PRECISED HARDY INEQUALITIES

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Abstract. The aim of this article is to present “precised” Hardy-type inequalities. Those inequalities are generalisations of the usual Hardy inequalities, their additional feature being that they are invariant under oscillations: when applied to highly oscillatory functions, both sides of the precised inequality are of the same order of magnitude. The proof relies on paradifferential calculus and Besov spaces. It is also adapted to the case of the Heisenberg group.

1. Introduction

The aim of this article is to prove a “precised” version of the Hardy inequalities [11], [12]. Those inequalities have some importance in Analysis (among other applications we can mention blow-up methods or the study of pseudodifferential operators with singular coefficients). Many works have been devoted to those inequalities, and our goal is first to provide an elementary proof of the standard Hardy inequality, and then to prove a precised inequality in the spirit of the precised Sobolev inequality proved in [10]. The setting will be both the classical $\mathbf{R}^N$ space, as well as the Heisenberg group $\mathbf{H}^d$ (for an application of the Hardy inequality on the Heisenberg group we refer for instance to [1]).

1.1. Elementary Hardy inequality.

The simple case of $\mathbf{R}^N$ with $N \geq 3$ with one derivative gives the following inequality:

$$\int_{\mathbf{R}^N} \frac{u^2(x)}{|x|^2}\, dx \leq C\|\nabla u\|_{L^2}^2. \tag{1.1}$$

In order to prove this inequality, it is enough to observe that we have

$$\frac{1}{|x|^2} = -\frac{1}{2} R\Big(\frac{1}{|x|^2}\Big) \quad \text{with} \quad R = x \cdot \nabla.$$

An integration by parts joint with the fact that the divergence of R is equal to N gives the result.

Let us now present the case of the Heisenberg group. The Heisenberg group $\mathbf{H}^d$ is the space $\mathbf{R}^{2d+1}$ endowed with the following product group law:

$$w \cdot w' = (x + x', y + y', s + s' + (y|x') - (y'|x))$$

where $w = (x, y, s)$ and $w' = (x', y', s')$. Let us notice that $\mathbf{H}^d$ is a non commutative group and that the inverse of w is $w^{-1} = (-x, -y, -s)$. The Lebesgue measure on $\mathbf{H}^d$ seen as $\mathbf{R}^{2d+1}$ is

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invariant by translation with respect to this law. We define the convolution of two functions by

$$f \star g(w') = \int_{\mathbf{H}^d} f(w'w^{-1})g(w)dw.$$

Let us emphasize that this convolution product is, as $\mathbf{H}^d$ itself, not commutative. We say that a vector field X is left invariant if $X(f(a\cdot)) = (Xf)(a\cdot)$. The Lie algebra of left invariant vector fields is spanned by the vector fields

$$X_j = \partial_{x_j} + y_j\partial_s\,,\ Y_j = \partial_{y_j} - x_j\partial_s \quad \text{with } j \in \{1,\dots,d\} \quad \text{and} \quad S = \partial_s = \frac{1}{2}[Y_j, X_j].$$

In all that follows, we shall denote by $\mathcal{Z}$ the family defined by $Z_j = X_j$ and $Z_{j+d} = Y_j$. Let us denote

$$\Delta_{\mathbf{H}} \stackrel{\text{def}}{=} \sum_{j=1}^{2d} Z_j^2 \quad \text{and for } \alpha \in \{1,\dots,2d\}^k, \quad \mathcal{Z}^\alpha \stackrel{\text{def}}{=} Z_{\alpha_1}\dots Z_{\alpha_k}. \tag{1.2}$$

One can associate Sobolev spaces to the system $\mathcal{Z}$ through the following definition.

Definition 1. *Let k be a non negative integer, we denote by $\dot{H}^k(\mathbf{H}^d)$ the homogeneous Sobolev space of order k which is the space of the functions u such that*

$$\|u\|^2_{\dot{H}^k(\mathbf{H}^d)} \stackrel{\text{def}}{=} \sum_{\alpha\in\{1,\dots,2d\}^k} \|\mathcal{Z}^\alpha u\|^2_{L^2(\mathbf{H}^d)} < \infty. \tag{1.3}$$

Let us also introduce the distance to the origin

$$\rho(w) \stackrel{\text{def}}{=} \big((|x|^2 + |y|^2)^2 + s^2\big)^{\frac{1}{4}} \quad \text{with} \quad w = (x, y, s)$$

and the dilation $\delta_\lambda(w) \stackrel{\text{def}}{=} (\lambda x, \lambda y, \lambda^2 s)$. Let us point out that the function ρ is homogenenous of degree 1 in the sense that

$$\rho \circ \delta_\lambda = \lambda\rho$$

and the vector fields Z_j change the homogeneity as

$$Z_j(f \circ \delta_\lambda) = \lambda(Z_j f) \circ \delta_\lambda.$$

Moreover, we have

$$|Z_j \rho^\sigma| \le C_\sigma \rho^{\sigma-1}. \tag{1.4}$$

Let us also introduce the homogeneous dimension $N = 2d + 2$ noticing that the Jacobian of the dilation δ_λ is λ^N. The Hardy inequality with one derivative in this context is

$$\int_{\mathbf{H}^d} \frac{u^2(w)}{\rho^2(w)} dw \le C\|\nabla_{\mathbf{H}} u\|^2_{L^2} \quad \text{where} \quad \nabla_{\mathbf{H}} u \stackrel{\text{def}}{=} (Z_1 u, \dots, Z_{2d} u).$$

The proof (as written for instance in [1]) of this inequality relies mainly on the fact that

$$\frac{1}{\rho^2} = -\frac{1}{2} R\Big(\frac{1}{\rho^2}\Big) \quad \text{with} \quad R \stackrel{\text{def}}{=} \sum_{j=1}^{d} (x_j X_j + y_j Y_j) + 2s\partial_s.$$

An integration by parts and the fact that $\operatorname{div} R = N$ essentially gives the result.

1.2. **More general Hardy inequalities.** Now we want to state Hardy inequalities with any number of derivatives less than $N/2$.

Theorem 1. *Let $s\in]0,N/2[$. There exists a constant C such that*

$$\int_{\mathbf{R}^N}\frac{|u|^2(x)}{|x|^{2s}}dx\leq C\|u\|^2_{\dot H^s(\mathbf{R}^N)}\quad\text{and}\quad\int_{\mathbf{H}^d}\frac{|u|^2(w)}{\rho^{2s}(w)}dw\leq C\|u\|^2_{\dot H^s(\mathbf{H}^d)}$$

where the spaces $\dot H^s$ are defined by complex interpolation.

Classically, the way of proving this consists in proving that the operators

$$\frac{1}{|x|^s}(-\Delta)^{-\frac{s}{2}}\quad\text{or}\quad\frac{1}{\rho^s}(-\Delta_{\mathbf{H}})^{-\frac{s}{2}}$$

are bounded on $L^2(\mathbf{R}^N)$ or $L^2(\mathbf{H}^d)$. The purpose of this paper is first to give a more direct proof of these inequalities, which will be the same for $\mathbf{R}^N$ or $\mathbf{H}^d$. Moreover, in the case of $\mathbf{R}^N$, let us apply the above Hardy inequality with $s=1$ to the family $(f_\varepsilon)_{\varepsilon>0}$ of functions defined by

$$f_\varepsilon(x)=e^{i\frac{x_1}{\varepsilon}}\theta(x)$$

where θ is a given function in the Schwartz class $\mathcal{S}(\mathbf{R}^N)$. The left-hand side of the inequality is obviously independent of ε and the right-hand side is of order ε^{-1}. The second purpose of this paper is to improve Hardy inequalities into inequalities which in particular will be invariant under the multiplication by oscillating functions like $e^{i\frac{(x|\omega)}{\varepsilon}}$.

This requires the introduction of Besov spaces of negative index and thus Littlewood Paley theory. In the case of $\mathbf{R}^N$, this is quite classical. In the case of the Heisenberg group, it was constructed by H. Bahouri, P. Gérard and C.-J. Xu in [2] (see also [3]). We can summarize this theory in the following properties, which hold regardless of the space which can be $\mathbf{R}^N$ or $\mathbf{H}^d$; one of the features of this paper is to write unified statements and proofs, which hold independently of the space. It is therefore natural to introduce unified notation. In the same way as on the Heisenberg group we have defined a family $\mathcal{Z}$ of vector fields, we will denote on $\mathbf{R}^N$

$$\text{for }\alpha\in\{1,\dots,N\}^k,\quad\mathcal{Z}^\alpha\stackrel{\text{def}}{=}X_{\alpha_1}\dots X_{\alpha_k},\quad\text{where}\quad X_{\alpha_j}\stackrel{\text{def}}{=}\partial_{x_{\alpha_j}}.$$

We will also use the following notation:

$$\forall w\in\mathbf{R}^N,\quad w^{-1}=-w,\quad\rho(w)\stackrel{\text{def}}{=}\left(\sum_{j=1}^N|w_j|^2\right)^{\frac12},\quad\text{and}\quad\forall a\in\mathbf{R},\ \delta_a w=aw.$$

Using that notation, the elements of Littlewood-Paley theory we will need are the following.

Both in the case of $\mathbf{R}^N$ and $\mathbf{H}^d$, there exists a family $(S_j)_{j\in\mathbf{Z}}$ of operators such that for any p belonging to $[1,\infty[$,

$$\forall u\in L^p,\ \lim_{j\to-\infty}\|S_ju\|_{L^p}=0\quad\text{and}\quad\lim_{j\to\infty}\|S_ju-u\|_{L^p}=0.\tag{1.5}$$

Moreover, for any multi-index α, there exists a constant C such that, for any $(p,q)\in[1,\infty]^2$ satisfying $p\leq q$, we have

$$\|\mathcal{Z}^\alpha S_ju\|_{L^q}\leq C2^{jN\left(\frac1p-\frac1q\right)+|\alpha|j}\|S_ju\|_{L^p}.\tag{1.6}$$

Moreover, if $\Delta_j \stackrel{\text{def}}{=} S_{j+1} - S_j$, *two integers* N_0 *and* N_1 *exist such that*

$$|j-j'| \geq N_0 \Longrightarrow \Big(\Delta_j \Delta_{j'} = 0 \quad \text{and} \quad \Delta_j \left(S_{j'-N_0} u \Delta_{j'} v\right) = 0\Big), \tag{1.7}$$

$$\Big(|k-k'| \leq N_0 \quad \text{and} \quad j \geq k + N_1\Big) \Longrightarrow \Delta_j \left(\Delta_k u \Delta_{k'} v\right) = 0. \tag{1.8}$$

For any positive integer k, *there exists a constant* C *such that, for any* $p \in [1,\infty]$,

$$\|\Delta_j u\|_{L^p} \leq C 2^{-2jk} \|(-\Delta)^k \Delta_j u\|_{L^p}. \tag{1.9}$$

The operators Δ_j *are of the form*

$$\Delta_j u = u \star h_j \quad \text{with} \quad h_j(w) = 2^{jN} h(\delta_{2^j} w) \quad \text{and} \quad h \in \mathcal{S}. \tag{1.10}$$

We remark that as Δ_j is a function of the Laplacian (resp. sublaplacian) on $\mathbf{R}^N$ (resp. $\mathbf{H}^d$), it commutes with the latter operator.

Definition 2. *Let* $s \in \mathbf{R}$ *be given, as well as* p *and* r, *two real numbers in the interval* $[1,\infty]$. *Then we define the space* $\dot{B}^s_{p,r}$ *of tempered distributions* u *such that*

$$\lim_{j\to-\infty} S_j u = 0 \quad \text{and} \quad \|u\|_{\dot{B}^s_{p,r}} \stackrel{\text{def}}{=} \left\|\left(2^{js}\|\Delta_j u\|_{L^p}\right)\right\|_{\ell^r(\mathbf{Z})} < \infty.$$

Let us notice that Inequality (1.6) implies immediately that, when $q \geq p$ and $r' \geq r$, we have

$$\|u\|_{\dot{B}^{s-N\left(\frac{1}{p}-\frac{1}{q}\right)}_{q,r'}} \leq C\|u\|_{\dot{B}^s_{p,r}}. \tag{1.11}$$

The result we will prove is the following. It is stated and proved indifferently in $\mathbf{R}^N$ and $\mathbf{H}^d$.

Theorem 2. *Let* s *be a real number in the interval* $]0, N/2[$ *and let* p *and* q *be two real numbers in* $[1,\infty]$ *such that*

$$2 \leq q < \frac{2N}{N-2s} < p \leq \infty.$$

There is a constant C *such that, for any function* $u \in \dot{B}^{s-N(\frac{1}{2}-\frac{1}{q})}_{q,2}$, *the following inequality holds:*

$$\left(\int \frac{|u(w)|^2}{\rho^{2s}(w)}\, dw\right)^{\frac{1}{2}} \leq C\|u\|^{\alpha}_{\dot{B}^{s-N\left(\frac{1}{2}-\frac{1}{p}\right)}_{p,2}} \|u\|^{1-\alpha}_{\dot{B}^{s-N\left(\frac{1}{2}-\frac{1}{q}\right)}_{q,2}} \quad \text{with} \quad \alpha = \frac{pq}{p-q}\Big(\frac{1}{q} - \frac{1}{2} + \frac{s}{N}\Big)\cdot$$

Let us remark that, when $p=\infty$ and $q=2$, the above theorem implies that

$$\left(\int \frac{|u(w)|^2}{\rho^{2s}(w)}\, dw\right)^{\frac{1}{2}} \leq C\|u\|^{\frac{2s}{N}}_{\dot{B}^{s-\frac{N}{2}}_{\infty,2}} \|u\|^{1-\frac{2s}{N}}_{\dot{H}^s}. \tag{1.12}$$

This inequality should be compared to the following similar result derived by P. Gérard, Y. Meyer and F. Oru in [10], in the case of the Sobolev inequalities on $\mathbf{R}^N$ (see [3] for the Heisenberg case), namely

$$\|u\|_{L^r} \leq C\|u\|^{\frac{2s}{N}}_{\dot{B}^{s-\frac{N}{2}}_{\infty,\infty}} \|u\|^{1-\frac{2s}{N}}_{\dot{H}^s} \quad \text{with} \quad \frac{1}{r} = \frac{1}{2} - \frac{s}{N}\cdot \tag{1.13}$$

The following result indicates the invariance of (1.12) and (1.13) under oscillations.

Proposition 1. *Let θ be a function in $\mathcal{S}$, p in $[1,\infty]$, σ in $]-N(1-1/p),+\infty[$ and ε_0 a positive real number. There exists a constant C such that the oscillatory function $f_\varepsilon(w) \stackrel{\text{def}}{=} \theta(w)e^{iw_1/\varepsilon}$ satisfies*

$$\forall \varepsilon \leq \varepsilon_0\,,\ \|f_\varepsilon\|_{\dot{B}^\sigma_{p,1}} \leq C\varepsilon^{-\sigma}. \tag{1.14}$$

This proposition implies immediately the following corollary.

Corollary 1. *There exists a family $(f_\varepsilon)_{\varepsilon>0}$ of smooth functions such that, for any s in $]0,N/2[$ and any $\beta > 2s/N$, we have*

$$\lim_{\varepsilon\to 0} \frac{\|f_\varepsilon\|_{L^{\frac{2N}{N-2s}}}}{\|f_\varepsilon\|^\beta_{\dot{B}^{s-\frac{N}{2}}_{\infty,\infty}}\|f_\varepsilon\|^{1-\beta}_{\dot{H}^s}} = +\infty \quad \text{and} \quad \lim_{\varepsilon\to 0} \frac{1}{\|f_\varepsilon\|^\beta_{\dot{B}^{s-\frac{N}{2}}_{\infty,\infty}}\|f_\varepsilon\|^{1-\beta}_{\dot{H}^s}} \int \frac{f_\varepsilon^2}{\rho^{2s}}dw = +\infty.$$

1.3. **Structure of the paper and idea of the proof.** The idea of the proof of Theorems 1 and 2 is to see them from a non linear point of view. More precisely, we write

$$\int \frac{u^2(w)}{\rho^{2s}(w)}dw = \langle \rho^{-2s}, u^2\rangle.$$

Then it is enough to prove that ρ^{-2s} and u^2 belongs to a pair of spaces in duality.

In the second section, we shall prove that ρ^{-2s} belongs to the space $\dot{B}^{N-2s}_{1,\infty}$. Then using product law, we shall conclude the proof of Theorem 1.

In the third section, we shall use paradifferential calculus to prove Theorem 2.

In the fourth section, we shall prove Proposition 1. We shall also investigate if it is possible to extend Corollary 1 for a family of non negative functions.

2. The behavior of negative powers of ρ

It is described by the following proposition.

Proposition 2. *Let s be a real number in the interval $]0,N/2[$. Then the function ρ^{-2s} belongs to the Besov space $\dot{B}^{N-2s}_{1,\infty}$.*

Proof of Proposition 2 Let us introduce a smooth compactly supported function χ which is identically equal to 1 near the unit ball and let us write

$$\rho^{-2s} = \rho_0 + \rho_1 \quad \text{with} \quad \rho_0 \stackrel{\text{def}}{=} \chi\rho^{-2s} \quad \text{and} \quad \rho_1 \stackrel{\text{def}}{=} (1-\chi)\rho^{-2s}.$$

It is obvious that $\rho^{-2s} \in L^1 + L^q$ with $q > N/2s$ which implies that $\lim\limits_{j\to-\infty} S_j\rho^{-2s} = 0$ in L^1+L^q. Then, the homogeneity of the function ρ gives

$$\begin{aligned} \Delta_j\rho^{-2s} &= 2^{jN}\rho^{-2s} \star h(\delta_{2^j}\cdot) \\ &= 2^{j(N+2s)}\rho^{-2s}(\delta_{2^j}\cdot) \star h(\delta_{2^j}\cdot) \\ &= 2^{2js}(\Delta_0\rho^{-2s})(\delta_{2^j}\cdot). \end{aligned}$$

Therefore $\|\Delta_j\rho^{-2s}\|_{L^1} = 2^{j(2s-N)}\|\Delta_0\rho^{-2s}\|_{L^1}$ which reduces the problem to proving that the function $\Delta_0\rho^{-2s}$ is in L^1. As ρ_0 is in L^1, $\Delta_0\rho_0$ is also in L^1 thanks to the continuity of the operator Δ_0 on Lebesgue spaces. In order to estimate ρ_1 in L^1, we shall use Inequality (1.9) to write that

$$\|\Delta_0\rho_1\|_{L^1} \leq C_k\|(-\Delta)^k\Delta_0\rho_1\|_{L^1} \leq C_k\|(-\Delta)^k\rho_1\|_{L^1}.$$

By the Leibniz formula, $(-\Delta)^k\rho_1 - (1-\chi)(-\Delta)^k\rho$ is a smooth compactly supported function. Then, we achieve the proof by using (1.4) and choosing k such that $2k > N - 2s$.

As an application, we shall prove Theorem 1. When u belongs to $\dot{H}^s$, then

$$u^2 \in \dot{B}^{2s-\frac{N}{2}}_{2,1} \quad \text{and} \quad \|u^2\|_{\dot{B}^{2s-\frac{N}{2}}_{2,1}} \leq C\|u\|^2_{\dot{H}^s}.$$

That result is classical in $\mathbf{R}^N$ and was proved in $\mathbf{H}^d$ by two of the authors in [3]. Now writing that

$$\langle\rho^{-2s}, u^2\rangle = \sum_{|j-j'|\leq N_0} \langle\Delta_j\rho^{-2s}, \Delta_{j'}u^2\rangle,$$

we infer, thanks to Proposition 2 and embeddings (1.11), that

$$\langle\rho^{-2s}, u^2\rangle \leq \|u\|^2_{H^s} \sum_{|j-j'|\leq N_0} 2^{-j\left(\frac{N}{2}-2s\right)} d_{j'} 2^{-j'\left(2s-\frac{N}{2}\right)} \quad \text{with} \quad (d_j)_{j\in\mathbf{Z}} \in \ell^1(\mathbf{Z}).$$

This proves Theorem 1.

Remark Let us point out that, in Theorem 1, the function ρ^{-2s} can be any function in $\dot{B}^{\frac{N}{2}-2s}_{2,\infty}$.

3. Paradifferential calculus and precised inequalities

In order to prove Theorem 2, let us recall the paraproduct algorithm introduced by J.-M. Bony in [4] in the case of $\mathbf{R}^N$ and by two of the authors in the case of $\mathbf{H}^d$ in [3]. In both cases, this allows to write that

$$u^2 = 2T_u u + R(u,u), \quad \text{with} \quad T_u u \stackrel{\text{def}}{=} \sum_j S_{j-N_0}u\Delta_j u \quad \text{and} \quad R(u,u) \stackrel{\text{def}}{=} \sum_{|j-j'|\leq N_0} \Delta_j u\Delta_{j'}u.$$

Using (1.7) and (1.6), we get

$$\begin{aligned}
\|\Delta_j T_u u\|_{L^\infty} &= \Big\|\sum_{|j-j'|\leq N_0} \Delta_j(S_{j'-N_0}u\Delta_{j'}u)\Big\|_{L^\infty}\\
&\leq \sum_{|j-j'|\leq N_0} \|S_{j'-N_0}u\|_{L^\infty}\|\Delta_{j'}u\|_{L^\infty}.
\end{aligned}$$

Now let us write that

$$2^{-j\left(\frac{N}{2}-s\right)}\|S_j u\|_{L^\infty} \leq \sum_{k\leq j-1} 2^{(j-k)\left(s-\frac{N}{2}\right)}2^{k\left(s-\frac{N}{2}\right)}\|\Delta_k u\|_{L^\infty}.$$

Young's inequality on series implies that

$$\|S_j u\|_{L^\infty} \leq Cc_j 2^{j\left(\frac{N}{2}-s\right)}\|u\|_{\dot{B}^{s-\frac{N}{2}}_{\infty,2}} \quad \text{with} \quad \sum_j c_j^2 = 1.$$

This gives

$$
\begin{aligned}
\|\Delta_j(T_u u)\|_{L^\infty} &\leq C\|u\|_{\dot B^{s-\frac{N}{2}}_{\infty,2}} \sum_{j'=j-N_0}^{j+N_0} 2^{j'(N-2s)} c_{j'} 2^{-j'\left(\frac{N}{2}-s\right)} \|\Delta_{j'} u\|_{L^\infty}\\
&\leq C\|u\|^2_{\dot B^{s-\frac{N}{2}}_{\infty,2}} 2^{j(N-2s)} \sum_{j'=j-N_0}^{j+N_0} d_{j'} \quad \text{with} \quad \sum_{j'} d_{j'} = 1.
\end{aligned}
$$

Thanks to (1.11), and Proposition 2, we have,

$$
\langle \rho^{-2s}, T_u u\rangle \leq C\|u\|^{2\alpha}_{\dot B^{s-N\left(\frac{1}{2}-\frac{1}{q}\right)}_{q,2}} \|u\|^{2-2\alpha}_{\dot B^{s-N\left(\frac{1}{2}-\frac{1}{p}\right)}_{p,2}} \tag{3.1}
$$

for any $0 \leq \alpha \leq 1$ and $p, q \geq 1$.

The estimate of $\langle \rho^{-2s}, R(u,u)\rangle$ relies on the following elementary interpolation lemma.

Proposition 3. *Let s be a real number in the interval $]0, N/2[$ and let p and q be two real numbers in $[1,\infty]$ such that*

$$
2 \leq q < \frac{2N}{N-2s} < p \leq \infty.
$$

There is a constant C such that for any functions f and g which belongs to $L^p \cap L^q$, we have

$$
\langle \rho^{-2s}, fg\rangle \leq C\|f\|^{\alpha}_{L^p}\|g\|^{\alpha}_{L^p}\|f\|^{1-\alpha}_{L^q}\|g\|^{1-\alpha}_{L^q} \quad \textit{with} \quad \alpha = \frac{pq}{p-q}\Big(\frac{1}{q} - \frac{1}{2} + \frac{s}{N}\Big)\cdot
$$

Proof of Proposition 3 Let us write that, for any positive R,

$$
\langle \rho^{-2s}, fg\rangle = I_1(R) + I_2(R) \quad \text{with} \quad I_1(R) \stackrel{\text{def}}{=} \int_{(\rho\leq R)} \frac{fg}{\rho^{2s}} dw \quad \text{and} \quad I_2(R) \stackrel{\text{def}}{=} \int_{(\rho\geq R)} \frac{fg}{\rho^{2s}} dw.
$$

The condition on p and q implies that ρ^{-2s} is locally $L^{\frac{p}{p-2}}$ and is $L^{\frac{q}{q-2}}$ outside any compact neighbourhood of 0. By Hölder's inequality, we infer that

$$
I_1(R) \leq \|\mathbf{1}_{(\rho\leq R)}\rho^{-2s}\|_{L^{\frac{p}{p-2}}} \|f\|_{L^p}\|g\|_{L^p} \quad \text{and} \quad I_2(R) \leq \|\mathbf{1}_{(\rho\geq R)}\rho^{-2s}\|_{L^{\frac{q}{q-2}}} \|f\|_{L^q}\|g\|_{L^q}.
$$

As the function ρ is homogeneous of order 1, we get, by the change of variable $w' = \delta_{R^{-1}} w$,

$$
\begin{aligned}
\|\mathbf{1}_{(\rho\leq R)}\rho^{-2s}\|_{L^{\frac{p}{p-2}}} &= R^{N-2s-\frac{2N}{p}} \|\mathbf{1}_{(\rho\leq 1)}\rho^{-2s}\|_{L^{\frac{p}{p-2}}} \quad \text{and}\\
\|\mathbf{1}_{(\rho\geq R)}\rho^{-2s}\|_{L^{\frac{q}{q-2}}} &= R^{N-2s-\frac{2N}{q}} \|\mathbf{1}_{(\rho\geq 1)}\rho^{-2s}\|_{L^{\frac{q}{q-2}}}.
\end{aligned}
$$

Thus we have, for any positive R,

$$
\langle \rho^{-2s}, fg\rangle \leq CR^{N-2s}\Big(R^{-\frac{2N}{p}}\|f\|_{L^p}\|g\|_{L^p} + R^{-\frac{2N}{q}}\|f\|_{L^q}\|g\|_{L^q}\Big).
$$

Choosing the optimal

$$
R = \left(\frac{\|f\|_{L^q}\|g\|_{L^q}}{\|f\|_{L^p}\|g\|_{L^p}}\right)^{\frac{pq}{2N(p-q)}}
$$

concludes the proof of the proposition.

Let us go back to the proof of Theorem 2. By definition of $R(u,u)$, we have

$$\langle \rho^{-2s}, R(u,u)\rangle = \sum_{|\ell|\leq N_0}\sum_{j\in\mathbf{Z}}\langle \rho^{-2s}, \Delta_j u \Delta_{j-\ell}u\rangle.$$

Proposition 3 implies that

$$\langle \rho^{-2s}, R(u,u)\rangle \leq \sum_{|\ell|\leq N_0}\sum_{j\in\mathbf{Z}}\Big(2^{2j(s-N(\frac12-\frac1p))}\|\Delta_j u\|_{L^p}\|\Delta_{j-\ell}u\|_{L^p}\Big)^{\alpha}$$
$$\times\left(2^{2j(s-N(\frac12-\frac1q))}\|\Delta_j u\|_{L^q}\|\Delta_{j-\ell}u\|_{L^q}\right)^{1-\alpha}.$$

By definition of the Besov norms, this implies that two series $(c_j)_{j\in\mathbf{Z}}$ and $(c'_j)_{j\in\mathbf{Z}}$ exist in the unit sphere of $\ell^2(\mathbf{Z})$ such that

$$\langle \rho^{-2s}, R(u,u)\rangle \leq C\|u\|^{2\alpha}_{B^{s-N(\frac12-\frac1p)}_{p,2}}\|u\|^{2(1-\alpha)}_{B^{s-N(\frac12-\frac1q)}_{q,2}}\sum_{|\ell|\leq N_0}\sum_{j\in\mathbf{Z}}(c_jc_{j-\ell})^{\alpha}(c'_jc'_{j-\ell})^{1-\alpha}.$$

From Hölder inequalities, it follows

$$\langle \rho^{-2s}, R(u,u)\rangle \leq C\|u\|^{2\alpha}_{B^{s-N(\frac12-\frac1p)}_{p,2}}\|u\|^{2(1-\alpha)}_{B^{s-N(\frac12-\frac1q)}_{q,2}}.$$

Together with (3.1), this gives Theorem 2.

4. Oscillations and fractal transforms in precised inequalities

The purpose of this section is to provide examples which show that the precised estimates are sharp. The first one deals with oscillating functions.

4.1. **Oscillations.** Here we want to prove Proposition 1, namely, by definition of Besov spaces, that for any function θ in $\mathcal{S}$, we have

$$\sum_j 2^{j\sigma}\|\Delta_j f_\varepsilon\|_{L^p} \leq C\varepsilon^{-\sigma} \quad\text{with}\quad f_\varepsilon(w) \stackrel{\text{def}}{=} e^{i\frac{w_1}{\varepsilon}}\theta(w). \tag{4.1}$$

We shall treat differently the high frequencies (indices j such that $2^j\varepsilon$ is greater than 1) and low frequencies (indices j such that $2^j\varepsilon$ is less than 1).

Let us first estimate the low frequencies. Denoting the vector $\widetilde{Z}_1 = \partial_{x_1}$ in the case of $\mathbf{R}^N$ and $\widetilde{Z}_1 = \partial_{x_1} - y_1\partial_s$ in the case of $\mathbf{H}^d$, we have

$$i\varepsilon\widetilde{Z}_1 e^{i\frac{w_1}{\varepsilon}} = -e^{i\frac{w_1}{\varepsilon}}.$$

By integration by parts, we get

$$\begin{aligned}
\Delta_j f_\varepsilon(w') &= (-i\varepsilon)^N 2^{jN}\int(-\widetilde{Z}_1)^N(e^{i\frac{w'_1-w_1}{\varepsilon}})\theta(w'w^{-1})h(\delta_{2^j}(w))dw\\
&= (-i\varepsilon)^N 2^{jN}\int e^{i\frac{w'_1-w_1}{\varepsilon}}(\widetilde{Z}_1)^N(\theta(w'w^{-1})h(\delta_{2^j}(w))dw\\
&= (-i\varepsilon)^N 2^{jN}\sum_{\ell=0}^{N}C_N^{\ell}\int e^{i\frac{w'_1-w_1}{\varepsilon}}\widetilde{Z}_1^{N-\ell}(\theta(w'w^{-1}))\widetilde{Z}_1^{\ell}(h(\delta_{2^j}(w))dw
\end{aligned}$$

where the vector field $\widetilde{Z}_1$ acts on the variable w. As

$$\widetilde{Z}_1(h(\delta_{2^j}(w)) = 2^j(\widetilde{Z}_1 h)(\delta_{2^j}(w)) \quad \text{and} \quad -\widetilde{Z}_1(\theta(w'w^{-1})) = (Z_1\theta)(w'w^{-1}),$$

we infer that

$$\begin{aligned} |\Delta_j f_\varepsilon(w')| &= \varepsilon^N 2^{jN}\Big|\sum_{\ell=0}^{N} C_N^\ell 2^{j\ell}(-1)^{N-\ell}\int e^{i\frac{w_1'-w_1}{\varepsilon}}(Z_1^{N-\ell}\theta)(w'w^{-1})(\widetilde{Z}_1^\ell h)(\delta_{2^j}(w))dw\Big| \\ &\leq \varepsilon^N 2^{jN}\sum_{\ell=0}^{N} C_N^\ell 2^{j\ell}\left(|Z_1^{N-\ell}\theta|\star|(\widetilde{Z}_1^\ell h)(\delta_{2^j}\cdot)|\right)(w'). \end{aligned}$$

Young inequalities imply that

$$\Big\||Z_1^{N-\ell}\theta|\star|(\widetilde{Z}_1^\ell h)(\delta_{2^j}\cdot)|\Big\|_{L^p} \leq \min\Big\{2^{-j\frac{N}{p}}\|Z_1^{N-\ell}\theta\|_{L^1}\|\widetilde{Z}_1^\ell h\|_{L^p}\,,\ 2^{-jN}\|Z_1^{N-\ell}\theta\|_{L^p}\|\widetilde{Z}_1^\ell h\|_{L^1}\Big\}.$$

Therefore, as $\sigma > -N\Big(1-\dfrac{1}{p}\Big)$,

$$\begin{aligned} \sum_{2^j\leq\frac{1}{\varepsilon}} 2^{j\sigma}\|\Delta_j f_\varepsilon\|_{L^p} &\leq C\varepsilon^N\bigg(\sum_{2^j\leq 1} 2^{j\left(\sigma+N\left(1-\frac{1}{p}\right)\right)} + \sum_{1\leq 2^j\leq\frac{1}{\varepsilon}} 2^{j(\sigma+N)}\bigg) \\ &\leq C\varepsilon^{-\sigma}. \end{aligned}$$

In order to estimate high frequencies, let us use (1.9). We get, for any non negative integer M,

$$\begin{aligned} \|\Delta_j f_\varepsilon\|_{L^p} &\leq C2^{j(N-2M)}\|((-\Delta)^M f_\varepsilon)\star h(\delta_{2^j}\cdot)\|_{L^p} \\ &\leq C2^{-2jM}\|(-\Delta)^M f_\varepsilon\|_{L^p}. \end{aligned}$$

The Leibniz formula implies that, for any $\varepsilon\in]0,\varepsilon_0]$, $\|(-\Delta)^M f_\varepsilon\|_{L^p}\leq C\varepsilon^{-2M}\|\theta\|_{L^p}$. Thus we infer, thanks to (1.6), that

$$\sum_{2^j\geq\frac{1}{\varepsilon}} 2^{j\sigma}\|\Delta_j f_\varepsilon\|_{L^p} \leq C\varepsilon^{-2M}\sum_{2^j\geq\frac{1}{\varepsilon}} 2^{j(\sigma-2M)}.$$

Choosing M such that $\sigma-2M$ is negative gives

$$\sum_{2^j\geq\frac{1}{\varepsilon}} 2^{j\sigma}\|\Delta_j f_\varepsilon\|_{L^p} \leq C\varepsilon^{-\sigma}. \tag{4.2}$$

This ends the proof of Proposition 1.

4.2. **Fractal transform and Besov norms.** In this subsection we will show that oscillations are not the sole responsible for the smallness of a Besov norm. Below we present another situation, of a sequence of *non negative* functions for which the L^p norms and the Besov norms are balanced as the family (f_ε) of Proposition 1. Again, we shall present statements and proofs common to the case of $\mathbf{R}^N$ and $\mathbf{H}^d$. In order to do so, let us define the distance d as

$$\forall (w,w')\in\mathbf{R}^N\times\mathbf{R}^N,\quad d(w,w') \stackrel{\text{def}}{=} \max_{1\leq j\leq N}|w_j-w_j'|$$

and, for any $(w,w')\in\mathbf{H}^d\times\mathbf{H}^d$,

$$d(w,w') \stackrel{\text{def}}{=} \max\Big\{\max_{1\leq j\leq d}|x_j-x_j'|\,,\ \max_{1\leq j\leq N}|y_j-y_j'|\,,\ |s-s'+(y'|x)-(y|x')|^{\frac{1}{2}}\Big\},$$

where in the case of $\mathbf{H}^d$ we have noted $w=(x,y,s)$ and $w'=(x',y',s')$. Let us denote by Q the ball for d centered at zero and of radius $1/2$. Now let us define the following quantities. Let D and L such that $D=L=N$ in the case of $\mathbf{R}^N$ and $D=N-1$ and $L=N+1$ in the case of $\mathbf{H}^d$. For J in $\{-1,1\}^L$, we define the point w_J of Q and the cube Q_J by

$$w_J \stackrel{\text{def}}{=} \delta_{\frac{3}{8}} J \quad \text{and} \quad Q_J \stackrel{\text{def}}{=} w_J \cdot \delta_{\frac{1}{4}} Q = \Big\{ w \,/\, d(w,w_J) \leq \frac{1}{8} \Big\}.$$

Omitted elementary computations show that

$$Q_J \subset Q \quad \text{and} \quad \Big(J \neq J' \Longrightarrow d(Q_J, Q_{J'}) \geq \frac{\sqrt{3}}{4} \Big). \tag{4.3}$$

Now let us define the transform T which duplicates (after dilation and translation) functions defined on Q.

Definition 3. *Let us denote by T the following transform*

$$T \left\{ \begin{array}{ccl} \mathcal{D}(Q) & \to & \mathcal{D}(Q) \\ f & \mapsto & Tf \stackrel{\text{def}}{=} 2^D \displaystyle\sum_{J \in \{-1,1\}^L} f_J \qquad \textit{with} \quad f_J(w) \stackrel{\text{def}}{=} f(\delta_4(w_J^{-1} w)). \end{array} \right.$$

For a subset A of Q , we denote by TA the set defined by

$$TA \stackrel{\text{def}}{=} \bigcup_{J \in \{-1,1\}^L} w_J \delta_{\frac{1}{4}} A.$$

Let us notice that $TA \subset Q$ and that Supp $(Tf) = T(\text{Supp } f)$. Let us also observe that, using (4.3), we have

$$\begin{aligned} \|Tf\|_{L^p}^p &= 2^{Dp} \sum_{J \in \{-1,1\}^L} \|f_J\|_{L^p}^p \\ &= 2^{Dp} \Big(\sum_{J \in \{-1,1\}^L} 2^{-2N} \Big) \|f\|_{L^p}^p \\ &= 2^{Dp+L-2N} \|f\|_{L^p}^p \\ &= 2^{D(p-1)} \|f\|_{L^p}^p. \end{aligned}$$

Thus, we have

$$\|Tf\|_{L^p} = 2^{D\left(1-\frac{1}{p}\right)} \|f\|_{L^p}. \tag{4.4}$$

The way T acts on Besov spaces is described by the following proposition.

Proposition 4. *For any $(p,r) \in [1,+\infty]^2$ and any σ in $\Big]-N\Big(1-\dfrac{1}{p}\Big), +\infty\Big[$, there exists a constant C such that*

$$\|Tf\|_{\dot{B}^\sigma_{p,r}} \leq 2^{D\left(1-\frac{1}{p}\right)+2\sigma} \|f\|_{\dot{B}^\sigma_{p,r}} + C \|f\|_{L^1}.$$

Proof of Proposition 4 For the sake of simplicity, we only prove this proposition in the case when $r=1$. By definition of the Besov norm, we have

$$\|Tf\|_{\dot{B}^\sigma_{p,1}} = T_1 f + T_2 f \quad \text{with} \quad T_1 f \stackrel{\text{def}}{=} \sum_{j \leq 0} 2^{j\sigma} \|\Delta_j Tf\|_{L^p} \quad \text{and} \quad T_2 f \stackrel{\text{def}}{=} \sum_{j \geq 1} 2^{j\sigma} \|\Delta_j Tf\|_{L^p}.$$

On the one hand, using Bernstein's inequality (1.6), the fact that $N\Big(1-\dfrac{1}{p}\Big)+\sigma>0$ and (4.4) with $p=1$, we get

$$\begin{aligned} T_1 f &\leq C\sum_{j\leq 0} 2^{j\sigma+jN\left(1-\frac{1}{p}\right)}\|\Delta_j Tf\|_{L^1}\\ &\leq C\|Tf\|_{L^1}\\ &\leq C\|f\|_{L^1}. \end{aligned}\tag{4.5}$$

The estimate on $T_2 f$ uses the special structure of TQ. Let us define the set

$$\widetilde{Q}\stackrel{\rm def}{=}\Big\{w\,/\,d(w,TQ)\leq\frac{1}{32}\Big\}=\bigcup_{J\in\{-1,1\}^L}\widetilde{Q}_J\quad\text{with}\quad \widetilde{Q}_J\stackrel{\rm def}{=}\Big\{w\,/\,d(w,Q_J)\leq\frac{1}{32}\Big\}.$$

Now let us write that $T_2 f = T_{21} f + T_{22} f$ with

$$T_{21} f\stackrel{\rm def}{=}\sum_{j\geq 1}2^{j\sigma}\|\Delta_j Tf\|_{L^p({}^c\widetilde{Q})}\quad\text{and}\quad T_{22} f\stackrel{\rm def}{=}\sum_{j\geq 1}2^{j\sigma}\|\Delta_j Tf\|_{L^p(\widetilde{Q})}.$$

Let us recall that

$$(\Delta_j Tf)(w')=2^{jN}\int Tf(w)h\left(\delta_{2^j}(w^{-1}w')\right)\,dw.$$

As h belongs to $\mathcal{S}$, we have, for any positive integer M, that $|h(w)|\leq C_M\,(1+\rho(w))^{-M}$. Thus, by homogeneity and by definition of ρ and d, we get, for all $(w,w')\in TQ\times\widetilde{Q}^c$,

$$\begin{aligned} \left|h(\delta_{2^j}(w^{-1}w'))\right| &\leq C_M\left(1+\rho(\delta_{2^j}(w^{-1}w'))\right)^{-N-1}\rho^{-M}(\delta_{2^j}(w^{-1}w'))\\ &\leq C_M\left(1+\rho(\delta_{2^j}(w^{-1}w'))\right)^{-N-1}2^{-jM}\rho^{-M}(w^{-1}w')\\ &\leq C_M 2^{-jM}\left(1+\rho(\delta_{2^j}(w^{-1}w'))\right)^{-N-1}. \end{aligned}$$

Using (4.4), we infer that, for any integer M,

$$\begin{aligned} \|\Delta_j Tf\|_{L^p({}^c\widetilde{Q})} &\leq C_M 2^{j\left(N\left(1-\frac{1}{p}\right)-M\right)}\|Tf\|_{L^1}\\ &\leq C_M 2^{j\left(N\left(1-\frac{1}{p}\right)-M\right)}\|f\|_{L^1}. \end{aligned}$$

Then, choosing M large enough, we infer

$$T_{21} f\leq C\|f\|_{L^1}.\tag{4.6}$$

Finally let us estimate $T_{22} f$. As $\widetilde{Q}$ is the disjoint union of the $\widetilde{Q}_J$, we get

$$\|\Delta_j Tf\|_{L^p(\widetilde{Q})}\leq 2^{\frac{L}{p}}\sup_{J\in\{-1,1\}^L}\|\Delta_j Tf\|_{L^p(\widetilde{Q}_J)}.\tag{4.7}$$

Let us first estimate $\|\Delta_j f_{J'}\|_{L^p(\widetilde{Q}_J)}$ for $J'\neq J$. We have, for all $w'\in\widetilde{Q}_J$,

$$(\Delta_j f_{J'})(w')=2^{jN}\int f_{J'}(w)h(\delta_{2^j}(w^{-1}w'))\,dw\,,$$

and in the integral, the distance $d(w,w')$ is greater than $1/32$. Then, reasoning as above we find that, for any positive integers M, there exists a constant C_M such that

$$J\neq J'\Longrightarrow\|\Delta_j f_{J'}\|_{L^p(\widetilde{Q}_J)}\leq C_M 2^{j\left(N\left(1-\frac{1}{p}\right)-M\right)}\|f_{J'}\|_{L^1}.\tag{4.8}$$

Then, let us observe that $\|\Delta_j f_J\|_{L^p(\widetilde{Q}_J)} \leq \|\Delta_j f_J\|_{L^p}$. Writing that

$$w^{-1}w' = \delta_{\frac{1}{4}}\Big(\big(\delta_4(w_J^{-1}w)\big)^{-1}\,\delta_4(w_J^{-1}w')\Big)$$

and changing variable $v = \delta_4(w_J^{-1}w)$ gives

$$(\Delta_j f_J)(w') = (\Delta_{j-2}f)\left(\delta_4(w_J^{-1}w')\right).$$

Thus $\|\Delta_j f_J\|_{L^p} = 2^{-\frac{2N}{p}}\|\Delta_{j-2}f\|_{L^p}$. Then using (4.7) and (4.8), we get by definition of T, that for all positive integers M,

$$\|\Delta_j f\|_{L^p(\widetilde{Q})} \leq 2^{D+\frac{L-2N}{p}}\|\Delta_{j-2}f\|_{L^p} + C_M 2^{j\left(N\left(1-\frac{1}{p}\right)-M\right)}\|f\|_{L^1}.$$

Thus by definition of $T_{22}f$, we get by choosing M large enough,

$$\begin{aligned} T_{22}f &\leq 2^{D\left(1-\frac{1}{p}\right)}\sum_{j\geq 1}2^{j\sigma}\|\Delta_{j-2}f\|_{L^p} + C_M\|f\|_{L^1}\sum_{j\geq 1}2^{j\left(\sigma+N\left(1-\frac{1}{p}\right)-M\right)}\\ &\leq 2^{D\left(1-\frac{1}{p}\right)+2\sigma}\|f\|_{\dot{B}^\sigma_{p,1}} + C\|f\|_{L^1}. \end{aligned}$$

Together with (4.5) and (4.6), this concludes the proof of the proposition.

Let us state the following corollary of Proposition 4.

Corollary 2. *For $(N-D)/2 < s < N/2$, there exists a sequence $(f_n)_{n\in\mathbf{N}}$ of non negative smooth and compactly supported functions such that, for any $\beta > 2s/N$,*

$$\lim_{n\to\infty}\frac{\|f_n\|_{L^{\frac{2N}{N-2s}}}}{\|f_n\|^\beta_{\dot{B}^{s-\frac{N}{2}}_{\infty,\infty}}\|f_n\|^{1-\beta}_{\dot{H}^s}} = +\infty.$$

Proof of Corollary 2 Let us consider a smooth compactly supported non negative function f_0 and let us define the sequence $(f_n)_{n\in\mathbf{N}}$ by $f_n = T^n f_0$. By iteration of the inequality of Proposition 4, we have

$$\|f_n\|_{B^\sigma_{p,q}} \leq 2^{n\left(D\left(1-\frac{1}{p}\right)+2\sigma\right)}\|f_0\|_{B^\sigma_{p,q}} + C\Big(\sum_{m=0}^{n-1}2^{m\left(D\left(1-\frac{1}{p}\right)+2\sigma\right)}\Big)\|f_0\|_{L^1}.$$

If $\sigma > -\dfrac{D}{2}\Big(1-\dfrac{1}{p}\Big)$, we deduce that

$$\|f_n\|_{B^\sigma_{p,q}} \leq C_{f_0}2^{n\left(D\left(1-\frac{1}{p}\right)+2\sigma\right)}$$

Applying this first with $\sigma = s - N/2$ and $p = q = \infty$ and then with $\sigma = s$ and $p = q = 2$ gives

$$\|f_n\|_{\dot{B}^{s-\frac{N}{2}}_{\infty,\infty}} \leq C_{f_0}2^{n(D-N+2s)} \quad\text{and}\quad \|f_n\|_{\dot{H}^s} \leq C_{f_0}2^{n\left(\frac{D}{2}+2s\right)}.$$

Assertion (4.4) claims that

$$\|f_n\|_{L^{\frac{2N}{N-2s}}} = 2^{nD\left(\left(\frac{1}{2}+\frac{s}{N}\right)\right)}\|f_0\|_{L^{\frac{2N}{N-2s}}}.$$

This concludes the proof of the corollary.

Remark Unfortunately, we cannot claim the same result for the precised Hardy inequality. Let us notice that the precised Hardy inequality has an obvious translation invariant generalization which is

$$\sup_a \int \frac{u^2(w)}{\rho^{2s}(a^{-1}w)} dw \leq C \|u\|^{\frac{2s}{N}}_{\dot{B}^{s-\frac{N}{2}}_{\infty,2}} \|u\|^{1-\frac{2s}{N}}_{\dot{H}^s}.$$

For the sequences used in the proof of Corollary 2, omitted computations show that, if s is greater than $\frac{1}{2}\Big(N - \frac{D}{2}\Big)$,

$$\sup_a \int \frac{f_n^2(w)}{\rho^{2s}(a^{-1}w)} dw \leq C 2^{n(D-N+2s)}.$$

This is exactly the same behavior as $\|f_n\|_{\dot{B}^{s-\frac{N}{2}}_{\infty,2}}$. We do not know if the exponent can be improved in (1.12) when we restrict to the cone of non negative functions.

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