can be predicted by a fluctuation-dissipation argument [33].

The convergence of the covariance of the fluctuations was proved for short times in [2] and extended to non-equilibrium states in [31]. Moreover, the Gaussian character of the limiting field and the fluctuating equation were conjectured in [14, 31, 32]. This conjecture was reconsidered and proved to be true in [6, 7], where the convergence of the full fluctuation process was obtained by using cumulant techniques, away from equilibrium, together with (much stronger) large deviation bounds.

All these results are severely limited to a small interval of time, exactly in the same way as for the validity of the nonlinear Boltzmann equation, as proved in [24]. This short time limitation was removed first for the covariance of the fluctuation field in [4], in the case of a two-dimensional gas of hard disks at equilibrium. The limiting covariance is governed by the Boltzmann equation, linearized around the Maxwellian distribution. In the companion paper [8], a more robust weak convergence method was introduced: taking advantage of the invariant measure, we discarded atypical dynamics (preventing the convergence) by localizing pathological behaviors and using a time decoupling. This allows to extend the previous result to arbitrary dimensions. The present paper elaborates upon the same strategy: we study the higher order moments of the fluctuation field and prove that they asymptotically factorize according to Gaussian rules.

1.1. **The model.** We consider here exactly the same setting as in [8], of which we recall the notations. The microscopic model consists of identical hard spheres of unit mass and of diameter ε. The motion of N such hard spheres is ruled by a system of ordinary differential equations, which are set in $(\mathbb{T}^d\times\mathbb{R}^d)^N$ where $\mathbb{T}^d$ is the unit d-dimensional periodic box with $d\geq 2$: writing $\mathbf{x}_i^\varepsilon\in\mathbb{T}^d$ for the position of the center of the particle labeled by i and $\mathbf{v}_i^\varepsilon\in\mathbb{R}^d$ for its velocity, one has

$$(1.1)\qquad \frac{d\mathbf{x}_i^\varepsilon}{dt}=\mathbf{v}_i^\varepsilon\,,\quad \frac{d\mathbf{v}_i^\varepsilon}{dt}=0\quad \text{as long as}\quad |\mathbf{x}_i^\varepsilon(t)-\mathbf{x}_j^\varepsilon(t)|>\varepsilon\quad\text{for}\quad 1\leq i\neq j\leq N\,,$$

with specular reflection at collisions:

$$\left.\begin{aligned}(\mathbf{v}_i^\varepsilon)' &:= \mathbf{v}_i^\varepsilon - \frac{1}{\varepsilon^2}(\mathbf{v}_i^\varepsilon - \mathbf{v}_j^\varepsilon)\cdot(\mathbf{x}_i^\varepsilon - \mathbf{x}_j^\varepsilon)\,(\mathbf{x}_i^\varepsilon - \mathbf{x}_j^\varepsilon)\\ \big(\mathbf{v}_j^\varepsilon\big)' &:= \mathbf{v}_j^\varepsilon + \frac{1}{\varepsilon^2}(\mathbf{v}_i^\varepsilon - \mathbf{v}_j^\varepsilon)\cdot(\mathbf{x}_i^\varepsilon - \mathbf{x}_j^\varepsilon)\,(\mathbf{x}_i^\varepsilon - \mathbf{x}_j^\varepsilon)\end{aligned}\right\}\quad \text{if } |\mathbf{x}_i^\varepsilon(t) - \mathbf{x}_j^\varepsilon(t)| = \varepsilon\,. \tag{1.2}$$

This flow does not cover all possible situations, as multiple collisions are excluded. But one can show (see [1]) that for almost every admissible initial configuration $(\mathbf{x}_i^{\varepsilon 0}, \mathbf{v}_i^{\varepsilon 0})_{1\leq i\leq N}$, there are neither multiple collisions, nor accumulation of collision times, so that the dynamics is globally well defined.

We will not be interested here in one specific realization of the dynamics, but rather in a statistical description. This is achieved by introducing a measure at time 0, on the phase space we now specify. The collections of N positions and velocities are denoted respectively by $X_N := (x_1, \dots, x_N)$ in $\mathbb{T}^{dN}$ and $V_N := (v_1, \dots, v_N)$ in $\mathbb{R}^{dN}$, and we set $Z_N := (X_N, V_N)$, with $Z_N = (z_1, \dots, z_N)$, $z_i = (x_i, v_i)$. A set of N particles is characterized by a random variable $\mathbf{Z}_N^{\varepsilon 0} = (\mathbf{z}_1^{\varepsilon 0}, \dots, \mathbf{z}_N^{\varepsilon 0})$, $\mathbf{z}_i^{\varepsilon 0} = \big(\mathbf{x}_i^{\varepsilon 0}, \mathbf{v}_i^{\varepsilon 0}\big)$ specifying the time-zero configuration in the phase space

$$\mathcal{D}_N^\varepsilon := \big\{ Z_N \in (\mathbb{T}^d \times \mathbb{R}^d)^N \,/\, \forall i \neq j\,, \quad |x_i - x_j| > \varepsilon \big\}\,, \tag{1.3}$$

and an evolution according to the deterministic flow (1.1)-(1.2) (well defined with probability 1)

$$t \mapsto \mathbf{Z}_N^\varepsilon(t) = \big(\mathbf{z}_1^\varepsilon(t), \dots, \mathbf{z}_N^\varepsilon(t)\big)\,, \qquad t > 0\,,$$

with $\mathbf{z}_i^\varepsilon(t) = (\mathbf{x}_i^\varepsilon(t), \mathbf{v}_i^\varepsilon(t))$.

To avoid spurious correlations due to a given total number of particles, we actually consider a grand canonical state (as in [23, 2]), living on the phase space

$$\mathcal{D}^\varepsilon := \bigcup_{N\geq 0} \mathcal{D}_N^\varepsilon$$

(notice that $\mathcal{D}_N^\varepsilon = \varnothing$ for N large). This means that the total number of particles is also a random variable, which we shall denote by $\mathcal{N}$. In the low density regime, referred to as the Boltzmann-Grad scaling, the density (average $\mathcal{N}$) is tuned by the parameter

$$\mu_\varepsilon := \varepsilon^{-(d-1)}\,,$$

ensuring that the mean free path between collisions is of order one [18].

More precisely, at equilibrium the probability density of finding N particles in Z_N is given by

$$\frac{1}{N!} W_N^{\varepsilon,\mathrm{eq}}(Z_N) := \frac{1}{\mathcal{Z}^\varepsilon}\frac{\mu_\varepsilon^N}{N!}\mathbf{1}_{\mathcal{D}_N^\varepsilon}(Z_N)\,\mathcal{M}^{\otimes N}(V_N)\,, \qquad \text{for } N = 0, 1, 2, \dots \tag{1.4}$$

with

$$\mathcal{M}(v) := \frac{1}{(2\pi)^{\frac{d}{2}}}\exp\Big(-\frac{|v|^2}{2}\Big)\,, \qquad \mathcal{M}^{\otimes N}(V_N) = \prod_{i=1}^N \mathcal{M}(v_i)\,,$$

and the partition function is given by

$$\mathcal{Z}^\varepsilon := 1 + \sum_{N\geq 1}\frac{\mu_\varepsilon^N}{N!}\int_{\mathcal{D}_N^\varepsilon}\mathcal{M}^{\otimes N}(V_N)dX_N\,dV_N = 1 + \sum_{N\geq 1}\frac{\mu_\varepsilon^N}{N!}\int_{\mathbb{T}^{dN}}\Big(\prod_{i\neq j}\mathbf{1}_{|x_i - x_j|>\varepsilon}\Big)dX_N\,. \tag{1.5}$$

Here and below, $\mathbf{1}_A$ will be the characteristic function of the set A, and we will also use the symbol $\mathbf{1}_*$ for the characteristic function of the set defined by condition $*$. Notice that, for notational convenience, we work with functions extended to zero outside $\mathcal{D}_N^\varepsilon$.

In the following, the probability of an event A with respect to the equilibrium measure (1.4) will be denoted $\mathbb{P}_\varepsilon^{\rm eq}(A)$, and $\mathbb{E}_\varepsilon^{\rm eq}$ will be the expected value. Definition (1.4) ensures that

$$\mu_\varepsilon^{-1}\mathbb{E}_\varepsilon^{\rm eq}(\mathcal{N}) \to 1$$

as $\mu_\varepsilon \to \infty$, as required.

1.2. **State of the art.** Consider the empirical density of the hard-sphere model:

$$\pi_t^\varepsilon := \frac{1}{\mu_\varepsilon}\sum_{i=1}^{\mathcal{N}} \delta_{\mathbf{z}_i^\varepsilon(t)} \,. \tag{1.6}$$

Under the initial grand canonical measure

$$\frac{1}{\mathcal{Z}^\varepsilon(f^0)}\frac{\mu_\varepsilon^N}{N!}\mathbf{1}_{\mathcal{D}_N^\varepsilon}(Z_N)\,(f^0)^{\otimes N}(Z_N)\,, \qquad \text{for } N=0,1,2,\dots \tag{1.7}$$

where f^0 is a smooth and fast (Gaussian) decaying density and $\mathcal{Z}^\varepsilon(f^0)$ the corresponding partition function, it has been proved by Lanford in [24] that, in the Boltzmann-Grad limit $\mu_\varepsilon \to \infty$, π_t^ε concentrates on the solution of the Boltzmann equation

$$\begin{cases} \partial_t f + v\cdot\nabla_x f = \displaystyle\int_{\mathbb{R}^d}\int_{\mathbb{S}^{d-1}}\Big(f(t,x,w')f(t,x,v') - f(t,x,w)f(t,x,v)\Big)\big((v-w)\cdot\omega\big)_+\,d\omega\,dw\,,\\ f(0,x,v) = f^0(x,v)\end{cases} \tag{1.8}$$

where the precollisional velocities (v',w') are defined by the scattering law

$$v' := v - \big((v-w)\cdot\omega\big)\,\omega\,, \qquad w' := w + \big((v-w)\cdot\omega\big)\,\omega\,. \tag{1.9}$$

More precisely, there exists a short time $T_L > 0$ depending only on f^0, such that for any test function $h : \mathbb{T}^d \times \mathbb{R}^d \to \mathbb{R}$ and any $\delta > 0$, $t \in [0, T_L]$,

$$\mathbb{P}_\varepsilon\left(\Big|\pi_t^\varepsilon(h) - \int_{\mathbb{T}^d\times\mathbb{R}^d} dz f(t,z)h(z)\Big| > \delta\right) \xrightarrow[\mu_\varepsilon\to\infty]{} 0\,, \tag{1.10}$$

which can be interpreted as a law of large numbers; see e.g. [20, 33, 12, 15, 28, 13, 5, 16, 17].

Using the invariance of the equilibrium measure (1.4), it is not hard to see that in our setting π_t^ε concentrates on $\mathcal{M}$, which is a stationary solution of the Boltzmann equation: for any test function $h : \mathbb{T}^d \times \mathbb{R}^d \to \mathbb{R}$ and any $\delta > 0$, $t \in \mathbb{R}$,

$$\mathbb{P}_\varepsilon^{\rm eq}\left(\Big|\pi_t^\varepsilon(h) - \int_{\mathbb{T}^d\times\mathbb{R}^d} dz \mathcal{M}(v)h(z)\Big| > \delta\right) \xrightarrow[\mu_\varepsilon\to\infty]{} 0\,. \tag{1.11}$$

Our purpose here is to study the fluctuations of the empirical density π_t^ε from its equilibrium value. In this regime, the collision operator in Eq. (1.8) is expected to reduce to the linearized operator (according to $f = \mathcal{M} + g$)

$$\begin{aligned}\mathcal{L}g &:= -v\cdot\nabla_x g\\ &\quad + \int_{\mathbb{R}^d\times\mathbb{S}^{d-1}}\big[\mathcal{M}(w')g(v') + g(w')\mathcal{M}(v') - \mathcal{M}(w)g(v) - g(w)\mathcal{M}(v)\big]\big((v-w)\cdot\omega\big)_+ dw\,d\omega\,.\end{aligned} \tag{1.12}$$

Moreover, such fluctuations should be of size $1/\sqrt{\mu_\varepsilon}$, which leads to define the fluctuation field $\zeta^{\varepsilon,\rm eq}$ by

$$\zeta_t^{\varepsilon,\rm eq}\big(h\big) := \sqrt{\mu_\varepsilon}\Big(\pi_t^\varepsilon(h) - \mathbb{E}_\varepsilon^{\rm eq}\big(\pi_t^\varepsilon(h)\big)\Big)\,, \tag{1.13}$$

for any test function h.

Making the analysis of (1.11) slightly more quantitative one easily proves that, for any given t, $\zeta_t^{\varepsilon,\mathrm{eq}}$ converges in law towards a Gaussian white noise ζ with zero mean and covariance

$$\mathbb{E}\Big(\zeta(h^{(1)})\zeta(h^{(2)})\Big) = \int_{\mathbb{T}^d\times\mathbb{R}^d} h^{(1)}(z)h^{(2)}(z)\mathcal{M}(v)dz\,. \tag{1.14}$$

Much more interesting is the analysis of time-correlations, i.e. of time-dependent products such as $\mathbb{E}_\varepsilon^{\mathrm{eq}}\big(\zeta_t^{\varepsilon,\mathrm{eq}}\big(h^{(1)}\big)\zeta_0^{\varepsilon,\mathrm{eq}}\big(h^{(2)}\big)\big)$, which involves an accurate understanding of the collisional processes in the hard-sphere dynamics. A local-in-time result for the covariance of the fluctuation field was obtained in [2], by a direct application of the method of [24] (as discussed below, this was extended in [4] and [8] to large times close to equilibrium). The techniques of dynamical clusters introduced in [7] allowed then to extend this short time convergence to moments of arbitrary order, as well as to establish the tightness property of the fluctuation process (see also [6] for a less technical presentation of this method).

Recall that (1.12) is well-defined in L^2. We also introduce

$$L^2_{\mathcal{M}} := \Big\{g:\mathbb{T}^d\times\mathbb{R}^d\to\mathbb{R}\,,\qquad \|g\|_{L^2_{\mathcal{M}}} := \Big(\int_{\mathbb{T}^d\times\mathbb{R}^d}|g|^2\,\mathcal{M}dxdv\Big)^{\frac12}<\infty\Big\}$$

and the Hilbert space indexed by $k\in\mathbb{Z}$

$$\mathcal{H}^k := \Big\{g:\mathbb{T}^d\times\mathbb{R}^d\to\mathbb{R}\,,\qquad \|g\|_{\mathcal{H}^k} := \big\|(\mathrm{Id}-\Delta_v+|v|^2-\Delta_x)^k g\big\|_{L^2_{\mathcal{M}}}<\infty\Big\}\,.$$

Theorem 1.1 (**Short time convergence of the fluctuating field**, [2, 7])**.** *Consider a system of hard spheres at equilibrium in a d-dimensional periodic box with $d\geq 2$. There exists a time $T^\star>0$ such that, in the Boltzmann-Grad limit $\mu_\varepsilon\to\infty$, the following properties hold true.*

(a) Let g^0 and h be two functions in $L^2_{\mathcal{M}}$. The covariance of the fluctuation field $(\zeta_t^{\varepsilon,\mathrm{eq}})_{t\in[0,T^\star]}$ converges:

$$\mathbb{E}_\varepsilon^{\mathrm{eq}}\Big[\zeta_0^{\varepsilon,\mathrm{eq}}(g^0)\;\zeta_t^{\varepsilon,\mathrm{eq}}(h)\Big]\to\int\mathcal{M}\,g(t)\,h\,dxdv\,, \tag{1.15}$$

where $\mathcal{M}g$ is the solution of the linearized Boltzmann equation $\partial_t\mathcal{M}g=\mathcal{L}\mathcal{M}g$, with $g_{|t=0}=g^0$.

(b) There exists $k>0$ such that the family of processes $(\zeta_t^{\varepsilon,\mathrm{eq}})_{t\in[0,T^\star]}$ is tight in the Skorokhod space $D\big([0,T^\star],\mathcal{H}^{-k}\big)$. More precisely,

$$\begin{aligned}&\lim_{\delta\to0^+}\lim_{\mu_\varepsilon\to\infty}\mathbb{P}_\varepsilon^{\mathrm{eq}}\Big[\sup_{\substack{|s-t|\leq\delta\\ s,t\in[0,T^\star]}}\big\|\zeta_t^{\varepsilon,\mathrm{eq}}-\zeta_s^{\varepsilon,\mathrm{eq}}\big\|_{-k}\geq\delta'\Big]=0\,,\qquad\forall\delta'>0\,,\\ &\lim_{A\to\infty}\lim_{\mu_\varepsilon\to\infty}\mathbb{P}_\varepsilon^{\mathrm{eq}}\Big[\sup_{t\in[0,T^\star]}\big\|\zeta_t^{\varepsilon,\mathrm{eq}}\big\|_{-k}\geq A\Big]=0\,.\end{aligned} \tag{1.16}$$

(c) The fluctuation field $(\zeta_t^{\varepsilon,\mathrm{eq}})_{t\in[0,T^\star]}$ converges in law to the (weak) solution of the fluctuating Boltzmann equation

$$d\zeta_t=\mathcal{L}\,\zeta_t\,dt+d\eta_t\,, \tag{1.17}$$

with initial datum (1.14), where $d\eta_t(x,v)$ is a stationary Gaussian noise.

The noise in the above equation is explicitly characterized (see [32]). It has zero mean and covariance (for all $T>0$)

$$\begin{aligned}&\mathbb{E}\Big(\int_0^T dt_1\int dz_1 h^{(1)}(z_1)\eta_{t_1}(z_1)\int_0^T dt_2\int dz_2\,h^{(2)}(z_2)\eta_{t_2}(z_2)\Big)\\ &\qquad=\frac12\int_0^T dt\int d\mu(z_1,z_2,\omega)\mathcal{M}(v_1)\,\mathcal{M}(v_2)\Delta h^{(1)}\,\Delta h^{(2)}\end{aligned} \tag{1.18}$$

denoting

$$d\mu(z_1,z_2,\omega) := \delta_{x_1-x_2}\big((v_1-v_2)\cdot\omega\big)_+d\omega\,dv_1\,dv_2dx_1dx_2$$

and defining for any h

$$\Delta h(z_1, z_2, \omega) := h(z_1') + h(z_2') - h(z_1) - h(z_2)\,,$$

where $z_i' := (x_i, v_i')$ with notation (1.9) for the velocities obtained upon scattering. Note that this noise is white in time and space, but correlated in velocities.

Weak solutions of Eq. (1.17) have been discussed in [7]. They are martingale solutions defined according to a classical procedure [19].

Remark 1.1. *Theorem* 1.1 *has been generalized out of equilibrium (see* [31] *for part (a) and* [7] *for parts (b)-(c)) for initial measures of type (1.7). The fluctuation field is still defined by Eq.* (1.13)*. The fluctuating Boltzmann equation is the linearized Boltzmann equation around the solution* $f(t)$ *of the Boltzmann equation with initial datum* f^0*, forced by a noise with a time-dependent covariance of the form* (1.18)*, with* $\mathcal{M}$ *replaced by* $f(t)$*.*

For more discussions on physical aspects of the fluctuation theory at low density, and on related mathematical results, we refer to [14, 22, 25, 29, 31, 32] *(see also* [7] *and the references therein).*

Using the invariant measure, it was shown in [4] in two space dimensions, and [8] in higher dimensions, that the convergence result (1.15) actually holds for all times.

Theorem 1.2 (**Long time convergence of the fluctuating field**, [8])**.** *Consider a system of hard spheres at equilibrium in a d-dimensional periodic box with* $d \geq 3$*. Let* g^0 *and* h *be two functions in* $L^2_{\mathcal{M}}$*. Then in the Boltzmann-Grad limit* $\mu_\varepsilon \to \infty$*, the covariance of the fluctuation field* $(\zeta_t^{\varepsilon,\mathrm{eq}})_{t\geq 0}$ *converges for all times*

$$\forall t \geq 0\,, \quad \mathbb{E}_\varepsilon^{\mathrm{eq}}\Big[\zeta_0^{\varepsilon,\mathrm{eq}}(g^0)\,\zeta_t^{\varepsilon,\mathrm{eq}}(h)\Big] \to \int \mathcal{M}\,g(t)\,h\,dxdv\,,$$

where $\mathcal{M}g$ *is the solution of the linearized Boltzmann equation* $\partial_t \mathcal{M}g = \mathcal{L}\mathcal{M}g$*, with* $g_{|t=0} = g^0$*.*

1.3. **Statement of the result.** Our goal in this paper is to build upon the techniques introduced in [8] to extend the validity of Theorem 1.1 to arbitrarily large times. To reach longer time scales, we devise a method of proof different from the one in [7]: we actually combine the cumulant technique of [7] (controlling locally the small correlations induced by the hard-sphere dynamics) with the weak convergence method introduced in [8], allowing to make an efficient use of the invariant measure and thus providing the long time convergence of the covariance of the fluctuation field.

We remark preliminarily, that at equilibrium, the tightness property (1.16) can be readily generalized to arbitrary times. Indeed splitting an arbitrary time interval $[0, \Theta]$ into subintervals of length $T^\star$, using a union bound and the time invariance of the equilibrium measure, we get

$$\mathbb{P}_\varepsilon^{\mathrm{eq}}\Big[\sup_{\substack{|s-t|\leq\delta \\ s,t\in[0,\Theta]}} \big\|\zeta_t^{\varepsilon,\mathrm{eq}} - \zeta_s^{\varepsilon,\mathrm{eq}}\big\|_{-k} \geq \delta'\Big] \leq 2\frac{\Theta}{T^\star}\mathbb{P}_\varepsilon^{\mathrm{eq}}\Big[\sup_{\substack{|s-t|\leq\delta \\ s,t\in[0,T^\star]}} \big\|\zeta_t^{\varepsilon,\mathrm{eq}} - \zeta_s^{\varepsilon,\mathrm{eq}}\big\|_{-k} \geq \delta'\Big]\,. \tag{1.19}$$

Thus the short time fluctuations in $[0, \Theta]$ can be controlled by (1.16), and the same is true for the norm $\|.\|_{-k}$ of the field.

Our goal here is thus to extend the result (c) of Theorem 1.1, and this will be done by going one step further in the weak convergence method of [8] looking at clusters of trajectories to identify the fluctuation structure, and to combine it with a suitable iteration procedure, allowing to extend the convergence result (1.15) to moments of the fluctuation field of order $P > 2$. We shall prove that such moments are vanishing for P odd, and converging to sums of products of covariances for P even, in agreement with the Wick rule. This, together with the tightness property, will imply the convergence to the fluctuating Boltzmann equation.

Theorem 1.3 (**Long time convergence of the fluctuating field).** *Consider a system of hard spheres at equilibrium in a d-dimensional periodic box with $d \geq 3$. Then, in the Boltzmann-Grad limit $\mu_\varepsilon \to \infty$, the fluctuation field $(\zeta_t^{\varepsilon,\mathrm{eq}})_{t\geq 0}$ converges in law for all times to the solution of the fluctuating Boltzmann equation*

$$d\zeta_t = \mathcal{L}\,\zeta_t\,dt + d\eta_t\,, \tag{1.20}$$

with initial datum (1.14).

Remark 1.2. *For simplicity we have chosen to state and prove the result in dimension $d \geq 3$ only: the geometric arguments needed to control pathological trajectories require indeed a specific treatment in the case of two space dimensions and we prefer to leave out these additional technicalities in this paper.*

Remark 1.3. *The moments of the fluctuation field remain actually under control on time intervals with size diverging slowly with ε, as $O\big(\log\log|\log\varepsilon|\big)$, as will be made clear by the quantitative convergence estimate in Proposition* 2.1. *Note that in this regime the hydrodynamical limit holds true as shown in* [4]. *The derivation of the fluctuating hydrodynamics of a Boltzmann gas is the subject of the third companion paper* [9] *(see also Chapter* 7 *of* [33] *for the general structure of the fluctuation theory).*

The proof of Theorem 1.3 requires several iterative steps, involving different time scales. A generalized fluctuation structure will first be obtained on a very small time scale δ in Section 3, and iterated to reach intermediate (small, of size τ) and macroscopic times in Sections 4 and 5. This requires in particular an elimination of small remainder terms encoding unlikely events, namely recollisions at scale δ and superexponential growth at scale τ. The main term of the iteration will be finally shown to converge to a Gaussian pairing in Section 6.

This intricated time sampling will be discussed first informally in Section 2 : we will see that at each time scale, remainders of very different nature are identified. It will be very important that all these remainders share a structure of products of fluctuation fields (as defined in (3.23)), for which we can establish general estimates on the expectation and covariance (see Sections 7-8).

2. Elements of strategy

The aim of this section is to provide an informal description of the procedure necessary to extend a convergence result of type (1.15), to arbitrary moments

$$I_P^{\varepsilon,\mathrm{eq}} := \mathbb{E}_\varepsilon^{\mathrm{eq}}\left[\prod_{p=1}^{P}\zeta_{\theta_p}^{\varepsilon,\mathrm{eq}}(h^{(p)})\right], \quad P \geq 3\,, \tag{2.1}$$

where $(\theta_1,\dots,\theta_P)$ is a collection of times with $0 = \theta_1 \leq \theta_2 \leq \dots \leq \theta_P =: \Theta$, and $(h^{(1)},\dots,h^{(P)})$ a collection of test functions. We shall restrict to L^∞ test functions from now on.

Our main result is the following. Here and in what follows we use the notation $A \ll 1$ to indicate that A goes to zero when μ_ε goes to infinity.

Proposition 2.1. *For any $P \geq 2$, denote by $\mathfrak{S}_P^{\mathrm{pairs}}$ the set of partitions of $\{1,\dots,P\}$ made only of pairs. Then asymptotically when $\mu_\varepsilon \to \infty$, the moments are determined by the covariances, in the following sense: for $\Theta \geq 1$ and $\tau > 0$ satisfying*

$$\Theta^{3(P-1)}\tau \ll 1 \quad \text{and} \quad \Theta/\left(\tau\log|\log\varepsilon|\right) \ll 1, \tag{2.2}$$

there holds uniformly in $\theta_1, \dots \theta_P \in [0, \Theta]$

$$
\begin{aligned}
&\left| \mathbb{E}_\varepsilon^{\mathrm{eq}} \left[\prod_{p=1}^P \zeta_{\theta_p}^{\varepsilon,\mathrm{eq}}(h^{(p)}) \right] - \sum_{\eta \in \mathfrak{S}_P^{\mathrm{pairs}}} \prod_{\{i,j\} \in \eta} \mathbb{E}_\varepsilon^{\mathrm{eq}} \left[\zeta_{\theta_i}^{\varepsilon,\mathrm{eq}}(h^{(i)}) \, \zeta_{\theta_j}^{\varepsilon,\mathrm{eq}}(h^{(j)}) \right] \right| \\
&\qquad \le \Big(\prod_{p=1}^P \| h^{(p)} \|_{L^\infty} \Big) \Big(C_P \tau^{1/2} \Theta^{(2P-1)/2} + (C_P \Theta)^{2^{\Theta/\tau}} \varepsilon^{1/8d} \Big)
\end{aligned}
\tag{2.3}
$$

for some constant C_P *depending only on* P*. Notice that if* P *is odd then* $\mathfrak{S}_P^{\mathrm{pairs}}$ *is empty and the product of moments is asymptotically 0.*

As the limit of the covariance has been computed in Theorem 1.1 and extended to the time interval $[0, \Theta]$ in [8], Proposition 2.1 fully determines the limiting moments which turn out to coincide with the Gaussian moments of the solution to the fluctuating Boltzmann equation (1.20). By the 'Moment Method' (see [3] Section 30, Theorem 30.1), this fully characterizes the limiting distribution. Combined with the tightness of the process (see (1.19) and (1.16)), this completes the proof of the convergence to the fluctuating Boltzmann equation stated in Theorem 1.3.

2.1. **The global pairing scheme.** For simplicity, we will assume that all evaluation times are different

$$
0 = \theta_1 < \theta_2 < \cdots < \theta_P =: \Theta .
$$

The idea is then to design an iteration scheme decreasing the parameter P, where each elementary step uses the weak convergence method introduced in [8], and which realizes progressively the pairing (Wick rule), up to small remainder terms.

Let us focus on the moments $I_P^{\varepsilon,\mathrm{eq}}$ defined in (2.1) and consider the first step of our iteration procedure. Our goal is to reduce the number of evaluation times by transforming the fluctuation at time θ_P into a sum of (more complicated) fluctuations at time θ_{P-1}

$$
I_P^{\varepsilon,\mathrm{eq}} = \sum_{m_P} \mathbb{E}_\varepsilon^{\mathrm{eq}} \left[\Big(\prod_{p=1}^{P-1} \zeta_{\theta_p}^{\varepsilon,\mathrm{eq}}(h^{(p)}) \Big) \zeta_{m_P,\theta_{P-1}}^{\varepsilon,\mathrm{eq}} \left(\phi_{\theta_P - \theta_{P-1}}^{(P)} \right) \right] ,
\tag{2.4}
$$

where $\zeta_{m_P,\theta_{P-1}}^{\varepsilon,\mathrm{eq}} \left(\phi_{\theta_P - \theta_{P-1}}^{(P)} \right)$ is a fluctuation of an observable involving m_P particles at time θ_{P-1} (the notation is defined below in (2.5)). The function $\phi_{\theta_P - \theta_{P-1}}^{(P)} = \phi_{\theta_P - \theta_{P-1}}^{(P)}[h^{(P)}](Z_{m_P})$ of m_P particles is given by the elementary test function $h^{(P)}$ pulled back, along the flow induced by the Duhamel formula, during a time $\theta_P - \theta_{P-1}$. We refer to Section 3 below for a detailed discussion, and focus now on the iteration procedure.

For this it is convenient to use the following extension of Definition (1.6) of the empirical distribution: for any integer $m \ge 1$, any time t and any test function H_m defined on $(\mathbb{T}^d \times \mathbb{R}^d)^m$, we set

$$
\pi_{m,t}^\varepsilon (H_m) := \frac{1}{\mu_\varepsilon^m} \sum_{(i_1, \dots, i_m)} H_m \Big(\mathbf{z}_{i_1}^\varepsilon(t), \dots, \mathbf{z}_{i_m}^\varepsilon(t) \Big) ,
$$

where the symbol $\sum_{(i_1,\dots,i_m)}$ indicates a sum over m-tuples of labels running from 1 to $\mathcal{N}$ which are all mutually different ($i_j \neq i_k$ for $k \neq j$). Note that $\pi_{1,t}^\varepsilon = \pi_t^\varepsilon$ according to (1.6), but $\pi_{m,t}^\varepsilon \neq (\pi_t^\varepsilon)^{\otimes m}$ for $m > 1$ since indices cannot be repeated. It will still be convenient to maintain a product notation by introducing a product symbol, the so-called "$\circledast$-product", which by definition takes into account non-repeated indices, so that we can write

$$
\pi_{m,t}^\varepsilon = (\pi_t^\varepsilon)^{\circledast m} .
$$

Finally we introduce the shorthand notation

$$\mathbb{E}_\varepsilon^{\rm eq}[H_m] := \mathbb{E}_\varepsilon^{\rm eq}[\pi^\varepsilon_{m,t}(H_m)]\,,$$

and extend Definition (1.13) of the fluctuation field:

$$\zeta^{\varepsilon,{\rm eq}}_{m,t}(H_m) := \sqrt{\mu_\varepsilon}\Big(\pi^\varepsilon_{m,t}(H_m) - \mathbb{E}_\varepsilon^{\rm eq}[H_m]\Big)\,. \tag{2.5}$$

Returning to (2.4) let us consider the product

$$\zeta^{\varepsilon,{\rm eq}}_{\theta_{P-1}}(h^{(P-1)})\zeta^{\varepsilon,{\rm eq}}_{m_P,\theta_{P-1}}\left(\phi^{(P)}_{\theta_P-\theta_{P-1}}\right)$$

and look at the repeated indices to decompose it into two contributions, with the above notation:

- a "⊛-product", which by definition takes into account the non-repeated indices,

$$\begin{aligned}
&\zeta^{\varepsilon,{\rm eq}}_{\theta_{P-1}}(h^{(P-1)}) \circledast \zeta^{\varepsilon,{\rm eq}}_{m_P,\theta_{P-1}}\left(\phi^{(P)}_{\theta_P-\theta_{P-1}}\right)\\
&= \mu_\varepsilon\left(\frac{1}{\mu_\varepsilon}\sum h^{(P-1)} - \mathbb{E}_\varepsilon^{\rm eq}[h^{(P-1)}]\right)\circledast\left(\frac{1}{\mu_\varepsilon^{m_P}}\sum \phi^{(P)}_{\theta_P-\theta_{P-1}} - \mathbb{E}_\varepsilon^{\rm eq}[\phi^{(P)}_{\theta_P-\theta_{P-1}}]\right)\\
&:= \mu_\varepsilon \pi^\varepsilon_{m_P+1,\theta_{P-1}}\left(h^{(P-1)}\otimes\phi^{(P)}_{\theta_P-\theta_{P-1}}\right) + \mu_\varepsilon\mathbb{E}_\varepsilon^{\rm eq}[h^{(P-1)}]\mathbb{E}_\varepsilon^{\rm eq}[\phi^{(P)}_{\theta_P-\theta_{P-1}}]\\
&\qquad - \mu_\varepsilon\mathbb{E}_\varepsilon^{\rm eq}[h^{(P-1)}]\pi^\varepsilon_{m_P,\theta_{P-1}}\left(\phi^{(P)}_{\theta_P-\theta_{P-1}}\right) - \mu_\varepsilon\pi^\varepsilon_{\theta_{P-1}}\left(h^{(P-1)}\right)\mathbb{E}_\varepsilon^{\rm eq}[\phi^{(P)}_{\theta_P-\theta_{P-1}}]
\end{aligned}$$

  which will be a new fluctuation to be analyzed at time θ_{P-1};
- a "contracted product"

$$\pi^\varepsilon_{m_P,\theta_{P-1}}\left(\psi^{(P,P-1)}\right),\quad \text{with}\quad \psi^{(P,P-1)} := \phi^{(P)}_{\theta_P-\theta_{P-1}}(Z_{m_P})\sum_{j=1}^{m_P} h^{(P-1)}(z_j)\,,$$

  which will essentially decouple from the rest of the weight and sum up to give the covariance

$$\mathbb{E}_\varepsilon^{\rm eq}[\zeta^{\varepsilon,{\rm eq}}_{\theta_{P-1}}(h^{(P-1)})\zeta^{\varepsilon,{\rm eq}}_{\theta_P}(h^{(P)})] = \sum_{m_P}\mathbb{E}_\varepsilon^{\rm eq}\left[\pi^\varepsilon_{m_P,\theta_{P-1}}\left(\psi^{(P,P-1)}\right)\right] + o(1)\,.$$

We obtain

$$\begin{aligned}
I_P^{\varepsilon,{\rm eq}} = \sum_{m_P}\mathbb{E}_\varepsilon^{\rm eq}&\left[\left(\prod_{p=1}^{P-2}\zeta^{\varepsilon,{\rm eq}}_{\theta_p}(h^{(p)})\right)\zeta^{\varepsilon,{\rm eq}}_{\theta_{P-1}}(h^{(P-1)})\circledast\zeta^{\varepsilon,{\rm eq}}_{m_P,\theta_{P-1}}\left(\phi^{(P)}_{\theta_P-\theta_{P-1}}\right)\right]\\
&+\mathbb{E}_\varepsilon^{\rm eq}[\zeta^{\varepsilon,{\rm eq}}_{\theta_{P-1}}(h^{(P-1)})\zeta^{\varepsilon,{\rm eq}}_{\theta_P}(h^{(P)})]I^{\varepsilon,{\rm eq}}_{P-2} + o(1)\,.
\end{aligned}$$

From the above discussion, one can guess the structure for the general term to be iterated at time θ_p

$$\prod_{\{j,j'\}\in\rho}\mathbb{E}_\varepsilon^{\rm eq}\left[\zeta^{\varepsilon,{\rm eq}}_{\theta_j}(h^{(j)})\zeta^{\varepsilon,{\rm eq}}_{\theta_{j'}}(h^{(j')})\right]\times\sum_{\mathbf{M}}\mathbb{E}_\varepsilon^{\rm eq}\left[\left(\prod_{u<p}\zeta^{\varepsilon,{\rm eq}}_{\theta_u}(h^{(u)})\right)\bigotimes_{i\in B}\zeta^{\varepsilon,{\rm eq}}_{m_i,\theta_p}\left(\phi^{(i)}_{\theta_i-\theta_p}\right)\right] \tag{2.6}$$

where :

- B is a subset of $\{p,\dots P\}$, ρ is a partition of $\{p,\dots P\}\smallsetminus B$ in pairs $\{j,j'\}$;
- $\mathbf{M} = (m_i)_{i\in B}$ records the number of variables; $\phi^{(p)}_0 = h^{(p)}$ and the other dual functions $\phi^{(i)}_{\theta_i-\theta_p}$ have been pulled back from $h^{(i)}$: each $\phi^{(i)}_{\theta_i-\theta_p}$ is a function of m_i variables containing the information on the backward transport of $h^{(i)}$ on $[\theta_p,\theta_i]$;
- the ⊛-product is defined as previously by avoiding repeated indices

$$\bigotimes_{i\in B}\zeta^{\varepsilon,{\rm eq}}_{m_i,\theta_p}\left(\phi^{(i)}_{\theta_i-\theta_p}\right) := \mu_\varepsilon^{|B|/2}\sum_{A\subset B}\pi^\varepsilon_{M_A,\theta_p}\left(\bigotimes_{i\in A}\phi^{(i)}_{\theta_i-\theta_p}\right)\prod_{j\in B\smallsetminus A}\mathbb{E}_\varepsilon^{\rm eq}[-\phi^{(j)}_{\theta_j-\theta_p}] \tag{2.7}$$

  where $M_A := \sum_{i\in A} m_i$.

As before, the fluctuation at time θ_p is first transformed into a sum of fluctuations at time θ_{p-1}, so that (2.6) can be rewritten :
(2.8)

$$\prod_{\{j,j'\}\in\rho} \mathbb{E}_\varepsilon^{\rm eq}\Big[\zeta^{\varepsilon,{\rm eq}}_{\theta_j}(h^{(j)})\zeta^{\varepsilon,{\rm eq}}_{\theta_{j'}}(h^{(j')})\Big] \times \sum_{\mathbf{M},\mathbf{N}} \mathbb{E}_\varepsilon^{\rm eq}\Bigg[\Big(\prod_{u<p}\zeta^{\varepsilon,{\rm eq}}_{\theta_u}(h^{(u)})\Big)\bigotimes_{i\in B}\zeta^{\varepsilon,{\rm eq}}_{m_i+n_i,\theta_{p-1}}\big(\phi^{(i)}_{\theta_i-\theta_{p-1}}\big)\Bigg]$$

where $\mathbf{N}=(n_i)_{i\in B}$ records the number of particles added when pulling back further the test functions indexed by $i\in B$ during the time interval $[\theta_{p-1},\theta_p]$. Then (by looking at repeated indices), either $p-1$ is added to B, or an element of B is randomly chosen and paired with $p-1$, adding a new pair in ρ. Finally going back down to $p=1$, the dominant term turns out to correspond to $B=\varnothing$ and $\rho\in\mathfrak{S}_P^{\rm pairs}$, describing all possible pairings of the fluctuation fields in (2.3).

2.2. **The pullback of test functions on a time $\delta\ll 1$.** By definition, the *"block"* $\phi^{(i)}_{\theta_i-\theta_p}$ will be obtained by pulling back $h^{(i)}$ according to a Duhamel series on $[\theta_p,\theta_i]$. We shall see that a proper definition of $\phi^{(i)}_{\theta_i-\theta_p}$ requires to describe all possible forward-in-time dynamics starting from a configuration Z_{m_i} at time θ_p, respecting suitable connection constraints. We encounter here a first difficulty due to an uncontrolled number of collisions in this forward dynamics. This issue appears already in the study of the covariance in [8] ($I_2^{\varepsilon,{\rm eq}}$ with the above notation), where we showed that the construction of dual functions is efficient if one performs a *conditioning* of the invariant measure: this conditioning ensures that all microscopic configurations have a controlled dynamical behaviour on an elementary time step of size δ, with $\varepsilon\ll\delta\ll 1$.

Given an integer $\gamma\in\mathbb{N}$, we call *microscopic cluster of size* γ a set $\mathcal{G}$ of γ particle configurations in $\mathbb{T}^d\times\mathbb{R}^d$ such that $(z,z')\in\mathcal{G}\times\mathcal{G}$ if and only if there are $z_1=z,z_2,\dots,z_\ell=z'$ in $\mathcal{G}$ such that

$$|x_i-x_{i+1}|\le 3\sqrt{\gamma}\mathbb{V}\delta\,,\quad \forall 1\le i\le \ell-1\,,$$

where $\mathbb{V}\in\mathbb{R}^+$ is related to an energy truncation. To fix ideas, we choose from now on the microscopic time scale δ, the intermediate time scale τ, the macroscopic time Θ, the energy cut-off $\mathbb{V}$ and the size of the cluster γ as follows:

$$\varepsilon\ll\delta\ll\tau\ll 1\ll\Theta=O\left(\log\log|\log\varepsilon|\right)\,,\qquad \gamma=4d\,,\quad \mathbb{V}=|\log\varepsilon|\,,\quad \delta=\varepsilon^{1-\frac{1}{2d}}\,. \tag{2.9}$$

Definition 2.2. *Given $\gamma\in\mathbb{N}$, we define the set Υ_N^ε as the set of initial configurations $\mathbf{Z}_N^{\varepsilon 0}$ in $\mathcal{D}_N^\varepsilon$ such that for any $p\in\{2,\dots,P\}$ and integers k,r such that $1\le k\le(\theta_p-\theta_{p-1})/\tau$ and $r\in[0,\tau/\delta]$, the configuration at time $\theta_p-(k-1)\tau-r\delta$ satisfies*

$$\forall 1\le j\le N\,,\quad |v_j|\le\mathbb{V}\,,$$

and any microscopic cluster of particles is of size at most γ.

On each elementary step, the hard sphere system will be shown to behave in essence as a collection of independent clusters of small size, and in particular the total number of collisions is under control. Note also that with the previous choices (2.9) of parameters, the conditioning by $\Upsilon_{\mathcal{N}}^\varepsilon$ is typical in the sense that the complement satisfies

$$\mathbb{P}_\varepsilon^{\rm eq}\big({}^c\Upsilon_{\mathcal{N}}^\varepsilon\big)\le\Theta\varepsilon^d\,. \tag{2.10}$$

We refer to [8, Section 6.1] for the proof of this result.

In the following, we shall mostly restrict our estimates to the sets Υ_N^ε, so it is useful to introduce the following notation: the probability of an event A with respect to the (unnormalized) conditioned measure is denoted by $\mathbb{P}_\varepsilon(A)$

$$\mathbb{P}_\varepsilon\left(A\right):=\mathbb{P}_\varepsilon^{\rm eq}\left(\Upsilon_N^\varepsilon\right)\mathbb{P}_\varepsilon^{\rm eq}\left(A\mid\Upsilon_{\mathcal{N}}^\varepsilon\right):=\mathbb{P}_\varepsilon^{\rm eq}\left(\Upsilon_N^\varepsilon\cap A\right)\,,$$

and $\mathbb{E}_\varepsilon$ is the corresponding expected value. The moments of the fluctuation field under such conditioning are written

$$I_P^\varepsilon := \mathbb{E}_\varepsilon \left[\prod_{p=1}^P \zeta_{\theta_p}^\varepsilon (h^{(p)}) \right]$$

where ζ^ε is the non-centered field defined by

$$\zeta_t^\varepsilon(h) := \sqrt{\mu_\varepsilon} \Big(\pi_t^\varepsilon(h) - \mathbb{E}_\varepsilon\big(\pi_t^\varepsilon(h)\big)\Big) .$$

Furthermore we use the notation

$$\mathbb{E}_\varepsilon[H_m] := \mathbb{E}_\varepsilon[\pi_{m,t}^\varepsilon(H_m)] ,$$

and

$$\zeta_{m,t}^\varepsilon(H_m) := \sqrt{\mu_\varepsilon}\Big(\pi_{m,t}^\varepsilon(H_m) - \mathbb{E}_\varepsilon[H_m]\Big) . \tag{2.11}$$

For future convenience we notice that

$$\zeta_{m,t}^\varepsilon(H_m) = \zeta_{m,t}^{\varepsilon,\mathrm{eq}}(H_m) + \sqrt{\mu_\varepsilon}\, \mathbb{E}_\varepsilon^{\mathrm{eq}}[\mathbf{1}_{{}^c\Upsilon_N^\varepsilon}\, \pi_{m,t}^\varepsilon(H_m)] . \tag{2.12}$$

The pairing mechanism described in Section 2.1 will be achieved with I_P^ε rather than $I_P^{\varepsilon,\mathrm{eq}}$, and the difference between I_P^ε and $I_P^{\varepsilon,\mathrm{eq}}$ will be shown to be of subleading order in (2.3), thanks to the estimate (2.10) (see Proposition 2.3 below).

The conditioning ensures that the pullback of the test functions can be performed efficiently, on very small time intervals of size $\delta = O(\varepsilon^{1-1/2d})$ (see Section 3). Each pullback will involve combinatorial factors, counting the number of trajectories compatible with a given Z_{m_i}, which cannot be iterated $O(1/\delta)$ times without leading to strong divergences. The idea is therefore to iterate only the principal terms, removing at each time step $O(\delta)$ all "non minimal correlations", up to reaching an intermediate time scale τ such that $\delta \ll \tau \ll 1$ (see Section 4).

2.3. **The factorization defect on a time $\tau \gg \delta$.** The second difficulty is that the pullback does not preserve exactly the factorized structure

$$\mathbb{E}_\varepsilon \left[\Big(\prod_{u<p} \zeta_{\theta_u}^\varepsilon (h^{(u)}) \Big) \bigotimes_{i\in B} \zeta_{m_i,\theta_p}^\varepsilon \big(\phi_{\theta_i-\theta_p}^{(i)}\big) \right] \neq \sum_{\mathbf{N}} \mathbb{E}_\varepsilon \left[\Big(\prod_{u<p} \zeta_{\theta_u}^\varepsilon (h^{(u)}) \Big) \bigotimes_{i\in B} \zeta_{m_i+n_i,\theta_{p-1}}^\varepsilon \big(\phi_{\theta_i-\theta_{p-1}}^{(i)}\big) \right] .$$

Let us start from an observable which is a product of blocks

$$\Phi_{\mathbf{M}}(Z_{\mathbf{M}}) := \prod_{i\in B} \phi^{(i)}\big(Z_{m_i}^{(j)}\big) , \tag{2.13}$$

denoting $\mathbf{M} = (m_i)_{i\in B}$ with $m_i \geq 1$. On an elementary time interval of size δ, these blocks $\phi^{(i)}$ are transported dynamically, which can lead to some dynamical correlations. Blocks are then connected into *"packets"* according to the dynamical correlations. Then, in order to keep a fluctuation structure, all variables have to be centered. The trivial packets containing only one block have already the fluctuation structure so no additional term appears. The other packets are called *"clustering"* since they contain at least two blocks, and their centering provides a contribution of the expectation. The goal of Section 3 is thus to establish the following identity, relating the fluctuation structure at time θ to that at time $\theta - \delta$ (with θ

chosen such that θ and $\theta - \delta$ belong to (θ_{p-1}, θ_p)):

$$\begin{aligned}
&\mathbb{E}_\varepsilon\Big[\Big(\prod_{u=1}^{p-1} \zeta^\varepsilon_{\theta_u}\big(h^{(u)}\big)\Big) \bigotimes_{i\in B} \zeta^\varepsilon_{m_i,\theta}\big(\phi^{(i)}\big)\Big] \\
&= \sum_{\mathbf{N}} \sum_{\substack{\eta_1\cup\eta_2 \text{ partition of } B\\ \eta_2 \text{ clustering}}} \prod_{q\leq|\eta_2|} \mu_\varepsilon^{1-\frac{|\eta_{2,q}|}{2}} \mathbb{E}_\varepsilon\Big[\phi_\delta^{(\eta_{2,q})}\Big] \\
&\quad\times \mathbb{E}_\varepsilon\Big[\Big(\prod_{u=1}^{p-1} \zeta^\varepsilon_{\theta_u}\big(h^{(u)}\big)\Big)\Big(\bigotimes_{q\leq|\eta_1|} \mu_\varepsilon^{\frac12-\frac{|\eta_{1,q}|}{2}} \zeta^\varepsilon_{M^\delta_{\eta_{1,q}},\theta-\delta}\Big(\phi_\delta^{(\eta_{1,q})}\Big)\Big)\Big].
\end{aligned} \tag{2.14}$$

where $M^\delta_{\eta_{1,q}}$ is the total number of particles in the packet $\eta_{1,q}$ at time $\theta - \delta$, and $\phi_\delta^{(\eta_{1,q})}$ is supported on configurations obtained by backward pseudo-trajectories forming a cluster $\eta_{1,q}$ (in a sense to be made precise in Section 3).

It will be useful in the sequel to interpret (2.14) by the following recipe (see Figure 1 page 11): between time θ and $\theta - \delta$

- some observables are grouped into clusters by dynamical constraints, which leads to the partition into packets η_1, η_2;
- observables are pulled back according to the Duhamel pseudo-trajectories compatible with the previous clustering conditions;
- some non trivial clusters (containing at least two packets) encoded in η_2 are expelled from the fluctuation, their expectation remaining as an independent factor in the product.

To iterate this procedure down to time $\theta - \tau$ with $\tau = R\delta \ll 1$, we need to extend formula (2.14) starting from packets and not only from blocks. We will therefore construct iteratively on each time step $[\theta - r\delta, \theta - (r-1)\delta]$ for $r = 1, \dots, R$ a sequence of nested partitions $\eta_1^{r-1} \hookrightarrow \eta_1^r \cup \eta_2^r$ with η_2^r corresponding to packets which are expelled from the main factorized structure, contributing only via their expectation, and η_1^r corresponding to packets contributing to the factorized structure via their fluctuations (see Figure 1).

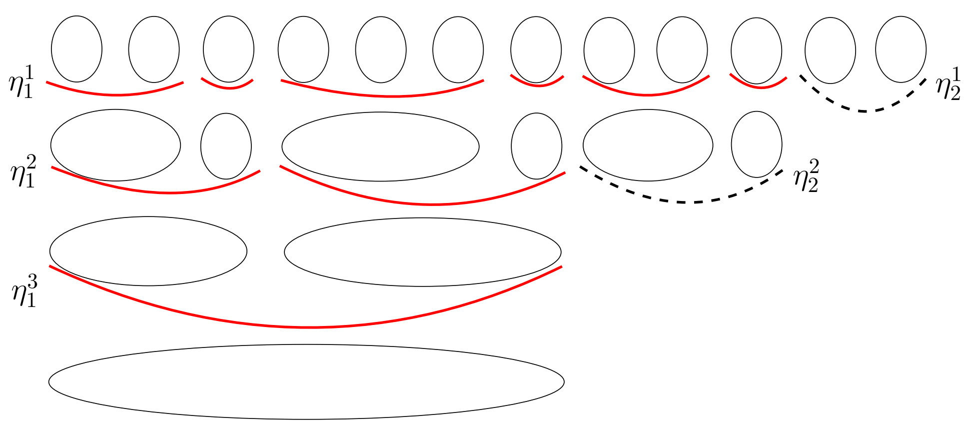

FIGURE 1. At time θ, the set B contains 12 blocks. The nested partitions are depicted and the dashed parts represent the expelled clustering cumulants. After 3 iterations, blocks 8 to 12 (numbered from left to right) have been expelled and the other blocks have merged into a single packet connected by dynamical constraints.

As assumed in (2.8) and as becomes apparent in view of the powers of μ_ε in (2.14), we expect that the leading order term corresponds to the single block type factorized structure, i.e. to $\cup_{r=1}^R \eta_1^r = \varnothing$ and η_2^R is the trivial partition in singletons. However, discarding clustering terms at each time $\theta - r\delta$ $(r \leq R)$ would generate diverging remainders since $1/\delta \gg 1$. We will therefore need to perform the full iteration (discarding only non minimal correlations as explained in Section 2.2) on some intermediate time scale $\tau \ll 1$, and then to combine all remainders due to clusterings on $[\theta - \tau, \theta]$ in a rather subtle way (see Section 4) to recover the single block type factorized structure (see Section 5). We refer to Figure 2 for a summary of the procedure.

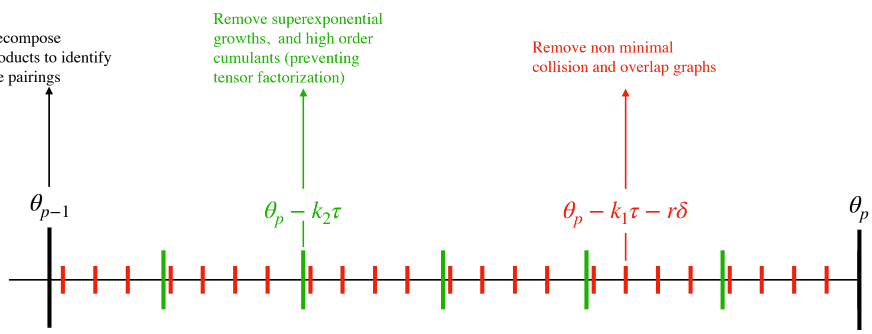

FIGURE 2. Double time-sampling of the interval (θ_{p-1}, θ_p) into pieces of size τ (in green) and subpieces of size δ (in red). At each time step of size δ, all non minimal correlations are discarded; at each time step of size τ, all other remainder terms are discarded (exponentially large collision trees, higher order clusters). Pairings are identified at each macroscopic time step θ_p, θ_{p-1}, etc.

2.4. **Control of the remainder terms.** All the remainder terms coming from these two samplings (at scales δ and τ) are controlled by decoupling the different times thanks to Hölder's inequality. This will rely on the moment estimates on the fluctuation field stated in the following proposition.

Proposition 2.3. *Let h be a function in $L^\infty(\mathbb{T}^d \times \mathbb{R}^d)$. Then for all $1 \leq p < \infty$ and for ε small enough, the moments of the fluctuation field (at equilibrium and under the conditioned measure) are bounded:*

$$\left| \mathbb{E}_\varepsilon^{\rm eq}\Big(\big(\zeta^{\varepsilon, \rm eq}(h) \big)^p \Big) \right| \leq C_p \|h\|_\infty^p \,, \tag{2.15}$$

$$\left| \mathbb{E}_\varepsilon\Big(\big(\zeta^{\varepsilon}(h) \big)^p \Big) \right| \leq C_p \|h\|_\infty^p \,, \tag{2.16}$$

where the constant $C_p > 0$ depends only on p. Moreover under the assumptions of Proposition 2.3

$$\left| I_P^{\varepsilon, \rm eq} - I_P^\varepsilon \right| \leq \Big(\prod_{p=1}^P \|h^{(p)}\|_{L^\infty} \Big) C_P \left(\Theta \varepsilon^d \right)^{1/2} \tag{2.17}$$

uniformly in $\theta_1, \dots \theta_p \in [0, \Theta]$.

The standard result (2.15) can be found in Proposition A.1 from [8], from which (2.16) and (2.17) will be derived in Section 8.

The key argument to implement the strategy is therefore to obtain estimates for the expectation and variance of $\otimes$ products defined by (2.7), proved in Sections 7 and 8. To derive these estimates it is necessary to have a precise description of the structure of the test functions $\phi^{(i)}_{\theta_i-\theta_p}(Z_{m_i})$. As will be made precise in Section 4, they are supported on "dynamical clusters" of m_i particles, called *forward clusters* below. This means that there exists a graph with m_i vertices, constructed by adding one edge each time two particles find themselves at a distance (equal or) less than ε during the time interval $[\theta_p, \theta_i]$.

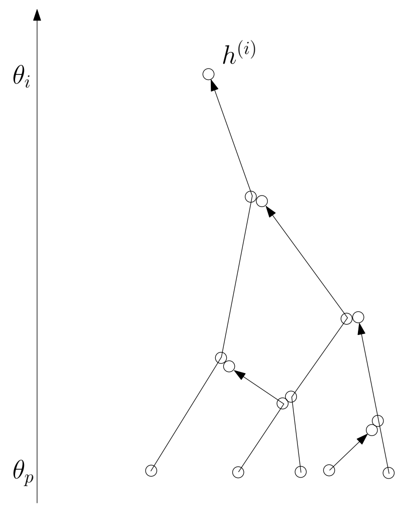

FIGURE 3. Forward cluster of $\phi^{(i)}_{\theta_i-\theta_p}$ associated with the pullback of $h^{(i)}$ during a time $\theta_i - \theta_p$, case of $m_i = 5$ hard spheres. The function $\phi^{(i)}$ is supported on a configuration Z_{m_i} such that, starting at time θ_p and going forward in time, particles encounter, and are progressively removed from the dynamics, until only one particle is left at time θ_i.

In particular they will be shown to satisfy roughly an estimate of type

$$\left|\phi^{(i)}_{\theta_i-\theta_p}(Z_{m_i})\right| \le C\frac{(C\mu_\varepsilon)^{m_i-1}}{m_i!}\sum \mathbf{1}_{Z_{m_i}\text{ forward cluster}}\,,$$

where we sum over all possible forward dynamics starting from Z_{m_i} as in Figure 3. Note that the size of the typical volume spanned by one particle in a finite time is μ_ε^{-1}, so that the volume of a cluster is typically $\mu_\varepsilon^{-m_i+1}$. The $1/m_i!$ is due to the symmetrisation by permutation of the particle labels. This estimate in turn will imply that

$$\mathbb{E}_\varepsilon\left[\left(\bigotimes_{i\in B}\zeta^\varepsilon_{m_i}\big(\phi^{(i)}\big)\right)^2\right] \le (C'\Theta)^{2M}\,,\quad M := \sum_{i\in B} m_i\,.$$

Moreover using the fact that error terms are supported on clusters with additional constraints, we will get some additional smallness, providing the expected control on the error terms at arbitrary times.

3. Preserving the fluctuation structure on small times

In this section we detail one part of the discussion of the previous section, namely how to transport the fluctuation structure between two time steps, thanks to the Duhamel formula. However in order to make sense of the Duhamel formula and its dual form uniformly in ε, we will actually not connect directly time θ_p to time θ_{p-1}, but rather introduce an iteration on infinitesimal time intervals (much smaller than Lanford's convergence time). This consists in transforming a weight at a time $\theta \in (\theta_{p-1}, \theta_p)$ in a (more complicated) weight at a time $\theta - \delta$

in (θ_{p-1},θ_p), for some very small $\delta>0$ tuned in (2.9). In what follows, we shall therefore focus only on the interval $(\theta-\delta,\theta)$: the precise statement requires some notation and is given at the end of this section (see Proposition 3.9 page 26).

The procedure relies on three ingredients: we fix $\theta=\theta_p-r\delta$ for some $r\in[0,(\theta_p-\theta_{p-1})/\delta]$.

- We first introduce the family of *correlation functions* $(G^\varepsilon_M)_{M\geq 1}$ at time $t\in[\theta-\delta,\theta]$, defined for any test function H_M of M variables by

$$\int G^\varepsilon_M(t,Z_M)H_M(Z_M)dZ_M := \mathbb{E}_\varepsilon\Big[\Big(\prod_{u=1}^{p-1}\zeta^\varepsilon_{\theta_u}\big(h^{(u)}\big)\Big)\pi^\varepsilon_{M,t}(H_M)\Big]. \tag{3.1}$$

  These correlation functions satisfy a hierarchy of linear evolution equations, the so-called BBGKY hierarchy, so that $G^\varepsilon_M(\theta)$ can be expressed as a Duhamel sum involving the correlation functions at time $\theta-\delta$ (see Section 3.1)

$$G^\varepsilon_M(\theta)=\sum_{N\geq 0}Q^\varepsilon_{M,N}(\delta)G^\varepsilon_{M+N}(\theta-\delta)\,, \tag{3.2}$$

  where the operator $Q^\varepsilon_{M,N}$ encodes transport and collisions.
- We then use a graphical representation of the elementary operator $Q^\varepsilon_{M,N}(\delta)$ in terms of "pseudo-trajectories" to define an "adjoint" operator (see Section 3.2).
- We finally recombine the contributions of the different correlation functions to identify a fluctuation structure at time $\theta-\delta$ (see Section 3.3).

Note that the last time interval when $r=\lfloor\frac{\theta_p-\theta_{p-1}}{\delta}\rfloor$ may be a little smaller than δ, but the very same arguments can be applied.

3.1. **The Duhamel iteration and its graphical representation.** In the grand canonical setting, (3.1) is equivalent to

$$G^\varepsilon_M(t,Z_M)=\frac{1}{\mu_\varepsilon^M}\sum_{n=0}^\infty\frac{1}{n!}\int_{(\mathbb{T}^d\times\mathbb{R}^d)^n}dz_{M+1}\dots dz_{M+n}\,W^\varepsilon_{M+n}(t,Z_{M+n})\,,\qquad t\in[0,\Theta]\,,$$

where the (signed, non-normalized) measure $\big(W^\varepsilon_N(t)\big)_{N\geq 1}$ is defined as follows.

On $[\theta_1,\theta_2]$, W^ε_N solves the Liouville equation

$$\partial_t W^\varepsilon_N+V_N\cdot\nabla_{X_N}W^\varepsilon_N=0\quad\text{ on }\ \mathcal{D}^\varepsilon_N\,, \tag{3.3}$$

with specular reflection on the boundary (and extending W^ε_N by zero outside $\mathcal{D}^\varepsilon_N$) and with initial data (see (1.4))

$$W^\varepsilon_N(\theta_1,Z_N):=\mathbf{1}_{\Upsilon^\varepsilon_N}W^{\varepsilon,\mathrm{eq}}_N(Z_N)\,\frac{1}{\sqrt{\mu_\varepsilon}}\left(\sum_{i=1}^N h^{(1)}(z_i)-\mu_\varepsilon\mathbb{E}_\varepsilon\big[h^{(1)}\big]\right).$$

Inductively for $p>2$, one solves again Eq. (3.3) on $[\theta_{p-1},\theta_p]$, with perturbed initial data

$$W^\varepsilon_N(\theta^+_{p-1},Z_N):=W^\varepsilon_N(\theta^-_{p-1},Z_N)\frac{1}{\sqrt{\mu_\varepsilon}}\left(\sum_{i=1}^N h^{(p-1)}(z_i)-\mu_\varepsilon\mathbb{E}_\varepsilon\big[h^{(p-1)}\big]\right),$$

where $\pm$ indicate the limits from the future/past.

3.1.1. *The Duhamel iteration.* By integration of the Liouville equation (3.3) for fixed ε, one obtains formally that for any integer M, the M-particle correlation function G^ε_M satisfies

$$\partial_t G^\varepsilon_M+V_M\cdot\nabla_{X_M}G^\varepsilon_M=C^\varepsilon_{M,M+1}G^\varepsilon_{M+1}\quad\text{ on }\quad\mathcal{D}^\varepsilon_M\,, \tag{3.4}$$

with specular boundary reflection as in (3.3). This is the well-known BBGKY hierarchy (see [11]), which is the elementary brick in the proof of Lanford's theorem for short times [24].

The operator $C^\varepsilon_{M,M+1}$ describes the collision between one "fresh" particle (labelled $M+1$) and one given particle $i \in \{1,\dots,M\}$:

$$C^\varepsilon_{M,M+1} G^\varepsilon_{M+1} := \sum_{i=1}^M C^{\varepsilon,i}_{M,M+1} G^\varepsilon_{M+1}$$

with

$$\begin{aligned}(C^{\varepsilon,i}_{M,M+1} G^\varepsilon_{M+1})(Z_M) := &\int_{\mathbb{S}^{d-1}\times\mathbb{R}^d} G^\varepsilon_{M+1}(Z_M^{\langle i\rangle}, x_i, v_i', x_i+\varepsilon\omega, u')\big((u-v_i)\cdot\omega\big)_+ \, d\omega\, du \\ &- \int_{\mathbb{S}^{d-1}\times\mathbb{R}^d} G^\varepsilon_{M+1}(Z_M, x_i+\varepsilon\omega, u)\big((u-v_i)\cdot\omega\big)_- \, d\omega\, du \\ = &\int_{\mathbb{S}^{d-1}\times\mathbb{R}^d} G^\varepsilon_{M+1}(Z_M^{\langle i\rangle}, x_i, v_i', x_i+\varepsilon\omega, u')\big((u-v_i)\cdot\omega\big)_+ \, d\omega\, du \\ &- \int_{\mathbb{S}^{d-1}\times\mathbb{R}^d} G^\varepsilon_{M+1}(Z_M, x_i-\varepsilon\omega, u)\big((u-v_i)\cdot\omega\big)_+ \, d\omega\, du\,,\end{aligned}$$

where (v_i', u') is recovered from (v_i, u) through the scattering law as in (1.9), and with the notation

$$Z_M^{\langle i\rangle} := (z_1,\dots,z_{i-1},z_{i+1},\dots,z_M)\,.$$

Now let us fix $\theta = \theta_p - r\delta$ for some $r \in [0, (\theta_p-\theta_{p-1})/\delta]$. Denote by S^ε_M the group associated with transport in $\mathcal{D}^\varepsilon_M$, with specular reflection on the boundary. By iteration of Duhamel's formula

$$G^\varepsilon_M(\theta) = S^\varepsilon_M(\delta) G^\varepsilon_M(\theta-\delta) + \int_{\theta-\delta}^\theta S^\varepsilon_M(\theta-t_1) C^\varepsilon_{M,M+1} G^\varepsilon_{M+1}(t_1)\, dt_1\,,$$

the solution G^ε_M of the hierarchy (3.4) can formally be expressed as a sum of operators acting on the data at time $\theta-\delta$:

$$G^\varepsilon_M(\theta) = \sum_{N\ge 0} Q^\varepsilon_{M,N}(\delta) G^\varepsilon_{M+N}(\theta-\delta)\,, \tag{3.5}$$

where we have defined

$$\begin{aligned}Q^\varepsilon_{M,N}(\delta) G^\varepsilon_{M+N}(\theta-\delta) := \int_{\theta-\delta}^\theta \int_{\theta-\delta}^{t_1} \cdots \int_{\theta-\delta}^{t_{N-1}} &S^\varepsilon_M(\theta-t_1) C^\varepsilon_{M,M+1} S^\varepsilon_{M+1}(t_1-t_2) \\ &\dots S^\varepsilon_{M+N}(t_N) G^\varepsilon_{M+N}(\theta-\delta)\, dt_N \dots dt_1\,.\end{aligned}$$

We stress that formula (3.5) is valid almost everywhere for a large class of measures $(W^\varepsilon_N(t))_{N\ge1}$, in spite of a collision operator being defined as a trace (see for instance [15, 30]).

3.1.2. *Backward pseudo-trajectories.* It is a standard procedure to translate the iterated Duhamel formula (3.5) in terms of (backward) *pseudo-trajectories*. We first encode the combinatorics of collisions in a graph $a = (a_j)_{1\le j\le N}$ where $a_j \in \{1,\dots,M+j-1\}$ denotes the label of the particle colliding with particle $M+j$ at its creation time. Note that the set of graphs a is a collection $\mathcal{A}_{M,N}$ of M binary trees with a total of N branchings. We define $\mathcal{A}^\pm_{M,N}$ the set of such *collision trees*, where each a_j is equipped with a sign $s_j \in \{-1,1\}$. Then, given such an a, as well as a configuration Z_M and collision parameters $(t_j,\omega_j,u_j)_{1\le j\le N}$ with $t_{N+1} = \theta-\delta < t_N < \dots < t_1 < \theta = t_0$, we define iteratively the pseudo-trajectory

$$\Psi^\varepsilon_{M,N} = \Psi^\varepsilon_{M,N}\Big(Z_M, a, (t_j,\omega_j,u_j)_{j=1,\dots,N}\Big)$$

as follows (denoting by $Z^\varepsilon_{M+i}(t)$ the coordinates of the pseudo-particles at time $t \in]t_{i+1}, t_i]$):

- starting from Z_M at time θ,
- transporting all existing particles backward on (t_j, t_{j-1}) (on $\mathcal{D}^\varepsilon_{M+j-1}$ with specular reflection at collisions),
- adding a new particle labeled $M+j$ at time t_j at position $x^\varepsilon_{a_j}(t_j) + \varepsilon s_j\omega_j$, and with velocity u_j,

- applying the scattering rule if $s_j > 0$.

We discard non admissible parameters for which the above procedure is ill-defined; in particular we exclude values of ω_j corresponding to an overlap of particles (two particles at mutual distance less than ε) as well as those such that $\omega_j \cdot \big(u_j - v^\varepsilon_{a_j}(t_j^+)\big) \le 0$. In the following we denote by $\mathcal{G}^\varepsilon_N(a, Z_M)$ the set of admissible parameters $(t_j, \omega_j, u_j)_{1\le j\le N}$ and by $Z^\varepsilon_{M+N}(\theta-\delta)$ the configuration at time $\theta - \delta$. With these notations, one gets the following geometric representation :

$$G^\varepsilon_M(\theta, Z_M) = \sum_{N\ge 0} \sum_{a\in\mathcal{A}^\pm_{M,N}} \int_{\mathcal{G}^\varepsilon_N(a,Z_M)} dT_N d\Omega_N dU_N \times \left(\prod_{j=1}^N s_j\Big(\big(u_j - v^\varepsilon_{a_j}(t_j^+)\big)\cdot\omega_j\Big)_+\right) G^\varepsilon_{M+N}\big(\theta-\delta, Z^\varepsilon_{M+N}(\theta-\delta)\big)\,,$$

where $(T_N, \Omega_N, U_N) := (t_j, \omega_j, u_j)_{1\le j\le N}$.

We recall the following classical notions of *collision* and *recollision* in a pseudo-trajectory.

Definition 3.1. *A collision is the addition of a fresh particle at distance ε from an existing particle (see the third item above), while a recollision involves two particles transported by the backward flow S^ε (in between two collision times).*

3.1.3. *Blocks and packets.* As mentioned in Section 2.3, when pulling back the product of fluctuation fields, various structures are involved. We refer to Figure 4 for a schematic description of the following definition.

Definition 3.2. *Given a particle labeled $j \in \{1,\dots,P\}$ and the number m_j of particles in the collision tree of j at some time, the associate block of particles is the set of all particles in the collision tree at that time. We denote by $Z^{(j)}_{m_j}$ the corresponding configuration.*

A packet of particles is a union of different blocks which have been connected dynamically at time t. These packets aggregate as the iteration described in Section 2.3 progresses. Given $B \subset \{1,\dots,P\}$, a partition ς of B and $i \le |\varsigma|$, we call C_i the packet of $M_{\varsigma_i} := \sum_{j\in\varsigma_i} m_j$ particles with configuration $(Z^{(j)}_{m_j})_{j\in\varsigma_i}$, connecting all the blocks j in ς_i.

For the elementary step of the iteration, we then have to consider functions of the form

$$\Phi_{\mathbf{M}}(Z_{\mathbf{M}}) := \prod_{i\le|\varsigma|} \phi^{(\varsigma_i)}\big(\{Z^{(j)}_{m_j}\}_{j\in\varsigma_i}\big) \tag{3.6}$$

where the functions $\phi^{(\varsigma_i)}$ will be assumed symmetric within each block $j \in \varsigma_i$. We have written $\mathbf{M} := (m_j)_{j\in B}$.

For a given $a \in \mathcal{A}^\pm_{M,N}$ with $M = \sum_{j\in B} m_j$, we denote by n_j the number of branchings in the collision tree issued from the block j on $(\theta-\delta, \theta)$. Note that the cardinal of the blocks and the packets vary with time. We also denote $\mathbf{N} := (n_j)_{j\in B}$ so that $\sum_{j\in B} n_j = N$. Finally we denote by $a^{(j)} \in \mathcal{A}^\pm_{m_j,n_j}$ the restriction of a to the block j. Then starting from (3.6) we can define

$$G^\varepsilon_M(\theta, Z_{\mathbf{M}}) = \sum_{\mathbf{N}} \sum_{a\in\mathcal{A}^\pm_{\mathbf{M},\mathbf{N}}} \int_{\mathcal{G}^\varepsilon_{\mathbf{N}}(a,Z_{\mathbf{M}})} dT_{\mathbf{N}} d\Omega_{\mathbf{N}} dU_{\mathbf{N}} \times \left(\prod_{j\in B}\prod_{\ell=1}^{n_j} s^{(j)}_\ell\Big(\big(u^{(j)}_\ell - v^\varepsilon_{a^{(j)}_\ell}(t^{(j)+}_\ell)\big)\cdot\omega^{(j)}_\ell\Big)_+\right) G^\varepsilon_{M+N}\big(\theta-\delta, Z^\varepsilon_{\mathbf{M}+\mathbf{N}}(\theta-\delta)\big)\,,$$

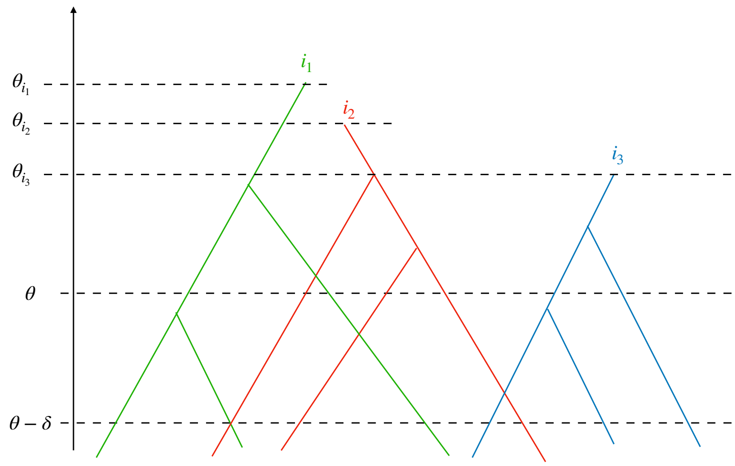

FIGURE 4. Three blocks indexed by i_1, i_2, i_3 consist of $m_{i_1} = 2$, $m_{i_2} = 3$ and $m_{i_3} = 2$ particles at time θ, at which time blocks i_1 and i_2 have merged into one packet. The remaining block merges with that packet to build one packet at time $\theta - \delta$.

where $(T_{\mathbf{N}}, \Omega_{\mathbf{N}}, U_{\mathbf{N}}) = \Big((t_\ell^{(j)}, \omega_\ell^{(j)}, u_\ell^{(j)})_{1\le \ell\le n_j}\Big)_{j\in B}$ and each of the $|B|$ sets $(t_\ell^{(j)})_{1\le\ell\le n_j}$ is ordered; recalling (3.1) and (3.5) and using the symmetry of correlation functions, one has

(3.7)
$$\begin{aligned}&\mathbb{E}_\varepsilon\Big[\Big(\prod_{u=1}^{p-1}\zeta^\varepsilon_{\theta_u}\big(h^{(u)}\big)\Big)\pi^\varepsilon_{M,\theta}(\Phi_{\mathbf{M}})\Big]\\ &\quad = \int G^\varepsilon_M(\theta, Z_{\mathbf{M}})\Phi_{\mathbf{M}}(Z_{\mathbf{M}})dZ_{\mathbf{M}} = \sum_{\mathbf{N}}\int \Big(Q^\varepsilon_{\mathbf{M},\mathbf{N}}(\delta)G^\varepsilon_{M+N}(\theta-\delta)\Big)(Z_{\mathbf{M}})\ \Phi_{\mathbf{M}}(Z_{\mathbf{M}})dZ_{\mathbf{M}}\,.\end{aligned}$$

Remark 3.3. *By the Fubini identity, one may equivalently prescribe an order on all the collision times* $(t_j)_{1\le j\le N}$ *corresponding to the trees* $a \in \mathcal{A}^\pm_{M,N}$*, or a partial order on all the times* $(t_\ell^{(j)})_{1\le\ell\le n_j}$ *for each* $j\in B$*, corresponding to the trees* $a\in\mathcal{A}^\pm_{\mathbf{M},\mathbf{N}}$*.*

3.2. **Pullback of observables.** The idea now is to take advantage of the geometric representation to construct the "adjoint" of the operator $Q^\varepsilon_{\mathbf{M},\mathbf{N}}(\delta)$, rewriting (3.7) as

$$\begin{aligned}\mathbb{E}_\varepsilon\Big[\Big(\prod_{u=1}^{p-1}\zeta^\varepsilon_{\theta_u}\big(h^{(u)}\big)\Big)\pi^\varepsilon_{M,\theta}(\Phi_{\mathbf{M}})\Big] \text{``}=\text{''} &\sum_{\mathbf{N}}\int G^\varepsilon_{M+N}(\theta-\delta)\Big(Q^{\varepsilon*}_{\mathbf{M},\mathbf{N}}(\delta)\Phi_{\mathbf{M}}\Big)dZ_{\mathbf{M}+\mathbf{N}}\\ \text{``}=\text{''} &\sum_{\mathbf{N}}\mathbb{E}_\varepsilon\Big[\Big(\prod_{u=1}^{p-1}\zeta^\varepsilon_{\theta_u}\big(h^{(u)}\big)\Big)\pi^\varepsilon_{M+N,\theta-\delta}\Big(Q^{\varepsilon*}_{\mathbf{M},\mathbf{N}}(\delta)\Phi_{\mathbf{M}}\Big)\Big].\end{aligned}$$

In other words, this means that we would like to change variables

$$(Z_{\mathbf{M}}, T_{\mathbf{N}}, \Omega_{\mathbf{N}}, U_{\mathbf{N}}) \mapsto Z^\varepsilon_{\mathbf{M}+\mathbf{N}}(\theta-\delta) \tag{3.8}$$

where $(T_{\mathbf{N}}, \Omega_{\mathbf{N}}, U_{\mathbf{N}}) = \Big((t_\ell^{(j)}, \omega_\ell^{(j)}, u_\ell^{(j)})_{1\le\ell\le n_j}\Big)_{j\in B}$. Unfortunately it is not true that the change of variables (3.8) is admissible in general, due to the presence of recollisions. Actually only recollisions involving particles of the same block are an issue, which leads to the following classification :

Definition 3.4. *A recollision is said to be* <u>*internal*</u> *if it involves two particles of the same block. It is said* <u>*external*</u> *if it involves two particles of different blocks.*

3.2.1. *Duality in absence of internal recollisions.* Let us first describe the duality argument in the case when there is no internal recollision, which is simpler. Setting $\mathbf{M} := (m_j)_{j\in B}$ the number of particles in the block j at time θ and $\mathbf{N} := (n_j)_{j\in B}$ the number of particles added to the block j between times θ and $\theta-\delta$, we denote by $Q^{\varepsilon 0}_{\mathbf{M},\mathbf{N}}$ the restriction of $Q^{\varepsilon}_{\mathbf{M},\mathbf{N}}$ to pseudo-trajectories without internal recollisions. We set, recalling (3.6) and (3.7),

$$I^0_{\mathbf{M},\mathbf{N}} := \int \Big(Q^{\varepsilon 0}_{\mathbf{M},\mathbf{N}}(\delta) G^{\varepsilon}_{M+N}(\theta-\delta)\Big)(Z_{\mathbf{M}})\, \Phi_{\mathbf{M}}(Z_{\mathbf{M}}) dZ_{\mathbf{M}} \,.$$

Given $a \in \mathcal{A}^{\pm}_{\mathbf{M},\mathbf{N}}$, consider the change of variables :

$$(Z_{\mathbf{M}}, T_{\mathbf{N}}, \Omega_{\mathbf{N}}, U_{\mathbf{N}}) \longmapsto Z^{\varepsilon}_{\mathbf{M}+\mathbf{N}}(\theta-\delta) \in \mathcal{R}^0_a \,, \tag{3.9}$$

where the configurations in $\mathcal{R}^0_a$ have to be compatible with pseudo-trajectories satisfying the following constraints on $(\theta-\delta,\theta)$:

(i) there are n_j particles added to the block j;
(ii) the addition of new particles in the block j is prescribed by $a^{(j)}$;
(iii) the pseudo-trajectory has no internal recollision.

Note that pseudo-trajectories compatible with $\mathcal{R}^0_a$ may have external recollisions (recall Definition 3.2). This change of variables is injective since the forward flow underlying (3.9) starting from $Z_{\mathbf{M}+\mathbf{N}} \in \mathcal{R}^0_a$ at $\theta-\delta$ can be defined in a unique way on $[\theta-\delta,\theta]$ as explained below.

Definition 3.5. *We say that two particles* <u>*encounter*</u> *when they approach at a distance ε at some time, in the forward flow.*

If two particles encounter at time t^-, their resulting configuration at time t^+ is then obtained as follows:

(a) if they belong to two different blocks, then their velocities are deflected according to the scattering law (1.2);
(b) if they are in the same block j, the collision is prescribed by the tree $a^{(j)}$, and one of the two particles disappears (it is removed) from the flow. The velocity of the remaining particle is updated by scattering, or not, according to the parameters $\mathbf{S}_j = (s^{(j)}_\ell)_{\ell\le n_j}$ encoded also by the tree $a^{(j)}$.

One can prove recursively that the jacobian of the inverse map (3.9) is

$$\frac{1}{\mu_\varepsilon^N} \prod_{j\in B} \prod_{\ell=1}^{n_j} \Big(\big(u^{(j)}_\ell - v^{\varepsilon}_{a^{(j)}_\ell}(t^{(j)+}_\ell)\big) \cdot \omega^{(j)}_\ell \Big)_+ \,.$$

Denoting by $Z^{\varepsilon}_{\mathbf{M}}(\theta, Z_{\mathbf{M}+\mathbf{N}})$ the configuration of the M particles at time θ starting from $Z_{\mathbf{M}+\mathbf{N}}$ in $\mathcal{R}^0_a$ at time $\theta-\delta$, one can therefore write

$$I^0_{\mathbf{M},\mathbf{N}} = \sum_{a\in\mathcal{A}^{\pm}_{\mathbf{M},\mathbf{N}}} \mu_\varepsilon^N \int_{\mathcal{R}^0_a} dZ_{\mathbf{M}+\mathbf{N}} G^{\varepsilon}_{M+N}(\theta-\delta, Z_{M+N}) \Phi_{\mathbf{M}}\big(Z^{\varepsilon}_{\mathbf{M}}(\theta, Z_{\mathbf{M}+\mathbf{N}})\big) \prod_{j\in B} \operatorname{sign}(\mathbf{S}_j) \,, \tag{3.10}$$

where the factor μ_ε^N comes from the jacobian, and $\operatorname{sign}(\mathbf{S}_j)$ is the product of all scatterings signs attached to the block j

$$\operatorname{sign}(\mathbf{S}_j) = \prod_{\ell=1}^{n_j} s^{(j)}_\ell \,.$$

Referring to (3.6), since $\Phi_{\mathbf{M}}$ is symmetric within blocks, we get a complete symmetrization within blocks by performing a partial symmetrization on the added particles : we will denote by $\mathfrak{S}^{n_j}_{m_j+n_j}$ the partitions of n_j+m_j particles in n_j (ordered) singletons and a block of size m_j. Then we can set

$$\Phi^0_{\mathbf{M},\mathbf{N}}(Z_{\mathbf{M}+\mathbf{N}}) := \mu_\varepsilon^N \sum_{\left(\sigma_j\in\mathfrak{S}^{n_j}_{n_j+m_j}\right)_{j\in B}} \sum_{a\in\mathcal{A}^\pm_{\mathbf{M},\mathbf{N}}} \Phi_{\mathbf{M}}\big(Z^\varepsilon_{\mathbf{M}}(\theta,Z_\sigma)\big)\mathbf{1}_{\{Z_\sigma\in\mathcal{R}^0_a\}}\prod_{j\in B}\left(\frac{m_j!}{(m_j+n_j)!}\operatorname{sign}(\mathbf{S}_j)\right),$$

where we have denoted for simplicity $Z^\varepsilon_{\mathbf{M}}(\theta,Z_\sigma)$ for the M remaining particles at time θ, and as a consequence (3.10) can be rewritten by duality, recalling (3.1),

$$I^0_{\mathbf{M},\mathbf{N}} = \mathbb{E}_\varepsilon\Big[\Big(\prod_{u=1}^{p-1}\zeta^\varepsilon_{\theta_u}\big(h^{(u)}\big)\Big)\pi^\varepsilon_{M+N,\theta-\delta}\big(\Phi^0_{\mathbf{M},\mathbf{N}}\big)\Big]. \tag{3.11}$$

Note that on each configuration $Z_{\mathbf{M}+\mathbf{N}}$, there exist at most 4^N different (σ,a) such that Z_σ belongs to $\mathcal{R}^0_a$. Indeed at each encounter in the forward dynamics (recall Definition 3.5), the particle which disappears has to be chosen, as well as a possible scattering. To fix these discrepancies, we introduce for each index $j\in B$ two sets of signs $\bar{\mathbf{S}}_j := (\bar s^{(j)}_\ell)_{1\le\ell\le n_j}$ and $\mathbf{S}_j := (s^{(j)}_\ell)_{1\le\ell\le n_j}$ which determine respectively which particle should be annihilated (say $\bar s^{(j)}_\ell = +$ if the particle with largest index remains, $\bar s^{(j)}_\ell = -$ if it disappears) and whether there is scattering ($s^{(j)}_\ell = +$) or not ($s^{(j)}_\ell = -$). Note that the signs $(s^{(j)}_\ell)_{1\le j\le n_i}$ are encoded in the collision tree $a^{(j)}$ while $(\bar s^{(j)}_\ell)_{1\le\ell\le n_j}$ are known if σ_j is given. We stress the fact that if two particles in different blocks encounter, there is no ambiguity on the dynamics: it corresponds to a recollision in the backward pseudo-trajectory hence there is always scattering (see Case (a) page 18). If we prescribe the sets $(\mathbf{S},\bar{\mathbf{S}}) := (\mathbf{S}_j,\bar{\mathbf{S}}_j)_{j\in B}$, then the mapping

$$(a,(\sigma_j)_{j\in B},Z_{\mathbf{M}},T_{\mathbf{N}},\Omega_{\mathbf{N}},U_{\mathbf{N}}) \longmapsto Z^\varepsilon_{\mathbf{M}+\mathbf{N}}(\theta-\delta) \tag{3.12}$$

restricted to pseudo-trajectories compatible with $(\mathbf{S},\bar{\mathbf{S}})$, is injective. This leads to defining

$$\Phi^0_{\mathbf{\Xi}^0}(Z_{\mathbf{M}+\mathbf{N}}) := \mu_\varepsilon^N\mathbf{1}_{\{Z_{\mathbf{M}+\mathbf{N}}\in\mathcal{R}^0_{\mathbf{S},\bar{\mathbf{S}}}\}}\,\Phi_{\mathbf{M}}\big(Z^\varepsilon_{\mathbf{M}}(\theta,Z_{\mathbf{M}+\mathbf{N}})\big)\prod_{j\in B}\left(\frac{m_j!}{(m_j+n_j)!}\operatorname{sign}(\mathbf{S}_j)\right), \tag{3.13}$$

where $\mathcal{R}^0_{\mathbf{S},\bar{\mathbf{S}}}$ is the set of configurations such that a forward flow with n_j annihilations in the block j and compatible with $(\mathbf{S},\bar{\mathbf{S}})$ exists, and where

$$\mathbf{\Xi}^0 := (\mathbf{M},\mathbf{N},\mathbf{S},\bar{\mathbf{S}}). \tag{3.14}$$

Our final result is then (3.11) with

$$\Phi^0_{\mathbf{M},\mathbf{N}} = \sum_{\mathbf{S},\bar{\mathbf{S}}}\Phi^0_{\mathbf{\Xi}^0},$$

where the sum over $\mathbf{S},\bar{\mathbf{S}}$ runs in $\{-1,1\}^{2n_j}, j\in B$.

Remark 3.6. *The symmetrisation over the labels of the particles, which was already an important argument in* [8]*, is a key step of the procedure: it is not apparent when looking at the expectation since the sum over the partial permutations compensates exactly the combinatorial factor* $m_j!/(m_j+n_j)!$*, on the other hand since the supports of the test functions are disjoint, it will be a true gain when computing the variance.*

3.2.2. *Duality: general case.* In the case when internal recollisions are allowed in backward pseudo-trajectories, the change of variables (3.8) is no longer injective and, in order to apply our strategy, we need to control the number of internal recollisions. The important fact is that thanks to the conditioning $\Upsilon^\varepsilon_{\mathcal{N}}$ introduced in Definition 2.2, the configuration at time $\theta-\delta$ has no microscopic cluster of more than γ particles, and the total energy of each microscopic cluster is at most $\gamma\mathbb{V}^2/2$ so that the variation of the relative distance between two particles from different clusters is at most $2\sqrt{\gamma}\,\mathbb{V}\delta$, which prevents any collision during the time lapse δ. Each cluster evolves therefore independently from the other clusters on the time interval $[\theta-\delta,\theta]$.

Furthermore the recollisions in each cluster cannot be due to periodicity since $\mathbb{V}\delta \ll 1$. Since the total number of collisions for a system of γ hard spheres in the whole space is finite (see Theorem 1.3 in [10] or [21]) say at most k_γ, each particle in a pseudo-trajectory cannot have more than $K_\gamma = \sum_{\ell=2}^{\gamma} k_\ell$ recollisions during the short amount of time δ. This crude upper bound on the number of recollisions takes into account the fact that the number of particles in a cluster may have varied on $[\theta-\delta,\theta]$ due the creation of new particles. We then associate with each particle i an index κ_i (less than K_γ) which is zero at time θ and increased by one each time the particle undergoes a recollision in the backward pseudo-dynamics. We denote by $\mathbf{K}_{\mathbf{M}+\mathbf{N}}$ the set of recollision indices $(\kappa_\ell^{(j)})_{\ell\le m_j+n_j}$ at time $\theta-\delta$. This new set of parameters enables us to recover the lost injectivity of (3.12). The construction of the forward dynamics starting from a configuration $Z_{\mathbf{M}+\mathbf{N}}$ is slightly more intricate. Fix a tree $a\in\mathcal{A}^{\pm}_{\mathbf{M},\mathbf{N}}$, a set of indices $\mathbf{K}_{\mathbf{M}+\mathbf{N}}$ and the starting configuration $Z_{\mathbf{M}+\mathbf{N}} = (Z^{(j)}_{m_j+n_j})_{j\in B}$. The forward flow is uniquely defined based on the following three possibilities, each time two particles encounter :

(a) either the two particles belong to two different blocks : in this case the particles are scattered and their indices are unchanged (this corresponds to an external recollision in the backward pseudo-trajectory);
(b) or the two particles belong to the same block and have a positive index: in this case also the particles are scattered, and their indices are decreased by 1 (this corresponds to an internal recollision in the backward pseudo-trajectory);
(c) or the two particles belong to the same block and one particle has zero index: one particle (with zero index) is annihilated and the other one is possibly scattered, as prescribed by the collision tree a. The indices are unchanged.

Finally we define, for each $a\in\mathcal{A}^{\pm}_{\mathbf{M},\mathbf{N}}$ and each $\mathbf{K}$ (we drop the index $\mathbf{M}+\mathbf{N}$ in $\mathbf{K}_{\mathbf{M}+\mathbf{N}}$ for the sake of readability in the sequel), the set $\mathcal{R}_{\mathbf{K},a}$ of configurations compatible with backward pseudo-trajectories having the following constraints:

(i) there are n_j particles added to the block j;
(ii) the addition of new particles is prescribed by $a^{(j)}$;
(iii) internal recollisions are compatible with $(\kappa_\ell^{(j)})_{\ell\le m_j+n_j}$ (coded in $\mathbf{K}$).

Notice that, denoting by $\mathbf{K}=\mathbf{0}$ the set of all null recollision indices, $\mathcal{R}_{\mathbf{0},a}=\mathcal{R}^0_a$. The change of variables, as in (3.9),

$$(Z_{\mathbf{M}},T_{\mathbf{N}},\Omega_{\mathbf{N}},U_{\mathbf{N}}) \longmapsto Z^{\varepsilon}_{\mathbf{M}+\mathbf{N}}(\theta-\delta)\in\mathcal{R}_{\mathbf{K},a} \tag{3.15}$$

is injective. Denoting as previously by $Z^{\varepsilon}_{\mathbf{M}}(\theta,Z_{\mathbf{M}+\mathbf{N}})$ the configuration of the M particles at time θ starting from $Z_{\mathbf{M}+\mathbf{N}}\in\mathcal{R}_{\mathbf{K},a}$ at time $\theta-\delta$, one can therefore write

$$\begin{aligned} I_{\mathbf{M},\mathbf{N}} &:= \int \Big(Q^{\varepsilon}_{\mathbf{M},\mathbf{N}}(\delta)G^{\varepsilon}_{M+N}(\theta-\delta)\Big)(Z_{\mathbf{M}})\,\Phi_{\mathbf{M}}(Z_{\mathbf{M}})dZ_{\mathbf{M}} \\ &= \sum_{a\in\mathcal{A}^{\pm}_{\mathbf{M},\mathbf{N}}}\sum_{\mathbf{K}}\mu_\varepsilon^N\int_{\mathcal{R}_{\mathbf{K},a}} dZ_{\mathbf{M}+\mathbf{N}}G^{\varepsilon}_{M+N}(\theta-\delta,Z_{\mathbf{M}+\mathbf{N}})\Phi_{\mathbf{M}}\big(Z^{\varepsilon}_{\mathbf{M}}(\theta,Z_{\mathbf{M}+\mathbf{N}})\big)\prod_{j\in B}\operatorname{sign}(\mathbf{S}_j)\,. \end{aligned} \tag{3.16}$$

As in (3.13), we can use the exchangeability of G^{ε}_{M+N} and $\phi^{(\varsigma_i)}$ to symmetrize partially the particles in each block j at time $\theta-\delta$, by summing over $\sigma_j\in\mathfrak{S}^{n_j}_{m_j+n_j}$. The mapping

$$(a,(\sigma_j)_{j\in B},Z_{\mathbf{M}},T_{\mathbf{N}},\Omega_{\mathbf{N}},U_{\mathbf{N}})\mapsto Z^{\varepsilon}_{\mathbf{M}+\mathbf{N}}(\theta-\delta)$$

is injective for any fixed $\mathbf{K}$ and $(\mathbf{S},\bar{\mathbf{S}})$, so one can define the δ-pullback of test functions

$$\sharp_\delta\Phi_{\mathbf{M}}(Z_{\mathbf{M}+\mathbf{N}}) := \Phi_{\mathbf{M}}\big(Z^{\varepsilon}_{\mathbf{M}}(\theta,Z_{\mathbf{M}+\mathbf{N}})\big) \tag{3.17}$$

where the configuration $Z^\varepsilon_{\mathbf{M}}(\theta, Z_{\mathbf{M}+\mathbf{N}})$ at time θ is obtained by the forward dynamics described above from the configuration $Z_{\mathbf{M}+\mathbf{N}}$ at time $\theta-\delta$. Finally we set

$$\Phi_{\boldsymbol{\Xi}}(Z_{\mathbf{M}+\mathbf{N}}) := \mu_\varepsilon^N \, \sharp_\delta \Phi_{\mathbf{M}}(Z_{\mathbf{M}+\mathbf{N}}) \, \mathbf{1}_{\{Z_{\mathbf{M}+\mathbf{N}} \in \mathcal{R}_{\mathbf{K},\mathbf{S},\bar{\mathbf{S}}}\}} \prod_{j\in B} \left(\frac{m_j!}{(m_j+n_j)!} \operatorname{sign}(\mathbf{S}_j) \right), \tag{3.18}$$

where $\mathcal{R}_{\mathbf{K},\mathbf{S},\bar{\mathbf{S}}}$ is the set of configurations such that a forward flow with n_j annihilations in the block j and compatible with $(\mathbf{K},\mathbf{S},\bar{\mathbf{S}})$ exists. We have denoted as in (3.14)

$$\boldsymbol{\Xi} := (\mathbf{M},\mathbf{N},\mathbf{K},\mathbf{S},\bar{\mathbf{S}})\,. \tag{3.19}$$

For $\mathbf{K}=\mathbf{0}$, note that $\mathcal{R}_{\mathbf{0},\mathbf{S},\bar{\mathbf{S}}} = \mathcal{R}^0_{\mathbf{S},\bar{\mathbf{S}}}$.

Finally set

$$\Phi_{\mathbf{M},\mathbf{N}} := \sum_{\mathbf{K},\mathbf{S},\bar{\mathbf{S}}} \Phi_{\boldsymbol{\Xi}}\,, \tag{3.20}$$

where the sum over $\mathbf{S},\bar{\mathbf{S}}$ runs in $\{-1,1\}^{2n_j}$, $j\in B$ and the sum over $\mathbf{K}$ runs in $\{0,\dots,K_\gamma\}^{m_j+n_j}$, with $j\in B$. Identity (3.16) can be rewritten by duality

$$\mathbb{E}_\varepsilon\Big[\Big(\prod_{u=1}^{p-1}\zeta^\varepsilon_{\theta_u}\big(h^{(u)}\big)\Big)\pi^\varepsilon_{M,\theta}\big(\Phi_{\mathbf{M}}\big)\Big] = \sum_{\mathbf{N}} \mathbb{E}_\varepsilon\Big[\Big(\prod_{u=1}^{p-1}\zeta^\varepsilon_{\theta_u}\big(h^{(u)}\big)\Big)\pi^\varepsilon_{M+N,\theta-\delta}\big(\Phi_{\mathbf{M},\mathbf{N}}\big)\Big]. \tag{3.21}$$

3.3. **Clustering structure.** Our aim is to study the transport of the factorization structure on an infinitesimal time interval $[\theta-\delta,\theta]$. We are interested in

$$\mathcal{I}_{\mathbf{M}} := \mathbb{E}_\varepsilon\Big[\Big(\prod_{u=1}^{p-1}\zeta^\varepsilon_{\theta_u}\big(h^{(u)}\big)\Big)\bigotimes_{i\le|\varsigma|}\zeta^\varepsilon_{M_{\varsigma_i},\theta}\big(\phi^{(\varsigma_i)}\big)\Big] \tag{3.22}$$

for a partition ς in packets of some $B\subset\{p,\dots,P\}$. As in (3.6), the function $\phi^{(\varsigma_i)}$ is evaluated at the configuration $(Z^{(j)}_{m_j})_{j\in\varsigma_i}$ at time θ. We recall that $M_{\varsigma_i} := \sum_{j\in\varsigma_i} m_j$. By analogy with (2.7), we define the $\circledast$-product for the conditioned fluctuation fields by discarding repeated indices

$$\bigotimes_{i\le|\varsigma|}\zeta^\varepsilon_{M_{\varsigma_i},\theta}\big(\phi^{(\varsigma_i)}\big) := \mu_\varepsilon^{|\varsigma|/2} \sum_{\alpha\subset\{1,\dots,|\varsigma|\}} \pi^\varepsilon_{M_\alpha,\theta}\Big(\bigotimes_{i\in\alpha}\phi^{(\varsigma_i)}\Big)\prod_{j\in\alpha^c}\mathbb{E}_\varepsilon[-\phi^{(\varsigma_j)}] \tag{3.23}$$

where $M_A = \sum_{i\in A} M_{\varsigma_i}$.

After using a pullback as in (3.21), we would like to recover a factorized structure with centered observables. This means that we need to decompose the functions $\Phi_{\boldsymbol{\Xi}}$ defined in (3.18) into products, and take care of the counterterms corresponding to contributions of different correlation functions. The technical procedure implementing this program is a cumulant decomposition of trajectories as devised in [7] (which we apply here to dual functions).

Let us first of all decompose the product

$$\mathcal{I}_{\mathbf{M}} = \mu_\varepsilon^{|\varsigma|/2} \sum_{\alpha\subset\{1,\dots,|\varsigma|\}} \Big(\prod_{i\in\alpha^c}\mathbb{E}_\varepsilon[-\phi^{(\varsigma_i)}]\Big) \times \mathbb{E}_\varepsilon\Big[\Big(\prod_{u=1}^{p-1}\zeta^\varepsilon_{\theta_u}\big(h^{(u)}\big)\Big)\pi^\varepsilon_{M_\alpha,\theta}(\Phi_\alpha)\Big], \tag{3.24}$$

denoting here $\mathbf{M}_\alpha = (m_j)_{i\in\alpha, j\in\varsigma_i}$, and $\Phi_\alpha = \bigotimes_{i\in\alpha}\phi^{(\varsigma_i)}$. To simplify notation, given the family $\mathbf{N}_\alpha = (n_j)_{i\in\alpha, j\in\varsigma_i}$ of added particles on $[\theta-\delta,\theta]$, we set $M^\delta_\alpha := M_\alpha + N_\alpha$ for the number of particles in the sets $(C_i)_{i\in\alpha}$ at time $\theta-\delta$. As in (3.21), we can write

$$\mathbb{E}_\varepsilon\Big[\Big(\prod_{u=1}^{p-1}\zeta^\varepsilon_{\theta_u}\big(h^{(u)}\big)\Big)\pi^\varepsilon_{M_\alpha,\theta}(\Phi_\alpha)\Big]$$
$$=\sum_{\mathbf{N}_\alpha}\int\Big(Q^\varepsilon_{\mathbf{M}_\alpha,\mathbf{N}_\alpha}(\delta)G^\varepsilon_{M_\alpha+N_\alpha}(\theta-\delta)\Big)(Z_{\mathbf{M}_\alpha})\ \Phi_\alpha(Z_{\mathbf{M}_\alpha})dZ_{\mathbf{M}_\alpha}$$
$$=\sum_{\mathbf{N}_\alpha}\mathbb{E}_\varepsilon\Big[\Big(\prod_{u=1}^{p-1}\zeta^\varepsilon_{\theta_u}\big(h^{(u)}\big)\Big)\pi^\varepsilon_{M^\delta_\alpha,\theta-\delta}\big(\Phi_{\mathbf{M}_\alpha,\mathbf{N}_\alpha}\big)\Big],$$

where as in (3.20)

$$\Phi_{\mathbf{M}_\alpha,\mathbf{N}_\alpha}:=\sum_{\mathbf{K}_\alpha,\mathbf{S}_\alpha,\bar{\mathbf{S}}_\alpha}\Phi_{\mathbf{\Xi}_\alpha}\,.$$

From now on to lighten further the notation we omit the dependence on the number of variables and set

$$\Phi_{\alpha,\delta}:=\Phi_{\mathbf{M}_\alpha,\mathbf{N}_\alpha}\,,$$

so that

$$\mathbb{E}_\varepsilon\Big[\Big(\prod_{u=1}^{p-1}\zeta^\varepsilon_{\theta_u}\big(h^{(u)}\big)\Big)\pi^\varepsilon_{M_\alpha,\theta}(\Phi_\alpha)\Big]=\sum_{\mathbf{N}_\alpha}\mathbb{E}_\varepsilon\Big[\Big(\prod_{u=1}^{p-1}\zeta^\varepsilon_{\theta_u}\big(h^{(u)}\big)\Big)\pi^\varepsilon_{M^\delta_\alpha,\theta-\delta}\big(\Phi_{\alpha,\delta}\big)\Big].\tag{3.25}$$

Note that in particular there holds

$$\mathbb{E}_\varepsilon[\phi^{(\varsigma_i)}]=\mathbb{E}_\varepsilon\Big[\pi^\varepsilon_{M_{\varsigma_i},\theta}\big(\phi^{(\varsigma_i)}\big)\Big]=\sum_{N_{\varsigma_i}}\mathbb{E}_\varepsilon\Big[\pi^\varepsilon_{M^\delta_{\varsigma_i},\theta-\delta}\big(\phi^{(\varsigma_i)}_\delta\big)\Big]=\sum_{\mathbf{N}_{\varsigma_i}}\mathbb{E}_\varepsilon\Big[\phi^{(\varsigma_i)}_\delta\Big],\tag{3.26}$$

where as above $\phi^{(\varsigma_i)}_\delta$ is defined after a summation over $\mathbf{K}_{\varsigma_i},\mathbf{S}_{\varsigma_i},\bar{\mathbf{S}}_{\varsigma_i}$.

Let us analyse $\Phi_{\alpha,\delta}$. Setting $\mathbf{M}^\delta_\alpha=\mathbf{M}_\alpha+\mathbf{N}_\alpha$, one has

$$\Phi_{\alpha,\delta}(Z_{\mathbf{M}^\delta_\alpha})=\sum_{\mathbf{K}_\alpha,\mathbf{S}_\alpha,\bar{\mathbf{S}}_\alpha}\Phi_{\mathbf{\Xi}_\alpha}(Z_{\mathbf{M}^\delta_\alpha})\,,\tag{3.27}$$

and the function $\Phi_{\alpha,\delta}$ is supported on configurations at time $\theta-\delta$ of the backward pseudo-trajectories corresponding to the packets $(C_i)_{i\in\alpha}$. We stress the fact that the variable decomposition among the blocks is still encoded in $Z_{\mathbf{M}^\delta_\alpha}$. If these pseudo-trajectories were evolving independently, then each of them would lead to a dual function $\phi^{(\varsigma_i)}_\delta$ and the product form would be exact. Even though this product form is the main part, there are further contributions due to dynamical correlations between the packets $(C_i)_{i\in\alpha}$ which we are going to analyze below.

We are going to group packets $(C_i)_{i\in\alpha}$ which are connected by (external) recollisions. Denote by $\mathcal{P}_\alpha$ the set of partitions of α. Given $\lambda\in\mathcal{P}_\alpha$, we restrict the change of variables (3.15) to the pseudo-trajectories such that a chain of recollisions occurs in each set $(\lambda_\ell)_{\ell\le|\lambda|}$, meaning that the graph with packets C_i as vertices and recollisions as edges has connected components specified by λ_ℓ. In particular, the pseudo-trajectories from two different connected components λ_{ℓ_1} and λ_{ℓ_2} do not approach, which we will denote by $\mathbf{1}_{\lambda_{\ell_1}\not\sim\lambda_{\ell_2}}$. Each connected component λ_ℓ will be called a *forest*. By extension, the blocks (and particles) of the associate packets will be said to belong to λ_ℓ. Denoting by $\mathcal{R}^\lambda_{\mathbf{K}_\alpha,\mathbf{S}_\alpha,\bar{\mathbf{S}}_\alpha}$ the corresponding restriction of $\mathcal{R}_{\mathbf{K}_\alpha,\mathbf{S}_\alpha,\bar{\mathbf{S}}_\alpha}$, we get the change of variables

$$(Z_{\mathbf{M}_\alpha},T_{\mathbf{N}_\alpha},\Omega_{\mathbf{N}_\alpha},U_{\mathbf{N}_\alpha})\longmapsto Z^\varepsilon_{\mathbf{M}^\delta_\alpha}(\theta-\delta)\in\mathcal{R}^\lambda_{\mathbf{K}_\alpha,\mathbf{S}_\alpha,\bar{\mathbf{S}}_\alpha}\,,\tag{3.28}$$

which is injective: by construction when two particles meet in the forward flow, they have to belong to the same forest λ_ℓ and the rule upon encounter (disappearance of a particle or not, scattering or not) is given by Definition 3.8 with parameters $\mathbf{K}_\alpha,\mathbf{S}_\alpha,\bar{\mathbf{S}}_\alpha$. In the

following we shall denote by $\mathcal{R}^{\lambda_\ell}_{\mathbf{K}_{\lambda_\ell},\mathbf{S}_{\lambda_\ell},\bar{\mathbf{S}}_{\lambda_\ell}}$ the set $\mathcal{R}^{\lambda}_{\mathbf{K}_\alpha,\mathbf{S}_\alpha,\bar{\mathbf{S}}_\alpha}$ restricted on a single forest. The definition (3.18) can be extended to dual functions with the constraint above:
(3.29)

$$\begin{aligned}\Phi_{\Xi_\alpha}(Z_{\mathbf{M}^\delta_\alpha}) &= \sum_{\lambda\in\mathcal{P}_\alpha} \mu_\varepsilon^{N_\alpha} \Phi_\alpha\big(Z^\varepsilon_{\mathbf{M}_\alpha}(\theta, Z_{\mathbf{M}^\delta_\alpha})\big) \mathbf{1}_{\{Z_{\mathbf{M}^\delta_\alpha}\in\mathcal{R}^\lambda_{\mathbf{K}_\alpha,\mathbf{S}_\alpha,\bar{\mathbf{S}}_\alpha}\}} \prod_{\substack{i\in\alpha\\ j\in\varsigma_i}} \left(\frac{m_j!}{(m_j+n_j)!}\operatorname{sign}(\mathbf{S}_j)\right)\\
&= \sum_{\lambda\in\mathcal{P}_\alpha} \prod_{\ell=1}^{|\lambda|} \Big(\Phi_{\lambda_\ell}\big(Z^\varepsilon_{\mathbf{M}_{\lambda_\ell}}(\theta, Z_{\mathbf{M}_{\lambda_\ell}+\mathbf{N}_{\lambda_\ell}})\big)\widetilde{\varphi}_{\lambda_\ell}(Z_{\mathbf{M}_{\lambda_\ell}+\mathbf{N}_{\lambda_\ell}})\Big) \times \prod_{\ell_1\neq\ell_2} \mathbf{1}_{\lambda_{\ell_1}\not\sim\lambda_{\ell_2}}(Z_{\mathbf{M}_\alpha+\mathbf{N}_\alpha})\\
&= \sum_{\lambda\in\mathcal{P}_\alpha} \prod_{\ell=1}^{|\lambda|} \Big((\sharp_\delta\Phi_{\lambda_\ell})\widetilde{\varphi}_{\lambda_\ell}\Big) \times \prod_{\ell_1\neq\ell_2} \mathbf{1}_{\lambda_{\ell_1}\not\sim\lambda_{\ell_2}}(Z_{\mathbf{M}_\alpha+\mathbf{N}_\alpha}),\end{aligned}$$

denoting by $(\sharp_\delta\Phi_{\lambda_\ell})$ the δ-pullback of Φ_{λ_ℓ} by the dynamics as in (3.17), and where the contribution of a forest is

$$\widetilde{\varphi}_{\lambda_\ell}(Z_{\mathbf{M}_{\lambda_\ell}+\mathbf{N}_{\lambda_\ell}}) := \mu_\varepsilon^{N_{\lambda_\ell}} \mathbf{1}_{\{Z_{\mathbf{M}_{\lambda_\ell}+\mathbf{N}_{\lambda_\ell}}\in\mathcal{R}^{\lambda_\ell}_{\mathbf{K}_{\lambda_\ell},\mathbf{S}_{\lambda_\ell},\bar{\mathbf{S}}_{\lambda_\ell}}\}} \prod_{\substack{i\in\lambda_\ell\\ j\in\varsigma_i}} \left(\frac{m_j!}{(m_j+n_j)!}\operatorname{sign}(\mathbf{S}_j)\right). \tag{3.30}$$

We have used the fact that the pseudo-trajectories associated with different forests do not intersect in order to decouple the Φ_{λ_ℓ}. The function $\widetilde{\varphi}_{\lambda_\ell}$ encodes in particular the correlations due to encounters between particles of different packets.

A correlation remains through the dynamical exclusion condition expressed by the constraint

$$\prod_{\ell_1\neq\ell_2} \mathbf{1}_{\lambda_{\ell_1}\not\sim\lambda_{\ell_2}}(Z_{\mathbf{M}_\alpha+\mathbf{N}_\alpha}),$$

encoding the fact that no encounter should occur between the particles in different forests λ_{ℓ_1} and λ_{ℓ_2}. We will expand this exclusion condition writing $\mathbf{1}_{\lambda_{\ell_1}\not\sim\lambda_{\ell_2}} = 1 - \mathbf{1}_{\lambda_{\ell_1}\sim\lambda_{\ell_2}}$, and defining the following notion.

Definition 3.7. *An overlap occurs between two forests $\lambda_{\ell_1}, \lambda_{\ell_2}$ (which is denoted by $\lambda_{\ell_1}\sim\lambda_{\ell_2}$) if two pseudo-particles from λ_{ℓ_1} and λ_{ℓ_2} find themselves at a distance less than ε one from the other at some time.*

Note that an overlap between two forests is a mathematical artefact to analyze the dynamical correlations. In particular, it does not modify the dynamics in the forests.

Definition 3.8 (Extended encounter rules). *Given a set of κ-indices $\mathbf{K}$, a set of signs $(\mathbf{S},\bar{\mathbf{S}})$ and a partition λ in forests, the forward flow starting from some configuration $Z_{\mathbf{M}+\mathbf{N}} = (Z^{(j)}_{m_j+n_j})_{j\in B}$ is reconstructed according to the following rules each time two particles encounter:*

- *either the two particles belong to two different forests : they do not see each other. The κ-indices are unchanged;*
- *or the two particles belong to two different blocks but to the same forest : they are scattered. The κ-indices are unchanged;*
- *or the two particles belong to the same block and have a positive κ-index: they are scattered. Both indices are decreased by 1;*
- *or the two particles belong to the same block and one particle has zero κ-index: one particle (with zero index) is annihilated. The label of the particle which is annihilated, and the possible scattering of the other colliding particle are prescribed by the signs $(\mathbf{S},\bar{\mathbf{S}})$. The other indices are unchanged.*

In order to identify all possible correlations, we introduce now a cumulant expansion of the non overlapping constraint

$$\prod_{\substack{\ell_1\neq\ell_2\\1\le\ell_1,\ell_2\le|\lambda|}} 1_{\lambda_{\ell_1}\not\sim\lambda_{\ell_2}} = \sum_{G\in\mathcal{G}_{|\lambda|}} \prod_{\{\ell_1,\ell_2\}\in E(G)} (-\mathbf{1}_{\lambda_{\ell_1}\sim\lambda_{\ell_2}}) = \sum_{\rho\in\mathcal{P}_{|\lambda|}} \prod_{q=1}^{|\rho|} \varphi_{\rho_q} \,, \tag{3.31}$$

where $\mathcal{G}_{|\lambda|}$ is the set of graphs G with $|\lambda|$ vertices, $E(G)$ denotes the set of edges of a graph G, and the cumulants are defined on the connected components ρ_q of $\{1,\dots,|\lambda|\}$ by

$$\varphi_{\rho_q} = \sum_{G'\in\mathcal{C}_{\rho_q}} \prod_{\{\ell_1,\ell_2\}\in E(G')} (-\mathbf{1}_{\lambda_{\ell_1}\sim\lambda_{\ell_2}}) \,, \tag{3.32}$$

denoting $\mathcal{C}_{\rho_q}$ the set of connected graphs with vertices ρ_q. In particular, the function φ_{ρ_q} is supported on clusters formed by overlapping forests.

Combining (3.29) with (3.31), we get

$$\Phi_{\mathbf{\Xi}_\alpha} = \sum_{\lambda\in\mathcal{P}_\alpha} \sum_{\rho\in\mathcal{P}_{|\lambda|}} \prod_{\ell=1}^{|\lambda|} \big(\sharp_\delta \Phi_{\lambda_\ell} \widetilde{\varphi}_{\lambda_\ell}\big) \prod_{q=1}^{|\rho|} \varphi_{\rho_q} \,,$$

denoting by $(\sharp_\delta \Phi_{\lambda_\ell})$ the δ-pullback of Φ_{λ_ℓ} by the dynamics as in (3.17). Exchanging the order of the sums, we end up with the following (scaled) cumulant expansion

$$\Phi_{\mathbf{\Xi}_\alpha} = \sum_{\eta\in\mathcal{P}_\alpha} \prod_{q=1}^{|\eta|} \mu_\varepsilon^{1-|\eta_q|} \phi^{(\eta_q)}_{\delta,\mathbf{\Xi}_{\eta_q}} \,, \tag{3.33}$$

where the (dual) cumulants are defined for any subset η_q of $\{1,\dots,|\varsigma|\}$ by

$$\phi^{(\eta_q)}_{\delta,\mathbf{\Xi}_{\eta_q}} := \mu_\varepsilon^{|\eta_q|-1} \sum_{\lambda\in\mathcal{P}_{\eta_q}} \Big(\prod_{\ell=1}^{|\lambda|} \sharp_\delta \Phi_{\lambda_\ell} \, \widetilde{\varphi}_{\lambda_\ell}(Z_{\mathbf{M}_{\lambda_\ell}+\mathbf{N}_{\lambda_\ell}}) \Big) \varphi_{\{\lambda_1,\dots,\lambda_{|\lambda|}\}} \,. \tag{3.34}$$

Recall that $\widetilde{\varphi}$ encodes the external recollisions between packets in each forest and keeps track of $\mathbf{K}_{\lambda_\ell}, \mathbf{S}_{\lambda_\ell}, \bar{\mathbf{S}}_{\lambda_\ell}$, while φ encodes the overlaps between forests. Finally we set

$$\phi^{(\eta_q)}_{\delta}(Z_{M^\delta_{\eta_q}}) := \sum_{\mathbf{K}_{\eta_q},\mathbf{S}_{\eta_q},\bar{\mathbf{S}}_{\eta_q}} \phi^{(\eta_q)}_{\delta,\mathbf{\Xi}_{\eta_q}} \tag{3.35}$$

so that, plugging (3.30) in (3.34) and denoting $\operatorname{sign}(\mathbf{S}_{\eta_q})$ the product of all scattering signs $\mathbf{S}_{\eta_q}$, we obtain

$$\begin{aligned} \phi^{(\eta_q)}_{\delta} = \mu_\varepsilon^{N_{\eta_q}+|\eta_q|-1} \left(\prod_{\substack{i\in\eta_q\\ j\in\varsigma_i}} \frac{m_j!}{(m_j+n_j)!} \right) \sum_{\mathbf{K}_{\eta_q},\mathbf{S}_{\eta_q},\bar{\mathbf{S}}_{\eta_q}} \sum_{\lambda\in\mathcal{P}_{\eta_q}} \operatorname{sign}(\mathbf{S}_{\eta_q}) \varphi_{\{\lambda_1,\dots,\lambda_{|\lambda|}\}} \\ \times \Big(\prod_{\ell=1}^{|\lambda|} (\sharp_\delta \Phi_{\lambda_\ell}) \mathbf{1}_{\{Z_{\mathbf{M}_{\lambda_\ell}+\mathbf{N}_{\lambda_\ell}}\in\mathcal{R}^{\lambda_\ell}_{\mathbf{K}_{\lambda_\ell},\mathbf{S}_{\lambda_\ell},\bar{\mathbf{S}}_{\lambda_\ell}}\}} \Big) . \end{aligned} \tag{3.36}$$

As a consequence of (3.27), (3.33) and (3.35), we finally obtain a cumulant decomposition for any collection α of packets

$$\Phi_{\alpha,\delta}(Z_{\mathbf{M}^\delta_\alpha}) = \sum_{\eta\in\mathcal{P}_\alpha} \prod_{q=1}^{|\eta|} \mu_\varepsilon^{1-|\eta_q|} \phi^{(\eta_q)}_{\delta} \,, \tag{3.37}$$

where $\mathbf{M}^\delta_\alpha = \mathbf{M}_\alpha + \mathbf{N}_\alpha$.

By definition, $\phi_\delta^{(\eta_q)}$ corresponds to the contribution of packets $(C_i)_{i\in\eta_q}$ which are completely connected dynamically by encounters. From the definition (3.24) of $\mathcal{I}_{\mathbf{M}}$, identities (3.25)-(3.26) and the cumulant decomposition (3.37), we arrive at

$$\mathcal{I}_{\mathbf{M}} = \mu_\varepsilon^{|\varsigma|/2} \sum_{\mathbf{N}} \sum_{\alpha\subset\{1,\dots,|\varsigma|\}} \Big(\prod_{i\in\alpha^c} \mathbb{E}_\varepsilon[-\phi_\delta^{(\varsigma_i)}]\Big) \times \sum_{\eta\in\mathcal{P}_\alpha} \mathbb{E}_\varepsilon\Big[\Big(\prod_{u=1}^{p-1} \zeta^\varepsilon_{\theta_u}\big(h^{(u)}\big)\Big)\pi^\varepsilon_{M^\delta_\alpha,\theta-\delta}\Big(\bigotimes_{q=1}^{|\eta|} \mu_\varepsilon^{1-|\eta_q|}\phi_\delta^{(\eta_q)}\Big)\Big].$$

Note that $\phi_\delta^{(\varsigma_i)}$ is indexed by a single set ς_i so that it is constructed without resorting to forests or overlaps. Moreover we can decompose each cumulant in the sum of its expectation and its fluctuation

$$\pi^\varepsilon_{M^\delta_{\eta_q},\theta-\delta}(\phi_\delta^{(\eta_q)}) = \mu_\varepsilon^{-\frac12}\zeta^\varepsilon_{M^\delta_{\eta_q},\theta-\delta}(\phi_\delta^{(\eta_q)}) + \mathbb{E}_\varepsilon[\phi_\delta^{(\eta_q)}]\,. \tag{3.38}$$

We obtain (cf. (3.23))

$$\mathcal{I}_{\mathbf{M}} = \mu_\varepsilon^{|\varsigma|/2} \sum_{\mathbf{N}} \sum_{\alpha\subset\{1,\dots,|\varsigma|\}} \sum_{\eta\in\mathcal{P}_\alpha} \Big(\prod_{i\in\alpha^c} \mathbb{E}_\varepsilon[-\phi_\delta^{(\varsigma_i)}]\Big) \times \sum_{\eta\in\mathcal{P}_\alpha} \sum_{I\subset\{1,\dots,|\eta|\}} \prod_{q\in I^c} \mu_\varepsilon^{1-|\eta_q|}\mathbb{E}_\varepsilon\Big[\phi_\delta^{(\eta_q)}\Big]\,\mathbb{E}_\varepsilon\Big[\Big(\prod_{u=1}^{p-1} \zeta^\varepsilon_{\theta_u}\big(h^{(u)}\big)\Big)\Big(\bigotimes_{q\in I} \mu_\varepsilon^{\frac12-|\eta_q|}\zeta^\varepsilon_{M^\delta_{\eta_q},\theta-\delta}(\phi_\delta^{(\eta_q)})\Big)\Big].$$

For a given α, we denote by I the set of observables contributing to the fluctuation field, and by I^c the set of observables contributing via their expectation. We then split I^c into two parts

$$I^c_1 := \{q\in I^c\,/\,|\eta_q| = 1\}\,,\quad I^c_2 := \{q\in I^c\,/\,|\eta_q|\geq 2\}\,.$$

We also define

$$\beta_- = \alpha^c\,,\quad \beta_+ = \bigcup_{q\in I^c_1}\eta_q\,,\quad \beta_1 = \bigcup_{q\in I}\eta_q\,,\quad \beta_2 = \bigcup_{q\in I^c_2}\eta_q\,,$$

and we denote (abusively) η_1 and η_2 the restriction of η to β_1 and β_2 respectively. By definition, η_2 has no singleton: recall that as defined in Section 2.3, such a partition is called *clustering*. Then by Fubini,

$$\begin{aligned}\mathcal{I}_{\mathbf{M}} = \mu_\varepsilon^{|\varsigma|/2} \sum_{\mathbf{N}} \sum_{\substack{\beta_1,\beta_2,\beta_-,\beta_+\\ \text{partition of }\{1,\dots,|\varsigma|\}}} \sum_{\eta_1\in\mathcal{P}_{\beta_1}} \sum_{\substack{\eta_2\in\mathcal{P}_{\beta_2}\\ \eta_2\ \text{clustering}}} \prod_{\ell_-\in\beta_-} \mathbb{E}_\varepsilon[-\phi_\delta^{(\ell_-)}]\\ \times \prod_{\ell_+\in\beta_+} \mathbb{E}_\varepsilon[\phi_\delta^{(\ell_+)}] \prod_{1\leq q\leq|\eta_2|} \mu_\varepsilon^{1-|\eta_{2,q}|}\mathbb{E}_\varepsilon[\phi_\delta^{(\eta_{2,q})}]\\ \times \mathbb{E}_\varepsilon\Big[\Big(\prod_{u=1}^{p-1} \zeta^\varepsilon_{\theta_u}\big(h^{(u)}\big)\Big)\Big(\bigotimes_{1\leq q\leq|\eta_1|} \mu_\varepsilon^{\frac12-|\eta_{1,q}|}\zeta^\varepsilon_{M^\delta_{\eta_{1,q}},\theta-\delta}(\phi_\delta^{(\eta_{1,q})})\Big)\Big].\end{aligned} \tag{3.39}$$

Fixing β_1 and β_2, we see that the sum over β_-,β_+ is zero as soon as $\beta_1\cup\beta_2 \neq \{1,\dots,|\varsigma|\}$. We find

$$\begin{aligned}\mathcal{I}_{\mathbf{M}} = \mu_\varepsilon^{|\varsigma|/2} \sum_{\mathbf{N}} \sum_{\substack{\beta_1,\beta_2\\ \text{partition of }\{1,\dots,|\varsigma|\}}} \sum_{\eta_1\in\mathcal{P}_{\beta_1}} \sum_{\substack{\eta_2\in\mathcal{P}_{\beta_2}\\ \eta_2\ \text{clustering}}} \prod_{1\leq q\leq|\eta_2|} \mu_\varepsilon^{1-|\eta_{2,q}|}\mathbb{E}_\varepsilon\Big[\phi_\delta^{(\eta_{2,q})}\Big]\\ \times \mathbb{E}_\varepsilon\Big[\Big(\prod_{u=1}^{p-1} \zeta^\varepsilon_{\theta_u}\big(h^{(u)}\big)\Big)\Big(\bigotimes_{1\leq q\leq|\eta_1|} \mu_\varepsilon^{\frac12-|\eta_{1,q}|}\zeta^\varepsilon_{M^\delta_{\eta_{1,q}},\theta-\delta}(\phi_\delta^{(\eta_{1,q})})\Big)\Big].\end{aligned}$$

Finally given η_1 and η_2 we decompose $|\varsigma| = \sum_q|\eta_{1,q}| + \sum_q|\eta_{2,q}|$ and we arrive at the following identity.

Proposition 3.9. *Consider a partition ς of a set $B \subset \{p, \dots, P\}$, indexing the test functions $(\phi^{(\varsigma_i)})_{i \le |\varsigma|}$ as in* (3.6). *Then for any $\theta = \theta_p - r\delta$ with $r \in [0, (\theta_p - \theta_{p-1})/\delta]$, there holds with notation* (3.36)

$$\begin{aligned}
&\mathbb{E}_\varepsilon\Big[\Big(\prod_{u=1}^{p-1} \zeta^\varepsilon_{\theta_u}\big(h^{(u)}\big)\Big) \bigotimes_{i \le |\varsigma|} \zeta^\varepsilon_{M_{\varsigma_i},\theta}\big(\phi^{(\varsigma_i)}\big)\Big] \\
&= \sum_{\mathbf{N}} \sum_{\substack{\eta_1 \cup \eta_2 \text{ partition of } \{1,\dots,|\varsigma|\} \\ \eta_2 \text{ clustering}}} \prod_{q=1}^{|\eta_2|} \mu_\varepsilon^{1 - \frac{|\eta_{2,q}|}{2}} \mathbb{E}_\varepsilon\Big[\phi_\delta^{(\eta_{2,q})}\Big] \\
&\qquad \times \mathbb{E}_\varepsilon\Big[\Big(\prod_{u=1}^{p-1} \zeta^\varepsilon_{\theta_u}\big(h^{(u)}\big)\Big)\Big(\bigotimes_{q=1}^{|\eta_1|} \mu_\varepsilon^{\frac{1}{2} - \frac{|\eta_{1,q}|}{2}} \zeta^\varepsilon_{M^\delta_{\eta_{1,q}},\theta-\delta}\Big(\phi_\delta^{(\eta_{1,q})}\Big)\Big)\Big].
\end{aligned} \tag{3.40}$$

The algebraic identity (3.40) extends formula (2.14) to take into account the structure of packets at time θ. The length δ of the time interval is limited only by the fact that we need to control the number of internal recollisions uniformly in ε (so that the sums over $\mathbf{K}_{\eta_{i,q}}$ defining $\phi_\delta^{(\eta_{i,q})}$ are finite). Extending this time interval would require to modify the conditioning, but then $\mathbf{1}_{(\Upsilon^\varepsilon_N)^c}$ would not be a negligible correction.

4. Extracting minimal cumulants

In this section, we aim at iterating Proposition 3.9 to pull back the fluctuation structure on an intermediate time scale τ such that $\delta \ll \tau \ll 1$. For the sake of simplicity, we choose τ such that $R := \tau/\delta$ is an integer. Let $\theta = \theta_p - k\tau$ for some integer k be such that $[\theta - \tau, \theta] \subset [\theta_{p-1}, \theta_p]$.

4.1. **Backward iterated clustering.** Let $B = \{b_\ell \mid \ell = 1, \dots, |B|\}$ be a subset of $\{p, \dots, P\}$ indexing the test functions $(\phi^{(i)})_{i \in B}$ at time θ. As explained in Section 2.3, the strategy is to iterate $R = \tau/\delta$ times the formula (3.40) down to time $\theta - \tau$. We therefore construct iteratively on each time step $[\theta - r\delta, \theta - (r-1)\delta]$ for $r = 1, \dots, R$ nested partitions $\eta_1^{r-1} \hookrightarrow \eta_1^r \cup \eta_2^r$ with η_2^r corresponding to (non trivial) packets which are expelled from the main factorized structure, contributing only via their expectation, and η_1^r corresponding to packets contributing to the factorized structure via their fluctuations.

Keeping track of all the intermediate η^r for $1 \le r \le R$ would imply rapidly growing combinatorics. Therefore, at time $\theta - r\delta$, we are going to identify cumulants of the form $\phi_{r\delta}^{(A)}$, with $A \subset B$ by taking into account all possible clustering dynamics leading to a given cluster indexed by A, whatever the successive partitions $\eta_1^1, \dots, \eta_1^r$: the identification of packets of packets with the union of these packets is therefore essential to gather all contributions. This identification will be key to control the combinatorics.

On the other hand, the combinatorics encoding all possible elementary pullbacks on $[\theta - r\delta, \theta - (r-1)\delta]$ leading to a given configuration at time $\theta - r\delta$, is also quite bad and by iteration will be out of control. Indeed, the number of possible $\mathbf{K}$ describing the forward dynamics on $[\theta - r\delta, \theta - (r-1)\delta]$ is $K_\gamma^{M_B^{r\delta}}$ (recalling $M_B^{r\delta} = M_B^{(r-1)\delta} + N_B^r$). In particular, we do not expect that formula (3.40) can be iterated brutally $O(\tau/\delta)$ times without having a strong divergence. The strategy to avoid this divergence will consist in retaining at each time step $[\theta - (r-1)\delta, \theta - r\delta]$, only "local minimal cumulants" defined by pulling back the observables along backward pseudo-trajectories such that

(i) internal recollisions (inside any block j) are forbidden, i.e. $\mathbf{K}_j = 0$;
(ii) recollisions between blocks in any given packet are forbidden;
(iii) the graph encoding the recollisions between packets of any given forest has to be minimally connected;

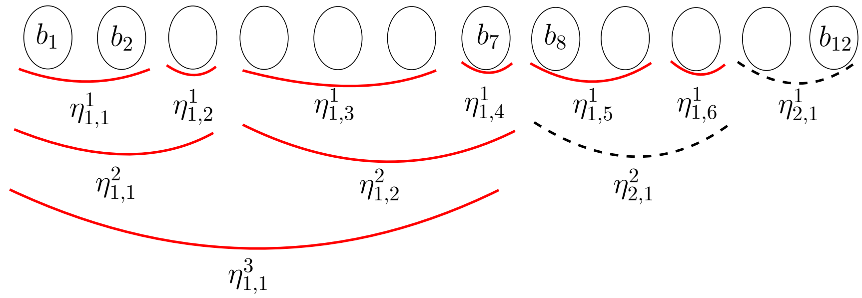

FIGURE 5. The nested partitions are depicted and the dashed parts represent the expelled clustering cumulants. By summing over all the possible intermediate decompositions of $\eta^3_{1,1} = \{b_1, \dots, b_7\}$, we will recover the cumulant $\phi_{3\delta}^{(\eta^3_{1,1})}$. The expelled cluster will be collected in the set $\rho^3 = \{\eta^1_{2,1}, \eta^2_{2,1}\}$.

(iv) the graph encoding the overlaps between the different forests has to be minimally connected.

We then define

$$\bar\phi_\delta^{(\eta_q)} := \mu_\varepsilon^{N_{\eta_q}+|\eta_q|-1} \left(\prod_{\substack{i\in\eta_q \\ j\in\varsigma_i}} \frac{m_j!}{(m_j+n_j)!} \right) \sum_{\mathbf{S}_{\eta_q}, \bar{\mathbf{S}}_{\eta_q}} \sum_{\lambda\in\mathcal{P}_{\eta_q}} \operatorname{sign}(\mathbf{S}_{\eta_q}) \bar\varphi_{\{\lambda_1,\dots,\lambda_{|\lambda|}\}} \times \Big(\prod_{\ell=1}^{|\lambda|} (\sharp_\delta \Phi_{\lambda_\ell}) \mathbf{1}_{\{Z_{\mathbf{M}_{\lambda_\ell}+\mathbf{N}_{\lambda_\ell}} \in \bar{\mathcal{R}}^{\lambda_\ell}_{\mathbf{S}_{\lambda_\ell}, \bar{\mathbf{S}}_{\lambda_\ell}}\}} \Big), \tag{4.1}$$

where $\bar{\mathcal{R}}^{\lambda_\ell}_{\mathbf{S}_{\lambda_\ell}, \bar{\mathbf{S}}_{\lambda_\ell}}$ is the set of configurations compatible with pseudo-trajectories in a forest satisfying (i)(ii)(iii), and $\bar\varphi$ is the restriction of φ to minimally connected graphs according to (iv). In particular, denoting by $\bar{\mathcal{R}}^{\lambda}_{\mathbf{S}_\lambda, \bar{\mathbf{S}}_\lambda}$ the resulting set of configurations, there holds

$$\bar\varphi_{\{\lambda_1,\dots,\lambda_{|\lambda|}\}} \mathbf{1}_{\{Z_{\mathbf{M}_\lambda+\mathbf{N}_\lambda} \in \bar{\mathcal{R}}^{\lambda}_{\mathbf{S}_\lambda, \bar{\mathbf{S}}_\lambda}\}} = (-1)^{|\lambda|-1} \mathbf{1}_{\{Z_{\mathbf{M}_\lambda+\mathbf{N}_\lambda} \in \bar{\mathcal{R}}^{\lambda}_{\mathbf{S}_\lambda, \bar{\mathbf{S}}_\lambda}\}} \tag{4.2}$$

Only the contribution of such local minimal cumulants in (3.40) will be iterated. Starting with $|B|$ blocks at time θ, we therefore obtain with this partial iteration "minimal cumulants" defined by pulling back the observables along backward pseudo-trajectories such that

(i) internal recollisions (inside any block j) are forbidden, i.e. $K_j = 0$;
(ii) the $|B|$ blocks are dynamically connected according to a graph with exactly $|B| - 1$ edges representing all encounters (corresponding to external recollisions and overlaps in the backward pseudo-trajectory). This graph is therefore minimally connected (meaning there are neither multiple edges nor cycles).

Remark 4.1. *Note that, by definition, particles which are in different packets at time $\theta - r\delta$ are independent on $[\theta - r\delta, \theta]$. In particular, even if they approach at a distance ε on $[\theta - r\delta, \theta]$, this is neither a recollision nor an overlap.*

4.2. **Iterated forward dynamics.** For minimal cumulants, the forward dynamics can be encoded by simpler combinatorics. Indeed to describe the partition in forests λ, it is enough to prescribe a sequence of signs $\mathcal{E} \in \{-1, +1\}^{|B|-1}$ encoding whether the $|B| - 1$ encounters between particles within different blocks have scattering (sign +1) or not (sign -1). Since we further assume that there is no internal encounter without annihilation, we end up with a "minimal forward dynamics" which is parametrized only by the sets of signs $(\mathbf{S}, \bar{\mathbf{S}})$ and $\mathcal{E}$.

Definition 4.2 (Minimal forward dynamics). *Given $1 \leq r \leq R$, let B be any subset of $\{1,\dots,P\}$, and consider integers $\mathbf{M}_B := (m_j)_{j\in B}$ and*

$$\mathbf{M}_B^{r\delta} := \mathbf{M}_B + \mathbf{N}_B := (m_j + n_j)_{j\in B}. \tag{4.3}$$

representing the particle numbers in the blocks respectively at times θ and $\theta - r\delta$.

A minimal forward dynamics on $[\theta - r\delta, \theta]$ starting from $Z_{\mathbf{M}_B^{r\delta}}$ is completely prescribed by two sequences $(\mathbf{S},\bar{\mathbf{S}}) = (\mathbf{S}_j,\bar{\mathbf{S}}_j)_{j\in B}$ with $(\mathbf{S}_j,\bar{\mathbf{S}}_j) \in \{-1,+1\}^{2n_j}$, and $\mathcal{E} \in \{-1,+1\}^{|B|-1}$. Moving forward in time, each time two particles approach at a distance ε,

- *if they belong to the same block j: the particle to be removed and the possible scattering are encoded by $\mathbf{S}_j, \bar{\mathbf{S}}_j$;*
- *if they belong to two different blocks: a (signed), minimally connected graph G_B with $|B|-1$ edges is constructed iteratively by adding an edge decorated with a sign according to $\mathcal{E}$ if the two particles are not already in the same connected component of the graph. If the two particles are in the same connected component of G_B, then*
  - *if the before-last edge in this connected component was created in a different time interval $[\theta - r'\delta, \theta - (r'-1)\delta]$, the particles are unaffected (this is actually not an encounter, see Remark 4.1);*
  - *if it occurs in the same time interval, the configuration is not admissible (by definition, the graph representing all encounters has to be minimal).*

We say that $Z_{\mathbf{M}_B^{r\delta}}$ is a minimal forward cluster *associated with $(\mathbf{S},\bar{\mathbf{S}},\mathcal{E})$ if the N_B annihilations occur, as well as the $|B|-1$ encounters making the graph G_B connected. This is denoted by $Z_{\mathbf{M}_B^{r\delta}} \in \mathcal{R}^{\min}_{\mathcal{E},\mathbf{S},\bar{\mathbf{S}}}$.*

The case when $B = \{i\}$ is reduced to one singleton is stressed by the denomination single minimal forward cluster*. Notice that a single minimal forward cluster is simply parametrised by $\mathbf{S},\bar{\mathbf{S}}$ (as $\mathcal{E}$ becomes irrelevant) and we will write $Z_{\mathbf{M}_B^{r\delta}} \in \mathcal{R}^{\min}_{\mathbf{S},\bar{\mathbf{S}}}$.*

Remark 4.3. *Note that the time intervals $[\theta - r'\delta, \theta - (r'-1)\delta]$ (with $r' \leq r$) when the different encounters happen are not prescribed in the definition above.*

With this definition of minimal forward dynamics, we obtain the following representation of minimal cumulants at time $\theta - r\delta$:

$$\bar{\phi}^{(B)}_{r\delta}(Z_{\mathbf{M}_B^{r\delta}}) := \mu_\varepsilon^{N_B+|B|-1}\left(\prod_{j\in B}\frac{m_j!}{M_j^{r\delta}!}\right)\sum_{\mathbf{S},\bar{\mathbf{S}},\mathcal{E}} \operatorname{sign}(\mathcal{E})\operatorname{sign}(\mathbf{S})\mathbf{1}_{Z_{\mathbf{M}_B^{r\delta}}\in\mathcal{R}^{\min}_{\mathcal{E},\mathbf{S},\bar{\mathbf{S}}}}\Big(\sharp_{r\delta}\bigotimes_{i\in B}\phi^{(i)}\Big), \tag{4.4}$$

where $\mathcal{R}^{\min}_{\mathcal{E},\mathbf{S},\bar{\mathbf{S}}}$ imposes that $Z_{\mathbf{M}_B^{r\delta}}$ is associated with a (unique) minimal forward dynamics on $[\theta - r\delta, \theta]$. As in (3.17), the pullback during a time $r\delta$ is given by

$$\sharp_{r\delta}\bigotimes_{i\in B}\phi^{(i)}(Z_{\mathbf{M}_B^{r\delta}}) := \bigotimes_{i\in B}\phi^{(i)}\big(Z^\varepsilon_{\mathbf{M}_B}(\theta, Z_{\mathbf{M}_B^{r\delta}})\big).$$

Remark 4.4. *Note that it is possible to prescribe a priori the numbers $(n_j^{r'})_{j\in B, r'\leq r}$ of particles annihilated at each time step in each tree, in which case there are additional "sampling" conditions on $\mathcal{R}^{\min}_{\mathcal{E},\mathbf{S},\bar{\mathbf{S}}}$.*

After iterating Proposition 3.9 up to time $\theta - r\delta$ for $1 \leq r \leq R$, we expect that the main contribution will be given by minimal cumulants of the form $\bar{\phi}^{(B)}_{r\delta}$. There are however additional terms at each time step $\theta - r'\delta$, for $1 \leq r' \leq r$, which will not be iterated; their contribution will be shown to be negligible in the limit. In order to analyse these error terms recursively for each r, we need a more general notion of forward cluster on $[\theta - r\delta, \theta]$, such that in the first interval $[\theta - r\delta, \theta - (r-1)\delta]$ the forward dynamics is not necessarily minimal but it is minimal starting at time $\theta - (r-1)\delta$.

Definition 4.5 (**Forward dynamics**). *Given $1 \le r \le R$, let B be any subset of $\{1,\dots,P\}$, and consider integers $\mathbf{M}_B := (m_j)_{j\in B}$, $\mathbf{M}_B^{(r-1)\delta} := \mathbf{N}_B^{<r} + \mathbf{M}_B := (m_j + n_j^{<r})_{j\in B}$ and $\mathbf{M}_B^{r\delta} := \mathbf{N}_B^r + \mathbf{N}_B^{<r} + \mathbf{M}_B = (m_j + n_j^{<r} + n_j^r)_{j\in B}$ representing the particle numbers in the blocks respectively at times θ, $\theta-(r-1)\delta$ and $\theta - r\delta$.*

A forward dynamics on $[\theta - r\delta, \theta]$ starting from $Z_{\mathbf{M}_B^{r\delta}}$ is completely prescribed by the following parameters :

- *for each $j \in B$, a sequence $(\mathbf{S}_j, \bar{\mathbf{S}}_j) \in \{-1,+1\}^{2n_j}$ encoding the particle to be removed and the possible scattering at each encounter between two particles of the same block j. The restriction to the time interval $[\theta - r\delta, \theta-(r-1)\delta]$ of these parameters is denoted by $(\mathbf{S}_j^r, \bar{\mathbf{S}}_j^r) \in \{-1,+1\}^{2n_j^r}$ and the other parameters by $(\mathbf{S}_j^{<r}, \bar{\mathbf{S}}_j^{<r}) \in \{-1,+1\}^{2n_j^{<r}}$;*
- *a partition ς of B, defining packets $(C_i)_{i\le|\varsigma|}$ at time $\theta-(r-1)\delta$;*
- *a partition $\lambda \in \mathcal{P}_{|\varsigma|}$ in forests on $[\theta - r\delta, \theta-(r-1)\delta]$;*
- *a multi-index $\mathbf{K} \in \{0,\dots,K_\gamma\}^{M_B^{r\delta}}$ encoding the number of internal encounters without annihilation per particle on $[\theta - r\delta, \theta-(r-1)\delta]$. Note that within the packet C_i, particles in different blocks are always scattered when they encounter during the time interval $[\theta - r\delta, \theta-(r-1)\delta]$;*
- *for each $i \le |\varsigma|$, a sequence $\mathcal{E}_i \in \{-1,+1\}^{|\varsigma_i|-1}$ encoding the types of encounters in the packet C_i on $[\theta-(r-1)\delta, \theta]$.*

We say that $Z_{\mathbf{M}_B^{r\delta}}$ is a forward cluster *associated with $(\mathbf{S}, \bar{\mathbf{S}}, \varsigma, \lambda, \mathbf{K}, (\mathcal{E}_i)_{i\le|\varsigma|})$ if all encounters occur in such a way that the graph coding these encounters is completely connected, modulo the identification of the m_j particles of each block j at time θ as a unique vertex.*

Remark 4.6. *The forward cluster $Z_{\mathbf{M}_B^{r\delta}}$ associated with $(\mathbf{S}, \bar{\mathbf{S}}, \varsigma, \lambda, \mathbf{K}, (\mathcal{E}_i)_{i\le|\varsigma|})$ is recovered by creating $|\varsigma|$ independent minimal forward clusters indexed by each ς_i up to time $\theta-(r-1)\delta$. The corresponding packets are linked dynamically to form a forward cluster at time $\theta - r\delta$.*

Remark 4.7. *Note that the partition into packets and the subpartition into forests are only prescribed on the first time interval $[\theta - r\delta, \theta-(r-1)\delta]$: thanks to the minimality assumption there is no need to prescribe those objects at each intermediate time step. All possible divergences are therefore concentrated on the first time interval, whose size δ was chosen for them to remain under control.*

Fix $n_j^r \le M_j^{r\delta}$ for any block j in B. Going back to the definition (3.36) of cumulants, the forward dynamics are related to a cumulant $\phi_{r\delta}^{(B)}$ which is obtained in terms of the minimal cumulants $\bar{\phi}_{(r-1)\delta}^{(\varsigma_i)}$ defined in (4.4) as follows

$$
\begin{aligned}
\phi_{r\delta}^{(B)}(Z_{\mathbf{M}_B^{r\delta}}) := \Big(\prod_{j\in B} \frac{M_j^{(r-1)\delta}!}{M_j^{r\delta}!}\Big) \sum_{\varsigma\in\mathcal{P}_B} \mu_\varepsilon^{N_B^r+|\varsigma|-1} \sum_{\substack{\mathbf{K},\mathbf{S}^r,\bar{\mathbf{S}}^r,\lambda \\ \varsigma\hookrightarrow\lambda}} \operatorname{sign}(\mathbf{S}^r)\varphi_{\{\lambda_1,\dots,\lambda_{|\lambda|}\}} \\
\times \prod_{\ell=1}^{|\lambda|} \Big(\sharp\delta \bigotimes_{\varsigma_i\subset\lambda_\ell} \bar{\phi}_{(r-1)\delta}^{(\varsigma_i)}\Big) \mathbf{1}_{\{Z_{\mathbf{M}_{\lambda_\ell}^{r\delta}} \in \mathcal{R}^{\lambda_\ell,\varsigma}_{\mathbf{K}_{\lambda_\ell},\mathbf{S}^r_{\lambda_\ell},\bar{\mathbf{S}}^r_{\lambda_\ell}}\}},
\end{aligned}
\tag{4.5}
$$

where for any $i \le |\varsigma|$, the minimal cumulant $\bar{\phi}_{(r-1)\delta}^{(\varsigma_i)}$ is coded by the parameters $\mathcal{E}_i$ and $(\mathbf{S}_j^{<r}, \bar{\mathbf{S}}_j^{<r})_{j\in\varsigma_i}$, and $\mathcal{R}^{\lambda_\ell,\varsigma}_{\mathbf{K}_{\lambda_\ell},\mathbf{S}^r_{\lambda_\ell},\bar{\mathbf{S}}^r_{\lambda_\ell}}$ is the set of configurations compatible with packets ς and forests (λ_ℓ) in the first time interval (cf. (3.28)).

4.3. **Discarding non minimal dynamics in the iteration.** Starting from general functions $\{\phi^{(i)}\}_{i\in B}$ at time θ, we have explained how they can be aggregated in (4.5) to form cumulants at time $\theta - r\delta$. In what follows, the structure of the functions $\{\phi^{(i)}\}_{i\in B}$ will become relevant. For $i \geq p$, we will assume that $\phi^{(i)} = \bar\phi^{(i)}$ is built from a *single* minimal cumulant obtained by the pullback of the test function $h^{(i)}$ during the time interval $[\theta, \theta_i]$

$$\bar\phi^{(i)}(Z_{m_i}) = \frac{\mu_\varepsilon^{m_i-1}}{m_i!} \sum_{\mathbf{S}_i^\theta, \bar{\mathbf{S}}_i^\theta} \mathrm{sign}(\mathbf{S}_i^\theta) \mathbf{1}_{Z_{m_i} \in \mathcal{R}^{\min}_{\mathbf{S}_i^\theta, \bar{\mathbf{S}}_i^\theta}} (\sharp_{\theta_i - \theta} h^{(i)}) , \tag{4.6}$$

where the signs $(\mathbf{S}_i^\theta, \bar{\mathbf{S}}_i^\theta) \in \{-1, 1\}^{2(m_i-1)}$ prescribe the encounters in the time interval $[\theta, \theta_i]$ in the same way as in Definition 4.2. From now on, we shall write $\bar\phi^{(i)}$ instead of $\phi^{(i)}$ to emphasize the minimality of the cumulant. This structure will be crucial for the geometric estimates in Sections 7 and 8.

The goal of this section is to prove the following approximate preservation of the fluctuation structure on the generic time interval $[\theta - \tau, \theta]$ (or on a possibly smaller one), discarding non minimal dynamics.

Proposition 4.8. *Fix $\theta = \theta_p - k\tau$ for some integer k such that $[\theta - \tau, \theta] \subset [\theta_{p-1}, \theta_p]$. Consider a set $B \subset \{p, \dots P\}$ and observables $(\bar\phi^{(i)})_{i\in B}$ supported on single minimal forward cluster at time θ as in* (4.6)*. There is a constant C_P depending only on P such that for $\delta \ll \tau = R\delta \ll 1$ the following holds, as $\mu_\varepsilon \to \infty$*

$$\begin{aligned} &\Big| \mathbb{E}_\varepsilon \Big[\Big(\prod_{u=1}^{p-1} \zeta^\varepsilon_{\theta_u}\big(h^{(u)}\big) \Big) \bigotimes_{i\in B} \zeta^\varepsilon_{m_i,\theta}\big(\bar\phi^{(i)}\big) \Big] \\ &\quad - \sum_{\mathbf{N}} \sum_{\substack{\eta\cup\rho \text{ partition of } B \\ \rho \text{ clustering}}} \prod_{q=1}^{|\rho|} \mu_\varepsilon^{1-\frac{|\rho_q|}{2}} \mathbb{E}_\varepsilon \Big[\bar\phi_\tau^{(\rho_q)} \Big] \times \mathbb{E}_\varepsilon \Big[\Big(\prod_{u=1}^{p-1} \zeta^\varepsilon_{\theta_u}\big(h^{(u)}\big) \Big) \Big(\bigotimes_{q=1}^{|\eta|} \mu_\varepsilon^{\frac12 - \frac{|\eta_q|}{2}} \zeta^\varepsilon_{M^\tau_{\eta_q}, \theta-\tau}\big(\bar\phi_\tau^{(\eta_q)}\big) \Big) \Big] \Big| \\ &\qquad \leq (C_P \Theta)^{M_B} \Big(\prod_{i\in B\cup\{1,\dots,p-1\}} \|h^{(i)}\|_{L^\infty} \Big) \varepsilon^{\frac{1}{8d}} , \end{aligned} \tag{4.7}$$

with notation (4.4)*, and with $M^\tau_{\eta_q}$ the total number of particles in the packet η_q at time $\theta - \tau$.*

The proof of this proposition is the content of the next two sections and proceeds in two steps: starting from

$$\mathcal{J}_{\mathbf{M}} := \mathbb{E}_\varepsilon \Big[\Xi_{p-1} \bigotimes_{i\in B} \zeta^\varepsilon_{m_i,\theta}\big(\bar\phi^{(i)}\big) \Big] \quad \text{with the short notation } \Xi_{p-1} := \prod_{u=1}^{p-1} \zeta^\varepsilon_{\theta_u}\big(h^{(u)}\big) , \tag{4.8}$$

we first extract iteratively the remainder terms (for which the minimality condition is violated), and identify the main part. This procedure produces a sum of $R := \frac{\tau}{\delta}$ error terms, which are estimated in Paragraph 4.4.

Proposition 4.9. *The fluctuation structure* (4.8) *is transported on $[\theta - \tau, \theta]$ according to*

$$\mathcal{J}_{\mathbf{M}} = \bar{\mathcal{J}}_{\mathbf{M}} + \sum_{r=1}^{R} \mathcal{R}^{\text{int}}_r , \tag{4.9}$$

where the principal part is given by

$$\bar{\mathcal{J}}_{\mathbf{M}} := \sum_{\mathbf{N}_B^\tau} \sum_{\substack{\eta^R \cup \rho^R \text{ partition of } B \\ \rho^R \text{ clustering}}} \prod_{q=1}^{|\rho^R|} \mu_\varepsilon^{1-\frac{|\rho_q^R|}{2}} \mathbb{E}_\varepsilon\Big[\bar{\phi}_\tau^{(\rho_q^R)}\Big] \times \mathbb{E}_\varepsilon\Big[\Xi_{p-1}\Big(\bigotimes_{q=1}^{|\eta^R|} \mu_\varepsilon^{\frac{1}{2}-\frac{|\eta_q^R|}{2}} \zeta^\varepsilon_{M_q^\tau,\theta-\tau}\Big(\bar{\phi}_\tau^{(\eta_q^R)}\Big)\Big)\Big] \tag{4.10}$$

with notation (4.4)*, the short notation* $M_q^{r\delta} := M_{\eta_q^r}^{r\delta}$ *for the total number of particles in* η_q^r *at time* $\theta - r\delta$*, and* $\mathbf{N}_B^{r\delta} = (n_j)_{j\in B}$ *the number of particles removed in each block during* $[\theta - r\delta, \theta]$*. The remainders are defined, with notation* (4.5)*, as the sums of*

$$\begin{aligned}
\mathcal{R}_r^{\text{int},1} &= \sum_{\mathbf{N}_B^{r\delta}} \sum_{\substack{\eta^r \cup \rho^r \text{ partition of } B \\ \rho^r \text{ clustering}}} \prod_{q=1}^{|\rho^r|} \mu_\varepsilon^{1-\frac{|\rho_q^r|}{2}} \mathbb{E}_\varepsilon\Big[\bar{\phi}_{r\delta}^{(\rho_q^r)}\Big] \\
&\quad \times \mathbb{E}_\varepsilon\left[\Xi_{p-1}\left(\bigotimes_{q=1}^{|\eta^r|} \mu_\varepsilon^{\frac{1}{2}-\frac{|\eta_q^r|}{2}} \zeta^\varepsilon_{M_q^{r\delta},\theta-r\delta}\Big(\phi_{r\delta}^{(\eta_q^r)}\Big) - \bigotimes_{q=1}^{|\eta^r|} \mu_\varepsilon^{\frac{1}{2}-\frac{|\eta_q^r|}{2}} \zeta^\varepsilon_{M_q^{r\delta},\theta-r\delta}\Big(\bar{\phi}_{r\delta}^{(\eta_q^r)}\Big)\right)\right], \\
\mathcal{R}_r^{\text{int},2} &= \sum_{\mathbf{N}_B^{r\delta}} \sum_{\substack{\eta^r \cup \rho^r \text{ partition of } B \\ \rho^r \text{ clustering}}} \left(\prod_{q=1}^{|\rho^r|} \mu_\varepsilon^{1-\frac{|\rho_q^r|}{2}} \mathbb{E}_\varepsilon\Big[\phi_{r\delta}^{(\rho_q^r)}\Big] - \prod_{q=1}^{|\rho^r|} \mu_\varepsilon^{1-\frac{|\rho_q^r|}{2}} \mathbb{E}_\varepsilon\Big[\bar{\phi}_{r\delta}^{(\rho_q^r)}\Big]\right) \\
&\quad \times \mathbb{E}_\varepsilon\Big[\Xi_{p-1}\Big(\bigotimes_{q=1}^{|\eta^r|} \mu_\varepsilon^{\frac{1}{2}-\frac{|\eta_q^r|}{2}} \zeta^\varepsilon_{M_q^{r\delta},\theta-r\delta}\Big(\phi_{r\delta}^{(\eta_q^r)}\Big)\Big)\Big].
\end{aligned} \tag{4.11}$$

Proof. Proposition 4.9 is proved recursively. More precisely we start with a decomposition of $\mathcal{J}_{\mathbf{M}}$ at step $r-1$ under the form

$$\mathcal{J}_{\mathbf{M}} = \bar{\mathcal{J}}_{\mathbf{M}}^{r-1} + \sum_{r'=1}^{r-1} \mathcal{R}_{r'}^{\text{int}},$$

where the main term is defined by

$$\bar{\mathcal{J}}_{\mathbf{M}}^{r-1} := \sum_{\mathbf{N}_B^{(r-1)\delta}} \sum_{\substack{\eta^{r-1} \cup \rho^{r-1} \text{ partition of } B \\ \rho^{r-1} \text{ clustering}}} \prod_q \mu_\varepsilon^{1-\frac{|\rho_q^{r-1}|}{2}} \mathbb{E}_\varepsilon\Big[\bar{\phi}_{(r-1)\delta}^{(\rho_q^{r-1})}\Big] \times \mathbb{E}_\varepsilon\Big[\Xi_{p-1}\Big(\bigotimes_q \mu_\varepsilon^{\frac{1}{2}-\frac{|\eta_q^{r-1}|}{2}} \zeta^\varepsilon_{M_q^{(r-1)\delta},\theta-(r-1)\delta}\Big(\bar{\phi}_{(r-1)\delta}^{(\eta_q^{r-1})}\Big)\Big)\Big].$$

For each q, we pull back the cumulants $\bar{\phi}_{(r-1)\delta}^{(\rho_q^{r-1})}$ and $\bar{\phi}_{(r-1)\delta}^{(\eta_q^{r-1})}$ on a time step δ and extract minimal clusters and remainder terms. A Fubini equality will enable us to replace the nested partitions by one partition $\eta^r \cup \rho^r$ thus completing the induction.

We first consider the expelled cumulants. As in (3.26), we have

$$\mathbb{E}_\varepsilon\Big[\bar{\phi}_{(r-1)\delta}^{(\rho_q^{r-1})}\Big] = \sum_{\mathbf{N}^r} \mathbb{E}_\varepsilon\Big[\phi_{r\delta}^{(\rho_q^{r-1}),\mathrm{t}}\Big],$$

where the superscript t in $\phi_{r\delta}^{(\rho_q^{r-1}),\mathrm{t}}$ stands for cumulants transported by the forward dynamics in the sense of Definition 4.5, without any clustering in the last time interval $[\theta - r\delta, \theta - (r-1)\delta]$.

By construction

$$\phi_{r\delta}^{(\rho_q^{r-1}),\mathrm{t}} := \mu_\varepsilon^{N_q^r}\left(\prod_{j\in\rho_q^{r-1}}\frac{M_j^{(r-1)\delta}!}{M_j^{r\delta}!}\right)\sum_{\mathbf{K},\mathbf{S}^r,\bar{\mathbf{S}}^r}\mathrm{sign}(\mathbf{S}^r)\Big(\sharp_\delta\bar\phi_{(r-1)\delta}^{(\rho_q^{r-1})}\Big)\mathbf{1}_{\{Z_{\mathbf{M}_q^{r\delta}}\in\mathcal{R}_{\mathbf{K},\mathbf{S},\bar{\mathbf{S}}}^{\rho_q^{r-1}}\}}\,, \tag{4.12}$$

denoting for simplicity by N_q^r the number of particles annihilated in ρ_q^{r-1} on $[\theta-r\delta,\theta-(r-1)\delta]$ so that the number of particles in ρ_q^{r-1} at $\theta-r\delta$ is $M_q^{r\delta}=M_{\rho_q^{r-1}}^{(r-1)\delta}+N_q^r$.

We turn now to the product of the fluctuation fields. The packets at time $\theta-(r-1)\delta$ are prescribed by η^{r-1}, and we will temporarily keep track of this fact by an additional superscript η^{r-1} in the pulled back observables :

$$\begin{aligned}&\mathbb{E}_\varepsilon\Big[\Xi_{p-1}\Big(\bigotimes_q\mu_\varepsilon^{\frac12-\frac{|\eta_q^{r-1}|}{2}}\zeta^\varepsilon_{M_q^{(r-1)\delta},\theta-(r-1)\delta}\big(\bar\phi_{(r-1)\delta}^{(\eta_q^{r-1})}\big)\Big)\Big]\\&=\sum_{\mathbf{N}^r}\sum_{\substack{\eta^{r-1}\hookrightarrow\eta_1^r\cup\eta_2^r\\ \eta_2^r\text{clustering}}}\prod_q\mu_\varepsilon^{1-\frac{|\eta_{2,q}^r|}{2}}\mathbb{E}_\varepsilon\Big[\phi_{r\delta}^{(\eta_{2,q}^r),\eta^{r-1}}\Big]\times\mathbb{E}_\varepsilon\Big[\Xi_{p-1}\Big(\bigotimes_q\mu_\varepsilon^{\frac12-\frac{|\eta_{1,q}^r|}{2}}\zeta^\varepsilon_{M_q^{r\delta},\theta-r\delta}\Big(\phi_{r\delta}^{(\eta_{1,q}^r),\eta^{r-1}}\Big)\Big)\Big]\,,\end{aligned}$$

where $\eta_1^r\cup\eta_2^r$ is a coarser partition than η^{r-1}, and each part of η_2^r contains at least two components of η^{r-1}: this term only appears if $|\eta^{r-1}|\geq 2$.

Our goal is to sum over all η^{r-1} compatible with a given coarser decomposition η^r, in order to retrieve cumulants at step r (see Figure 6). Let us first introduce

$$\begin{aligned}\phi_{r\delta}^{(\eta_{1,q}^r)} &:= \Big(\prod_{j\in\eta_{1,q}^r}\frac{M_j^{(r-1)\delta}!}{M_j^{r\delta}!}\Big)\sum_{\substack{\varsigma,\lambda\\ \varsigma\hookrightarrow\lambda\hookrightarrow\eta_{1,q}^r}}\mu_\varepsilon^{N^r+|\varsigma|-1}\sum_{\mathbf{K},\mathbf{S}^r,\bar{\mathbf{S}}^r}\mathrm{sign}(\mathbf{S}^r)\varphi_{\{\lambda_1,\dots,\lambda_{|\lambda|}\}}\\&\qquad\times\prod_{\ell=1}^{|\lambda|}\Big(\sharp_\delta\bigotimes_{\varsigma_j\subset\lambda_\ell}\bar\phi_{(r-1)\delta}^{(\varsigma_j)}\Big)\mathbf{1}_{\{Z_{\mathbf{M}_{\lambda_\ell}^{r\delta}}\in\mathcal{R}^{\lambda_\ell}_{\mathbf{K}_{\lambda_\ell},\mathbf{S}_{\lambda_\ell}}\}}\\ \phi_{r\delta}^{(\eta_{2,q}^r),c} &:= \Big(\prod_{j\in\eta_{2,q}^r}\frac{M_j^{(r-1)\delta}!}{M_j^{r\delta}!}\Big)\sum_{\substack{\varsigma,\lambda\\ \varsigma\hookrightarrow\lambda\hookrightarrow\eta_{2,q}^r\\ |\varsigma|\geq2}}\mu_\varepsilon^{N^r+|\varsigma|-1}\sum_{\mathbf{K},\mathbf{S}^r,\bar{\mathbf{S}}^r}\mathrm{sign}(\mathbf{S}^r)\varphi_{\{\lambda_1,\dots,\lambda_{|\lambda|}\}}\\&\qquad\times\prod_{\ell=1}^{|\lambda|}\Big(\sharp_\delta\bigotimes_{\varsigma_j\subset\lambda_\ell}\bar\phi_{(r-1)\delta}^{(\varsigma_j)}\Big)\mathbf{1}_{\{Z_{\mathbf{M}_{\lambda_\ell}^{r\delta}}\in\mathcal{R}^{\lambda_\ell}_{\mathbf{K}_{\lambda_\ell},\mathbf{S}_{\lambda_\ell}}\}}\end{aligned} \tag{4.13}$$

denoting by N^r the total number of particles annihilated on $[\theta-r\delta,\theta-(r-1)\delta]$ in the packets. Observe that, with respect to Eq. (3.36), this definition contains an additional sum over previous packets ς at time $\theta-(r-1)\delta$. Now we can apply Fubini's equality to sum over η^{r-1} thanks to the above definitions. We define indeed $\rho^r:=\rho^{r-1}\cup\eta_2^r$ and $\eta^r:=\eta_1^r$. The sets ρ^{r-1} and η_2^r play symmetric roles and the cumulants $\phi_{r\delta}^{(\rho_q^{r-1}),\mathrm{t}}$ and $\phi_{r\delta}^{(\eta_{2,q}^r),c}$ can be combined into one cumulant $\phi_{r\delta}^{(\rho_q^r)}$. Now we sum (4.3) over η^{r-1} and use (4.13) to recover

$$\bar{\mathcal{J}}_{\mathbf{M}}^{r-1}=\sum_{\mathbf{N}_B^r}\sum_{\substack{\eta^r\cup\rho^r\text{ partition of }B\\ \rho^r\text{ clustering}}}\left(\prod_q\mu_\varepsilon^{1-\frac{|\rho_q^r|}{2}}\mathbb{E}_\varepsilon\Big[\phi_{r\delta}^{(\rho_q^r)}\Big]\right)\times\mathbb{E}_\varepsilon\Big[\Xi_{p-1}\Big(\bigotimes_q\mu_\varepsilon^{\frac12-\frac{|\eta_q^r|}{2}}\zeta^\varepsilon_{M_{\eta_q^r}^{r\delta},\theta-r\delta}\Big(\phi_{r\delta}^{(\eta_q^r)}\Big)\Big)\Big]\,.$$

We finally extract from the main term the minimal cumulants. We restrict the support of the cumulants $\phi_{r\delta}^{(\sigma)}$ (with $\sigma=\eta_q^r$ or ρ_q^r) to configurations such that the graph recording all the encounters on the time interval $[\theta-r\delta,\theta-(r-1)\delta]$ has no cycle (nor multiple edge). In particular there is no internal encounter without annihilation inside the blocks so that the sum over $\mathbf{K}$ in (4.13) disappears, and by definition, all "admissible" partitions $\varsigma = (\varsigma_i)_{1\leq i\leq|\varsigma|}$ of σ into packets are characterized by the graph recording the encounters between the ς_i which must be minimally connected. This local minimal dynamics in $[\theta - r\delta, \theta - (r-1)\delta]$ is then entirely prescribed by the signs $\mathbf{S}^r, \bar{\mathbf{S}}^r$ encoding encounters between particles of the same tree on the time interval, and signs $\mathcal{E}^r \in \{-1,+1\}^{|\varsigma|-1}$.

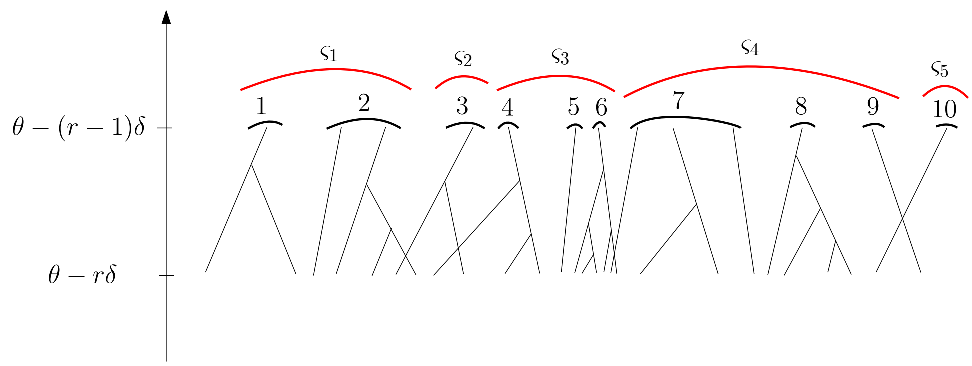

FIGURE 6. A schematic picture of generic contribution to the minimal cumulant $\bar{\phi}^{(\sigma)}_{r\delta}$, in the case of a σ consisting of 10 blocks (for simplicity $\sigma = \{1, 2, \cdots, 10\}$) grouped into $s = 5$ packets. An admissible partition ς is represented in red : it imposes some dynamical constraints after $\theta-(r-1)\delta$. The dynamics in the depicted time interval is prescribed by $\mathcal{E}^r, \mathbf{S}^r, \bar{\mathbf{S}}^r$. Note that two blocks of the same packet ς_i should not collide or overlap on $[\theta - r\delta, \theta - (r-1)\delta]$.

For such an admissible ς, there is a unique partition λ in forests and (4.2) holds. Thus the restriction of $\phi^{(\sigma)}_{r\delta}$ to minimal local forward dynamics is given by

$$
\begin{aligned}
\bar{\phi}^{(\sigma)}_{r\delta} &= \sum_{s\leq|\sigma|} \mu_\varepsilon^{N^r_\sigma+s-1} \sum_{\mathcal{E}^r,\mathbf{S}^r,\bar{\mathbf{S}}^r} \operatorname{sign}(\mathcal{E}^r)\,\operatorname{sign}(\mathbf{S}^r) \\
&\qquad \times \sum_{\substack{\text{admissible } \varsigma \\ |\varsigma|=s}} \mathbf{1}_{Z_{\mathbf{M}^{r\delta}_\sigma} \in \mathcal{R}^{\mathrm{loc\,min},\varsigma}_{\mathcal{E}^r,\mathbf{S}^r,\bar{\mathbf{S}}^r}} \prod_{i=1}^s \prod_{j\in\varsigma_i} \frac{M_j^{(r-1)\delta}!}{M_j^{r\delta}!} \big(\sharp_\delta \bar{\phi}^{(\varsigma_i)}_{(r-1)\delta}\big)\,,
\end{aligned}
\tag{4.14}
$$

where $\mathcal{R}^{\mathrm{loc\,min},\varsigma}_{\mathcal{E}^r,\mathbf{S}^r,\bar{\mathbf{S}}^r}$ is the set of all configurations leading to minimal forward dynamics on the time interval $[\theta - r\delta, \theta - (r-1)\delta]$ compatible with $\mathcal{E}^r \in \{-1,1\}^{s-1}$, $\mathbf{S}^r$, $\bar{\mathbf{S}}^r$ and ς.

Now by the induction assumption, $\bar{\phi}^{(\varsigma_i)}_{(r-1)\delta}$ is a minimal cumulant given by formula (4.4) at time $\theta - (r-1)\delta$

$$
\begin{aligned}
\bar{\phi}^{(\varsigma_i)}_{(r-1)\delta}(Z_{M^{(r-1)\delta}_{\varsigma_i}}) =& \mu_\varepsilon^{N^{<r}_{\varsigma_i}+|\varsigma_i|-1} \left(\prod_{j\in\varsigma_i} \frac{m_j!}{M_j^{(r-1)\delta}!} \right) \\
& \sum_{\bar{\mathbf{S}}^{<r}_i,\mathbf{S}^{<r}_i,\mathcal{E}^{<r}_i} \operatorname{sign}(\mathcal{E}^{<r}_i)\,\operatorname{sign}(\mathbf{S}^{<r}_i) \mathbf{1}_{Z_{M^{(r-1)\delta}_{\varsigma_i}} \in \mathcal{R}^{\min}_{\mathcal{E}^{<r}_i,\mathbf{S}^{<r}_i,\bar{\mathbf{S}}^{<r}_i}} \Big(\sharp_{(r-1)\delta} \bigotimes_{j\in\varsigma_i} \phi^{(j)}\Big)\,,
\end{aligned}
$$

where $N^{<r}_{\varsigma_i}$ denotes the number of particles annihilated on the full time interval $[\theta-(r-1)\delta, \theta]$ and $\mathcal{E}^{<r}_i, \mathbf{S}^{<r}_i, \bar{\mathbf{S}}^{<r}_i$ is the collection of all signs on that same time interval. Plugging this formula into (4.14) (and using again Fubini's identity to pass from the parametrization in terms of $\big(\mathcal{E}^r, \mathbf{S}^r, \bar{\mathbf{S}}^r, \big(\bar{\mathbf{S}}^{<r}_i, \mathbf{S}^{<r}_i, \mathcal{E}^{<r}_i\big)_i\big)$ to a global parametrization $\big(\mathcal{E}, \mathbf{S}, \bar{\mathbf{S}}\big)$), we recover formula (4.4)

at time $\theta - r\delta$. Thus we obtain the expected decomposition at $\theta - r\delta$:

$$\bar{\mathcal{J}}_{\mathbf{M}}^{r-1} = \sum_{\mathbf{N}_B^r} \sum_{\substack{\eta^r \cup \rho^r \\ \rho^r}} \sum_{\substack{\text{partition of } B \\ \text{clustering}}} \Big(\prod_q \mu_\varepsilon^{1-\frac{|\rho_q^r|}{2}} \mathbb{E}_\varepsilon\Big[\bar{\phi}_{r\delta}^{(\rho_q^r)}\Big]\Big) \mathbb{E}_\varepsilon\Big[\Xi_{p-1}\Big(\bigotimes_q \mu_\varepsilon^{\frac{1}{2}-\frac{|\eta_q^r|}{2}} \zeta^\varepsilon_{M^{r\delta}_{\eta_q^r},\theta-r\delta}\Big(\bar{\phi}_{r\delta}^{(\eta_q^r)}\Big)\Big)\Big] + \mathcal{R}_r^{\rm int}\,.$$

Proposition 4.9 is proved. □

4.4. **Estimates of the remainders.** We now establish the following estimates for the remainders, which thanks to Proposition 4.9 imply Proposition 4.8 as an immediate corollary.

Proposition 4.10. *Under the assumptions of Proposition* 4.8*, there is a constant C_P depending only on P such that the remainder $\mathcal{R}_r^{\rm int}$ defined in* (4.11) *satisfies the following estimate :*

$$\Big|\sum_{r=1}^R \mathcal{R}_r^{\rm int}\Big| \le (C_P\Theta)^{M_B}\Big(\prod_{i\in B\cup\{1,\dots,p-1\}} \|h^{(i)}\|_{L^\infty}\Big)\varepsilon^{\frac{1}{8d}}\,,$$

with $R = \tau/\delta$ and $M_B = \sum_{i\in B} m_i$.

In our argument, the specific form of the function Ξ_{p-1} (see (4.8)) will be irrelevant. Only the following two features are needed :

- Ξ_{p-1} depends on the particle configurations before (and at) time θ_{p-1};
- Ξ_{p-1} has a uniformly bounded variance.

Notice indeed that by Hölder's inequality,

$$\mathbb{E}_\varepsilon[\Xi_{p-1}^2] = \mathbb{E}_\varepsilon\Big[\Big(\prod_{u=1}^{p-1}\zeta^\varepsilon_{\theta_u}\big(h^{(u)}\big)\Big)^2\Big] \le \prod_{u=1}^{p-1}\mathbb{E}_\varepsilon\Big[\zeta^\varepsilon_{\theta_u}\big(h^{(u)}\big)^{2(p-1)}\Big]^{\frac{1}{(p-1)}} \le C_p\prod_{u=1}^{p-1}\|h^{(u)}\|_\infty^2\,, \tag{4.15}$$

and thus Ξ_{p-1} is bounded in L^2, by Proposition 2.3 which ensures that the fluctuation fields are bounded in $L^{2(p-1)}$.

The key ingredient will be the following lemma controlling the expectation and variance of cumulants based on their clustering structure. It will be proved in Sections 7 and 8: see Section 7 for the proof of (4.16), Paragraph 8.1 for the proof of (4.17) and Paragraph 8.2 for the proof of (4.18).

Lemma 4.11. *Let $B\subset\{p,p+1,\dots,P\}$. Consider observables $(\bar{\phi}^{(i)})_{i\in B}$, supported on single minimal forward clusters as in* (4.6) *at time $\theta = \theta_p - k\tau$. Let $\sigma = (\sigma_i)_{i\le|\sigma|}$ be a collection of packets in B, and the corresponding cumulants $\phi_{r\delta}^{(\sigma_i)}$ defined by* (4.5)*. Denote by M_{σ_i} the number of particles in σ_i at time θ, by $N_{\sigma_i}^r$ the number of particles to be removed in the last time interval $[\theta - r\delta, \theta-(r-1)\delta]$, by $N_{\sigma_i}^{<r}$ the number of other particles to be removed in $[\theta-(r-1)\delta,\theta]$ and by $M_{\sigma_i}^{r\delta} = M_{\sigma_i}+N_{\sigma_i}^r+N_{\sigma_i}^{<r}$ the total number of particles in σ_i at time $\theta-r\delta$. Then there is constant $C_P>0$ depending on P and K_γ such that such that*

$$\mathbb{E}_\varepsilon\Big[\Big|\phi_{r\delta}^{(\sigma_i)}\Big|\Big] \le C_P\Big(\prod_{j\in\sigma_i}\|h^{(j)}\|_{L^\infty}\Big)\,(C_P\Theta)^{M_{\sigma_i}-|\sigma_i|}(C_P\delta)^{N^r_{\sigma_i}}(C_P\tau)^{N^{<r}_{\sigma_i}+|\sigma_i|-1}\,, \tag{4.16}$$

and

$$\begin{aligned}\Big|\mathbb{E}_\varepsilon\Big[\bigotimes_{i\le|\sigma|}\zeta^\varepsilon_{M^{r\delta}_{\sigma_i},\theta-r\delta}\big(\phi_{r\delta}^{(\sigma_i)}\big)\Big]\Big| &\le C_P\varepsilon\prod_{i\le|\sigma|}\Big(\prod_{j\in\sigma_i}\|h^{(j)}\|_{L^\infty}\Big)\\ &\quad\times(C_P\Theta)^{M_{\sigma_i}-|\sigma_i|}\Big((C_P\delta)^{N^r_{\sigma_i}}(C_P\tau)^{N^{<r}_{\sigma_i}+|\sigma_i|-1}\Big)^{1/2}\,,\end{aligned} \tag{4.17}$$

as well as

(4.18)
$$\mathbb{E}_\varepsilon\Big[\Big(\bigotimes_{i\le|\sigma|}\zeta^\varepsilon_{M^{r\delta}_{\sigma_i},\theta-r\delta}\big(\phi^{(\sigma_i)}_{r\delta}\big)\Big)^2\Big]\le C_P\prod_{i\le|\sigma|}\Big(\prod_{j\in\sigma_i}\|h^{(j)}\|_{L^\infty}\Big)^2 \times\Big((C_P\Theta)^{2M_{\sigma_i}+N^r_{\sigma_i}+N^{<r}_{\sigma_i}-|\sigma_i|}(C_P\delta)^{N^r_{\sigma_i}}(C_P\tau)^{N^{<r}_{\sigma_i}+|\sigma_i|-1}\Big).$$

The same estimates hold for the minimal cumulants $\bar\phi^{(\sigma_i)}_{r\delta}$ *defined by* (4.4).

Recall that the remainders $\mathcal{R}^{\rm int}_r$ in (4.11) are due to the non minimal dynamics. The smallness will come from the fact that at least one factor in the product of expectations, or one fluctuation field involves a dynamical graph with a cycle (or multiple dynamical edge)

$$\phi^{(\sigma_i),{\rm cyc}}_{r\delta}:=\phi^{(\sigma_i)}_{r\delta}-\bar\phi^{(\sigma_i)}_{r\delta}\,. \tag{4.19}$$

We will therefore also need the following lemma, proved in Section 7 for (4.20) and Section 8.2 for (4.21).

Lemma 4.12. *Consider observables* $(\bar\phi^{(i)})_{i\in B}$ *at time* θ *supported on single minimal forward cluster as in* (4.6). *Define* $\phi^{(\sigma_i),{\rm cyc}}_{r\delta}$ *by* (4.19). *Then, with the notations of Lemma* 4.11, *we have that for any* $i\le|\sigma|$

$$\mathbb{E}_\varepsilon\Big[\big|\phi^{(\sigma_i),{\rm cyc}}_{r\delta}\big|\Big]\le C_P\Big(\prod_{j\in\sigma_i}\|h^{(j)}\|_{L^\infty}\Big) \times\varepsilon\delta|\log\varepsilon|(|\log\varepsilon|\Theta)^{2d+4}(C_P\Theta)^{M_{\sigma_i}-|\sigma_i|}(C_P\delta)^{(N^r_{\sigma_i}-1)_+}(C_P\tau)^{(N^{<r}_{\sigma_i}+|\sigma_i|-2)_+}, \tag{4.20}$$

and

$$\begin{aligned}&\mathbb{E}_\varepsilon\Big[\Big(\zeta^\varepsilon_{M^{r\delta}_{\sigma_i},\theta-r\delta}\big(\phi^{(\sigma_i),{\rm cyc}}_{r\delta}\big)\bigotimes\Big(\bigotimes_{j\ne i}\zeta^\varepsilon_{M^{r\delta}_{\sigma_j},\theta-r\delta}\big(\phi^{(\sigma_j)}_{r\delta}\big)\Big)\Big)^2\Big]\\ &\le C_P\prod_{i\le|\sigma|}\Big(\prod_{j\in\sigma_i}\|h^{(j)}\|_{L^\infty}\Big)^2\varepsilon\delta|\log\varepsilon|(|\log\varepsilon|\Theta)^{2d+4}(C_P\Theta)^{M_{\sigma_i}+M^{r\delta}_{\sigma_i}-|\sigma_i|}(C_P\tau)^{(N^{<r}_{\sigma_i}+|\sigma_i|-2)_+}\\ &\quad\times(C_P\delta)^{(N^r_{\sigma_i}-1)_+}\prod_{j\ne i}\Big((C_P\Theta)^{2M_{\sigma_j}+N^r_{\sigma_j}+N^{<r}_{\sigma_j}-|\sigma_j|}(C_P\delta)^{N^r_{\sigma_j}}(C_P\tau)^{N^{<r}_{\sigma_j}+|\sigma_j|-1}\Big).\end{aligned} \tag{4.21}$$

The same estimates hold when replacing the cumulants $\phi^{(\sigma_j)}_{r\delta}$ *by the minimal cumulants* $\bar\phi^{(\sigma_j)}_{r\delta}$ *defined by* (4.4).

Proof of Proposition 4.10. Using the homogeneity, we can assume without loss of generality that $\|h^{(i)}\|_{L^\infty}\le1$ so we do not keep track of $\|h^{(i)}\|_{L^\infty}$ in the estimates. Recall the definition (4.11) of $\mathcal{R}^{\rm int}_r$. The fluctuation terms in $\mathcal{R}^{\rm int}_r$ can be decoupled from the function Ξ_{p-1} by using the Cauchy-Schwarz estimate, leading in the case of $\mathcal{R}^{{\rm int},1}_r$ to

$$\begin{aligned}\big|\mathcal{R}^{{\rm int},1}_r\big|&\le\sum_{\mathbf{N}^{r\delta}_B}\ \sum_{\substack{\eta^r\cup\rho^r\ \text{partition of } B\\ \rho^r\ \text{clustering}}}\prod_{q=1}^{|\rho^r|}\mu_\varepsilon^{1-\frac{|\rho^r_q|}{2}}\mathbb{E}_\varepsilon\Big[\bar\phi^{(\rho^r_q)}_{r\delta}\Big]\mathbb{E}_\varepsilon^{\frac12}\big[\Xi^2_{p-1}\big]\\ &\quad\times\mathbb{E}_\varepsilon^{\frac12}\Bigg[\Bigg(\bigotimes_{q=1}^{|\eta^r|}\mu_\varepsilon^{\frac12-\frac{|\eta^r_q|}{2}}\zeta^\varepsilon_{M^{r\delta}_q,\theta-r\delta}\Big(\phi^{(\eta^r_q)}_{r\delta}\Big)-\bigotimes_{q=1}^{|\eta^r|}\mu_\varepsilon^{\frac12-\frac{|\eta^r_q|}{2}}\zeta^\varepsilon_{M^{r\delta}_q,\theta-r\delta}\Big(\bar\phi^{(\eta^r_q)}_{r\delta}\Big)\Bigg)^2\Bigg]\\ &\le C_P\sum_{\mathbf{N}^{r\delta}_B}\ \sum_{\substack{\eta^r\cup\rho^r\ \text{partition of } B\\ \rho^r\ \text{clustering}}}\prod_{q=1}^{|\rho^r|}\mathbb{E}_\varepsilon\Big[\bar\phi^{(\rho^r_q)}_{r\delta}\Big]\mathbb{E}_\varepsilon^{\frac12}\Bigg[\Bigg(\bigotimes_{q=1}^{|\eta^r|}\zeta^\varepsilon_{M^{r\delta}_q,\theta-r\delta}\Big(\phi^{(\eta^r_q)}_{r\delta}\Big)-\bigotimes_{q=1}^{|\eta^r|}\zeta^\varepsilon_{M^{r\delta}_q,\theta-r\delta}\Big(\bar\phi^{(\eta^r_q)}_{r\delta}\Big)\Bigg)^2\Bigg]\end{aligned}$$

thanks to (4.15) and the fact that $\mu_\varepsilon^{1-|\eta_q^r|} \le 1$. Then, since there is at least one factor $\phi_{r\delta}^{(\eta_q^r),\mathrm{cyc}} = \phi_{r\delta}^{(\eta_q^r)} - \bar\phi_{r\delta}^{(\eta_q^r)}$, using (4.16) and (4.21) we find that $\mathcal{R}_r^{\mathrm{int},1}$ is bounded by

$$C_P(\varepsilon\delta|\log\varepsilon|)^{1/2}(|\log\varepsilon|\Theta)^{d+2}\,(C_P\delta)^{N_{\rho^r}^r}(C_P\tau)^{N_{\rho^r}^{<r}+\sum_i|\rho_i^r|-|\rho^r|}(C_P\Theta)^{M_{\rho^r}-\sum_i|\rho_i^r|}$$
$$\times(C_P\Theta)^{(M_{\eta^r}-\sum_i|\eta_i^r|+M_{\eta^r}^{r\delta})/2}(C_P\delta)^{(N_{\eta^r}^r-1)_+/2}(C_P\tau)^{(N_{\eta^r}^{<r}+\sum_i|\eta_i^r|-|\eta|-1)_+/2}\,.$$

The reasoning is similar for $\mathcal{R}_r^{\mathrm{int},2}$, using (4.18) and (4.20). Summing over $(\mathbf{N}^{r'})_{r'\le r}$, then over $r \le R = \frac{\tau}{\delta}$, we get

$$\Big|\sum_{r=1}^R \mathcal{R}_r^{\mathrm{int}}\Big| \le (C_P\Theta)^{M_B}(\varepsilon\delta|\log\varepsilon|)^{1/2}(|\log\varepsilon|\Theta)^{d+2}\frac{\tau}{\delta} \le (C_P\Theta)^{M_B}\varepsilon^{\frac{1}{8d}}\,, \tag{4.22}$$

with the choice $\delta = \varepsilon^{1-\frac{1}{2d}}$ in (2.9) and τ satisfying (2.2). This concludes the proof of Proposition 4.10, and thus of Proposition 4.8.

Notice that the choice of the parameter δ is an optimisation between the fact that δ has to be small so that Υ_N^ε is a typical event and the necessity for δ to be larger than ε for the estimate (4.22) to converge to 0. □

5. Almost-preserving of the fluctuation structure

5.1. **Subexponential clusters.** In Proposition 4.8, we proved that the fluctuation structure at a time θ can be pulled back to time $\theta-\tau$ up to small error terms. We now want to iterate this formula to pull back the fluctuation on any macroscopic time interval $[\theta_{p-1},\theta_p]$ $(2\le p\le P)$.

For this, we choose the parameter τ so that for all $i\in[1,P], (\theta_i-\theta_{i-1})/\tau$ is not an integer. Each time interval $\big[\theta_{i-1},\theta_i\big]$ is cut into $k_i = [(\theta_i-\theta_{i-1})/\tau]+1$ slices (of size τ, except for the last slice $\big[\theta_{i-1},\theta_i-(k_i-1)\tau\big]$ which is smaller due to this assumption on τ). This leads to a decomposition of $[0,\Theta]$ into $K_P := \sum_{i=2}^P k_i$ slices, denoted $I_\ell = [\tau_{\ell+1},\tau_\ell]$ (in decreasing order): thus $I_1 := (\Theta-\tau,\Theta), I_2 = (\Theta-2\tau,\Theta-\tau)\dots.$

In particular, we introduce the decreasing sequence of integers $\{\kappa_p\}_{p\le P}$ such that

$$\theta_p = \tau_{\kappa_p}\quad\text{and we set}\quad \mathbb{L}_p = \{\kappa_p,\dots,\kappa_{p-1}-1\}. \tag{5.1}$$

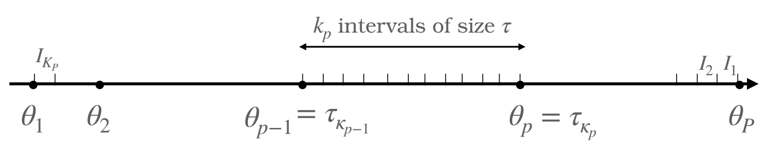

Figure 7. The time interval $[\theta_1,\theta_P]$ is split into K_P intervals of smaller size τ denoted by $I_1,\dots,I_{K_P}$ and ranked in a decreasing order.

In the next definition, we are going to strengthen the notion of single minimal forward cluster introduced in Definition 4.2.

Definition 5.1 (Subexponential cluster). *Let $i\in\{1,\dots,P\}$ and $\tau_\ell<\theta_i=\tau_{\kappa_i}$ be given. We consider a single minimal forward cluster during the time interval $[\tau_\ell,\theta_i]$, originating from a single particle. This cluster is said to be subexponential at τ_ℓ if on each time interval I_k for $\kappa_i\le k<\ell$ the number of annihilations n_i^k in the forward dynamics is less than 2^k.*

The corresponding single minimal cumulant $\bar\phi^{(i)}$ is defined as in (4.6) as the pullback during $[\tau_\ell,\theta_i]$ of the function $h^{(i)}$, with the appropriate subexponential restrictions on the annihilation numbers.

Note that the reference time for the subexponential growth is chosen to be Θ rather than θ_i as one might have expected (since the backward flow only starts at time θ_i). As will be apparent later (see Proposition 5.6), the reason for this choice is that the contribution of one single superexponential cluster must be small enough to compensate the size of all other subexponential clusters, so the reference time has to be the same for all $i \in \{1,\dots,P\}$.

The main result of this section is the following: it shows that the fluctuation structure involving single subexponentials cumulants is preserved on any macroscopic time interval $[\theta_{p-1},\theta_p]$. Its proof is the goal of the following paragraphs.

Proposition 5.2. *Given a subset B of $\{p,\dots,P\}$, consider for each $i \in B$ a single minimal cumulant $\bar\phi^{(i)}$ of m_i variables, supported on a subexponential cluster at time $\theta_p = \tau_{\kappa_p}$ as in Definition 5.1. Then denoting by $\mathbf{N}^\ell := (n_i^\ell)_{i\in B}$ the number of particles annihilated in each block on the time step ℓ, and by $M_i^{\kappa_{p-1}} = m_i + \sum_{\ell\in\mathbb{L}_p} n_i^\ell$ the number of particles at time θ_{p-1}, there holds*

$$\Big|\mathbb{E}_\varepsilon\Big[\Xi_{p-1}\Big(\bigotimes_{i\in B}\zeta^\varepsilon_{m_i,\theta_p}(\bar\phi^{(i)})\Big)\Big] - \sum_{\substack{(\mathbf{N}^\ell)_{\ell\in\mathbb{L}_p}\\ \text{subexp}}} \mathbb{E}_\varepsilon\Big[\Xi_{p-1}\Big(\bigotimes_{i\in B}\zeta^\varepsilon_{M_i^{\kappa_{p-1}},\theta_{p-1}}(\bar\phi^{(i)}_{\theta_p-\theta_{p-1}})\Big)\Big]\Big|$$
$$\le C_P\Big(\prod_{i\in B\cup\{1,\dots,p-1\}}\|h^{(i)}\|_{L^\infty}\Big)\Big((C_P\Theta)^{P\cdot 2^{\kappa_p}}\,\varepsilon^{\frac{1}{8d}} + (C_P\Theta^{2P-1}\tau)^{2^{\kappa_{p-1}}}\Big)\,,$$

where $\bar\phi^{(i)}_{\theta_p-\theta_{p-1}}$ is supported on a (single minimal) subexponential cluster at time θ_{p-1}.

Using the homogeneity, we can assume without loss of generality that $\|h^{(i)}\|_{L^\infty} \le 1$ so from now on we no longer keep track of $\|h^{(i)}\|_{L^\infty}$ in the estimates.

5.2. **The main term and the remainders on a small time step.** Let us start by considering one time step $I_\ell = [\tau_{\ell+1},\tau_\ell] \subset [\theta_{p-1},\theta_p]$. Given a subset B of $\{p,\dots,P\}$, consider for each $i \in B$ a function $\bar\phi^{(i)}$ of m_i^ℓ variables, supported on a (single minimal) subexponential cluster at time τ_ℓ. Define

$$\mathcal{I}^\ell_{\mathbf{M}^\ell} := \mathbb{E}_\varepsilon\Big[\Xi_{p-1}\bigotimes_{i\in B}\zeta^\varepsilon_{m_i^\ell,\tau_\ell}\big(\bar\phi^{(i)}\big)\Big]\,.$$

Let us apply Proposition 4.8: we are going to show that asymptotically as $\mu_\varepsilon \to \infty$, the fluctuation structure at time $\tau_{\ell+1}$ is similar to the fluctuation structure at time τ_ℓ. For this, consider the principal part $\bar{\mathcal{I}}^\ell_{\mathbf{M}^\ell}$ defined in Proposition 4.9, but now on the time interval I_ℓ

$$\bar{\mathcal{I}}^\ell_{\mathbf{M}^\ell} := \sum_{\mathbf{N}^\ell}\sum_{\substack{\eta\cup\rho \text{ partition of } B\\ \rho \text{ clustering}}}\prod_q \mu_\varepsilon^{1-\frac{|\rho_q|}{2}}\mathbb{E}_\varepsilon\Big[\bar\phi^{(\rho_q)}_{|I_\ell|}\Big]\times\mathbb{E}_\varepsilon\Big[\Xi_{p-1}\Big(\bigotimes_q \mu_\varepsilon^{\frac12-\frac{|\eta_q|}{2}}\zeta^\varepsilon_{M^{\ell+1}_{\eta_q},\tau_{\ell+1}}\Big(\bar\phi^{(\eta_q)}_{|I_\ell|}\Big)\Big)\Big]\,,$$

where

$$\Big|\mathcal{I}^\ell_{\mathbf{M}^\ell} - \bar{\mathcal{I}}^\ell_{\mathbf{M}^\ell}\Big| \le (C_P\Theta)^{M^\ell_B}\varepsilon^{\frac{1}{8d}} \tag{5.2}$$

by Propositions 4.9-4.10, and split in turn $\bar{\mathcal{I}}^\ell_{\mathbf{M}^\ell}$ into different contributions

$$\bar{\mathcal{I}}^\ell_{\mathbf{M}^\ell} = \mathcal{I}^{\ell+1}_{\mathbf{M}^{\ell+1}} + \mathcal{I}^{\exp,\ell}_{\mathbf{M}} + \mathcal{I}^{2,\ell}_{\mathbf{M}} + \mathcal{I}^{>,\ell}_{\mathbf{M}}$$

which are defined below. We have written $M^{\ell+1}_{\eta_q}$ for the number of particles in η_q at time $\tau_{\ell+1}$, and $\mathbf{N}^\ell$ the number of particles annihilated on $[\tau_{\ell+1},\tau_\ell]$.

The main contribution is a product where each of the $|B|$ terms has evolved independently ($|\rho| = \varnothing$, η consisting only of singletons) in a controlled way

$$\mathcal{I}^{\ell+1}_{\mathbf{M}^{\ell+1}} := \sum_{\mathbf{N}^\ell \text{ subexp}}\mathbb{E}_\varepsilon\Big[\Xi_{p-1}\Big(\bigotimes_{i\in B}\zeta^\varepsilon_{m_i^\ell+n_i^\ell,\tau_{\ell+1}}\big(\bar\phi^{(i)}_{|I_\ell|}\big)\Big)\Big]\,,$$

where the sum is restricted to subexponential $\mathbf{N}^\ell$, meaning that for all $i \in B$, $n_i^\ell \le 2^\ell$. The function $\bar\phi^{(i)}_{|I_\ell|}$ is thus supported on a (single minimal) subexponential cluster at time $\tau_{\ell+1}$ as in Definition 5.1. We stress the fact that $\mathcal{I}^{\ell+1}_{\mathbf{M}^{\ell+1}}$ has the same product structure as $\mathcal{I}^\ell_{\mathbf{M}^\ell}$.

The remainder $\bar{\mathcal{I}}^\ell_{\mathbf{M}^\ell} - \mathcal{I}^{\ell+1}_{\mathbf{M}^{\ell+1}}$ is split into the following three terms:

— the higher order cumulants

(5.3)
$$\mathcal{I}^{>,\ell}_{\mathbf{M}} := \sum_{\substack{\eta\cup\rho \text{ partition of } B\\ \rho \text{ clustering}}} \sum_{\mathbf{N}^\ell_\rho} \mathbf{1}^{>} \prod_q \mu_\varepsilon^{1-\frac{|\rho_q|}{2}} \mathbb{E}_\varepsilon\Big[\bar\phi^{(\rho_q)}_{|I_\ell|}\Big] \times \sum_{\mathbf{N}^\ell_\eta} \mathbb{E}_\varepsilon\Big[\Xi_{p-1}\Big(\bigotimes_q \mu_\varepsilon^{\frac12-\frac{|\eta_q|}{2}} \zeta^\varepsilon_{M^{\ell+1}_{\eta_q},\tau_{\ell+1}}\Big(\bar\phi^{(\eta_q)}_{|I_\ell|}\Big)\Big)\Big],$$

where $\mathbf{1}^{>}$ indicates that either at least one η_q has more than one element, or at least one ρ_q has more than two elements;

— a term collecting pair cumulants in ρ

$$\mathcal{I}^{2,\ell}_{\mathbf{M}} := \sum_{\substack{\eta\cup\rho \text{ partition of } B\\ \eta \text{ singletons}, \rho \text{ pairs}}} \sum_{\mathbf{N}^\ell_\rho} \prod_q \mathbb{E}_\varepsilon\Big[\bar\phi^{(\rho_q)}_{|I_\ell|}\Big] \times \sum_{\mathbf{N}^\ell_\eta} \mathbb{E}_\varepsilon\Big[\Xi_{p-1}\Big(\bigotimes_q \zeta^\varepsilon_{M^{\ell+1}_q,\tau_{\ell+1}}\Big(\bar\phi^{(q)}_{|I_\ell|}\Big)\Big)\Big],$$

— a term with only single minimal cumulants but at least one has a superexponential growth

$$\mathcal{I}^{\exp,\ell}_{\mathbf{M}} := \sum_{\mathbf{N}^\ell \text{superexp}} \mathbb{E}_\varepsilon\Big[\Xi_{p-1}\Big(\bigotimes_{i\in B} \zeta^\varepsilon_{m_i^\ell+n_i^\ell,\tau_{\ell+1}}\Big(\bar\phi^{(i)}_{|I_\ell|}\Big)\Big)\Big], \tag{5.4}$$

where the sum is restricted to superexponential $\mathbf{N}^\ell$, meaning that at least one $i \in B$ satisfies that $n_i^\ell > 2^\ell$.

The following Paragraphs 5.3, 5.4 and 5.5 of this section consist in proving that the terms $\mathcal{I}^{>,\ell}_{\mathbf{M}}$, $\mathcal{I}^{2,\ell}_{\mathbf{M}}$ and $\mathcal{I}^{\exp,\ell}_{\mathbf{M}}$ are small. As a consequence of Propositions 5.4, 5.5, 5.6 and of (5.2), we deduce the following result on the time step $[\tau_{\ell+1}, \tau_\ell]$.

Proposition 5.3. *The following estimate holds:*

$$\Big|\bar{\mathcal{I}}^\ell_{\mathbf{M}^\ell} - \mathcal{I}^{\ell+1}_{\mathbf{M}^{\ell+1}}\Big| \le (C_P\Theta)^{2^\ell P} \varepsilon^{\frac{1}{8d}} + \big(C_P\Theta^{2P-1}\tau\big)^{2^{\ell-1}}\,.$$

Proof of Proposition 5.2. Using repeatedly the results of (5.2) and Proposition 5.3, which transports the fluctuation structure on any intermediate interval I_ℓ, we can recover the fluctuation structure on the longer time interval $[\theta_{p-1}, \theta_p]$.

Recall that $\theta_p = \tau_{\kappa_p}$ with the notation (5.1). We consider the following fluctuation structure at time θ_p

$$\mathcal{I}_{\mathbf{M}} := \mathbb{E}_\varepsilon\Big[\Xi_{p-1}\Big(\bigotimes_{i\in B} \zeta^\varepsilon_{m_i,\theta_p}(\bar\phi^{(i)})\Big)\Big],$$

where B is a subset of $\{p,\dots,P\}$ and for each $i \in B$, $\bar\phi^{(i)}$ is a function of m_i variables, supported on a subexponential minimal cluster at time θ_p.

Using repeatedly Proposition 5.3 on I_ℓ for ℓ in $\mathbb{L}_p$, we get that

$$\mathcal{I}_{\mathbf{M}} = \sum_{\substack{(\mathbf{N}^\ell)_{\ell\in\mathbb{L}_p}\\ \text{subexp}}} \mathbb{E}_\varepsilon\Big[\Xi_{p-1}\Big(\bigotimes_{i\in B} \zeta^\varepsilon_{M_i^{\kappa_{p-1}},\theta_{p-1}}(\bar\phi^{(i)}_{\theta_p-\theta_{p-1}})\Big)\Big] + \sum_{\ell\in\mathbb{L}_p} \mathcal{R}^\ell_{\mathbf{M}},$$

where for each $i \in B$, $\bar\phi^{(i)}_{\theta_p-\theta_{p-1}}$ is a function of $M_i^{\kappa_{p-1}} := m_i + \sum_{\ell\in\mathbb{L}_p} n_i^\ell$ variables, supported on a subexponential minimal single cluster at time θ_{p-1}.

The remainders $\mathcal{R}^\ell_{\mathbf{M}}$ come from the terms which are neglected at each step : big clusters, vanishing pairings, and superexponential terms, as well as the remainder terms $\sum_r \mathcal{R}^{\rm int}_r$. By Propositions 4.10 (see Eq. (5.2)) and 5.3, we get at each step ℓ

$$\mathcal{R}^\ell_{\mathbf{M}} \le (C_P\Theta)^{2^\ell P}\,\varepsilon^{\frac{1}{8d}} + \big(C_P\Theta^{2P-1}\tau\big)^{2^{\ell-1}} \,.$$

Note that the exponential factor takes also into account the sum over all choices of $(\mathbf{N}^k)_{k\le\ell}$ since $2^{1+\dots+\ell} \le 2^{\ell^2}$. □

5.3. **Removing big clusters.** We treat here the high order cumulants collected in $\mathcal{I}^{>,\ell}_{\mathbf{M}}$ (see (5.3)). These cumulants describe dynamical correlations which are negligible at the scale of the fluctuations. More precisely, we have the following result.

Proposition 5.4. *The remainder accounting for big clusters satisfies the following estimate:*

$$\Big|\mathcal{I}^{>,\ell}_{\mathbf{M}}\Big| \le (C_P\Theta)^{2^\ell P}\,\mu_\varepsilon^{-\frac12}\,\tau^{\frac12}\,. \tag{5.5}$$

Proof. By construction each cumulant $\bar\phi^{(\rho_q)}_{|I_\ell|}$ appearing in (5.3) is supported on a minimal cluster and the product of the expectations can be estimated by Lemma 4.11:

$$\Big|\mathbb{E}_\varepsilon\Big[\bar\phi^{(\rho_q)}_{|I_\ell|}\Big]\Big| \le C\,(C_P\Theta)^{M^\ell_{\rho_q}-|\rho_q|}\,(C_P\tau)^{N^\ell_{\rho_q}+|\rho_q|-1}, \tag{5.6}$$

where $N^\ell_{\rho_q}$ is the total number of particles annihilated in ρ_q during I_ℓ and $M^\ell_{\rho_q} = \sum_{i\in\rho_q} m^\ell_i$. The functions $\bar\phi^{(i)}$ at time τ_ℓ are supported on (single minimal) subexponential clusters whose sizes are bounded by

$$m^\ell_i \le 2^\ell \quad \text{so that} \quad \sum_q M^\ell_{\rho_q} \le 2^\ell P. \tag{5.7}$$

In the inequality (5.6), the power of τ keeps track of the total number $N^\ell_{\rho_q}$ of annihilated particles and of the $|\rho_q|-1$ clusterings in the time interval I_ℓ. Since $\tau \ll 1$, the sums with respect to $\mathbf{N}^\ell_{\rho_q}$ converge and a factor $\tau^{|\rho_q|-1}$ remains. We get

$$\Bigg|\sum_{\mathbf{N}^\ell_\rho}\prod_q \mu_\varepsilon^{1-\frac{|\rho_q|}{2}}\,\mathbb{E}_\varepsilon\Big[\bar\phi^{(\rho_q)}_{|I_\ell|}\Big]\Bigg| \le (C_P\Theta)^{2^\ell P}\prod_q \mu_\varepsilon^{1-\frac{|\rho_q|}{2}}\,\tau^{|\rho_q|-1}\,. \tag{5.8}$$

If one of the ρ_q is not a pair then $|\rho_q|/2 \ge 3/2$ and this leads to an additional decay in $\tau\mu_\varepsilon^{-\frac12}$.

We turn now to the estimate of the part of $\mathcal{I}^{>,\ell}_{\mathbf{M}}$ which is weighted by a product of fluctuation fields. By Hölder's inequality and Lemma 4.11, we have that

$$\begin{aligned}
&\Bigg|\mathbb{E}_\varepsilon\Big[\Xi_{p-1}\Big(\bigotimes_q \mu_\varepsilon^{\frac12-\frac{|\eta_q|}{2}}\,\zeta^\varepsilon_{M^{\ell+1}_{\eta_q},\tau_{\ell+1}}\Big(\bar\phi^{(\eta_q)}_{|I_\ell|}\Big)\Big)\Big]\Bigg| \\
&\le \prod_{u=1}^{p-1}\mathbb{E}_\varepsilon\Big[\zeta^\varepsilon\big(h^{(u)}\big)^{2(p-1)}\Big]^{\frac{1}{2(p-1)}}\,\mathbb{E}_\varepsilon\Big[\Big(\bigotimes_q \mu_\varepsilon^{\frac12-\frac{|\eta_q|}{2}}\,\zeta^\varepsilon_{M^{\ell+1}_{\eta_q},\tau_{\ell+1}}\Big(\bar\phi^{(\eta_q)}_{|I_\ell|}\Big)\Big)^2\Big]^{\frac12} \\
&\le C_P\prod_q (C_P\Theta)^{M^\ell_{\eta_q}+\frac12 N^\ell_{\eta_q}-\frac12|\eta_q|}\,(C_P\tau)^{\frac{N^\ell_\eta}{2}}\,\Big(\frac{C_P\tau}{\mu_\varepsilon}\Big)^{\frac{|\eta_q|}{2}-\frac12}\,,
\end{aligned} \tag{5.9}$$

where the moments of the fluctuation field are bounded by Proposition 2.3. Summing over $\mathbf{N}^\ell_\eta$ gives

$$\sum_{\mathbf{N}^\ell_\eta}\Bigg|\mathbb{E}_\varepsilon\Big[\Xi_{p-1}\Big(\bigotimes_q \mu_\varepsilon^{\frac12-\frac{|\eta_q|}{2}}\,\zeta^\varepsilon_{M^{\ell+1}_{\eta_q},\tau_{\ell+1}}\Big(\bar\phi^{(\eta_q)}_{|I_\ell|}\Big)\Big)\Big]\Bigg| \le (C_P\Theta)^{2^\ell P}\prod_q\Big(\frac{C\tau}{\mu_\varepsilon}\Big)^{\frac{|\eta_q|}{2}-\frac12}\,. \tag{5.10}$$

In particular if one η_q satisfies $|\eta_q| > 1$, we gain at least one power of $\mu_\varepsilon^{-1/2}\tau^{1/2}$. Combining (5.10) with (5.8), we recover

$$\left|\mathcal{I}^{>,\ell}_{\mathbf{M}}\right| \leq (C_P\Theta)^{2^\ell P}\, \mu_\varepsilon^{-\frac{1}{2}}\, \tau^{\frac{1}{2}}$$

and (5.5) follows thanks to the fact that $\tau \leq 1$. We stress that the combinatorial factors arising from partitioning $B \subset \{1,\dots,P\}$ into ρ,η depend only on P. Proposition 5.5 is proved. □

5.4. **Control of pair cumulants at equilibrium.** If ρ is made only of pairs and η of singletons, then (5.8) and (5.10) do not provide any decay as a power of μ_ε. In fact out of equilibrium, these pairings contribute to the covariance and they were first analysed in [31] in terms of a recollision operator (see also [7]). Instead at equilibrium, these terms vanish in the limit $\mu_\varepsilon \to \infty$ due to a symmetry property of the limiting measure. Thus to avoid the bookkeeping exercise of tracking these terms in the iteration, we prefer to show that they do not contribute in the equilibrium regime considered in this paper.

Proposition 5.5. *The remainder accounting for pair cumulants is estimated as follows :*

$$\left|\mathcal{I}^{2,\ell}_{\mathbf{M}}\right| \leq (C_P\Theta)^{2^\ell P}\, \varepsilon^{\frac{1}{8d}}\,. \tag{5.11}$$

Proof. The key estimate is to show that for pairs the expectation $\mathbb{E}_\varepsilon\Big[\bar\phi^{(\rho_q)}_{|I_\ell|}\Big]$ vanishes in the limit when $\mu_\varepsilon \to \infty$. To fix ideas, we consider a clustering ρ_q of the form $\{1,2\}$ connecting the single minimal cumulants $\bar\phi^{(1)}$, $\bar\phi^{(2)}$ supported on subexponential clusters at time τ_ℓ by an encounter on $I_\ell = [\tau_{\ell+1},\tau_\ell]$. Let us show that

$$\Big|\sum_{\mathbf{N}^\ell_{\{1,2\}}} \mathbb{E}_\varepsilon\Big[\bar\phi^{(\{1,2\})}_{|I_\ell|}\Big]\Big| \leq (C_P\Theta)^{m_1^\ell+m_2^\ell}\, \varepsilon^{\frac{1}{8d}}\,, \tag{5.12}$$

from which Proposition 5.5 follows immediately by summing over the partitions and taking into account the size (5.7) of the subexponential clusters at time τ_ℓ.

Using Proposition 4.8 with $|B| = 2, p = 1$ on the time interval I_ℓ leads to the explicit decomposition

$$\begin{aligned}\mathbb{E}_\varepsilon\Big[\bigotimes_{i=1}^{2}\zeta^\varepsilon_{m_i^\ell,\tau_\ell}\big(\bar\phi^{(i)}\big)\Big] = \sum_{\mathbf{N}^\ell_{\{1,2\}}} \mathbb{E}_\varepsilon\Big[\bar\phi^{(\{1,2\})}_{|I_\ell|}\Big] + \sum_{n_1^\ell,n_2^\ell}\mathbb{E}_\varepsilon\Big[\bigotimes_{i=1}^{2}\zeta^\varepsilon_{m_i^\ell+n_i^\ell,\tau_{\ell+1}}\big(\bar\phi^{(i)}_{|I_\ell|}\big)\Big]\\ + O\Big((C_P\Theta)^{m_1^\ell+m_2^\ell}\varepsilon^{\frac{1}{8d}}\Big).\end{aligned} \tag{5.13}$$

In this way, estimating the expectation of the pair correlations $\bar\phi^{(\{1,2\})}_{|I_\ell|}$ can be achieved by controlling the two $\circledast$-products : one at time τ_ℓ and the other one at time $\tau_{\ell+1}$. By construction $\bar\phi^{(i)}_{|I_\ell|}$ is supported on a single minimal cluster, thus by (4.17) in Lemma 4.11, the expectation is small

$$\left|\sum_{\mathbf{N}^\ell_{\{1,2\}}} \mathbb{E}_\varepsilon\Big[\bigotimes_{i=1}^{2}\zeta^\varepsilon_{m_i^\ell+n_i^\ell,\tau_{\ell+1}}\big(\bar\phi^{(i)}_{|I_\ell|}\big)\Big]\right| \leq (C_P\Theta)^{m_1^\ell+m_2^\ell}\sum_{n_1^\ell,n_2^\ell}(C_P\tau)^{(n_1^\ell+n_2^\ell)/2}\,\varepsilon\,. \tag{5.14}$$

The expectation $\mathbb{E}_\varepsilon\Big[\circledast_{i=1}^2\zeta^\varepsilon_{m_i,\tau_\ell}\big(\bar\phi^{(i)}\big)\Big]$ can be estimated in the same way. Since $\tau \ll 1$, summing over n_1^ℓ, n_2^ℓ completes the proof of (5.12). □

5.5. **Removing superexponential collision trees.** In this section, we estimate dynamical flows exhibiting a superexponential number of annihilations, namely $\mathcal{I}_{\mathbf{M}}^{\exp,\ell}$ defined in (5.4). The result is the following.

Proposition 5.6. *The remainder corresponding to superexponential clusters is estimated as follows :*

$$\left|\mathcal{I}_{\mathbf{M}}^{\exp,\ell}\right| \leq \left(C_P \Theta^{2P-1}\tau\right)^{2^{\ell-1}} .$$

Proof. Compared to the previous sections, the control of $\mathcal{I}_{\mathbf{M}}^{\exp,\ell}$ requires a more careful description of the functions $\phi^{(i)}$, taking into account the time sampling. They are supported on subexponential clusters at time τ_ℓ (see Definition 5.1). As a consequence each function depends at most on 2^ℓ particles and the total number of particles at time τ_ℓ is at most $|B|\,2^\ell$.

By definition of $\mathcal{I}_{\mathbf{M}}^{\exp,\ell}$, there is $i \in B$ such that on I_ℓ, the number n_i^ℓ of annihilated particles associated with i is larger than 2^ℓ. This means that on a time step of size τ, at least half of the particles (up to the factor $|B|$) are removed.

By Hölder's inequality as in (4.15) and by (4.18) (without microscopic time splitting), we then have that

$$\begin{aligned}\left|\mathcal{I}_{\mathbf{M}}^{\exp,\ell}\right| &\leq \sum_{\mathbf{N}^\ell \text{ superexp}} \left|\mathbb{E}_\varepsilon\Big[\Xi_{p-1}\Big(\bigotimes_{i\in B}\zeta^\varepsilon_{m_i^\ell+n_i^\ell,\tau_{\ell+1}}(\bar\phi^{(i)}_{|I_\ell|})\Big)\Big]\right| \\ &\leq \sum_{\mathbf{N}^\ell \text{ superexp}} \prod_{u=1}^{p-1}\mathbb{E}_\varepsilon\Big[\zeta^\varepsilon\big(h^{(u)}\big)^{2(p-1)}\Big]^{\frac{1}{2(p-1)}}\,\mathbb{E}_\varepsilon\Big[\Big(\bigotimes_{i\in B}\zeta^\varepsilon_{m_i^\ell+n_i^\ell,\tau_{\ell+1}}(\bar\phi^{(i)}_{|I_\ell|})\Big)^2\Big]^{\frac12} \\ &\leq C_P \sum_{\mathbf{N}^\ell \text{ superexp}} \prod_{i\in B}\Big((C_P\Theta)^{2m_i^\ell+n_i^\ell-1}\tau^{n_i^\ell}\Big)^{1/2} \\ &\leq \Big(C_P\Theta^{2|B|-1}\tau\Big)^{2^\ell/2} .\end{aligned}$$

Proposition 5.6 follows using that $|B| \leq P$. □

6. Asymptotics of the principal term

As explained in Section 2, our goal is to pull back the test functions in time in order to build pairings and establish the Wick factorisation of the moments. In Section 5, we have been able to pull back minimal cumulants from one sampling time θ_p to the next θ_{p-1}. In order to proceed to the next step and reach θ_{p-2}, one has to take into account the new structure of the expectation at time θ_{p-1} after the multiplication by the function $h^{(p-1)}$. This will induce the pairing mechanism identified in Section 2 which will be quantified in this section.

In Section 6.1, we analyse the repeated indices at time θ_{p-1} when taking the product of the fluctuation field $\zeta^\varepsilon_{\theta_{p-1}}(h^{(p-1)})$ with the $\circledast$-product obtained in Proposition 5.2 from the iteration. The induction for the derivation of Proposition 2.1 is completed in Section 6.2.

6.1. **Asymptotic pairing.** Recall that at time θ_{p-1}, for any $i \in \{p,\dots,P\}$, the function $\bar\phi^{(i)}_{\theta_i-\theta_{p-1}}$ is the pullback of the test function $h^{(i)}$ on $[\theta_{p-1},\theta_i]$ (in the sense of Definition 5.1). We now study the product

$$\begin{aligned}&\zeta^\varepsilon_{\theta_{p-1}}(h^{(p-1)}) \times \Big(\bigotimes_{i\in B}\zeta^\varepsilon_{m_i^{\kappa_{p-1}},\theta_{p-1}}(\bar\phi^{(i)}_{\theta_i-\theta_{p-1}})\Big) \\ &= \mu_\varepsilon^{(|B|+1)/2}\Big(\pi^\varepsilon_{\theta_{p-1}}\big(h^{(p-1)}\big) - \mathbb{E}_\varepsilon\big[h^{(p-1)}\big]\Big)\Bigg(\sum_{A\subset B}\prod_{i\in B\setminus A}\mathbb{E}_\varepsilon\Big[-\bar\phi^{(i)}_{\theta_i-\theta_{p-1}}\Big]\pi^\varepsilon_{M_A^{\kappa_{p-1}},\theta_{p-1}}\Big(\bigotimes_{i\in A}\bar\phi^{(i)}_{\theta_i-\theta_{p-1}}\Big)\Bigg),\end{aligned}$$

where

$$m_i^{\kappa_{p-1}} \leq 2^{\kappa_{p-1}}, \tag{6.1}$$

as by definition, the number of particles added in a subexponential cluster on I_ℓ is smaller than 2^ℓ. In particular, the following crude bound holds : $M_A^{\kappa_{p-1}} = \sum_{i\in A} m_i^{\kappa_{p-1}} \leq 2^{\kappa_{p-1}}|A|$. For the sake of readability, we will omit the superscript κ_{p-1} in the rest of this paragraph.

We split the sum in $\pi^\varepsilon_{\theta_{p-1}}\big(h^{(p-1)}\big)$ according to the repeated indices : when the index does not appear in the sum $\pi^\varepsilon_{M_A,\theta_{p-1}}$, we get a $\circledast$-product, else we get a contracted product:

$$\begin{aligned}
\zeta^\varepsilon_{\theta_{p-1}}(h^{(p-1)}) \times \Big(\circledast_{i\in B} \zeta^\varepsilon_{m_i,\theta_{p-1}}(\bar\phi^{(i)}_{\theta_i-\theta_{p-1}})\Big) \\
&= \mu_\varepsilon^{(|B|+1)/2} \sum_{A\subset B} \Bigg(\prod_{i\in B\cup\{p-1\}\setminus A} \mathbb{E}_\varepsilon\Big[-\bar\phi^{(i)}_{\theta_i-\theta_{p-1}}\Big]\Bigg) \pi^\varepsilon_{M_A,\theta_{p-1}}\Big(\bigotimes_{i\in A} \bar\phi^{(i)}_{\theta_i-\theta_{p-1}}\Big) \\
&+ \mu_\varepsilon^{(|B|+1)/2} \sum_{A\subset B} \Bigg(\prod_{i\in B\setminus A} \mathbb{E}_\varepsilon\Big[-\bar\phi^{(i)}_{\theta_i-\theta_{p-1}}\Big]\Bigg) \pi^\varepsilon_{M_A+1,\theta_{p-1}}\Big(h^{(p-1)} \bigotimes_{i\in A} \bar\phi^{(i)}_{\theta_i-\theta_{p-1}}\Big) \\
&+ \mu_\varepsilon^{(|B|-1)/2} \sum_{A\subset B} \Bigg(\prod_{i\in B\setminus A} \mathbb{E}_\varepsilon\Big[-\bar\phi^{(i)}_{\theta_i-\theta_{p-1}}\Big]\Bigg) \pi^\varepsilon_{M_A,\theta_{p-1}}\Big(\sum_{j\in A} \psi^{(j,p-1)} \bigotimes_{\substack{i\in A\\ i\neq j}} \bar\phi^{(i)}_{\theta_i-\theta_{p-1}}\Big),
\end{aligned}$$

where $\bar\phi_0^{(p-1)} := h^{(p-1)}$ and

$$\psi^{(j,p-1)}(Z_{m_j}) := \bar\phi^{(j)}_{\theta_j-\theta_{p-1}}(Z_{m_j}) \sum_{\ell=1}^{m_j} h^{(p-1)}(z_\ell)\,. \tag{6.2}$$

Using the definition of the $\circledast$-product, we get the identity

$$\begin{aligned}
\zeta^\varepsilon_{\theta_{p-1}}(h^{(p-1)}) \times \Big(\circledast_{i\in B} \zeta^\varepsilon_{m_i,\theta_{p-1}}(\bar\phi^{(i)}_{\theta_i-\theta_{p-1}})\Big) = \circledast_{i\in B\cup\{p-1\}} \zeta^\varepsilon_{m_i,\theta_{p-1}}(\bar\phi^{(i)}_{\theta_i-\theta_{p-1}}) \\
+ \mu_\varepsilon^{(|B|-1)/2} \sum_{j\in B} \sum_{\tilde A\subset B\setminus\{j\}} \Big(\prod_{i\in(B\setminus\{j\})\setminus\tilde A} \mathbb{E}_\varepsilon\Big[-\bar\phi^{(i)}_{\theta_i-\theta_{p-1}}\Big]\Big) \pi^\varepsilon_{M_{\tilde A}+m_j,\theta_{p-1}}\Big(\psi^{(j,p-1)} \bigotimes_{i\in\tilde A} \bar\phi^{(i)}_{\theta_i-\theta_{p-1}}\Big)\,.
\end{aligned}$$

Decomposing $\mu_\varepsilon^{-m_j} \sum \psi^{(j,p-1)}$ in its expectation plus a fluctuation as in (3.38)

$$\pi^\varepsilon_{m_j,\theta_{p-1}}(\psi^{(j,p-1)}) = \mathbb{E}_\varepsilon[\psi^{(j,p-1)}] + \mu_\varepsilon^{-\frac12} \zeta^\varepsilon_{m_j,\theta_{p-1}}(\psi^{(j,p-1)})\,,$$

we finally obtain, using again the definition of the $\circledast$-product,

$$\begin{aligned}
\zeta^\varepsilon_{\theta_{p-1}}(h^{(p-1)}) \times \Big(\circledast_{i\in B} \zeta^\varepsilon_{m_i,\theta_{p-1}}(\bar\phi^{(i)}_{\theta_i-\theta_{p-1}})\Big) &= \circledast_{i\in B\cup\{p-1\}} \zeta^\varepsilon_{m_i,\theta_{p-1}}(\bar\phi^{(i)}_{\theta_i-\theta_{p-1}}) \\
&+ \sum_{j\in B} \mathbb{E}_\varepsilon[\psi^{(j,p-1)}] \circledast_{i\neq j} \zeta^\varepsilon_{m_i,\theta_{p-1}}(\bar\phi^{(i)}_{\theta_i-\theta_{p-1}}) \\
&+ \mu_\varepsilon^{-1/2} \sum_{j\in B} \zeta^\varepsilon_{m_j,\theta_{p-1}}(\psi^{(j,p-1)}) \circledast \Big(\circledast_{i\neq j} \zeta^\varepsilon_{m_i,\theta_{p-1}}(\bar\phi^{(i)}_{\theta_i-\theta_{p-1}})\Big)\,.
\end{aligned} \tag{6.3}$$

As the function $\psi^{(j,p-1)}$ has the same cluster structure of $\bar\phi^{(j)}_{\theta_j-\theta_{p-1}}$ (with only a slightly different weight), the inequalities of Lemma 4.11 apply also for $\psi^{(j,p-1)}$. Proceeding as in (5.9), we apply Hölder's inequality and Lemma 4.11 to show that the last term in the previous decomposition has a vanishing contribution in the limit

$$\mu_\varepsilon^{-1/2} \left| \mathbb{E}_\varepsilon\Big[\Xi_{p-2} \Big(\sum_{j\in B} \zeta^\varepsilon_{m_j^{\kappa_{p-1}},\theta_{p-1}}(\psi^{(j,p-1)}) \circledast_{i\neq j} \zeta^\varepsilon_{m_i^{\kappa_{p-1}},\theta_{p-1}}(\bar\phi^{(i)}_{\theta_i-\theta_{p-1}})\Big)\Big]\right| \leq \mu_\varepsilon^{-1/2} (C_P\Theta)^M,$$

with $M = \sum_{i\in B} m_i \le 2^{\kappa_p} P$ and $\Xi_{p-2} := \prod_{u=1}^{p-2} \zeta^\varepsilon_{\theta_u}\big(h^{(u)}\big)$ was introduced in (4.8).

Thus we obtain the following result.

Proposition 6.1. *For any $j \ge p$, let $\bar\phi^{(j)}_{\theta_j-\theta_{p-1}}$ be the generic subexponential dynamical cluster from the expansion of $h^{(j)}$, and denote $\psi^{(j,p-1)}$ its contraction with $h^{(p-1)}$ defined by (6.2). We set $\phi^{(p-1)}_0 = h^{(p-1)}$. Then*

$$\begin{aligned}\Bigg|\mathbb{E}_\varepsilon\Big[\Xi_{p-1}\Big(\bigotimes_{i\in B}\zeta^\varepsilon_{m_i^{\kappa_{p-1}},\theta_{p-1}}(\bar\phi^{(i)}_{\theta_i-\theta_{p-1}})\Big)\Big] - \mathbb{E}_\varepsilon\Big[\Xi_{p-2}\Big(\bigotimes_{i\in B\cup\{p-1\}}\zeta^\varepsilon_{m_i^{\kappa_{p-1}},\theta_{p-1}}(\bar\phi^{(i)}_{\theta_i-\theta_{p-1}})\Big)\Big]\\ - \sum_{j\in B}\mathbb{E}_\varepsilon[\psi^{(j,p-1)}]\mathbb{E}_\varepsilon\Big[\Xi_{p-2}\Big(\bigotimes_{i\in B\setminus\{j\}}\zeta^\varepsilon_{m_i^{\kappa_{p-1}},\theta_{p-1}}(\bar\phi^{(i)}_{\theta_i-\theta_{p-1}})\Big)\Big]\Bigg| \le C_P\mu_\varepsilon^{-1/2}(C_P\Theta)^{2^{\kappa_p}P}.\end{aligned}\tag{6.4}$$

The second term in (6.4) has the required structure to be pulled back up to time θ_{p-2} by considering cumulants indexed by the larger set $B\cup\{p-1\}$. The sum in (6.4) involves the product of $\mathbb{E}_\varepsilon\left[\psi^{(j,p-1)}\right]$, which will be linked to a covariance in Corollary 6.2, and a $\circledast$-product which has the right structure to be pulled back up to time θ_{p-2} by considering cumulants indexed by the smaller set $B\setminus\{j\}$.

Corollary 6.2. *For any $j \ge p$, denote by $\bar\phi^{(j)}_{\theta_j-\theta_{p-1}}$ the subexponential minimal cumulants issued from $h^{(j)}$, and denote by $\psi^{(j,p-1)}$ their contraction with $h^{(p-1)}$ defined by (6.2). Then, one has*

$$\Bigg|\sum_{\substack{\mathbf{N}_j\\ \text{subexp}}}\mathbb{E}_\varepsilon\left[\psi^{(j,p-1)}\right] - \mathbb{E}_\varepsilon\Big[\zeta^\varepsilon_{\theta_{p-1}}(h^{(p-1)})\,\zeta^\varepsilon_{\theta_j}(h^{(j)})\Big]\Bigg| \le (C_P\Theta)^{2^{\kappa_p}}\varepsilon^{\frac{1}{8d}} + (C_P\Theta^3\tau)^{2^{\kappa_{p-1}}}.$$

In particular, one has the uniform bound

$$\Bigg|\sum_{\substack{\mathbf{N}_j\\ \text{subexp}}}\mathbb{E}_\varepsilon\left[\psi^{(j,p-1)}\right]\Bigg| \le C\,.$$

Proof of Corollary 6.2. Using repeatedly Proposition 5.2 with only two test functions $h^{(p-1)}, h^{(j)}$ but with the same time sampling I_ℓ, we get

$$\begin{aligned}\Bigg|\mathbb{E}_\varepsilon\Big[\zeta^\varepsilon_{\theta_{p-1}}\big(h^{(p-1)}\big)\zeta^\varepsilon_{\theta_j}(h^{(j)})\Big)\Big] - \sum_{\substack{\mathbf{N}_j\\ \text{subexp}}}\mathbb{E}_\varepsilon\Big[\zeta^\varepsilon_{\theta_{p-1}}\big(h^{(p-1)}\big)\zeta^\varepsilon_{m_j^{\kappa_{p-1}},\theta_{p-1}}(\bar\phi^{(j)}_{\theta_j-\theta_{p-1}})\Big)\Big]\Bigg|\\ \le (C_P\Theta)^{2^{\kappa_p}}\varepsilon^{\frac{1}{8d}} + (C_P\Theta^3\tau)^{2^{\kappa_{p-1}}}\quad.\end{aligned}$$

Then, from Proposition 6.1 with $B = \{j\}$, we get

$$\begin{aligned}\Bigg|\mathbb{E}_\varepsilon\Big[\zeta^\varepsilon_{\theta_{p-1}}\big(h^{(p-1)}\big)\zeta^\varepsilon_{m_j^{\kappa_{p-1}},\theta_{p-1}}(\bar\phi^{(j)}_{\theta_j-\theta_{p-1}})\Big)\Big]\\ - \mathbb{E}_\varepsilon\Big[\zeta^\varepsilon_{\theta_{p-1}}\big(h^{(p-1)}\big)\circledast\zeta^\varepsilon_{m_j^{\kappa_{p-1}},\theta_{p-1}}(\bar\phi^{(j)}_{\theta_j-\theta_{p-1}})\Big] - \mathbb{E}_\varepsilon[\psi^{(j,p-1)}]\Bigg| \le C\,\mu_\varepsilon^{-1/2}(C\Theta)^{2^{\kappa_p}}.\end{aligned}$$

It remains to sum over the subexponential annihilation numbers $\mathbf{N}_j = (n_j^\ell)_{\kappa_j<\ell\leq\kappa_{p-1}-1}$. Since $\mathbf{N}_j$ takes at most $2^{1+\dots+2^{\kappa_p}}$ values, the error terms remain under control and we get

$$\left|\mathbb{E}_\varepsilon\Big[\zeta^\varepsilon_{\theta_{p-1}}\big(h^{(p-1)}\big)\zeta^\varepsilon_{\theta_j}\big(h^{(j)}\big)\Big] - \sum_{\substack{\mathbf{N}_j\\ \text{subexp}}} \mathbb{E}_\varepsilon\big[\psi^{(j,p-1)}\big] - \sum_{\substack{\mathbf{N}_j\\ \text{subexp}}} \mathbb{E}_\varepsilon\Big[\zeta^\varepsilon_{\theta_{p-1}}\big(h^{(p-1)}\big) \circledast \zeta^\varepsilon_{m_j^{\kappa_{p-1}},\theta_{p-1}}\big(\bar\phi^{(j)}_{\theta_j-\theta_{p-1}}\big)\Big]\right|$$
$$\leq (C_P\Theta)^{2^{\kappa_p}}\varepsilon^{\frac{1}{8d}} + (C_P\Theta^3\tau)^{2^{\kappa_{p-1}}}\,.$$

By (5.14), we find that the first term in the right-hand side vanishes in the limit :

$$\left|\sum_{\substack{\mathbf{N}_j\\ \text{subexp}}} \mathbb{E}_\varepsilon\Big[\zeta^\varepsilon_{\theta_{p-1}}\big(h^{(p-1)}\big) \circledast \zeta^\varepsilon_{m_j^{\kappa_{p-1}},\theta_{p-1}}\big(\bar\phi^{(j)}_{\theta_j-\theta_{p-1}}\big)\Big]\right| \leq (C_P\Theta)^{2^{\kappa_p+1}}\varepsilon\,.$$

This completes Corollary 6.2. □

In the following section, we are going to iterate these propositions in order to decompose the moments of the field as a product of covariances and some remainder terms.

6.2. **Proof of Proposition 2.1 : convergence of the moments.** We are now going to combine the previous results to prove Proposition 2.1. We proceed by induction and at time θ_p, we assume that the following decomposition holds, with notation (6.2):

$$\begin{aligned}\mathbb{E}_\varepsilon\Big[\prod_{u=1}^P \zeta^\varepsilon_{\theta_u}\big(h^{(u)}\big)\Big] \sim &\sum_{\substack{B\cup B^c=\{p,\dots,P\}\\ B^c\cap B=\varnothing}} \sum_{\eta_p\in\mathfrak{S}^{\text{pairs}}_{B^c}} \prod_{\substack{\{i,j\}\in\eta_p\\ i<j}} \Big(\sum_{\substack{\mathbf{N}_j\\ \text{subexp}}} \mathbb{E}_\varepsilon\big[\psi^{(j,i)}\big]\Big)\\ &\times \sum_{\substack{\mathbf{N}_B\\ \text{subexp}}} \mathbb{E}_\varepsilon\Big[\Big(\prod_{u=1}^{p-1}\zeta^\varepsilon_{\theta_u}\big(h^{(u)}\big)\Big)\Big(\bigcircledast_{i\in B}\zeta^\varepsilon_{m_i^{\kappa_{p-1}},\theta_{p-1}}\big(\bar\phi^{(i)}_{\theta_i-\theta_{p-1}}\big)\Big)\Big],\end{aligned}\tag{6.5}$$

where $\sim$ means that the difference is bounded by $(C_P\Theta)^{P\cdot 2^{\kappa_P}}\varepsilon^{\frac{1}{8d}} + (C_P\Theta^{2P-1}\tau)^{1/2}$. Notice that the decomposition is valid at time Θ with $B = \{P\}$ and $B^c = \varnothing$.

Given $B, \eta_p \in \mathfrak{S}^{\text{pairs}}_{B^c}$, we are going to apply the procedure described in the previous sections to expand the expectation in the second line of (6.5) and to derive the induction relation at time θ_{p-1}. Combining (6.4) and (6.5), we obtain

$$\begin{aligned}\mathbb{E}_\varepsilon\Big[\prod_{u=1}^P \zeta^\varepsilon_{\theta_u}\big(h^{(u)}\big)\Big] \sim &\sum_{\substack{B\cup B^c=\{p,\dots,P\}\\ B^c\cap B=\varnothing}} \sum_{\eta_p\in\mathfrak{S}^{\text{pairs}}_{B^c}} \prod_{\substack{\{i,j\}\in\eta_p\\ i<j}} \Big(\sum_{\substack{\mathbf{N}_j\\ \text{subexp}}} \mathbb{E}_\varepsilon\big[\psi^{(j,i)}\big]\Big)\\ &\times \sum_{\substack{\mathbf{N}_B\\ \text{subexp}}} \mathbb{E}_\varepsilon\Big[\Big(\prod_{u=1}^{p-2}\zeta^\varepsilon_{\theta_u}\big(h^{(u)}\big)\Big)\Big(\bigcircledast_{i\in B\cup\{p-1\}}\zeta^\varepsilon_{m_i^{\kappa_{p-1}},\theta_{p-1}}\big(\bar\phi^{(i)}_{\theta_i-\theta_{p-1}}\big)\Big)\Big]\\ &+ \sum_{\substack{B\cup B^c=\{p,\dots,P\}\\ B^c\cap B=\varnothing}} \sum_{\eta_p\in\mathfrak{S}^{\text{pairs}}_{B^c}} \prod_{\substack{\{i,j\}\in\eta_p\\ i<j}} \Big(\sum_{\substack{\mathbf{N}_j\\ \text{subexp}}} \mathbb{E}_\varepsilon\big[\psi^{(j,i)}\big]\Big)\\ &\times \sum_{j'\in B}\left(\sum_{\substack{\mathbf{N}_{j'}\\ \text{subexp}}} \mathbb{E}_\varepsilon\big[\psi^{(j',p-1)}\big]\right) \sum_{\substack{\mathbf{N}_{B\smallsetminus\{j'\}}\\ \text{subexp}}} \mathbb{E}_\varepsilon\Big[\Big(\prod_{u=1}^{p-2}\zeta^\varepsilon_{\theta_u}\big(h^{(u)}\big)\Big)\Big(\bigcircledast_{i\in B\smallsetminus\{j'\}}\zeta^\varepsilon_{m_i^{\kappa_{p-1}},\theta_{p-1}}\big(\bar\phi^{(i)}_{\theta_i-\theta_{p-1}}\big)\Big)\Big]\,.\end{aligned}\tag{6.6}$$

In the first contribution, since there is no new pairing, we set $\eta_{p-1} = \eta_p$ and the product form holds now on the set $B\cup\{p-1\}\subset\{p-1,\dots,P\}$. For the second contribution, we define the new set $\eta_{p-1} = \eta_p\cup\{(p-1,j')\}$ with the additional pair. The $\circledast$-product at time θ_{p-1} holds on the reduced set $B\smallsetminus\{j'\}$.

Thus the induction assumption (6.5) is also valid at θ_{p-1} and it can be iterated up to time θ_1:

$$\mathbb{E}_\varepsilon\Big[\Big(\prod_{u=1}^{P}\zeta^\varepsilon_{\theta_u}\big(h^{(u)}\big)\Big)\Big] \sim \sum_{B\subset\{1,\dots,P\}}\sum_{\eta_1\in\mathfrak{S}^{\text{pairs}}_{B^c}}\prod_{\substack{\{i,j\}\in\eta_1\\ i<j}}\Big(\sum_{\substack{\mathbf{N}_j\\ \text{subexp}}}\mathbb{E}_\varepsilon\big[\psi^{(j,i)}\big]\Big)\sum_{\substack{\mathbf{N}_B\\ \text{subexp}}}\mathbb{E}_\varepsilon\Big[\bigotimes_{i\in B}\Big(\zeta^\varepsilon_{m_i^{\kappa_1},\theta_1}\big(\bar\phi^{(i)}_{\theta_i-\theta_1}\big)\Big)\Big].$$

As the induction is applied only P times, the difference remains bounded by $(C_P\Theta)^{P\cdot 2^{\kappa_P}}\varepsilon^{\frac{1}{8d}} + (C_P\Theta^{2P-1}\tau)^{1/2}$.

By (4.17) in Lemma 4.11, the terms for which $B\neq\varnothing$ can be neglected. Thus the only remaining term is $B=\varnothing$ and η_1 is an element of $\mathfrak{S}^{\text{pairs}}_P$:

$$\mathbb{E}_\varepsilon\Big[\prod_{u=1}^{P}\zeta^\varepsilon_{\theta_u}\big(h^{(u)}\big)\Big] \sim \sum_{\eta_1\in\mathfrak{S}^{\text{pairs}}_P}\prod_{\substack{\{i,j\}\in\eta_1\\ i<j}}\Big(\sum_{\substack{\mathbf{N}_j\\ \text{subexp}}}\mathbb{E}_\varepsilon\big[\psi^{(j,i)}\big]\Big). \tag{6.7}$$

Recall that $\sim$ means that the difference is bounded by $(C_P\Theta)^{P\cdot 2^{K_P}}\varepsilon^{\frac{1}{8d}} + (C_P\Theta^{2P-1}\tau)^{1/2}$, so the factorisation estimate (6.7) is quantitative and remains valid for (slowly) diverging times.

Identifying the covariances with Corollary 6.2 concludes the proof of Proposition 2.1. □

7. Geometric estimates

In this section, we adapt previous results from [8] to the present context, allowing to prove the first parts of Lemma 4.11 and Lemma 4.12: we control the cluster functions by establishing the bound on the expectation (4.16) as well as the smallness estimate (4.20) for non minimal clusters.

In this section, since we will consider only one packet σ_i, we will drop the index i to lighten the notation, as well as the time dependence in the test functions. We thus consider a collection $(\bar\phi^{(j)})_{j\in\sigma}$ of single minimal cumulants originating from single particles at times θ_j as in (4.6). These cumulants are aggregated on $[\theta-r\delta,\theta]$ as in (4.5) to form the cumulant $\phi^{(\sigma)}$ at time $\theta-r\delta$ supported on forward clusters as in Definition 4.5.

We therefore have $|\sigma|$ blocks of cardinalities $\mathbf{M}^{r\delta}=(M_j^{r\delta})_{j\in\sigma}$ at time $\theta-r\delta$. We denote by

- $\mathbf{N}^r$ the number of annihilations in the different blocks on $\mathcal{I}_\delta := [\theta-r\delta,\theta-(r-1)\delta]$;
- $\mathbf{N}^{<r}$ the number of annihilations in the different blocks on $\mathcal{I}_\tau := [\theta-(r-1)\delta,\theta]$;

and define $N^r=\sum_{j\in\sigma}N_j^r$, $N^{<r}=\sum_{j\in\sigma}N_j^{<r}$, $M^{r\delta}=M+N^r+N^{<r}=\sum_{j\in\sigma}M_j^{r\delta}$.

To determine the forward dynamics on $[\theta-r\delta,\Theta]$, we also need to fix as in Definition 4.5

- $\mathbf{K}\in\{0,\dots,K_\gamma\}^{M^{r\delta}}$ counting for each particle the internal encounters without annihilation on $\mathcal{I}_\delta$ (recall that the number of such encounters is under control by construction of $\phi^{(\sigma)}$);
- $(\mathbf{S},\bar{\mathbf{S}})\in\{1,-1\}^{2(M^{r\delta}-|\sigma|)}$ prescribing the encounters with annihilation on $\mathcal{I}_\delta\cup\mathcal{I}_\tau\cup\mathcal{I}_\theta$, among which we denote by $(\mathbf{S}^r,\bar{\mathbf{S}}^r)\in\{1,-1\}^{2N^r}$ the signs prescribing the encounters on $\mathcal{I}_\delta$, $(\mathbf{S}^{<r},\bar{\mathbf{S}}^{<r})\in\{1,-1\}^{2N^{<r}}$ those prescribing the encounters on $\mathcal{I}_\tau$ and finally $(\mathbf{S}^\theta,\bar{\mathbf{S}}^\theta)\in\{1,-1\}^{2(M-|\sigma|)}$ the signs prescribing the encounters on $\mathcal{I}_\theta$;
- a partition ς of σ prescribing the packets at time $\theta-(r-1)\delta$, as well as a partition λ of these packets in forests, prescribing the external encounters on $\mathcal{I}_\delta$;
- $\mathcal{E}=(\mathcal{E}_i)$ where for each packet ς_i, $\mathcal{E}_i\in\{1,-1\}^{|\varsigma_i|-1}$ prescribing the external encounters on $\mathcal{I}_\tau$.

With this notation, plugging (4.4) into (4.5) provides

$$\phi^{(\sigma)}_{r\delta}(Z_{\mathbf{M}^{r\delta}}) = \Big(\prod_{j\in\sigma}\frac{M_j^{(r-1)\delta}!}{M_j^{r\delta}!}\Big)\sum_{\varsigma\in\mathcal{P}_\sigma}\mu_\varepsilon^{N^r+|\varsigma|-1}\sum_{\substack{\mathbf{K},\mathbf{S}^r,\bar{\mathbf{S}}^r,\lambda\\ \varsigma\hookrightarrow\lambda}}\operatorname{sign}(\mathbf{S}^r)\varphi_{\{\lambda_1,\dots,\lambda_{|\lambda|}\}}$$

$$\times\prod_{\ell=1}^{|\lambda|}\Big(\sharp_\delta\bigotimes_{\varsigma_i\subset\lambda_\ell}\bar\phi^{(\varsigma_i)}_{(r-1)\delta}\Big)\mathbf{1}_{\{Z_{\mathbf{M}^{r\delta}_{\lambda_\ell}}\in\mathcal{R}^{\lambda_\ell}_{\mathbf{K}_{\lambda_\ell},\mathbf{S}^r_{\lambda_\ell},\bar{\mathbf{S}}^r_{\lambda_\ell}}\}}$$

$$=\left(\prod_{j\in\sigma}\frac{m_j!}{M_j^{r\delta}!}\right)\mu_\varepsilon^{N^r+N^{<r}+|\sigma|-1}\sum_{\varsigma\in\mathcal{P}_\sigma}\sum_{\substack{\mathbf{K},\mathbf{S}^{\le r},\bar{\mathbf{S}}^{\le r},\mathcal{E},\lambda\\ \varsigma\hookrightarrow\lambda}}\operatorname{sign}(\mathcal{E})\operatorname{sign}(\mathbf{S}^{\le r})\varphi_{\{\lambda_1,\dots,\lambda_{|\lambda|}\}}$$

$$\times\mathbf{1}_{\{Z_{\mathbf{M}^{r\delta}}\text{ forward cluster associated with }(\mathbf{S}^{\le r},\bar{\mathbf{S}}^{\le r},\varsigma,\lambda,\mathbf{K},\mathcal{E})\}}\prod_{\ell=1}^{|\lambda|}\prod_{\varsigma_i\subset\lambda_\ell}\Big(\sharp_{(r-1)\delta}\bigotimes_{j\in\varsigma_i}\bar\phi^{(j)}\Big)$$

where we wrote $(\mathbf{S}^{\le r},\bar{\mathbf{S}}^{\le r})=(\mathbf{S}^{<r},\mathbf{S}^r,\bar{\mathbf{S}}^{<r},\bar{\mathbf{S}}^r)$, and $\operatorname{sign}(\mathbf{S}^{\le r})$ for the product of all the components of $\mathbf{S}^{\le r}$. On the other hand by the assumption (4.6) on the structure of the test functions, one has

$$\bar\phi^{(j)}(Z_{m_j})=\frac{\mu_\varepsilon^{m_j-1}}{m_j!}\sum_{\mathbf{S}^\theta_j,\bar{\mathbf{S}}^\theta_j}\operatorname{sign}(\mathbf{S}^\theta_j)\mathbf{1}_{Z_{m_j}\in\mathcal{R}^{\min}_{\mathbf{S}^\theta_j,\bar{\mathbf{S}}^\theta_j}}\big(\sharp_{\theta_j-\theta}h^{(j)}\big)\,, \tag{7.1}$$

Recall that we can assume without loss of generality that $\|h^{(j)}\|_\infty\le 1$ for all $j\in\sigma$. Since $\mathbf{1}_{\Upsilon^\varepsilon_N}\le 1$, inequalities (4.16) and (4.20) follow from their counterparts at equilibrium

$$\mathbb{E}^{\rm eq}_\varepsilon\Big[\Big|\phi^{(\sigma)}\Big|\Big]\le C_P(|\sigma|K_\gamma)^{M^{r\delta}}\,(C\Theta)^{M-|\sigma|}(C\delta)^{N^r}(C\tau)^{N^{<r}+|\sigma|-1}\,, \tag{7.2}$$

and

$$\mathbb{E}^{\rm eq}_\varepsilon\Big[\Big|\phi^{(\sigma),{\rm cyc}}\Big|\Big]\le\varepsilon\delta|\log\varepsilon|(\Theta|\log\varepsilon|)^{2d+4}\,C_P(|\sigma|K_\gamma)^{M^{r\delta}}(C\Theta)^{M-|\sigma|}(C\delta)^{(N^r-1)_+}(C\tau)^{(N^{<r}+|\sigma|-2)_+}\,, \tag{7.3}$$

which we now prove.

Estimating roughly the L^∞ norm of the cumulant $\varphi_{\{\lambda_1,\dots\lambda_{|\lambda|}\}}$ by $|\lambda|!$ (note that $|\lambda|!\le|\sigma|!$), and using that

$$\frac{M^{r\delta}!}{\prod_{j\in\sigma}M_j^{r\delta}!}\le|\sigma|^{M^{r\delta}}\,, \tag{7.4}$$

we infer that

$$|\phi^{(\sigma)}(Z_{\mathbf{M}^{r\delta}})|\le|\sigma|^{M^{r\delta}}\Big(\prod_{j\in\sigma}\|h^{(j)}\|_\infty\Big)\frac{\mu_\varepsilon^{M^{r\delta}-1}}{M^{r\delta}!}\sum_{\bar{\mathbf{S}},\mathbf{S},\mathbf{K},\varsigma,\lambda,\mathcal{E}}|\lambda|!\,\mathbf{1}_{\{Z_{\mathbf{M}^{r\delta}}\text{ forward cluster}\}}\,, \tag{7.5}$$

where the forward cluster in the indicator refers to the dynamics in $[\theta-r\delta,\Theta]$ constructed as described above and depends on the whole set of global parameters $\bar{\mathbf{S}},\mathbf{S},\mathbf{K},\varsigma,\lambda,\mathcal{E}$. Note that the cardinality of such parameters is bounded by $C_P(CK_\gamma)^{M^{r\delta}}$ for some pure constant C, and C_P depending only on P.

We now need to describe more precisely the support of $\phi^{(\sigma)}$, i.e. to extract from the cluster structure some "independent" geometric conditions. To show Inequality (7.2) on $\mathbb{E}^{\rm eq}_\varepsilon\Big[\Big|\phi^{(\sigma)}\Big|\Big]$

it is enough to prove that the size of the support is controlled by

(7.6)
$$\sup_{x_{M^{r\delta}}\in\mathbb{T}^d}\int \mathbf{1}_{\{Z_{\mathbf{M}^{r\delta}} \text{ forward cluster}\}}\mathcal{M}^{\otimes M^{r\delta}}(V_{M^{r\delta}})\,dX_{M^{r\delta}-1}dV_{M^{r\delta}} \leq C\frac{M^{r\delta}!}{\mu_\varepsilon^{M^{r\delta}-1}}(C\Theta)^{M-|\sigma|}(C\delta)^{N^r}(C\tau)^{N^{<r}+|\sigma|-1}$$

for some pure constant $C>0$, uniformly in $(\bar{\mathbf{S}},\mathbf{S},\mathbf{K},\varsigma,\lambda,\mathcal{E})$. Indeed, assuming (7.6), we deduce from (7.5) that

(7.7)
$$\begin{aligned}\mathbb{E}_\varepsilon^{\rm eq}\Big[\big|\phi^{(\sigma)}\big|\Big] &= \int G^{\varepsilon,\rm eq}_{M^{r\delta}}\,|\phi^{(\sigma)}|\,dZ_{M^{r\delta}}\\ &\leq |\sigma|^{M^{r\delta}}\frac{\mu_\varepsilon^{M^{r\delta}-1}}{M^{r\delta}!}\prod_{j\in\sigma}\|h^{(j)}\|_\infty\sum_{\bar{\mathbf{S}},\mathbf{S},\mathbf{K},\varsigma,\lambda,\mathcal{E}}|\lambda|!\int G^{\varepsilon,\rm eq}_{M^{r\delta}}\,\mathbf{1}_{\{Z_{\mathbf{M}^{r\delta}} \text{ forward cluster}\}}\,dZ_{M^{r\delta}}\\ &\leq |\sigma|^{M^{r\delta}}\frac{\mu_\varepsilon^{M^{r\delta}-1}}{M^{r\delta}!}\prod_{j\in\sigma}\|h^{(j)}\|_\infty\sum_{\bar{\mathbf{S}},\mathbf{S},\mathbf{K},\varsigma,\lambda,\mathcal{E}}|\lambda|!\int \mathcal{M}^{\otimes M^{r\delta}}\,\mathbf{1}_{\{Z_{\mathbf{M}^{r\delta}} \text{ forward cluster}\}}\,dZ_{M^{r\delta}}\\ &\leq C_P\,|\sigma|^{M^{r\delta}}\,(CK_\gamma)^{M^{r\delta}}(C\Theta)^{M-|\sigma|}(C\delta)^{N^r}(C\tau)^{N^{<r}+|\sigma|-1}\,,\end{aligned}$$

where the sums over the parameters $\bar{\mathbf{S}},\mathbf{S},\mathbf{K},\varsigma,\lambda,\mathcal{E}$ have been bounded by $(CK_\gamma)^{M^{r\delta}}$ and combinatorial factors depending only on P. Notice that, to compute the expectation, we have made use of the correlation functions of the invariant measure (1.4), which we recall:

$$G^{\varepsilon,\rm eq}_M(Z_M) := \frac{\mathcal{M}^{\otimes M}}{\mathcal{Z}^\varepsilon}\sum_{n=0}^\infty\frac{\mu_\varepsilon^n}{n!}\int_{(\mathbb{T}^d\times\mathbb{R}^d)^n}dz_{M+1}\dots dz_{M+n}\,\mathbf{1}_{\mathcal{D}^\varepsilon_{M+n}}(Z_{M+n})\,\mathcal{M}^{\otimes n}\,,\qquad M=1,2,\cdots$$

with $\mathcal{Z}^\varepsilon$ given by (1.5). Since $\mathbf{1}_{\mathcal{D}^\varepsilon_{M+n}}(Z_{M+n})\leq\mathbf{1}_{\mathcal{D}^\varepsilon_M}(Z_M)\mathbf{1}_{\mathcal{D}^\varepsilon_n}(z_{M+1},\dots,z_{M+n})$, these correlation functions satisfy the pointwise bound $G^{\varepsilon,\rm eq}_M(Z_M)\leq\mathcal{M}^{\otimes M}$, which justifies the third line of (7.7). This concludes the derivation of inequality (7.2) and therefore of (4.16).

The proof of (7.6) follows the strategy of Lemma 4.2 in [8]. We adapt it to this new framework.

Definition 7.1 (Forward tree)**.** *A forward tree* $T_< = (q_i,\bar q_i)_{1\leq i\leq M^{r\delta}-1}$ *is constructed by recording in increasing order of times, denoted by* t_i*, the encounters of the forward dynamics (recall Definition* 3.5*) which do not create any cycle (nor multiple edge). These encounters are said to be clustering.*

Note that $q_i,\bar q_i$ are generic notations for the indexes of the two particles involved in the i-th encounter, they can of course take several times the same value.

Even though the encounters can be of different nature, they lead to similar geometric constraints in the forward dynamics and they are coded in the same way in terms of the dual variables. The type of each link $(q_i,\bar q_i)$ (with or without annihilation, with or without scattering) is encoded in the set of parameters $\varsigma,\lambda,\mathbf{S},\bar{\mathbf{S}},\mathbf{K},\mathcal{E}$. Then,

$$\sum_{\varsigma,\lambda,\mathbf{S},\bar{\mathbf{S}},\mathbf{K},\mathcal{E}}\mathbf{1}_{\{Z_{\mathbf{M}^{r\delta}} \text{ forward cluster}\}} = \sum_{\varsigma,\lambda,\mathbf{S},\bar{\mathbf{S}},\mathbf{K},\mathcal{E}}\;\sum_{T_<\in\mathcal{T}^<_{M^{r\delta}}}\mathbf{1}_{\{Z_{\mathbf{M}^{r\delta}}\in\mathcal{R}^{\rm comp}_{T_<}\}},$$

where $\mathcal{R}^{\rm comp}_{T_<}$ is the set of configurations compatible with $(\varsigma,\lambda,\mathbf{S},\bar{\mathbf{S}},\mathbf{K},\mathcal{E},T_<)$, and $\mathcal{T}^<_{M^{r\delta}}$ stands for the set of all ordered trees with $M^{r\delta}-1$ edges. The above sum over ordered trees corresponds to a partition, meaning that for any given $Z_{\mathbf{M}^{r\delta}}$, at most one term is non zero. Note, for future reference (see Section 8 below), that (7.5) implies

$$|\phi^{(\sigma)}(Z_{\mathbf{M}^{r\delta}})| \leq C_P\, |\sigma|^{M^{r\delta}} \Big(\prod_{j\in\sigma} \|h^{(j)}\|_\infty \Big) \frac{\mu_\varepsilon^{M^{r\delta}-1}}{M^{r\delta}!} \sum_{\bar{\mathbf{S}},\mathbf{S},\mathbf{K},\varsigma,\lambda,\mathcal{E}} \ \sum_{T_\prec \in \mathcal{T}^\prec_{M^{r\delta}}} \mathbf{1}_{\{Z_{\mathbf{M}^{r\delta}} \in \mathcal{R}^{\mathrm{comp}}_{T_\prec}\}} \,. \tag{7.8}$$

We then need to integrate over the variables $Z_{M^{r\delta}}$ restricted to the set $\mathcal{R}^{\mathrm{comp}}_{T_\prec}$. This set is a collection of constraints which are not independent one from the other. However, exploiting the ordering of edges in $T_\prec$, we can identify a sequence of "independent" variables (see Definition 7.2 below). The basic idea is that, when we follow the dynamics forward in time, each new edge corresponds to an encounter involving at least one *new* variable. A convenient way to proceed is by using as new variables the relative positions between particles realizing encounters, keeping all velocities fixed. More precisely, given an admissible tree $T_\prec$, let us define the relative positions at time $\theta - r\delta$

$$\hat{x}_i := x_{q_i} - x_{\bar{q}_i} \,. \tag{7.9}$$

Given the relative positions $(\hat{x}_s)_{s<i}$ and the velocities $V_{M^{r\delta}}$, we fix a forward flow with clustering encounters at times $t_1 < \cdots < t_{i-1}$. By construction, q_i and $\bar{q}_i$ belong to two forward pseudo-trajectories that have not interacted yet. In other words, q_i and $\bar{q}_i$ do not belong to the same connected component in the graph $G_{i-1} := (q_j, \bar{q}_j)_{1\leq j\leq i-1}$. Inside each connected component, relative positions are fixed by the previous constraints, and one degree of freedom remains. Therefore we can vary $\hat{x}_i$ so that an encounter at time t_i occurs between q_i and $\bar{q}_i$ (moving rigidly the corresponding connected components). This encounter condition defines recursively the sets $\mathcal{B}_{T_\prec,i}(\hat{x}_1, \dots, \hat{x}_{i-1}, V_{M^{r\delta}})$ prescribing the constraints on $\hat{x}_i$.

Definition 7.2. *We say that the sets $(\mathcal{B}_{T_\prec,i})_{i\leq M^{r\delta}-1}$ are* <u>*sequentially independent*</u> *if for all i the set $\mathcal{B}_{T_\prec,i}$ is defined by constraints depending only on $\hat{x}_1, \dots, \hat{x}_{i-1}, V_{M^{r\delta}}$.*

Suppose that the time t_i belongs to the set $\mathcal{I} \in \{\mathcal{I}_\delta, \mathcal{I}_\tau, \mathcal{I}_\theta\}$. If the particles q_i and $\bar{q}_i$ move in straight lines, then the measure of $\mathcal{B}_{T_\prec,i}$ can be estimated by

$$|\mathcal{B}_{T_\prec,i}| \leq \frac{C}{\mu_\varepsilon} |v^\varepsilon_{q_i}(t^+_{i-1}) - v^\varepsilon_{\bar{q}_i}(t^+_{i-1})| \int \mathbf{1}_{t_i\in\mathcal{I}}\, \mathbf{1}_{t_i\geq t_{i-1}} dt_i \,. \tag{7.10}$$

Thus, by a Cauchy-Schwarz inequality there holds

$$\sum_{q_i,\bar{q}_i} |\mathcal{B}_{T_\prec,i}| \leq \frac{C}{\mu_\varepsilon} \big(V^2_{M^{r\delta}} + M^{r\delta} \big) M^{r\delta} \int \mathbf{1}_{t_i\in\mathcal{I}}\, \mathbf{1}_{t_i\geq t_{i-1}} dt_i \,. \tag{7.11}$$

Note however that, by definition of the forward tree, particles $q_i, \bar{q}_i$ may encounter on $[t_{i-1}, t_i]$ with other particles from their respective connected component in the graph G_{i-1}. In this case the particle trajectories are piecewise affine but a bound similar to (7.10) is obtained by summing over all the portions of the trajectory and the upper bound (7.11) still holds (see Section 8.1 of [7] for details).

At this point we proceed as in [8] (the proof of Lemma 4.1 therein contains the same computation that follows, except for the time sampling condition $t_i \in \mathcal{I}$ appearing in (7.10)-(7.11)).

We first apply the change of variables

$$X_{M^{r\delta}-1} \longrightarrow \hat{X}_{M^{r\delta}-1}$$

and observe that, for any fixed $x_{M^{r\delta}}$, this is a map of translations with $dX_{M^{r\delta}-1} = d\hat{X}_{M^{r\delta}-1}$. The constraints, imposed by the $M^{r\delta} - 1$ encounters, can be evaluated one after the other following the order prescribed above. Hence by Fubini's theorem

$$
\begin{aligned}
\sum_{T_\prec \in \mathcal{T}^\prec_{M^{r\delta}}} \int d\hat X_{M^{r\delta}-1} \prod_{i=1}^{M^{r\delta}-1} \mathbf{1}_{\mathcal{B}_{T_\prec,i}} &\le \sum_{T_\prec \in \mathcal{T}^\prec_{M^{r\delta}}} \int d\hat x_1 \mathbf{1}_{\mathcal{B}_{T_\prec,1}} \int d\hat x_2 \cdots \int d\hat x_{M^{r\delta}-1} \mathbf{1}_{\mathcal{B}_{T_\prec,M^{r\delta}-1}} \\
&\le \Big(\frac{C}{\mu_\varepsilon}\Big)^{M^{r\delta}-1} \big(V^2_{M^{r\delta}} + M^{r\delta}\big)^{M^{r\delta}-1} (M^{r\delta})^{M^{r\delta}-1} \int dt_1 \dots dt_{M^{r\delta}-1} \mathbf{1}_{\rm samp},
\end{aligned}
\tag{7.12}
$$

where $\mathbf{1}_{\rm samp}$ is the constraint on the encounter times respecting the sampling. Retaining only the information on the number of encounters in each time interval, we get by integrating over these ordered times an upper bound of the form

$$
\frac{\delta^{N^r}}{N^r!} \frac{\tau^{N^{<r}+|\sigma|-1}}{(N^{<r}+|\sigma|-1)!} \frac{\Theta^{M^{r\delta}}}{M!} \le \frac{3^{M^{r\delta}-1}}{(M^{r\delta}-1)!} \delta^{N^r} \tau^{N^{<r}+|\sigma|-1} \Theta^{M^{r\delta}} ,
\tag{7.13}
$$

where we used the inequality

$$
\frac{(M^{r\delta}-1)!}{N^r!\,(N^{<r}+|\sigma|-1)!\,M!} \le 3^{M^{r\delta}-1}.
$$

Up to a factor $C^{M^{r\delta}}$, the factorial $(M^{r\delta}-1)!$ compensates the factor $(M^{r\delta})^{M^{r\delta}}$ in (7.12). Furthermore, in any dimension, for any R, N

$$
\sup_V \Big\{ \exp\big(-\frac{1}{8}|V|^2\big) \, (|V|^2+R)^N \Big\} \le C^N e^R N^N.
\tag{7.14}
$$

After integrating the velocities with respect to the measure $\mathcal{M}^{\otimes M^{r\delta}}$, we deduce from the previous inequality that the term $\big(V^2_{M^{r\delta}} + M^{r\delta}\big)^{M^{r\delta}}$ gives another factor of order $(M^{r\delta})^{M^{r\delta}}$ which leads, up to a factor $C^{M^{r\delta}}$, to the term $M^{r\delta}!$ in (7.6). This completes the proof of (7.6) and therefore of the inequality (7.2). □

We turn now to the proof of Inequality (7.3). We proceed as for (7.6) and our purpose is to show that

$$
\begin{aligned}
&\sup_{x_{M^{r\delta}} \in \mathbb{T}^d} \int \mathbf{1}_{\{Z_{\mathbf{M}^{r\delta}} \text{ non minimal forward cluster}\}} \mathcal{M}^{\otimes M^{r\delta}} (V_{M^{r\delta}})\, dX_{M^{r\delta}-1} dV_{M^{r\delta}} \\
&\qquad \le C \frac{M^{r\delta}!}{\mu_\varepsilon^{M^{r\delta}-1}} \varepsilon\delta |\log\varepsilon| (\Theta|\log\varepsilon|)^{2d+4} \, (C\Theta)^{M-|\sigma|} (C\delta)^{(N^r-1)_+} (C\tau)^{(N^{<r}+|\sigma|-2)_+} ,
\end{aligned}
\tag{7.15}
$$

for some pure constant $C>0$, uniformly in $(\bar{\mathbf{S}}, \mathbf{S}, \mathbf{K}, \varsigma, \lambda, \mathcal{E})$.

By construction, there is at least one encounter violating the minimality, and therefore at least one clustering in $\mathcal{I}_\delta$. The support estimate (7.6) can be refined by using that the graph encoding all encounters has strictly more than $(M^{r\delta}-1)$ edges (i.e. it strictly contains the forward tree $T_\prec$), which means that there will be at least one cycle in this graph. This reinforces one of the geometric conditions on the sets $\mathcal{B}_{T_\prec,i}$ (see (7.10)), leading to the following estimate

$$
\begin{aligned}
&\sum_{T_\prec \in \mathcal{T}^\prec_{M^{r\delta}}} \int \mathcal{M}^{\otimes M^{r\delta}} (V_{M^{r\delta}})\, dV_{M^{r\delta}} \int d\hat X_{M^{r\delta}-1} \mathbf{1}_{\rm cycle} \prod_{i=1}^{M^{r\delta}-1} \mathbf{1}_{\mathcal{B}_{T_\prec,i}} \\
&\qquad \le \Big(\frac{C}{\mu_\varepsilon}\Big)^{M^{r\delta}-1} \varepsilon|\log\varepsilon| (\mathbb{V}\Theta)^{2d+4} (M^{r\delta})^{2+2M^{r\delta}} \int_{\theta-r\delta}^{\theta-(r-1)\delta} dt_1 \cdots \int_{t_{M^{r\delta}-2}}^{\Theta} dt_{M^{r\delta}-1} \mathbf{1}_{\rm samp} .
\end{aligned}
\tag{7.16}
$$

We refer to [8] for the proof of this estimate (see Eq. (5.12) in [8], which is derived under the same assumptions on the sets $\mathcal{B}_{T_\prec,i}$ except for the minor difference in the time sampling condition $t_i \in \mathcal{I}$). We recall the choices (2.9) for $\mathbb{V}$ and Θ. Then integrating over the simplex in time represented by $\mathbf{1}_{\rm samp}$ leads to (7.15), where the contribution δ comes from the first edge of the forward tree. Since we did not track the nature of this edge, the terms $(N^r-1)_+$ and $(N^{<r}+|\sigma|-2)_+$ have been adjusted to take into account all the possibilities.

This concludes the proof of (7.15), hence of (7.3). □

8. Expectation and variance of $\circledast$-products

The aim of this section is to control the expectation and variance of $\circledast$-products in order to complete the proofs of Lemma 4.11 and Lemma 4.12.

Without loss of generality, we suppose from now on that the sets σ_i are indexed by $i \in \{1,\dots,q\}$.

We start by proving the estimates on the equilibrium measures (in Paragraphs 8.1 and 8.2), and then show in Paragraph 8.3 how to conclude to Estimates (4.17), (4.18) and (4.21).

8.1. **Expectation of centered $\circledast$-products.**

In this section, we prove the following inequality:

$$\left|\mathbb{E}_\varepsilon^{\rm eq}\left[\bigcircledast_{i=1}^q \zeta^{\varepsilon,{\rm eq}}_{M_i^{r\delta}}\big(\phi^{(\sigma_i)}\big)\right]\right| \le C_q\varepsilon \prod_{i=1}^q (|\sigma_i|K_\gamma)^{M_i^{r\delta}} \times \Big((C\Theta)^{M_i-|\sigma_i|}(C\delta)^{N_i^r}(C\tau)^{N_i^{<r}+|\sigma_i|-1}\Big)\,. \tag{8.1}$$

Inequality (8.1) follows from the control on the structure of the test functions $\phi^{(\sigma_i)}$ given by Eq. (7.5). Notice that in the latter estimate, the function on the right-hand side is invariant by translations (simultaneous of the $M_i^{r\delta}$ particles in the space $\mathbb{T}^d$), and bounded in an L^1–weighted norm (by (7.6)). Using these two ingredients we shall now prove that

$$\begin{aligned}\left|\mathbb{E}_\varepsilon^{\rm eq}\left[\bigcircledast_{i=1}^q \zeta^{\varepsilon,{\rm eq}}_{M_i^{r\delta}}\big(\phi^{(\sigma_i)}\big)\right]\right| &\le C_q\,\varepsilon\prod_{i=1}^q \frac{(C|\sigma_i|\mu_\varepsilon)^{M_i^{r\delta}-1}}{M_i^{r\delta}!}\\ &\quad\times\sum_{\bar{\mathbf{S}},\mathbf{S},\mathbf{K},\varsigma,\lambda,\mathcal{E}}\ \sup_{x_{M_i^{r\delta}}\in\mathbb{T}^d}\int \mathbf{1}_{\{Z_{\mathbf{M}_i^{r\delta}}\text{ forward cluster}\}}\mathcal{M}^{\otimes M_i^{r\delta}}dX_{M_i^{r\delta}-1}dV_{M_i^{r\delta}}\end{aligned} \tag{8.2}$$

where the sum is taken over the collection $\big(\varsigma,\lambda,\mathbf{K},\mathbf{S},\bar{\mathbf{S}},\mathcal{E}\big)$ parametrising the clusters and $\mathbf{M}_i^{r\delta}$ codes the cardinalities of the blocks in σ_i (as in (7.5)). The small factor ε will be obtained by a cluster expansion of the invariant measure, tracing the small correlations between different fluctuation fields. Eq. (8.1) follows then from (8.2) and (7.6).

In order to establish (8.2), we first use the definition of $\circledast$-product given by (2.7) and write

$$\bigcircledast_{i=1}^q\left(\frac{1}{\mu_\varepsilon^{M_i^{r\delta}}}\sum\phi^{(\sigma_i)}-\mathbb{E}_\varepsilon^{\rm eq}[\phi^{(\sigma_i)}]\right)=\sum_{A\subset\{1,\dots,q\}}\pi^\varepsilon_{M_A^{r\delta}}\left(\Phi_{\mathbf{M}_A^{r\delta}}\right)\prod_{j\in A^c}\mathbb{E}_\varepsilon^{\rm eq}[-\phi^{(\sigma_j)}]\,,$$

with $M_A^{r\delta}=\sum_{j\in A}M_j^{r\delta}$, $\mathbf{M}_A^{r\delta}=(M_j^{r\delta})_{j\in A}$ and $\Phi_{\mathbf{M}_A^{r\delta}}:=\otimes_{j\in A}\phi^{(\sigma_j)}$ and where A^c is the complement of A in $\{1,\dots,q\}$. Then,

$$\begin{aligned}\mathbb{E}_\varepsilon^{\rm eq}\left[\bigcircledast_{i=1}^q\zeta^{\varepsilon,{\rm eq}}_{M_i^{r\delta}}\left(\phi^{(\sigma_i)}\right)\right]&=\frac{1}{\mathcal{Z}^\varepsilon}\mu_\varepsilon^{q/2}\sum_{A\subset\{1,\dots,q\}}\prod_{j\in A^c}\mathbb{E}_\varepsilon^{\rm eq}[-\phi^{(\sigma_j)}]\\ &\times\sum_{p\ge0}\frac{\mu_\varepsilon^p}{p!}\int dZ_{M_A^{r\delta}}d\bar Z_p\,\Phi_{\mathbf{M}_A^{r\delta}}(Z_{M_A^{r\delta}})\mathbf{1}_{\mathcal{D}^\varepsilon_{M_A^{r\delta}+p}}(Z_{M_A^{r\delta}},\bar Z_p)\mathcal{M}^{\otimes(M_A^{r\delta}+p)}(V_{M_A^{r\delta}},\bar V_p)\,.\end{aligned}$$

We decompose $Z_{M_A^{r\delta}}$ in $|A|$ subconfigurations $(Z^{(i)}_{M_i^{r\delta}})_{i\in A}$ (each one containing possibly several blocks). We then use a cluster expansion of the exclusion $\mathbf{1}_{\mathcal{D}^\varepsilon_{M_A^{r\delta}+p}}$, representing each $Z^{(i)}_{M_i^{r\delta}}$ by one vertex, and $(\bar z_j)_{1\leq j\leq p}$ as p separate vertices. We denote by $d(y,y^*)$ the minimum relative distance (in position) between elements $y,y^*\in S_{|A|+p} := \{(Z^{(i)}_{M_i^{r\delta}})_{i\in A}, \bar z_1,\dots \bar z_p\}$.

We define the cumulants

$$\varphi(Z_{M_A^{r\delta}}, \bar z_1,\dots \bar z_p) := \sum_{G\in\mathcal{C}_{S_{|A|+p}}} \prod_{\{y,y^*\}\in E(G)} (-\mathbf{1}_{d(y,y^*)\leq\varepsilon})\,,$$

and more generally for any subpart of $Y\subset S_{|A|+p}$

$$\varphi(Y) = \sum_{G\in\mathcal{C}_Y} \prod_{\{y,y^*\}\in E(G)} (-\mathbf{1}_{d(y,y^*)\leq\varepsilon})\,, \tag{8.3}$$

denoting by $\mathcal{C}_Y$ the connected graphs with vertices in Y and by $E(G)$ the edges of the graph G. Notice that this definition is analogous to the one used to treat the dynamical correlations in (3.31), but now the exclusion is static

$$\prod_{\substack{y\neq y^*\\ y,y^*\in Y}} \mathbf{1}_{d(y,y^*)>\varepsilon} = \sum_{G\in\mathcal{G}_Y} \prod_{\{y,y^*\}\in E(G)} (-\mathbf{1}_{d(y,y^*)\leq\varepsilon}) = \sum_{\rho\in\mathcal{P}_Y} \prod_{q=1}^{|\rho|} \varphi(\rho_q)\,,$$

where $\varphi(\rho_q)$ is defined by (8.3).

We then have the following cumulant expansion

$$\begin{aligned}\mathbf{1}_{\mathcal{D}^\varepsilon_{M_A^{r\delta}+p}}\Big(Z_{M_A^{r\delta}},\bar Z_p\Big) &= \Big(\prod_{i\in A}\mathbf{1}_{\mathcal{D}^\varepsilon_{M_i^{r\delta}}}(Z^{(i)}_{M_i^{r\delta}})\Big)\Bigg(\prod_{\substack{y,y^*\in S_{|A|+p}\\ y\neq y^*}} \mathbf{1}_{d(y,y^*)>\varepsilon}\Bigg)\Big(Z_{M_A^{r\delta}},\bar Z_p\Big)\\ &= \Big(\prod_{i\in A}\mathbf{1}_{\mathcal{D}^\varepsilon_{M_i^{r\delta}}}(Z^{(i)}_{M_i^{r\delta}})\Big)\sum_{\bar\sigma_0\subset\{1,\dots,p\}}\mathbf{1}_{\mathcal{D}^\varepsilon_{|\bar\sigma_0|}}(\bar Z_{\bar\sigma_0})\sum_{\eta\in\mathcal{P}_A}\sum_{\substack{\bar\sigma_1,\dots,\bar\sigma_{|\eta|}\\ \cup_{i=0}^{|\eta|}\bar\sigma_i=\{1,\dots,p\}\\ \bar\sigma_i\cap\bar\sigma_{i'}=\emptyset, i\neq i'}}\prod_{i=1}^{|\eta|}\varphi(Z_{\eta_i},\bar Z_{\bar\sigma_i}),\end{aligned}$$

where $\mathcal{P}_A$ is the set of partitions of A, and $Z_{\eta_i} = \Big(Z^{(j)}_{M_j^{r\delta}}\Big)_{j\in\eta_i}$. Note that the $\bar\sigma_i$ may be empty (in particular all $\bar\sigma_i$ are empty if $|\bar\sigma_0| = p$). Using the symmetry in the exchange of particle labels, we get, denoting $n_i := |\bar\sigma_i|$,

$$\binom{p}{n_0}\binom{p-n_0}{n_1}\dots\binom{p-n_0-\dots-n_{|\eta|-1}}{n_{|\eta|}} = \frac{p!}{n_0!\,\dots\,n_{|\eta|}!}$$

choices for the repartition of the background particles $\bar Z_p$.

Then, using the definition of the partition function $\mathcal{Z}^\varepsilon$, we obtain

$$\begin{aligned}&\frac{1}{\mathcal{Z}^\varepsilon}\sum_{p\geq0}\frac{\mu_\varepsilon^p}{p!}\int d\bar Z_p\,\mathbf{1}_{\mathcal{D}^\varepsilon_{M_A^{r\delta}+p}}\Big(Z_{M_A^{r\delta}},\bar Z_p\Big)\mathcal{M}^{\otimes p}(\bar V_p)\\ &\quad= \frac{1}{\mathcal{Z}^\varepsilon}\Big(\prod_{i\in A}\mathbf{1}_{\mathcal{D}^\varepsilon_{M_i^{r\delta}}}(Z^{(i)}_{M_i^{r\delta}})\Big)\sum_{\eta\in\mathcal{P}_A}\sum_{p\geq0}\sum_{\substack{n_0,\dots,n_{|\eta|}\geq0\\ \sum n_i=p}}\Big(\frac{\mu_\varepsilon^{n_0}}{n_0!}\int\mathcal{M}^{\otimes n_0}\mathbf{1}_{\mathcal{D}^\varepsilon_{n_0}}\Big(\bar Z_{n_0}\Big)d\bar Z_{n_0}\Big)\\ &\qquad\qquad\times\prod_{i=1}^{|\eta|}\frac{\mu_\varepsilon^{n_i}}{n_i!}\int\mathcal{M}^{\otimes n_i}\varphi(Z_{\eta_i},\bar Z_{n_i})d\bar Z_{n_i}\end{aligned}$$

$$= \Big(\prod_{i\in A}\mathbf{1}_{\mathcal{D}^\varepsilon_{M_i^{r\delta}}}(Z^{(i)}_{M_i^{r\delta}})\Big)\sum_{\eta\in\mathcal{P}_A}\prod_{\ell=1}^{|\eta|}\Bigg(\sum_{n_\ell\geq0}\frac{\mu_\varepsilon^{n_\ell}}{n_\ell!}\int\mathcal{M}^{\otimes n_\ell}\varphi(Z_{\eta_\ell},\bar Z_{n_\ell})d\bar Z_{n_\ell}\Bigg)\,. \tag{8.4}$$

By Fubini's equality, we finally get that

$$\mathbb{E}_\varepsilon^{\rm eq}\Big[\bigotimes_{i=1}^q \zeta^{\varepsilon,{\rm eq}}_{M_i^{r\delta}}\left(\phi^{(\sigma_i)}\right)\Big] = \mu_\varepsilon^{q/2}\sum_{A\subset\{1,\dots,q\}}\Big(\prod_{j\in A^c}\mathbb{E}_\varepsilon^{\rm eq}[-\phi^{(\sigma_j)}]\Big)$$
$$\times\sum_{\eta\in\mathcal{P}_A}\prod_{\ell=1}^{|\eta|}\Big[\sum_{n_\ell\geq 0}\frac{\mu_\varepsilon^{n_\ell}}{n_\ell!}\int \mathcal{M}^{\otimes\left(M^{r\delta}_{\eta_\ell}+n_\ell\right)}\varphi(Z_{\eta_\ell},\bar Z_{n_\ell})\Big(\prod_{j\in\eta_\ell}\phi^{(\sigma_j)}\mathbf{1}_{\mathcal{D}^\varepsilon_{M_j^{r\delta}}}\left(Z^{(j)}_{M_j^{r\delta}}\right)\Big)d\bar Z_{n_\ell}dZ_{\eta_\ell}\Big].$$

By definition, if one part η_ℓ is a singleton, say $\{j\}$, we find that the corresponding factor of the product is (using again Eq. (8.4) with $A=\{j\}$)

$$\sum_{n_\ell\geq 0}\frac{\mu_\varepsilon^{n_\ell}}{n_\ell!}\int \mathcal{M}^{\otimes\left(M^{r\delta}_{j}+n_\ell\right)}\varphi(Z_{M_j^{r\delta}},\bar Z_{n_\ell})\,\phi^{(\sigma_j)}\mathbf{1}_{\mathcal{D}^\varepsilon_{M_j^{r\delta}}}\left(Z_{M_j^{r\delta}}\right)d\bar Z_{n_\ell}dZ_{M_j^{r\delta}} = \mathbb{E}_\varepsilon^{\rm eq}\big[\phi^{(\sigma_j)}\big]. \tag{8.5}$$

We will therefore split any partition η of A in a union of singletons $\{j\}$ for $j\in A\smallsetminus B$, and a partition $\tilde\eta$ of B with no singleton. In particular, we have that $\tilde\eta$ has a number of parts $|\tilde\eta|\leq\frac12|B|$. Thus absorbing the sum over singletons as in (3.39), we get that

$$\begin{aligned}\mathbb{E}_\varepsilon^{\rm eq}\Big[\bigotimes_{i=1}^q \zeta^{\varepsilon,{\rm eq}}_{M_i^{r\delta}}\left(\phi^{(\sigma_i)}\right)\Big] &= \mu_\varepsilon^{q/2}\sum_{B\subset\{1,\dots,q\}}\sum_{A\subset B^c}(-1)^{|B^c|-|A|}\Big(\prod_{j\in B^c}\mathbb{E}_\varepsilon^{\rm eq}[\phi^{(\sigma_j)}]\Big)\\
&\times\sum_{\eta\in(\mathcal{P}_B)^*}\prod_{\ell=1}^{|\eta|}\Big[\sum_{n_\ell\geq 0}\frac{\mu_\varepsilon^{n_\ell}}{n_\ell!}\int \mathcal{M}^{\otimes\left(M^{r\delta}_{\eta_\ell}+n_\ell\right)}\varphi(Z_{\eta_\ell},\bar Z_{n_\ell})\Big(\prod_{j\in\eta_\ell}\phi^{(\sigma_j)}\mathbf{1}_{\mathcal{D}^\varepsilon_{M_j^{r\delta}}}\left(Z^{(j)}_{M_j^{r\delta}}\right)\Big)d\bar Z_{n_\ell}dZ_{\eta_\ell}\Big]\\
&= \mu_\varepsilon^{q/2}\sum_{\eta\in\left(\mathcal{P}_{\{1,\dots,q\}}\right)^*}\prod_{\ell=1}^{|\eta|}\Big[\sum_{n_\ell\geq 0}\frac{\mu_\varepsilon^{n_\ell}}{n_\ell!}\int \mathcal{M}^{\otimes\left(M^{r\delta}_{\eta_\ell}+n_\ell\right)}\varphi(Z_{\eta_\ell},\bar Z_{n_\ell})\Big(\prod_{j\in\eta_\ell}\phi^{(\sigma_j)}\mathbf{1}_{\mathcal{D}^\varepsilon_{M_j^{r\delta}}}\left(Z^{(j)}_{M_j^{r\delta}}\right)\Big)d\bar Z_{n_\ell}dZ_{\eta_\ell}\Big]\end{aligned}\tag{8.6}$$

where $(\mathcal{P}_A)^*$ stands for the partitions without singletons of a set A.

Recall that the cumulants defined by (8.3) can be controlled by the tree inequality (see e.g. [26, 27])

$$|\varphi(Y)|\leq\sum_{T\in\mathcal{T}_Y}\prod_{\{y,y^*\}\in E(T)}\mathbf{1}_{d(y,y^*)\leq\varepsilon}\,,\tag{8.7}$$

where $\mathcal{T}_Y$ is the set of minimally connected graphs (trees) with vertices in Y. Thus inside each connected component η_ℓ, a tree connects the $|\eta_\ell|$ vertices $Z^{(j)}_{M_j^{r\delta}}$ and the n_ℓ background particles (where each edge corresponds to the distance being smaller than ε).

For a given tree T, let $d_1,\dots,d_{|\eta_\ell|+n_\ell}$ be the degrees of the graph (number of edges per vertex). Integrating with respect to $\bar Z_{n_\ell},Z_{\eta_\ell}$ leads to

$$\begin{aligned}&\int \mathcal{M}^{\otimes\left(M^{r\delta}_{\eta_\ell}+n_\ell\right)}\prod_{\{y,y^*\}\in E(T)}\mathbf{1}_{d(y,y^*)\leq\varepsilon}\Big(\prod_{j\in\eta_\ell}\left|\phi^{(\sigma_j)}\right|\mathbf{1}_{\mathcal{D}^\varepsilon_{M_j^{r\delta}}}\left(Z^{(j)}_{M_j^{r\delta}}\right)\Big)d\bar Z_{n_\ell}dZ_{\eta_\ell}\\
&\quad\leq C_q\left(C'\varepsilon^d\right)^{|\eta_\ell|+n_\ell-1}\prod_{j\in\eta_\ell}(M_j^{r\delta})^{d_j}\\
&\qquad\times\prod_{j\in\eta_\ell}\frac{(C|\sigma_j|\mu_\varepsilon)^{M_j^{r\delta}-1}}{M_j^{r\delta}!}\sum_{\bar{\mathbf{S}},\mathbf{S},\mathbf{K},\varsigma,\lambda,\mathcal{E}}\sup_{x_{M_j^{r\delta}}\in\mathbb{T}^d}\int\mathbf{1}_{\{Z_{\mathbf{M}_j^{r\delta}}\text{ forward cluster}\}}\mathcal{M}^{\otimes M_j^{r\delta}}dX_{M_j^{r\delta}-1}dV_{M_j^{r\delta}}\,,\end{aligned}\tag{8.8}$$

where the constant $C'>0$ depends only on d.

To justify (8.8), we first notice that for each j in η_ℓ, the subconfiguration $X^{(j)}_{M_j^{r\delta}}$ covers a volume of order $M_j^{r\delta}\varepsilon^d$. Thus overlapping two such configurations indexed by j,j' leads to a factor $M_j^{r\delta} M_{j'}^{r\delta} \varepsilon^d$, and overlapping the subconfiguration j and a single particle to a factor $M_j^{r\delta}\varepsilon^d$, while overlapping two particles leads to a simple factor ε^d. Therefore, overall each edge of the tree brings a factor ε^d, and each subconfiguration $X^{(j)}_{M_j^{r\delta}}$ brings a factor $M_j^{r\delta}$ per edge attached to the vertex j of the tree. Furthermore the integral in the last line is a consequence of (7.5) and of the translation invariance of the indicator functions of forward clusters. Indeed the tree T encoding the static overlaps imposes a geometrical constraint only on the position of a single particle in each $X^{(j)}_{M_j^{r\delta}}$, say $x^{(j)}_{M_j^{r\delta}}$. Therefore by Fubini, we can first fix the variables $\left(\hat X^{(j)}_{M_j^{r\delta}-1}, V^{(j)}_{M_j^{r\delta}}\right)$ defined by (7.9) in such a way that the dynamical constraints are satisfied, integrate the variables $\left(\bar X_{n_\ell}, \left(x^{(j)}_{M_j^{r\delta}}\right)_j\right)$ according to the tree structure, and then integrate with respect to $\left(\hat X^{(j)}_{M_j^{r\delta}-1}, V^{(j)}_{M_j^{r\delta}}\right)$ for all j. This leads to (8.8).

There are $(n-2)!/\prod_j (d_j-1)!$ trees of size n with specified vertex degrees (see e.g. Lemma 2.4.1 in [7]), so that summing (8.8) over all trees leads to the combinatorial factor

$$(|\eta_\ell| + n_\ell - 2)! \sum_{d_1,\cdots,d_{|\eta_\ell|+n_\ell}\geq 1} \left(\prod_{j\in\eta_\ell} \frac{(M_j^{r\delta})^{d_j}}{(d_j-1)!}\right)\left(\prod_{j\notin\eta_\ell} \frac{1}{(d_j-1)!}\right) \leq |\eta_\ell|! n_\ell! 2^{|\eta_\ell|+n_\ell} \left(\prod_{j\in\eta_\ell} M_j^{r\delta} e^{M_j^{r\delta}}\right) e^{n_\ell} \,.$$

Thus, enlarging the constants C_q, C, C' from line to line and recalling that $M^{r\delta} = \sum_{i=1}^q M_i^{r\delta}$, we deduce that

(8.9)

$$\begin{aligned}\left|\mathbb{E}_\varepsilon^{\rm eq}\left[\bigotimes_{i=1}^q \zeta^{\varepsilon,{\rm eq}}_{M_i^{r\delta}}\left(\phi^{(\sigma_i)}\right)\right]\right| &\leq C_q\, C'^{M^{r\delta}} \mu_\varepsilon^{q/2} \sum_{\eta\in\left(\mathcal{P}_{\{1,\dots,q\}}\right)^*} \prod_{\ell=1}^{|\eta|} (C'\varepsilon^d)^{|\eta_\ell|-1} \sum_{n_\ell\geq 0} (C'\mu_\varepsilon\varepsilon^d)^{n_\ell} \\ &\quad\times \prod_{i=1}^q \frac{(C|\sigma_i|\mu_\varepsilon)^{M_i^{r\delta}-1}}{M_i^{r\delta}!} \sum_{\bar{\mathbf{S}},\mathbf{S},\mathbf{K},\varsigma,\lambda,\mathcal{E}} \sup_{x_{M_i^{r\delta}}\in\mathbb{T}^d} \int \mathbf{1}_{\{Z_{M_i^{r\delta}} \text{ forward cluster}\}} \mathcal{M}^{\otimes M_i^{r\delta}} dX_{M_i^{r\delta}-1} dV_{M_i^{r\delta}} \\ &\leq C_q\, C'^{M^{r\delta}} \left(\mu_\varepsilon\varepsilon^d\right)^{q/2} \\ &\quad\times \prod_{i=1}^q \frac{(C|\sigma_i|\mu_\varepsilon)^{M_i^{r\delta}-1}}{M_i^{r\delta}!} \sum_{\bar{\mathbf{S}},\mathbf{S},\mathbf{K},\varsigma,\lambda,\mathcal{E}} \sup_{x_{M_i^{r\delta}}\in\mathbb{T}^d} \int \mathbf{1}_{\{Z_{M_i^{r\delta}} \text{ forward cluster}\}} \mathcal{M}^{\otimes M_i^{r\delta}} dX_{M_i^{r\delta}-1} dV_{M_i^{r\delta}} \\ &\leq C_q\, C'^{M^{r\delta}} \varepsilon \\ &\quad\times \prod_{i=1}^q \frac{(C|\sigma_i|\mu_\varepsilon)^{M_i^{r\delta}-1}}{M_i^{r\delta}!} \sum_{\bar{\mathbf{S}},\mathbf{S},\mathbf{K},\varsigma,\lambda,\mathcal{E}} \sup_{x_{M_i^{r\delta}}\in\mathbb{T}^d} \int \mathbf{1}_{\{Z_{M_i^{r\delta}} \text{ forward cluster}\}} \mathcal{M}^{\otimes M_i^{r\delta}} dX_{M_i^{r\delta}-1} dV_{M_i^{r\delta}} \,,\end{aligned}$$

where in the second inequality we used that ε is small to sum the series and that for $|\eta| \leq q/2$, $q\geq 2$

$$\mu_\varepsilon^{q/2} \prod_{\ell=1}^{|\eta|} \left(\varepsilon^d\right)^{|\eta_\ell|-1} = \mu_\varepsilon^{q/2}\left(\varepsilon^d\right)^{q-|\eta|} \leq \left(\mu_\varepsilon\varepsilon^d\right)^{q/2} \,.$$

Equation (8.2) is proved. □

8.2. **Variance of $\otimes$-products.** We aim at proving the bound (4.18), let us compute

$$\begin{aligned}\mathbb{E}_\varepsilon^{\rm eq}\Big[\Big(\bigotimes_{i=1}^q \zeta^\varepsilon_{M_i^{r\delta}}\big(\phi^{(\sigma_i)}\big)\Big)^2\Big] &= \mathbb{E}_\varepsilon^{\rm eq}\left[\mu_\varepsilon^q\left(\bigotimes_{i=1}^q\left(\frac{1}{\mu_\varepsilon^{M_i^{r\delta}}}\sum \phi^{(\sigma_i)} - \mathbb{E}_\varepsilon[\phi^{(\sigma_i)}]\right)\right)^2\right] \\ &= \sum_{\substack{A\subset\{1,\dots,q\}\\ A'\subset\{1',\dots,q'\}}} \prod_{j\in A^c\cup(A')^c} \mathbb{E}_\varepsilon[-\phi^{(\sigma_j)}] \times \sum_{\ell=0}^{\min(M_A^{r\delta},M_{A'}^{r\delta})} \mathcal{E}_{A,A',\ell}\end{aligned} \tag{8.10}$$

where the joint expectation with ℓ repeated indices is denoted by

$$\mathcal{E}_{A,A',\ell} := \frac{1}{\mu_\varepsilon^{M_A^{r\delta}+M_{A'}^{r\delta}-q}}\mathbb{E}_\varepsilon^{\rm eq}\left[\sum_{\left|(i_1,\dots,i_{M_A^{r\delta}})\cap(i'_1,\dots,i'_{M_{A'}^{r\delta}})\right|=\ell} \Phi_{\mathbf{M}_A^{r\delta}}\big(\mathbf{z}^\varepsilon_{i_1},\dots,\mathbf{z}^\varepsilon_{i_{M_A^{r\delta}}}\big)\Phi_{\mathbf{M}_{A'}^{r\delta}}\big(\mathbf{z}^\varepsilon_{i'_1},\dots,\mathbf{z}^\varepsilon_{i'_{M_{A'}^{r\delta}}}\big)\right]$$

with notations introduced above, which we recall again: $M_A^{r\delta} = \sum_{j\in A} M_j^{r\delta}$, $\mathbf{M}_A^{r\delta} = (M_j^{r\delta})_{j\in A}$, $\Phi_{\mathbf{M}_A^{r\delta}} = \bigotimes_{j\in A}\phi^{(\sigma_j)}$ and $M_{A'}^{r\delta} = \sum_{j\in A'} M_j^{r\delta}$, $\mathbf{M}_{A'}^{r\delta} = (M_j^{r\delta})_{j\in A'}$, $\Phi_{\mathbf{M}_{A'}^{r\delta}} = \bigotimes_{j\in A'}\phi^{(\sigma_j)}$. Denoting by Λ and Λ' the subsets of indices selecting the ℓ contracted variables, we get

$$\begin{aligned}\mathcal{E}_{A,A',\ell} = \frac{\mu_\varepsilon^{q-\ell}}{\mathcal{Z}^\varepsilon} &\sum_{\substack{\Lambda\subset\{1,\dots,M_A^{r\delta}\}\\ \Lambda'\subset\{1,\dots,M_{A'}^{r\delta}\}\\ |\Lambda|=|\Lambda'|=\ell}} \sum_{\chi_\ell:\Lambda\mapsto\Lambda'} \sum_{p\geq 0}\frac{\mu_\varepsilon^p}{p!}\int dZ_{M_A^{r\delta}} dZ'_{M_{A'}^{r\delta}} d\bar Z_p \delta_{\chi_\ell}(Z_{M_A^{r\delta}}, Z'_{M_{A'}^{r\delta}}) \\ &\times \Phi_{\mathbf{M}_A^{r\delta}}(Z_{M_A^{r\delta}})\Phi_{\mathbf{M}_{A'}^{r\delta}}(Z'_{M_{A'}^{r\delta}})\mathbf{1}_{\mathcal{D}^\varepsilon_{M_A^{r\delta}+M_{A'}^{r\delta}-\ell+p}}\mathcal{M}^{\otimes(M_A^{r\delta}+M_{A'}^{r\delta}-\ell+p)},\end{aligned} \tag{8.11}$$

where the injective map $\chi_\ell:\Lambda\mapsto\Lambda'$ encodes the repetition of the indices in $Z_{M_A^{r\delta}}, Z'_{M_{A'}^{r\delta}}$

$$\delta_{\chi_\ell} = \prod_{j\in\Lambda}\delta_{z_j - z'_{\chi_\ell(j)}}\,.$$

A factor $\mu_\varepsilon^{-\ell}$ is gained from these repetitions.

Step 1. A graph structure with $M_A^{r\delta}+M_{A'}^{r\delta}-\ell$ vertices, depicted in Figure 8, can be extracted from the constraints $\Phi_{\mathbf{M}_A^{r\delta}}, \Phi_{\mathbf{M}_{A'}^{r\delta}}, \chi_\ell$ in (8.11) :

- the dynamical constraints corresponding to $(Z^{(j)}_{M_j^{r\delta}})_{j\in A}$ and $(Z^{(j)}_{M_j^{r\delta}})_{j\in A'}$, coded by the functions $\phi^{(\sigma_j)}$ according to (7.5), lead to vertices forming $|A|+|A'|$ connected orange packets;
- the constraint χ_ℓ from the ℓ repetitions in the variables is represented by green lines (contractions) in Figure 8. The vertices linked by green lines correspond in fact to the same repeated variable, and are therefore identified.

We consider a partition η of $A\cup A'$ into s connected components $\eta_1,\dots,\eta_s$ (represented in blue in Figure 8): in each component η_i, all orange packets are connected by green lines. Denote by ℓ_i the number of green lines in the component η_i. By definition

$$\ell_i \geq |\eta_i|-1, \qquad \sum_{i=1}^s \ell_i = \ell\,,$$

where $|\eta_i|$ is the number of orange packets in η_i. We denote by $\delta_{\chi_{\ell_i}}$ the identification of particles restricted to η_i, so that

$$\delta_{\chi_\ell} = \prod_{i=1}^s \delta_{\chi_{\ell_i}}\,,$$

and recall that a green line can only join a packet $j\in A$ and a packet $j\in A'$.

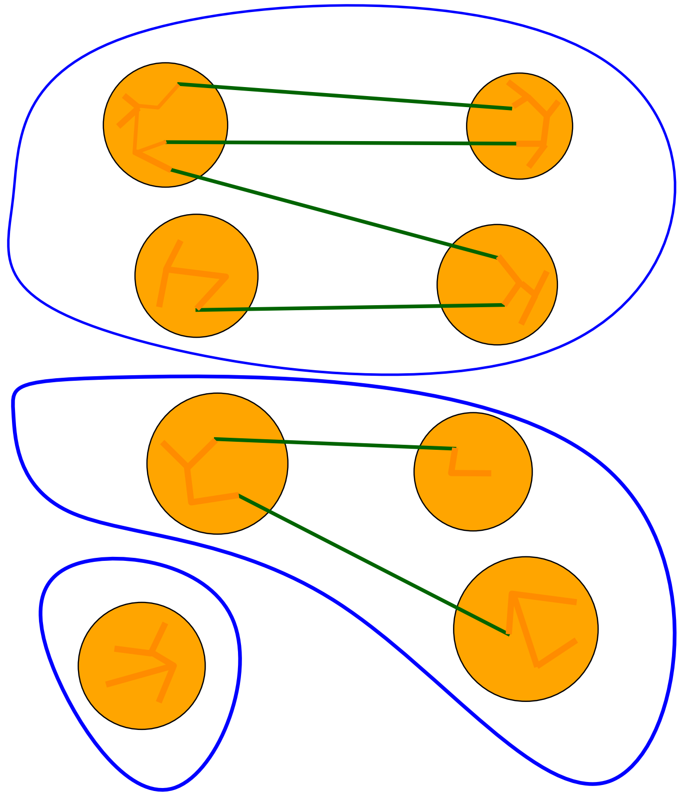

FIGURE 8. A partition η with $|A| = |A'| = 4$ and $s = 3$. The first component is $\eta_1 = \{1, 2, 1', 2'\}$, the second component is $\eta_2 = \{3, 3', 4'\}$ and the third component $\eta_3 = \{4\}$. The number of green lines is $\ell_1 = 4$, $\ell_2 = 2$ and $\ell_3 = 0$.

Recall that $Z_{\eta_i} = Z^{(\eta_i)}_{M^{r\delta}_{\eta_i}}$ is the collection of particles in the connected component η_i, where

$$M^{r\delta}_{\eta_i} = \sum_{j \in A \cap \eta_i} M^{r\delta}_j + \sum_{j \in A' \cap \eta_i} M^{r\delta}_j \,.$$

We should keep in mind now that, as ℓ_i particles of A' are identified with particles in A, then the total number of particles is actually $M^{r\delta}_{\eta_i} - \ell_i$.

Step 2. We need now to define new forward tree graphs to be associated with each component η_i. We denote by T_j for $j \in A$ ($j \in A'$), the orange forward tree describing the cluster structure of $\phi^{(\sigma_j)}$ used in estimate (7.8), coding the geometric constraints on the configuration $Z^{(j)}_{M^{r\delta}_j}$ associated with the forward dynamics in terms of minimally connected graphs (we drop here the index $\prec$, but remember that the graphs are equipped with an ordering of edges). We recall that, in such forward dynamics, each configuration of $M^{r\delta}_j$ particles is partitioned in blocks, and that the cardinalities of such blocks are coded in the notation $\mathbf{M}^{r\delta}_j$ in (7.8); the component η_i inherits then a partition in blocks $\mathbf{M}^{r\delta}_{\eta_i}$. In each connected component η_i, we extract a minimally connected graph T_{η_i} on the set of vertices $Z^{(\eta_i)}_{M^{r\delta}_{\eta_i}}$ (equipped with an ordering of edges) by means of the following procedure (see also Figure 9). We collect first all edges coming from T_j for $j \in A \cap \eta_i$ (note that this cannot produce any cycle by definition). This will form the skeleton of the graph and will be denoted by $T^{1A}_{\eta_i}$. Then we look in turn at the edges in the remaining orange forward tree graphs T_j with $j \in A' \cap \eta_i$. Following the ordering, we keep only edges that do not produce cycles after identification of vertices linked by green lines (note that this peeling is unique). We end up with a forward tree $T^A_{\eta_i}$ on the

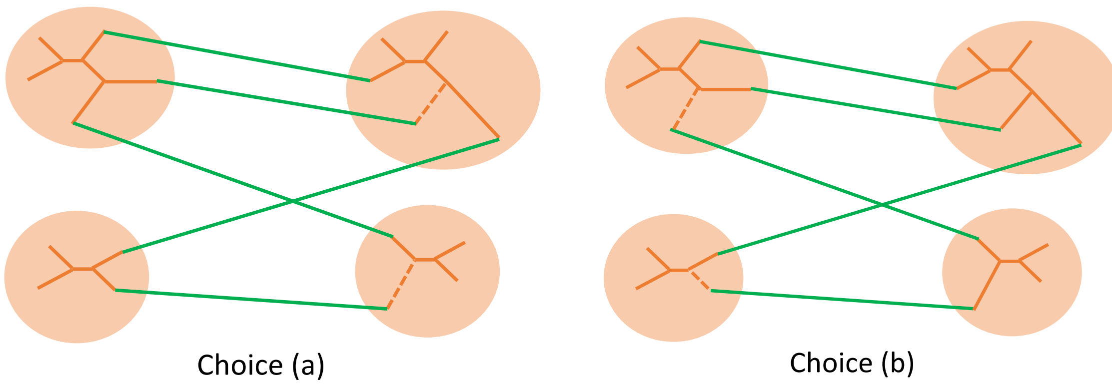

FIGURE 9. Two sets A (on the left in both cases (a) and (b)) and A' (on the right in both cases (a) and (b)) are connected by green lines representing identification of particles. (a) The skeleton is the set of all orange edges in $\eta_i \cap A$, and the minimally connected graph $T^A_{\eta_i}$ is obtained by discarding the orange edges in A' which create cycles in the graph (dotted orange edges). (b) The skeleton is the set of all orange edges in $\eta_i \cap A'$, and the minimally connected graph $T^{A'}_{\eta_i}$ is obtained by discarding the orange edges in A which create cycles in the graph (dotted orange edges).

set of vertices $Z^{(\eta_i)}_{M^{r\delta}_{\eta_i}}$ encoding some of the dynamical constraints of the orange forward trees, which will produce small factors. We stress that by construction, the admissible tree graphs $T^A_{\eta_i}$ depend on $\Lambda, \Lambda', \chi_\ell$ and on the arbitrary choice of constructing the skeleton over $A \cap \eta_i$. The superscript A in $T^A_{\eta_i}$ is a shortened notation reminding us of this dependence.

Thus in (8.11), for each component η_i we have a product of test functions controlled by the following estimate, which extends (7.8) to the case of products with repeated variables:

$$
\begin{aligned}
&\mu_\varepsilon^{-\ell_i} \delta_{\chi_{\ell_i}} \Big| \prod_{j \in \eta_i} \phi^{(\sigma_j)} \Big| \Big(Z_{\mathbf{M}^{r\delta}_{\eta_i}} \Big) \\
&\leq C_q \Bigg(\prod_{j \in \eta_i} |\sigma_j|^{M^{r\delta}_j} \Bigg) |\eta_i|^{M^{r\delta}_{\eta_i}} \frac{\mu_\varepsilon^{M^{r\delta}_{\eta_i} - |\eta_i| - \ell_i}}{M^{r\delta}_{\eta_i}!} \, \delta_{\chi_{\ell_i}} \sum_{\substack{\varsigma,\, \lambda,\, \mathbf{S},\, \bar{\mathbf{S}},\, \mathbf{K},\, \mathcal{E} \\ \varsigma', \lambda', \mathbf{S}', \bar{\mathbf{S}}', \mathbf{K}', \mathcal{E}'}} \sum_{T^A_{\eta_i}} \mathbf{1}_{\{ Z_{\mathbf{M}^{r\delta}_{\eta_i}} \in \mathcal{R}^{\text{comp}}_{T^A_{\eta_i}} \}}
\end{aligned}
\tag{8.12}
$$

where $\big(\varsigma,\, \lambda,\, \mathbf{S},\, \bar{\mathbf{S}},\, \mathbf{K},\, \mathcal{E}\big)$ and $\big(\varsigma', \lambda', \mathbf{S}', \bar{\mathbf{S}}', \mathbf{K}', \mathcal{E}'\big)$ are the whole collections of variables which are necessary to parametrise the orange clusters in A and in A' respectively. Here, $\mathcal{R}^{\text{comp}}_{T^A_{\eta_i}}$ is the corresponding set of compatible configurations for a given ordered tree graph on $M^{r\delta}_{\eta_i} - \ell_i$ vertices. Therefore for each orange edge in $T^A_{\eta_i}$, there exist two particles which will be dynamically constrained by encounters, according to the specified forward dynamics (and respecting the time sampling).

Step 3. To control the background particles $\bar{Z}_p$ in (8.11), we use a cluster expansion of the exclusion $\mathbf{1}_{\mathcal{D}^\varepsilon_{M^{r\delta}_A + M^{r\delta}_{A'} - \ell + p}}$ as in (8.4). We consider now $\Big(Z^{(\eta_i)}_{M^{r\delta}_{\eta_i}} \Big)_{1 \leq i \leq s}$ as s blocks represented each by one vertex, and $(\bar z_j)_{1\le j\le p}$ as p separate vertices. We then have

$$\begin{aligned}\frac{1}{\mathcal{Z}^\varepsilon}\sum_{p\ge 0}\frac{\mu_\varepsilon^p}{p!}\int d\bar Z_p \mathbf{1}_{\mathcal{D}^\varepsilon_{M_A^{r\delta}+M_{A'}^{r\delta}-\ell+p}}(Z_{M_A^{r\delta}}, Z'_{M_{A'}^{r\delta}}, \bar Z_p)\mathcal{M}^{\otimes p}(\bar V_p)\\ = \prod_{i=1}^s \mathbf{1}_{\mathcal{D}^\varepsilon_{M_{\eta_i}^{r\delta}-\ell_i}}\left(Z^{(\eta_i)}_{M_{\eta_i}^{r\delta}}\right)\sum_{\omega\in\mathcal{P}_s}\prod_{u=1}^{|\omega|}\left(\sum_{n_u\ge 0}\frac{\mu_\varepsilon^{n_u}}{n_u!}\int \mathcal{M}^{\otimes n_u}\varphi(Z_{\omega_u}, \bar Z_{n_u})d\bar Z_{n_u}\right),\end{aligned} \tag{8.13}$$

where $Z_{\omega_u} = \left(Z^{(\eta_i)}_{M_{\eta_i}^{r\delta}}\right)_{i\in\omega_u}$.

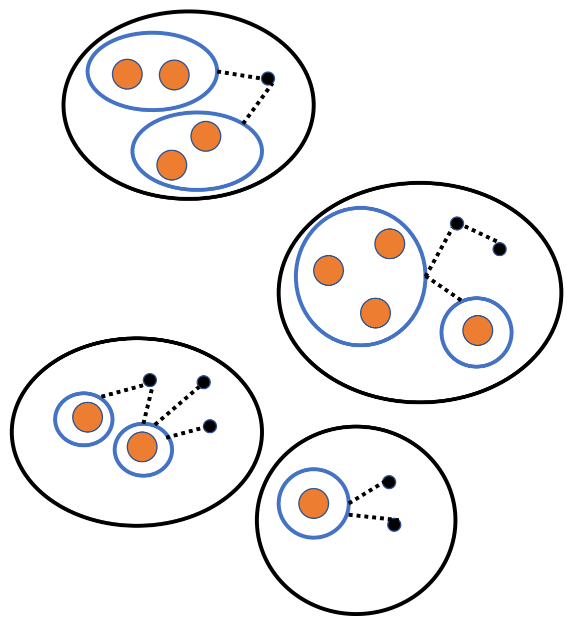

FIGURE 10. The s components $\eta_1,\dots,\eta_s$ (represented in blue) are grouped in $|\omega|$ parts (represented in black) according to the partition ω, and each of these parts ω_u is provided with an arbitrary number n_u of background particles (black dots). In each part, because of the tree inequality, all vertices are connected by a tree (represented by the dotted black lines).

Step 4. By Fubini, we finally get from (8.10), (8.11) and (8.13) that

$$\begin{aligned}&\mathbb{E}_\varepsilon^{\mathrm{eq}}\left[\mu_\varepsilon^q\left(\bigotimes_{j=1}^q\left(\frac{1}{\mu_\varepsilon^{M_j^{r\delta}}}\sum\phi^{(\sigma_j)}-\mathbb{E}_\varepsilon[\phi^{(\sigma_j)}]\right)\right)^2\right] = \mu_\varepsilon^q\sum_{\substack{A\subset\{1,\dots,q\}\\ A'\subset\{1',\dots,q'\}}}\prod_{j\in A^c\cup(A')^c}\mathbb{E}_\varepsilon[-\phi^{(\sigma_j)}]\\ &\times\sum_{\omega\in\mathcal{P}_{A\cup A'}}\prod_{u=1}^{|\omega|}\Bigg[\sum_{n_u\ge 0}\frac{\mu_\varepsilon^{n_u}}{n_u!}\sum_{\eta\in\mathcal{P}_{\omega_u}}\sum_{\substack{\ell_i\ge|\eta_i|-1\\ 1\le i\le|\eta|}}\sum_{\substack{\Lambda_i,\Lambda'_i,\chi_{\ell_i}\\ 1\le i\le|\eta|}}\\ &\times\int\mathcal{M}^{\otimes(M_{\omega_u}^{r\delta}-\sum_{i=1}^{|\eta|}\ell_i+n_u)}\varphi(Z_{\eta_1}, Z_{\eta_2},\dots,\bar Z_{n_u})\prod_{i=1}^{|\eta|}\Big(\mathbf{1}_{\mathcal{D}^\varepsilon_{M_{\eta_i}^{r\delta}-\ell_i}}\mu_\varepsilon^{-\ell_i}\delta_{\chi_{\ell_i}}\prod_{j\in\eta_i}\phi^{(\sigma_j)}dZ_{\eta_i}\Big)d\bar Z_{n_u}\Bigg].\end{aligned}$$

The set ω_u corresponds to $M_{\omega_u}^{r\delta}-\sum_{i=1}^{|\eta|}\ell_i$ particles, to which n_u background particles are added, as depicted in Figure 10. Here we denote abusively by $\eta = \{\eta_1,\dots,\eta_{|\eta|}\}$ the generic partition of one ω_u. There are $|\eta|$ components in the partition η and, in each component η_i, we recall that $\ell_i \geq |\eta_i|-1$ denotes the number of green edges and $\delta_{\chi_{\ell_i}}$ the identification of the ℓ_i particles in Λ'_i with the ℓ_i particles in Λ_i.

Using (8.5) and proceeding as in (8.6), we split any partition ω of $A \cup A'$ in a union of singletons $\{j\}$ for $j \in (A \smallsetminus B) \cup (A' \smallsetminus B')$, and a partition $\tilde{\omega}$ of $B \cup B'$ with no singleton (and at most $\frac{1}{2}(|B| + |B'|)$ parts). Compared with the previous situation, we cannot absorb the sum over singletons due to the defect of centering, and we have (noticing that $\ell_j = 0$ for singletons)

$$\mathbb{E}_\varepsilon^{\mathrm{eq}}\Bigg[\mu_\varepsilon^q\Bigg(\bigotimes_{j=1}^{q}\Bigg(\frac{1}{\mu_\varepsilon^{M_j^{r\delta}}}\sum\phi^{(\sigma_j)} - \mathbb{E}_\varepsilon\big[\phi^{(\sigma_j)}\big]\Bigg)\Bigg)^2\Bigg]$$

$$\begin{aligned}
&= \mu_\varepsilon^q \sum_{\substack{B\subset\{1,\dots,q\}\\ B'\subset\{1',\dots,q'\}}}\Bigg(\prod_{j\in^c B\cup^c B'}\Big(\mathbb{E}_\varepsilon^{\mathrm{eq}}\big[\phi^{(\sigma_j)}\big] - \mathbb{E}_\varepsilon\big[\phi^{(\sigma_j)}\big]\Big)\Bigg)\sum_{\omega\in(\mathcal{P}_{B\cup B'})^*}\prod_{u=1}^{|\omega|}\Bigg[\sum_{\eta\in\mathcal{P}_{|\omega_u|}}\sum_{\substack{\ell_i\geq|\eta_i|-1\\ 1\leq i\leq|\eta|}}\sum_{\substack{\Lambda_i,\Lambda'_i,\chi_{\ell_i}\\ 1\leq i\leq|\eta|}} \\
&\qquad\times\sum_{n_u\geq 0}\frac{\mu_\varepsilon^{n_u}}{n_u!}\int\mathcal{M}^{\otimes(M_{\omega_u}^{r\delta}-\sum_{i=1}^{|\eta|}\ell_i+n_u)}\varphi(Z_{\eta_1},\dots,Z_{\eta_{|\eta|}},\bar{Z}_{n_u}) \\
&\qquad\times\prod_{i=1}^{|\eta|}\Big(\mathbf{1}_{\mathcal{D}^\varepsilon_{M_{\eta_i}^{r\delta}-\ell_i}}\mu_\varepsilon^{-\ell_i}\delta_{\chi_{\ell_i}}\prod_{j\in\eta_i}\phi^{(\sigma_j)}dZ_{\eta_i}\Big)d\bar{Z}_{n_u}\Bigg].
\end{aligned} \tag{8.14}$$

To estimate from above the latter formula, some of the constraints on the clustering structure can be forgotten. Indeed we know from Step 2 that, to each component η_i and each χ_{ℓ_i}, we can associate a minimally connected graph $T^B_{\eta_i}$, encoding dynamical constraints associated with orange edges: see Eq. (8.12) and Figure 9. The next and final step will be to integrate these dynamical constraints. At this stage, the assumptions from Lemma 4.11 on the cumulant structure will become relevant to describe precisely the set $\mathcal{R}^{\mathrm{comp}}_{T^B_{\eta_i}}$ including the time sampling.

First of all, we proceed by estimating the integral over the background particles as already done in Section 8.1. For each ω_u and $\eta \in \mathcal{P}_{\omega_u}$, the functions φ can be controlled by the tree inequality (8.7), this time applied over the vertices $(Z_{\eta_i})_{1\leq i\leq|\eta|}$ (considered as subconfigurations) and the n_u background particles. Using (8.12) and the translation invariance, we obtain (as in (8.8)-(8.9)) that the term in the last two lines in (8.14) is bounded in absolute value by

$$\begin{aligned}
&C_P C_P^{M_\eta^{r\delta}}(C\varepsilon^d)^{|\eta|-1}\prod_{i=1}^{|\eta|}\frac{\mu_\varepsilon^{M_{\eta_i}^{r\delta}-|\eta_i|-\ell_i}}{M_{\eta_i}^{r\delta}!} \\
&\sum_{\substack{\varsigma,\,\lambda,\,\mathbf{S},\,\bar{\mathbf{S}},\,\mathbf{K},\,\mathcal{E}\\ \varsigma',\lambda',\mathbf{S}',\bar{\mathbf{S}}',\mathbf{K}',\mathcal{E}'}}\sum_{T^B_{\eta_i}}\sup\int\mathbf{1}_{\mathcal{D}^\varepsilon_{M_{\eta_i}^{r\delta}-\ell_i}}(Z_{\eta_i})\,\mathcal{M}^{\otimes(M_{\eta_i}^{r\delta}-\ell_i)}\mathbf{1}_{\{Z_{\mathbf{M}_{\eta_i}^{r\delta}}\in\mathcal{R}^{\mathrm{comp}}_{T^B_{\eta_i}}\}}\delta_{\chi_{\ell_i}}dZ_{\eta_i}\,.
\end{aligned} \tag{8.15}$$

As in (8.8), (8.12), the first sum is taken over the whole collections parametrising the forward dynamics (and the sums over such parameters can be bounded by $C_P^{M_\eta^{r\delta}}$ and combinatorial factors depending only on $q \leq P$); moreover the supremum is taken over one single positional variable and, for brevity, Z'_{η_i} is the configuration Z_{η_i} deprived of such variable.

The remaining integral is now estimated in a similar way as in the proof of (7.6) above. We may follow the same strategy devised in Section 7, ordering the orange edges in time in a way to respect the sampling, and identifying a sequence of independent degrees of freedom which can be progressively estimated; see (7.9)-(7.14). However, particles in the skeleton play a special role, as explained in what follows.

Recall that, by Step 2, the tree $T^B_{\eta_i}$ is constructed asymmetrically so that the union of the skeletons $\bigcup_i T^{1B}_{\eta_i}$ records all the dynamical constraints in $(\phi^{(\sigma_j)})_{j\in B}$ (Figures 8-9). For these edges, we proceed exactly as in Section 7 and recover a bound of the form (7.12)-(7.13) taking into account the time sampling. Instead, the orange edges which are outside the skeleton are estimated more crudely, discarding the dynamical constraints associated with the sampling. This leads to the estimate

$$\begin{aligned}\sum_{T^B_{\eta_i}} \sup \int \mathbf{1}_{\mathcal{D}^\varepsilon_{M^{r\delta}_{\eta_i}-\ell_i}}(Z_{\eta_i})\, \mathcal{M}^{\otimes(M^{r\delta}_{\eta_i}-\ell_i)} \mathbf{1}_{\{Z_{\mathbf{M}^{r\delta}_{\eta_i}} \in \mathcal{R}^{\text{comp}}_{T^B_{\eta_i}}\}} \delta_{\chi_{\ell_i}} dZ_{\eta_i} \leq C\left(\frac{1}{\mu_\varepsilon}\right)^{M^{r\delta}_{\eta_i}-\ell_i-1} \\ \times \prod_{j\in\eta_i\cap B} \left(M^{r\delta}_j\right)^{2(M^{r\delta}_j-1)} \frac{3^{M^{r\delta}_j-1}}{\left(M^{r\delta}_j-1\right)!} (C_P\delta)^{N^r_{\sigma_j}} (C_P\tau)^{N^{<r}_{\sigma_j}+|\sigma_j|-1} (C_P\Theta)^{M_j-|\sigma_j|} \\ \times \left(M^{r\delta}_{\eta_i\cap B'}\right)^{2(M^{r\delta}_{\eta_i\cap B'}-\ell_i)} \frac{(C_P\Theta)^{M^{r\delta}_{\eta_i\cap B'}-\ell_i}}{\left(M^{r\delta}_{\eta_i\cap B'}-\ell_i\right)!}\end{aligned} \tag{8.16}$$

Note that one factor $\left(M^{r\delta}_j\right)^{M^{r\delta}_j-1}$ is compensated by the factorial at the denominator in the second line, and the same can be said for one factor $\left(M^{r\delta}_{\eta_i\cap B'}\right)^{M^{r\delta}_{\eta_i\cap B'}-\ell_i}$ in the third line. Hence, enlarging the constants, we get

$$\begin{aligned}\sum_{T^B_{\eta_i}} \sup \int \mathbf{1}_{\mathcal{D}^\varepsilon_{M^{r\delta}_{\eta_i}-\ell_i}}(Z_{\eta_i})\, \mathcal{M}^{\otimes(M^{r\delta}_{\eta_i}-\ell_i)} \mathbf{1}_{\{Z_{\mathbf{M}^{r\delta}_{\eta_i}} \in \mathcal{R}^{\text{comp}}_{T^B_{\eta_i}}\}} \delta_{\chi_{\ell_i}} dZ_{\eta_i} \leq C\left(\frac{1}{\mu_\varepsilon}\right)^{M^{r\delta}_{\eta_i}-\ell_i-1} \\ \times \left(M^{r\delta}_{\eta_i}\right)^{M^{r\delta}_{\eta_i}} \left(M^{r\delta}_{\eta_i\cap B'}\right)^{-\ell_i} \left(\prod_{j\in\eta_i\cap B} (C_P\delta)^{N^r_{\sigma_j}} (C_P\tau)^{N^{<r}_{\sigma_j}+|\sigma_j|-1} (C_P\Theta)^{M_j-|\sigma_j|}\right) (C_P\Theta)^{M^{r\delta}_{\eta_i\cap B'}-\ell_i}\end{aligned} \tag{8.17}$$

where the factor $\left(M^{r\delta}_{\eta_i}\right)^{M^{r\delta}_{\eta_i}}$ compensates, up to geometric terms, the factorial in (8.15). On the other hand, the number of possible contractions at ℓ_i fixed is

$$\sum_{\substack{\Lambda_i,\Lambda'_i,\chi_{\ell_i}\\ 1\leq i\leq|\eta|}} 1 = \binom{M^{r\delta}_{\eta_i\cap B}}{\ell_i}\binom{M^{r\delta}_{\eta_i\cap B'}}{\ell_i}\ell_i! \leq 2^{M^{r\delta}_{\eta_i\cap B}} \left(M^{r\delta}_{\eta_i\cap B'}\right)^{\ell_i} \tag{8.18}$$

which compensates the factor $\left(M^{r\delta}_{\eta_i\cap B'}\right)^{-\ell_i}$ in (8.17). Therefore by (8.14), (8.15), (8.17) and (8.18) we deduce that (recalling $\sum_i |\eta_i| = |B| + |B'|$)

$$\begin{aligned}\mathbb{E}^{\text{eq}}_\varepsilon\left[\mu^q_\varepsilon\left(\bigotimes_{j=1}^q\left(\frac{1}{\mu_\varepsilon^{M^{r\delta}_j}}\sum \phi^{(\sigma_j)} - \mathbb{E}_\varepsilon[\phi^{(\sigma_j)}]\right)\right)^2\right] \\ \leq C_P\mu^q_\varepsilon \sum_{B\subset\{1,\dots,q\}} \sum_{B'\subset\{1',\dots,q'\}} \mu_\varepsilon^{-|B|-|B'|}\left(\prod_{j\in{}^cB\cup{}^cB'} \left|\mathbb{E}^{\text{eq}}_\varepsilon[\mathbf{1}_{{}^c\Upsilon^\varepsilon_{\mathcal{N}}}\phi^{(\sigma_j)}]\right|\right) \sum_{\omega\in(\mathcal{P}_{B\cup B'})^*} \prod_{u=1}^{|\omega|} \sum_{\eta\in\mathcal{P}_{\omega_u}} \\ \times \left(C\varepsilon^d\mu_\varepsilon\right)^{|\eta|-1} \mu_\varepsilon \left(\prod_{j\in\eta^B_i} (C_P\delta)^{N^r_{\sigma_j}} (C_P\tau)^{N^{<r}_{\sigma_j}+|\sigma_j|-1} (C_P\Theta)^{M_j-|\sigma_j|}\right) (C_P\Theta)^{M^{r\delta}_{\eta_i\cap B'}}\end{aligned} \tag{8.19}$$

for some constant C_P as in the statement of Lemma 4.11 and some pure constant C. As $\eta_i \cap B$ can be replaced by $\eta_i \cap B'$ by symmetry, we also deduce that

(8.20)
$$\mathbb{E}_\varepsilon^{\rm eq}\left[\mu_\varepsilon^q\left(\bigotimes_{j=1}^q\left(\frac{1}{\mu_\varepsilon^{M_j^{r\delta}}}\sum\phi^{(\sigma_j)}-\mathbb{E}_\varepsilon[\phi^{(\sigma_j)}]\right)\right)^2\right]$$
$$\le C_P\mu_\varepsilon^q\sum_{B\subset\{1,\dots,q\}}\sum_{B'\subset\{1',\dots,q'\}}\mu_\varepsilon^{-|B|-|B'|}\left(\prod_{j\in{}^cB\cup{}^cB'}\left|\mathbb{E}_\varepsilon^{\rm eq}[\mathbf{1}_{{}^c\Upsilon_{\mathcal N}^\varepsilon}\phi^{(\sigma_j)}]\right|\right)\sum_{\omega\in\mathcal{P}^*_{B\cup B'}}\prod_{u=1}^{|\omega|}\sum_{\eta\in\mathcal{P}_{\omega_u}}$$
$$\times\left(C'\varepsilon^d\mu_\varepsilon\right)^{|\eta|-1}\mu_\varepsilon\left(\prod_{j\in\eta_i}(C_P\delta)^{N^r_{\sigma_j}}\left(C_P\tau\right)^{N^{<r}_{\sigma_j}+|\sigma_j|-1}\left(C_P\Theta\right)^{2M_j+N^{<r}_{\sigma_j}+N^r_{\sigma_j}-|\sigma_j|}\right)^{1/2}.$$

Recall that $C\varepsilon^d\mu_\varepsilon = C\varepsilon$, so that we obtain a rough upper bound

(8.21)
$$\prod_{u=1}^{|\omega|}\sum_{\eta\in\mathcal{P}_{\omega_u}}\left(C'\varepsilon^d\mu_\varepsilon\right)^{|\eta|-1}\mu_\varepsilon\left(\prod_{j\in\eta_i}(C_P\delta)^{N^r_{\sigma_j}}\left(C_P\tau\right)^{N^{<r}_{\sigma_j}+|\sigma_j|-1}\left(C_P\Theta\right)^{2M_j+N^{<r}_{\sigma_j}+N^r_{\sigma_j}-|\sigma_j|}\right)^{1/2}$$
$$\le\mu_\varepsilon^{|\omega|}\left(\prod_{j\in B\cup B'}(C_P\delta)^{N^r_{\sigma_j}}\left(C_P\tau\right)^{N^{<r}_{\sigma_j}+|\sigma_j|-1}\left(C_P\Theta\right)^{2M_j+N^{<r}_{\sigma_j}+N^r_{\sigma_j}-|\sigma_j|}\right)^{1/2}.$$

We are left with the cost of the conditioning in the singletons

(8.22)
$$\mathbb{E}_\varepsilon^{\rm eq}[\mathbf{1}_{{}^c\Upsilon_{\mathcal N}^\varepsilon}\phi^{(\sigma_j)}]\equiv\mathbb{E}_\varepsilon^{\rm eq}[\mathbf{1}_{{}^c\Upsilon_{\mathcal N}^\varepsilon}\pi^\varepsilon_{M^{r\delta}_{\sigma_j}}(\phi^{(\sigma_j)})]$$

which, recalling that $\pi^\varepsilon_{M^{r\delta}_{\sigma_j}}(\phi^{(\sigma_j)}) = \mathbb{E}_\varepsilon^{\rm eq}\Big[\pi^\varepsilon_{M^{r\delta}_{\sigma_j}}(\phi^{(\sigma_j)})\Big]+\mu_\varepsilon^{-1/2}\zeta^{\varepsilon,{\rm eq}}_{M^{r\delta}_{\sigma_j}}\big(\phi^{(\sigma_j)}\big)$, is bounded as follows:

(8.23)
$$\left|\mathbb{E}_\varepsilon^{\rm eq}[\mathbf{1}_{{}^c\Upsilon_{\mathcal N}^\varepsilon}\pi^\varepsilon_{M^{r\delta}_{\sigma_j}}(\phi^{(\sigma_j)})]\right| = \left|\mathbb{P}_\varepsilon^{\rm eq}\Big[{}^c\Upsilon^\varepsilon_{\mathcal N}\Big]\mathbb{E}_\varepsilon^{\rm eq}\Big[\phi^{(\sigma_j)}\Big]+\mu_\varepsilon^{-1/2}\mathbb{E}_\varepsilon^{\rm eq}\Big[\mathbf{1}_{{}^c\Upsilon_{\mathcal N}^\varepsilon}\zeta^{\varepsilon,{\rm eq}}_{M^{r\delta}_{\sigma_j}}\big(\phi^{(\sigma_j)}\big)\Big]\right|$$
$$\le\mathbb{P}_\varepsilon^{\rm eq}\Big[{}^c\Upsilon^\varepsilon_{\mathcal N}\Big]\mathbb{E}_\varepsilon^{\rm eq}\Big[\Big|\phi^{(\sigma_j)}\Big|\Big]+\mu_\varepsilon^{-1/2}\mathbb{P}_\varepsilon^{\rm eq}\Big[{}^c\Upsilon^\varepsilon_{\mathcal N}\Big]^{1/2}\mathbb{E}_\varepsilon^{\rm eq}\Big[\Big(\zeta^{\varepsilon,{\rm eq}}_{M^{r\delta}_{\sigma_j}}\big(\phi^{(\sigma_j)}\big)\Big)^2\Big]^{1/2}$$
$$\le C_P\Big[\Theta\varepsilon^d(C_P\Theta)^{M_j-|\sigma_j|}(C_P\delta)^{N^r_{\sigma_j}}(C_P\tau)^{N^{<r}_{\sigma_j}+|\sigma_j|-1}$$
$$+\mu_\varepsilon^{-1/2}\left(\Theta\varepsilon^d\right)^{1/2}\Big((C_P\Theta)^{2M_j+N^r_{\sigma_j}+N^{<r}_{\sigma_j}-|\sigma_j|}(C_P\delta)^{N^r_{\sigma_j}}(C_P\tau)^{N^{<r}_{\sigma_j}+|\sigma_j|-1}\Big)^{1/2}\Big].$$

In the last step we used (2.10), together with (7.2) to bound the first term, and the estimate (4.18) at equilibrium in the case of one single factor to bound the second term (this estimate has been proved in [8] and follows also from the previous computation). Notice that the second term is dominant for μ_ε large (and $\Theta > 1$). We will actually only keep the rough estimate

(8.24)
$$\left|\mathbb{E}_\varepsilon^{\rm eq}[\mathbf{1}_{{}^c\Upsilon_{\mathcal N}^\varepsilon}\pi^\varepsilon_{M^{r\delta}_{\sigma_j}}(\phi^{(\sigma_j)})]\right|\le\mu_\varepsilon^{-1/2}\Big((\Theta\varepsilon^d)(C_P\Theta)^{2M_j+N^r_{\sigma_j}+N^{<r}_{\sigma_j}-|\sigma_j|}(C_P\delta)^{N^r_{\sigma_j}}(C_P\tau)^{N^{<r}_{\sigma_j}+|\sigma_j|-1}\Big)^{1/2}.$$

When inserting this into (8.20)-(8.21), we obtain the following power counting

(8.25)
$$\mu_\varepsilon^q\mu_\varepsilon^{-|B|-|B'|}\mu_\varepsilon^{|\omega|}\mu_\varepsilon^{-(|{}^cB|+|{}^cB'|)/2}=\mu_\varepsilon^{-(|B|+|B'|)/2}\mu_\varepsilon^{|\omega|}\le 1$$

because the partition in ω has no singleton. Notice from (8.24) that each contribution in ${}^cB,{}^cB'$ has an additional factor $\varepsilon^{d/2}$ so that the leading order terms in the power counting (8.25) are associated with $|B| = |B'| = q$ and with partitions ω which are pairing of the sets, as expected from the Gaussian asymptotics.

In conclusion, we arrive to

$$\mathbb{E}_\varepsilon^{\mathrm{eq}}\left[\mu_\varepsilon^q\left(\bigotimes_{j=1}^q\left(\frac{1}{\mu_\varepsilon^{M_j^{r\delta}}}\sum\phi^{(\sigma_j)}-\mathbb{E}_\varepsilon[\phi^{(\sigma_j)}]\right)\right)^2\right] \tag{8.26}$$
$$\leq C_P\prod_{j=1}^q\left((C_P\Theta)^{2M_j+N_{\sigma_j}^{<r}+N_{\sigma_j}^r-|\sigma_j|}(C_P\delta)^{N_{\sigma_j}^r}(C_P\tau)^{N_{\sigma_j}^{<r}+|\sigma_j|-1}\right).$$

A similar proof leads to the estimate

$$\mathbb{E}_\varepsilon^{\mathrm{eq}}\left[\mu_\varepsilon^q\left(\bigotimes_{\substack{j=1\\j\neq i}}^q\left(\frac{1}{\mu_\varepsilon^{M_j^{r\delta}}}\sum\phi^{(\sigma_j)}-\mathbb{E}_\varepsilon[\phi^{(\sigma_j)}]\right)\bigotimes\left(\frac{1}{\mu_\varepsilon^{M_i^{r\delta}}}\sum\phi^{(\sigma_i),\mathrm{cyc}}-\mathbb{E}_\varepsilon[\phi^{(\sigma_i),\mathrm{cyc}}]\right)\right)^2\right] \tag{8.27}$$
$$\leq C_P\varepsilon\delta|\log\varepsilon|(\Theta|\log\varepsilon|)^{2d+4}(C_P\Theta)^{2M_i+N_{\sigma_i}^{<r}+N_{\sigma_i}^r-|\sigma_i|}(C_P\delta)^{(N_{\sigma_i}^r-1)_+}(C_P\tau)^{(N_{\sigma_i}^{<r}+|\sigma_i|-2)_+}$$
$$\times\prod_{j\neq i}(C_P\Theta)^{2M_j+N_{\sigma_j}^{<r}+N_{\sigma_j}^r-|\sigma_j|}(C_P\delta)^{N_{\sigma_j}^r}(C_P\tau)^{N_{\sigma_j}^{<r}+|\sigma_j|-1}.$$

To obtain (8.27), we repeat the above argument leading to (8.26). In what follows, we only discuss the main differences. In the derivation of the product bound (8.12), we used the elementary estimates (7.4), (7.8) where the cluster structure is given by minimally connected graphs. In the case of (8.27), one factor of type $\phi^{(\sigma_i),\mathrm{cyc}}$ is present which satisfies the different estimate

$$|\phi^{(\sigma_i),\mathrm{cyc}}(Z_{\mathbf{M}_i^{r\delta}})|\leq C_P\,|\sigma_i|^{M_i^{r\delta}}\frac{\mu_\varepsilon^{M_i^{r\delta}-1}}{M_i^{r\delta}!}\sum_{\bar{\mathbf{S}},\mathbf{S},\mathbf{K},\varsigma,\lambda,\mathcal{E}}\;\sum_{T_\prec\in\mathcal{T}^\prec_{M_i^{r\delta}}}\mathbf{1}_{\{Z_{\mathbf{M}_i^{r\delta}}\in\mathcal{R}^{\mathrm{comp,rec}}_{T_\prec}\}}.$$

Here the set $\mathcal{R}^{\mathrm{comp,rec}}_{T_\prec}$ is defined as the set $\mathcal{R}^{\mathrm{comp}}_{T_\prec}$, with the additional constraint that the graph encoding all encounters in the forward dynamics should contain at least one edge on $\mathcal{I}_\delta$ and at least one cycle. The construction in step 2 proceeds then as before, but the set $\mathcal{R}^{\mathrm{comp}}$ in (8.12) is replaced by $\mathcal{R}^{\mathrm{comp,rec}}$ if the factor $\phi^{(\sigma_i),\mathrm{cyc}}$ belongs to the skeleton ($i\in A$). This leads to a formula as (8.14) where, depending on B,B', we distinguish several possibilities:

- i belongs to B and B'. The estimates (8.16)-(8.17) are then improved by applying (7.16) (instead of (7.12)), which uses the reinforced geometric condition on the cycle to bring an additional small factor $\varepsilon\delta|\log\varepsilon|(\Theta|\log\varepsilon|)^{2d+4}$. One gets then a contribution as in the right hand side of (8.20) with such an additional smallness.
- i belongs to B and ${}^cB'$ (or viceversa). Similarly, (8.20) is modified by a small factor $(\varepsilon\delta|\log\varepsilon|(\Theta|\log\varepsilon|)^{2d+4})^{1/2}$. An even smaller factor $(\varepsilon^{2d-1}\Theta\varepsilon\delta|\log\varepsilon|(\Theta|\log\varepsilon|)^{2d+4})^{1/2}$ is produced by the estimate of $\mathbb{E}_\varepsilon^{\mathrm{eq}}[\mathbf{1}_{{}^c\Upsilon^\varepsilon_\mathcal{N}}\phi^{(\sigma_i),\mathrm{cyc}}]$ (performed as in (8.23)), thanks to (7.3) and (4.21) in the case of one single factor.
- i belongs to ${}^cB$ and ${}^cB'$. Then we have two factors $\mathbb{E}_\varepsilon^{\mathrm{eq}}[\mathbf{1}_{{}^c\Upsilon^\varepsilon_\mathcal{N}}\phi^{(\sigma_i),\mathrm{cyc}}]$ estimated as previously.

In all the cases we end up with a gain $\varepsilon\delta|\log\varepsilon|(\Theta|\log\varepsilon|)^{2d+4}$, which proves (8.27).

8.3. **Conclusion of the proofs.** In this section, we shall derive (4.17), (4.18) and (4.21) from the analogous results obtained above under the equilibrium measure. Finally we will prove Proposition 2.3.

Proof of (4.17). Recalling (2.12) there holds

$$\begin{aligned}\Big|\mathbb{E}_\varepsilon\Big[\bigotimes_{i=1}^q \zeta^\varepsilon_{M_i^r\delta}\big(\phi^{(\sigma_i)}\big)\Big]\Big| &= \Big|\mathbb{E}_\varepsilon\Big[\bigotimes_{i=1}^q\Big(\zeta^{\varepsilon,\mathrm{eq}}_{M_i^r\delta}\big(\phi^{(\sigma_i)}\big)+\sqrt{\mu_\varepsilon}\,\mathbb{E}^{\mathrm{eq}}_\varepsilon[\mathbf{1}_{{}^c\Upsilon^\varepsilon_{\mathcal N}}\,\pi^\varepsilon_{M_i^r\delta}(\phi^{(\sigma_i)})]\Big)\Big]\Big|\\ &\leq \sum_{A\subset\{1,\dots,q\}}\Big(\Big|\mathbb{E}^{\mathrm{eq}}_\varepsilon\Big[\bigotimes_{i\in A}\zeta^{\varepsilon,\mathrm{eq}}_{M_i^r\delta}\big(\phi^{(\sigma_i)}\big)\Big]\Big|+\Big|\mathbb{E}^{\mathrm{eq}}_\varepsilon\Big[\mathbf{1}_{{}^c\Upsilon^\varepsilon_{\mathcal N}}\bigotimes_{i\in A}\zeta^{\varepsilon,\mathrm{eq}}_{M_i^r\delta}\big(\phi^{(\sigma_i)}\big)\Big]\Big|\Big)\\ &\qquad\times\prod_{j\in A^c}\sqrt{\mu_\varepsilon}\,\Big|\mathbb{E}^{\mathrm{eq}}_\varepsilon[\mathbf{1}_{{}^c\Upsilon^\varepsilon_{\mathcal N}}\,\pi^\varepsilon_{M_j^r\delta}(\phi^{(\sigma_j)})]\Big|\,.\end{aligned}$$

The second line can be bounded by (8.23), while the first term in the first line is bounded by (8.1). Finally the second term in the first line is bounded by

$$\begin{aligned}\mathbb{E}^{\mathrm{eq}}_\varepsilon\Big[\mathbf{1}_{{}^c\Upsilon^\varepsilon_{\mathcal N}}\otimes_{i\in A}\zeta^{\varepsilon,\mathrm{eq}}_{M_i^r\delta}\big(\phi^{(\sigma_i)}\big)\Big] &\leq \mathbb{P}^{\mathrm{eq}}_\varepsilon\Big[{}^c\Upsilon^\varepsilon_{\mathcal N}\Big]^{1/2}\mathbb{E}^{\mathrm{eq}}_\varepsilon\Big[\Big(\otimes_{i\in A}\zeta^{\varepsilon,\mathrm{eq}}_{M_i^r\delta}\big(\phi^{(\sigma_i)}\big)\Big)^2\Big]^{1/2}\\ &\leq \big(\Theta\varepsilon^d\big)^{1/2}C_P\prod_{j\in A}\Big((C_P\Theta)^{2M_j+N^{<r}_{\sigma_j}+N^r_{\sigma_j}-|\sigma_j|}(C_P\delta)^{N^r_{\sigma_j}}(C_P\tau)^{N^{<r}_{\sigma_j}+|\sigma_j|-1}\Big)^{1/2}\end{aligned}$$

where we used (2.10) and the analogue of (8.26) in the simpler case of centered fluctuations. Using that $\big(\Theta\varepsilon^d\big)^{1/2}\ll\varepsilon$ for $d>2$ we obtain that

$$\Big|\mathbb{E}_\varepsilon\Big[\bigotimes_{i=1}^q\zeta^\varepsilon_{M_i^r\delta}\big(\phi^{(\sigma_i)}\big)\Big]\Big|\leq C_q\varepsilon\prod_{i=1}^q\Big((C_P\Theta)^{2M_i+N^{<r}_{\sigma_i}+N^r_{\sigma_i}-|\sigma_i|}(C_P\delta)^{N^r_{\sigma_i}}(C_P\tau)^{N^{<r}_{\sigma_i}+|\sigma_i|-1}\Big)^{1/2}\,,\tag{8.28}$$

which concludes the proof. □

Proof of (4.18) *and* (4.21). Both estimates follow immediately as

$$\mathbb{E}_\varepsilon\Big[\Big(\bigotimes_{i=1}^q\zeta^\varepsilon_{M_i^r\delta}\big(\phi^{(\sigma_i)}\big)\Big)^2\Big]\leq\mathbb{E}^{\mathrm{eq}}_\varepsilon\Big[\Big(\bigotimes_{i=1}^q\zeta^\varepsilon_{M_i^r\delta}\big(\phi^{(\sigma_i)}\big)\Big)^2\Big]\,.$$

□

Proof of Proposition 2.3. Proceeding as before,

$$\begin{aligned}\Big|\mathbb{E}_\varepsilon\Big(\big(\zeta^\varepsilon(h)\big)^p\Big)\Big| &= \Big|\mathbb{E}_\varepsilon\Big(\big(\zeta^{\varepsilon,\mathrm{eq}}(h)+\sqrt{\mu_\varepsilon}\,\mathbb{E}^{\mathrm{eq}}_\varepsilon[\mathbf{1}_{{}^c\Upsilon^\varepsilon_{\mathcal N}}\,\pi^\varepsilon(h)]\big)^p\Big)\Big|\\ &\leq\sum_{k=0}^p\binom{p}{k}\Big(\Big|\mathbb{E}^{\mathrm{eq}}_\varepsilon\Big(\big(\zeta^{\varepsilon,\mathrm{eq}}(h)\big)^k\Big)\Big|+\Big|\mathbb{E}^{\mathrm{eq}}_\varepsilon\Big(\mathbf{1}_{{}^c\Upsilon^\varepsilon_{\mathcal N}}\big(\zeta^{\varepsilon,\mathrm{eq}}(h)\big)^k\Big)\Big|\Big)\,\mu_\varepsilon^{\frac{p-k}{2}}\,\Big|\mathbb{E}^{\mathrm{eq}}_\varepsilon[\mathbf{1}_{{}^c\Upsilon^\varepsilon_{\mathcal N}}\,\pi^\varepsilon(h)]\Big|^{p-k}\\ &\leq\sum_{k=0}^p\binom{p}{k}\Big(C_k\|h\|^k_\infty+\big(\Theta\varepsilon^d\big)^{1/2}\sqrt{C_{2k}}\|h\|^k_\infty\Big)\\ &\qquad\times\mu_\varepsilon^{\frac{p-k}{2}}\Big(\big(\Theta\varepsilon^d\big)\frac{\mathbb{E}^{\mathrm{eq}}_\varepsilon[\mathcal N]}{\mu_\varepsilon}\|h\|_\infty+\mu_\varepsilon^{-1/2}\big(\Theta\varepsilon^d\big)^{1/2}\sqrt{C_2}\|h\|_\infty\Big)^{p-k}\\ &\leq C_p\|h\|^p_\infty\end{aligned}$$

for some constant $C_p>0$, where in the second inequality we used (2.15) (derived in Proposition A.1 from [8]) and (2.10). This proves (2.16).

To prove (2.17), we write

$$\mathbb{E}_\varepsilon\Big[\prod_{p=1}^P\zeta^\varepsilon_{\theta_p}(h^{(p)})\Big]=\sum_{A\subset\{1,\cdots,P\}}\mathbb{E}_\varepsilon\Big[\prod_{p\in A}\zeta^{\varepsilon,\mathrm{eq}}_{\theta_p}(h^{(p)})\Big]\mu_\varepsilon^{|A^c|/2}\Big(\prod_{p\in A^c}\mathbb{E}^{\mathrm{eq}}_\varepsilon[\mathbf{1}_{{}^c\Upsilon^\varepsilon_{\mathcal N}}\,\pi^\varepsilon(h^{(p)})]\Big)\,,$$

from which we get

$$\begin{aligned}\left|I_P^{\varepsilon,\mathrm{eq}} - I_P^\varepsilon\right| &= \left|\mathbb{E}_\varepsilon^{\mathrm{eq}}\left[\prod_{p=1}^P \zeta_{\theta_p}^{\varepsilon,\mathrm{eq}}(h^{(p)})\right] - \mathbb{E}_\varepsilon\left[\prod_{p=1}^P \zeta_{\theta_p}^{\varepsilon}(h^{(p)})\right]\right| \\ &\leq \left|\mathbb{E}_\varepsilon^{\mathrm{eq}}\left[\mathbf{1}_{{}^c\Upsilon_\mathcal{N}^\varepsilon} \prod_{p=1}^P \zeta_{\theta_p}^{\varepsilon,\mathrm{eq}}(h^{(p)})\right]\right| \\ &\qquad + \sum_{\substack{A\subset\{1,\cdots,P\}\\ A^c\neq\varnothing}} \left|\mathbb{E}_\varepsilon\left[\prod_{p\in A} \zeta_{\theta_p}^{\varepsilon,\mathrm{eq}}(h^{(p)})\right]\mu_\varepsilon^{|A^c|/2}\left(\prod_{p\in A^c}\mathbb{E}_\varepsilon^{\mathrm{eq}}\left[\mathbf{1}_{{}^c\Upsilon_\mathcal{N}^\varepsilon}\pi^\varepsilon(h^{(p)})\right]\right)\right| .\end{aligned}$$

Using once again (2.10) and (8.23), together with Hölder's inequality to bound the moments of fluctuation fields, one deduces the estimate

$$\left|I_P^{\varepsilon,\mathrm{eq}} - I_P^\varepsilon\right| \leq C_P\left(\Theta\varepsilon^d\right)^{1/2} .$$

This concludes the proof of Proposition 2.3. □

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