ON THE PROPAGATION OF OCEANIC WAVES DRIVEN BY A STRONG MACROSCOPIC FLOW

ISABELLE GALLAGHER, THIERRY PAUL, AND LAURE SAINT-RAYMOND

Abstract. In this work we study oceanic waves in a shallow water flow subject to strong wind forcing and rotation, and linearized around an inhomogeneous (non zonal) stationary profile. This extends the study [2], where the profile was assumed to be zonal only and where explicit calculations were made possible due to the 1D setting.

Here the diagonalization of the system, which allows to identify Rossby and Poincaré waves, is proved by an abstract semi-classical approach. The dispersion of Poincaré waves is also obtained by a more abstract and more robust method using Mourre estimates. Only some partial results however are obtained concerning the Rossby propagation, as the two dimensional setting complicates very much the study of the dynamical system.

1. Introduction

This paper is a continuation of [2] (see also [1]) so before discussing the matter of this paper (in Section 2) let us review the contents of that work. We shall start by recalling briefly the model, then we shall explain the methods and results obtained in [2] and discuss their limitations.

1.1. The model.

The goal of [2] is to understand, through the study of a toy model, the persistence of oceanic eddies observed long past by physicists among which [4, 5, 6, 11, 12], who gave heuristic arguments to explain their formation due both to wind forcing and to convection by a macroscopic current.

The ocean is considered in this toy model as an incompressible, inviscid fluid with free surface submitted to gravitation and wind forcing, and we further make the following classical assumptions: we assume that the density of the fluid is homogeneous $\rho = \rho_0 =$ constant, that the pressure law is given by the hydrostatic approximation $p = \rho_0 g z$, and that the motion is essentially horizontal and does not depend on the vertical coordinate. This leads to the so-called shallow water approximation.

For the sake of simplicity, the effects of the interaction with the boundaries are not discussed and the model is purely horizontal with the longitude x_1 and the latitude x_2 both in $\mathbf{R}$.

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The evolution of the water height h and velocity v is then governed by the shallow-water equations with Coriolis force

$$\begin{aligned} \partial_t(\rho_0 h) + \nabla\cdot(\rho_0 h v) = 0 \\ \partial_t(\rho_0 h v) + \nabla\cdot(\rho_0 h v\otimes v) + \omega(\rho_0 h v)^\perp + \rho_0 g h \nabla h = \rho_0 h\tau \end{aligned} \tag{1.1}$$

where ω denotes the vertical component of the Earth rotation vector Ω, $v^\perp := (-v_2, v_1)$, g is the gravity and τ is the - stationary - forcing responsible for the macroscopic flow. The vertical component of the Earth rotation is therefore $\Omega\sin(x_2/R)$, where R is the radius of the Earth; note that it is classical in the physical literature to consider the linearization of ω (known as the betaplane approximation) $\omega(x_2) = \Omega x_2/R$. We consider general functions ω in the sequel, with some restrictions that are be made precise later.

We consider small fluctuations (η, u) around the stationary solution $(\bar h, \bar v)$ satisfying

$$\bar h = \text{constant}, \quad \nabla\cdot(\bar v\otimes\bar v) + \omega\bar v^\perp = \tau, \quad \operatorname{div}\bar v = 0.$$

In [2] the study is restricted to the case of a shear flow, in the sense that $\bar v(x) = (\bar v_1(x_2), 0)$, with $\bar v_1$ a smooth, compactly supported function. Some orders of magnitude and scalings allow to transform the previous system into the following one:

$$\begin{aligned} \partial_t\eta + \frac{1}{\varepsilon}\nabla\cdot u + \bar u\cdot\nabla\eta + \varepsilon^2\nabla\cdot(\eta u) = 0\,, \\ \partial_t u + \frac{1}{\varepsilon^2} b u^\perp + \frac{1}{\varepsilon}\nabla\eta + \bar u\cdot\nabla u + u\cdot\nabla\bar u + \varepsilon^2 u\cdot\nabla u = 0\,, \end{aligned} \tag{1.2}$$

where $b := \omega/|\Omega|$ and with ε of the order of both Fr^2 and $\mathrm{Ro}^{1/2}$, where the Froude number Fr and the Rossby number Ro are nondimensional parameters measuring respectively the influence of gravity and of the Coriolis force. For the large scale motions under consideration, we have typically $\mathrm{Fr}^2 := (v_0\ell_0)/(t_0 g\delta h) \sim 0.1$ and $\mathrm{Ro} := 1/(t_0|\Omega|) \sim 0.01$, where $t_0 \sim 10^6\,\mathrm{s}\,(\sim 0,38\,\text{months})$ and $\ell_0 \sim 10^4\,\mathrm{km}$ are the typical time and length scales, $\delta h = (h-\bar h)/\eta \ \sim 1\,\mathrm{m}$ is the typical height fluctuation, and $v_0 \sim 0.1\mathrm{ms}^{-1}$ is the typical velocity fluctuation: $u = (v-\bar v)/v_0$.

1.2. **Methods and results in** [2]**.** Most of the analysis in [2] concerns the linear version of (1.2), namely the following system:

$$\varepsilon^2 i\partial_t \mathbf{v} + A(x_2, \varepsilon D, \varepsilon)\mathbf{v} = 0 \qquad \mathbf{v} = (v_0, v_1, v_2), \tag{1.3}$$

where $D := \frac{1}{i}\partial$, and the linear propagator is given by

$$A(x_2,\varepsilon D,\varepsilon) := i\begin{pmatrix} \varepsilon\bar u_1\varepsilon\partial_1 & \varepsilon\partial_1 & \varepsilon\partial_2 \\ \varepsilon\partial_1 & \varepsilon\bar u_1\varepsilon\partial_1 & -b(x_2)+\varepsilon^2\bar u_1' \\ \varepsilon\partial_2 & b(x_2) & \varepsilon\bar u_1\varepsilon\partial_1 \end{pmatrix}\,,$$

The first step of the analysis consists in diagonalizing (approximately) the system (1.3). The computation of a kind of **characteristic polynomial** associated with (1.3), in symbolic form, allows to construct three symbols the quantization of which provides three scalar propagators (this will be explained more explicitly below).

Two of those propagators, called Poincaré propagators, are then proved to satisfy dispersive estimates; that result relies on a spectral analysis (usual semi-classical theory does not operate here due to the very large time scales at play) using **global quantum normal forms**, which requires that b has at most one, non degenerate critical value and which also uses very much the fact that the motion is translation-invariant in x_1. A stationary phase argument on the spectral decomposition of any solution to the Poincaré propagation gives the result: Poincaré modes exit any compact set in finite time.

The last propagator is the Rossby one, which is one order of magnitude (in ε) smaller than the Poincaré modes. This allows to analyse the propagation by semi-classical analysis tools. In particular the precise study of the dynamical system associated with those waves, which is an **integrable system** due to translation invariance in x_1, allows to derive a condition on the initial microlocalization of the solution which guarantees that the Rossby waves are trapped for all times in a compact set.

Those results on the linear system (1.3) can finally be transposed to the original system (1.2) due to the high power of ε in front of the nonlinearity, and due to the semi-classical setting, which allows to exhibit vector fields which almost-commute with the linear operator $A(x_2, \varepsilon D, \varepsilon)$.

1.3. **Limitations of the methods of [2].** The restriction which is the most used in the analysis described briefly in the previous paragraph is the fact that the stationary flow $\bar{u}$ is a shear flow of the type $\bar{u} = (\bar{u}_1(x_2), 0)$. Indeed

- It allows to Fourier-transform in the direction x_1, which makes the diagonalization procedure much easier;
- It simplifies the spectral analysis of Poincaré waves, again due to the Fourier transform (in particular the dual variable ξ_1 is fixed during the propagation, and there is a wave-like behaviour in x_1);
- It allows the Rossby dynamical system to be integrable, which is a tremendous help in the analysis.

An additional restriction in the previous arguments is that in order to prove the dispersion of Poincaré waves, the rotation amplitude b should have at most one, non degenerate critical value: this allows to use a Bohr-Sommerfeld quantization argument to compute the eigenvalues of the Poincaré operator. This assumption on b is not really restrictive from the physical point of view. On the other hand, it is important for physical reasons to consider 2D convection flows.

1.4. **On the nonlinear term.** As explained above, most of the analysis in [2] is concerned with the linear system (1.3). In order to transpose the linear results to the nonlinear setting, one uses the following arguments (along with the fact that the coupling is vanishing when ε goes to zero):

- Uniform existence which is obtained via an almost-commutation result;
- Bilinear estimates in anisotropic semi-classical spaces;
- A Gronwall lemma, which requires an $L^\infty(\mathbf{R}^2)$ bound on the linear solution. This is not known in general, due to the bad Sobolev embeddings in semi-classical settings, so the nonlinear result is proved for vanishing couplings only.

It is important to notice that none of those three steps require that $\bar{u}$ is a shear flow. In the whole of this paper we shall therefore only focus on the linear equation, and leave to the reader the transposition to the nonlinear equation, using the above steps.

2. Main result of this paper and strategy of the proof

2.1. **The model.** In this paper we shall be concerned with the linear system

$$\varepsilon^2 i\partial_t \mathbf{v} + A(x,\varepsilon D,\varepsilon)\mathbf{v} = 0 \qquad \mathbf{v} = (v_0, v_1, v_2), \tag{2.1}$$

where the linear propagator is given by

$$A(x,\varepsilon D,\varepsilon) := i\begin{pmatrix} \varepsilon\bar{u}\cdot\varepsilon\nabla & \varepsilon\partial_1 & \varepsilon\partial_2 \\ \varepsilon\partial_1 & \varepsilon\bar{u}\cdot\varepsilon\nabla + \varepsilon^2\partial_1\bar{u}_1 & -b+\varepsilon^2\partial_2\bar{u}_1 \\ \varepsilon\partial_2 & b+\varepsilon^2\partial_1\bar{u}_2 & \varepsilon\bar{u}\cdot\varepsilon\nabla + \varepsilon^2\partial_2\bar{u}_2 \end{pmatrix}. \tag{2.2}$$

We shall assume throughout the paper that b is smooth, with a symbol-like behaviour: for all $\alpha \in \mathbf{N}$, there is a constant C_α such that for all $x_2 \in \mathbf{R}$,

$$|b^{(\alpha)}(x_2)| \leq C_\alpha\big(1+b^2(x_2)\big)^{\frac{1}{2}}. \tag{2.3}$$

We shall further assume that

$$\lim_{|x_2|\to\infty} b^2(x_2) = \infty,$$

and that b^2 has only non degenerate critical points.

We shall also suppose that the initial data is microlocalized in some compact set $\mathcal{C}$ of $T^*\mathbf{R}^2$ satisfying

$$\mathcal{C}\cap\{\xi_1^2+\xi_2^2+b^2(x_2)=0\}=\emptyset \tag{2.4}$$

or actually rather

$$\mathcal{C}\cap\{\xi_1=0\}=\emptyset\,. \tag{2.5}$$

We shall prove that assumption (2.4) is propagated by the flow, while (2.5) is propagated only by the Poincaré component. We recall (see for instance [2], Appendix B) that a function f is microlocalized in a compact set $\mathcal{C}$ of $T^*\mathbf{R}^2$ if for any (x_0,ξ_0) in the complement of $\mathcal{C}$ in $\mathbf{R}^4$ (we shall identify $T^*\mathbf{R}^2$ to $\mathbf{R}^4$ in the following), there is a smooth function χ_0, bounded as well as all its derivatives and equal to one at (x_0,ξ_0), satisfying

$$\|\mathrm{Op}_\varepsilon^W(\chi_0)u_0\|_{L^2(\mathbf{R}^2)} = O(\varepsilon^\infty), \tag{2.6}$$

where $\mathrm{Op}_\varepsilon^W$ denotes the Weyl quantization:

$$\mathrm{Op}_\varepsilon^W(\chi_0)u_0(x) := \frac{1}{(2\pi\varepsilon)^4}\int e^{i(x-y)\cdot\xi/\varepsilon}\chi_0(\frac{x+y}{2},\xi)u_0(y)\,dyd\xi. \tag{2.7}$$

We also recall that (2.6) means that for any $N\in\mathbf{N}$, there are ε_0 and C such that

$$\forall\varepsilon\in]0,\varepsilon_0],\quad \|\mathrm{Op}_\varepsilon^W(\chi_0)u_0\|_{L^2(\mathbf{R}^2)} \leq C\varepsilon^N.$$

In the following, to simplify some formulations, we shall denote by $(\mu)\mathrm{Supp}_\star f$ the projection of the (micro)support of f onto the $\star=0$ axis, where $\star$ represents an element of $\{x_1,x_2,\xi_1,\xi_2\}$.

2.2. **Statement of the main result and organization of the paper.** Let us state the main theorem proved in this paper.

Theorem 1. *Let* $\mathbf{v}_{\varepsilon,0}$ *be a family of initial data, microlocalized in a compact set* $\mathcal{C}$ *satisfying Assumption (2.5). For any parameter* $\varepsilon > 0$*, denote by* $\mathbf{v}_\varepsilon$ *the associate solution to (2.1). Then for all* $t \geq 0$ *one can write* $\mathbf{v}_\varepsilon(t)$ *as the sum of a "Rossby" vector field and a "Poincaré" vector field:* $\mathbf{v}_\varepsilon(t) = \mathbf{v}_\varepsilon^R(t) + \mathbf{v}_\varepsilon^P(t)$*, satisfying the following properties:*

(1) $\mu Supp\, \mathbf{v}_\varepsilon^R(t)$ *and* $\mu Supp\, \mathbf{v}_\varepsilon^P(t)$ *satisfy (2.4) for all times.*
(2) *For any compact set* Ω *in* $\mathbf{R}^2$*, one has*

$$\forall t > 0, \quad \|\mathbf{v}_\varepsilon^P(t)\|_{L^2(\Omega)} = O(\varepsilon^\infty).$$

(3) $\mu Supp_{x_2} \mathbf{v}_\varepsilon^R(t)$ *lies in a bounded subset of* $\mathbf{R}$ *uniformly in time.*

Compared to [2], the main difficulties are due to the presence of a x_1-dependent underlying flow $\bar{u}$. The diagonalization of the system (exhibiting Rossby and Poincaré-type waves, with very different qualitative features) must be revised, and obtained in a less explicit way. Moreover the proof of (2) in Theorem 1, namely the dispersion of Poincaré waves can also not be proved in the same way (note that it is not assumed here that b^2 has at most one non degenerate critical value). Finally the trapping of Rossby waves seems much harder to obtain since the underlying dynamical system decouples no more; the behaviour of the Rossby waves is therefore much less precise than in [2].

Let us explain our strategy here, compared with that in [2] described above.

2.2.1. *The diagonalization.* The construction of the Rossby and Poincaré modes is not as direct as in [2] due to the lack of translation invariance in x_1. We choose therefore to follow a more abstract way to recover those modes in Section 3, which relies on **semi-classical analysis, and normal forms** (instead of explicit computations as in [2]). Finding the propagators associated with those modes requires a microlocalisation assumption of the type (2.4), in order for the eigenvalues of the matrix of principal symbols to be well separated (see for instance [10] for a related result). The diagonalization result is therefore in this paragraph conditional to the fact that the solution to the propagation equation is correctly microlocalized (that corresponds to Point 1 of Theorem 1).

2.2.2. *Dispersion of Poincaré waves and propagation of the nondegeneracy assumption (2.5).* In order to prove (2) in Theorem 1 we again rely on a more abstract, and more efficient method than that followed in [2]. It is based on Mourre estimates and on Assumption (2.5) on the initial data: we start by proving, by a semi-classical argument, that after a very short time (of the order of ε) the support in x_1 of the solution escapes the support of $\bar{u}$. Then we use **Mourre estimates** to prove that the solution remains outside the support of $\bar{u}$ for all times, and actually escapes any compact set in x_1 in finite time (to prove this last point we use the fact that the equation reduces to a translation-invariant equation in x_1 since the support of the solution is outside the support in x_1 of $\bar{u}$). This allows finally to check that the nondegeneracy assumption (2.5) does hold for all times. This analysis is achieved in Section 4.

2.2.3. *Study of Rossby waves and propagation of the nondegeneracy assumption (2.4).* In Section 5 we first prove that the nondegeneracy assumption (2.4) does hold during the propagation of Rossby waves. That is due to semi-classical analysis, by the study of the dynamical system associated with those waves. The study of that system is also the key to the proof of Point (3), which is also proved in Section 5.

3. Reduction to scalar propagators

In this section we shall construct three operators T_+, T_- and T_R diagonalizing $A(x, \varepsilon D, \varepsilon)$.

We shall start by proving a general diagonalization result, and at the end we shall apply the general result to our context.

Before stating the general result, let us give some notation. A semi-classical symbol is a function $a = a(x, \xi; \varepsilon)$ defined on $\mathbf{R}^{2d} \times]0, \varepsilon_0]$ for some $\varepsilon_0 > 0$, which depends smoothly on (x, ξ) and such that for any $\alpha \in \mathbf{N}^{2d}$ and any compact set $\mathcal{K} \subset \mathbf{R}^{2d}$, there is a constant C such that for any $((x, \xi), \varepsilon) \in \mathcal{K} \times]0, \varepsilon_0]$,

$$|\partial^\alpha a((x, \xi), \varepsilon)| \leq C.$$

We shall consider the Weyl quantization of such symbols, as recalled in (2.7): for all u is in $\mathcal{D}(\mathbf{R}^d)$,

$$\mathrm{Op}_\varepsilon^W(a)u(x) := \frac{1}{(2\pi\varepsilon)^d} \int e^{i(x-y)\cdot\xi/\varepsilon} a(\frac{x+y}{2}, \xi) u(y)\, dy d\xi.$$

We shall denote the principal symbol of a pseudo-differential operator A by $\sigma_p(A)$. We shall say that a pseudodifferential operator $\mathrm{Op}_\varepsilon^W(a)$ is supported in a set $\mathcal{K}$ if for any smooth function χ equal to one in a neighborhood of $\mathcal{K}$ one has $a\chi = a$.

Finally we shall say that a matrix is pseudodifferential if each of its entries is a pseudodifferential operator.

Let us first prove the following general result.

Theorem 2. *Let $\mathcal{K}$ be a compact subset of $\mathbf{R}^{2d}$, and consider a $N \times N$ hermitian pseudodifferential matrix $A_\varepsilon = A(x, \varepsilon D, \varepsilon)$, supported in $\mathcal{K}$. Assume that*

- *the (matrix) principal symbol of $A(x, \varepsilon D, 0)$, denoted by $\mathcal{A}_0$, is diagonalizable, in the sense that there are some unitary and diagonal matrices of symbols, $\mathcal{U}$ and $\mathcal{D}$, such that*

$$\mathcal{U}^{-1} \mathcal{A}_0 \mathcal{U} = \mathcal{D},$$

- *the eigenvalues $(\delta_1(x, \xi), \ldots, \delta_N(x, \xi))$ satisfy*

$$\forall i \neq j, \quad \inf_{(x,\xi) \in \mathcal{K}} |\delta_i(x, \xi) - \delta_j(x, \xi)| \geq C > 0. \tag{3.1}$$

Then there exists a family of unitary and diagonal pseudodifferential operators V_ε and D_ε supported in $\mathcal{K}$, such that:

$$V_\varepsilon^* A_\varepsilon V_\varepsilon = D_\varepsilon + O(\varepsilon^\infty), \quad V_\varepsilon^* V_\varepsilon = I + O(\varepsilon^\infty). \tag{3.2}$$

Moreover one has

$$D_\varepsilon = D_0 + \varepsilon D_1 + O(\varepsilon^2), \tag{3.3}$$

where $D_0 = \mathrm{Op}_\varepsilon^W(\mathcal{D})$ *and the principal symbol of* D_1 *is given by*

$$\mathcal{D}_1 = \sigma_p(D_1) = \ diag\left(\sigma_p(\widetilde{\Delta}_1 - \frac{D_0 I_1 + I_1 D_0}{2})\right)$$

with the notations

$$\widetilde{\Delta}_1 = \frac{1}{\varepsilon}\left(\mathrm{Op}_\varepsilon^W(\mathcal{U}^*)A_\varepsilon \mathrm{Op}_\varepsilon^W(\mathcal{U}) - D_0\right), \qquad (3.4)$$
$$I_1 = \frac{1}{\varepsilon}\left(\mathrm{Op}_\varepsilon^W(\mathcal{U}^*)\mathrm{Op}_\varepsilon^W(\mathcal{U}) - I\right).$$

More explicitly, let us denote by $a_{ij}(x,\xi)$ *the matrix elements of* $\mathcal{A}_1$*, subsymbol of* $A(x,\varepsilon D,\varepsilon)$ *defined by:*

$$\mathcal{A}_1 := \sigma_p\left(\partial_\varepsilon A\right)$$

and by $u_{nj}(x,\xi)$*,* $j = 1\dots d$*, the coordinates of any unit eigenvector of* $\mathcal{A}_0(x,\xi)$ *of eigenvalue* $\delta_n(x,\xi)$*. We have*
(3.5)

$$(\mathcal{D}_1)_{nn} = \sum_{j,k=1\dots d}\left(\Im\left(\overline{u_{jn}}\{a_{jk},u_{kn}\}\right) + \frac{a_{jk}\{\overline{u_{jn}},u_{kn}\}}{2i}\right) + (\mathcal{U}^*\mathcal{A}_1\mathcal{U})_{nn}) + \frac{1}{2i}\sum_{j=1}^{d}\delta_n\{u_{jn},\overline{u_{jn}}\},$$

where $\{f,g\} := \nabla_\xi f \nabla_x g - \nabla_x f \nabla_\xi g$ *is the Poisson bracket on* $T^*\mathbf{R}^d$*.*

Here and in all the sequel, we say that a pseudo-differential operator V is unitary if it satisfies

$$V^*V = I + O(\varepsilon^\infty).$$

The proof is divided into two parts: in Section 3.1 we present the formal construction and in Section 3.2 we show that the symbols of the various operators formally constructed are indeed symbols. Finally Section 3.3 is devoted to the case of the matrix given by (2.2).

3.1. **The formal construction.** The proof of Theorem 2 is a combination of semiclassical and perturbation methods. Let us start by defining

$$U_0 = \mathrm{Op}_\varepsilon^W(\mathcal{U}).$$

Elementary properties of the Weyl quantization imply then that $U_0^* A_\varepsilon U_0 = D_0 + O(\varepsilon)$.

The following proposition shows that one can construct a unitary pseudodifferential operator U_∞ such that

$$U_\infty^* A_\varepsilon U_\infty = D_0 + O(\varepsilon).$$

Lemma 3.1. *Let* U *be a pseudodifferential matrix such that* $U^*U = I + \varepsilon I_1$*, where* I *is the identity. Then one can find* $V \sim \sum_{k=0}^{\infty} \varepsilon^k V_k$ *such that*

$$(U + \varepsilon V)^*(U + \varepsilon V) = I + O(\varepsilon^\infty). \qquad (3.6)$$

Proof. Let us denote $V_0 := -\dfrac{1}{2}UI_1$. On easily checks that $(U+\varepsilon V_0)^*(U+\varepsilon V_0) = I + O(\varepsilon^2)$. Indeed

$$\begin{aligned}(U+\varepsilon V_0)^*(U+\varepsilon V_0) &= U^*U - \frac{\varepsilon}{2}(I_1U^*U + U^*UI_1) + O(\varepsilon^2)\\ &= I + \varepsilon I_1 - \varepsilon I_1 + O(\varepsilon^2).\end{aligned}$$

Then one concludes by iteration. □

That lemma allows to define the pseudo-differential operator of (semiclassical) order 0

$$\Delta_1 = \frac{1}{\varepsilon}\left(U_\infty^* A_\varepsilon U_\infty - D_0\right),$$

where U_∞ is a unitary operator.

Now our aim is to find a unitary operator V_∞ (up to $O(\varepsilon^\infty)$) such that

$$(U_\infty V_\infty)^* A_\varepsilon (U_\infty V_\infty) = D_\infty + O(\varepsilon^\infty),$$

where $D_\infty = D_0 + \varepsilon D_1 + \dots$ is a diagonal matrix satisfying the conclusions of the theorem.

We shall write $V_\infty = e^{i\varepsilon W}$, with W selfadjoint (so V_∞ thus constructed is automatically unitary). We recall that if W is a pseudodifferential operator, then so is $e^{i\varepsilon W}$ (simply by writing $e^{i\varepsilon W} \sim \sum_0^\infty \frac{(i\varepsilon)^k}{k!} W^k$).

We look for W under the form $W \sim \displaystyle\sum_0^\infty \varepsilon^k W_k$, and compute the W_k recursively. Since

$$V_\infty^*(D_0+\varepsilon\Delta_1)V_\infty = (D_0+\varepsilon\Delta_1) + i\varepsilon[(D_0+\varepsilon\Delta_1), W] + \frac{(i\varepsilon)^2}{2}[[(D_0+\varepsilon\Delta_1), W], W] + \dots$$

we see that, if W_1 satisfies

$$i[D_0, W_1] + \Delta_1 = D_1 + O(\varepsilon),\ D_1 \text{ diagonal}, \tag{3.7}$$

then we have that

$$e^{-i\varepsilon W_1}(D_0+\varepsilon\Delta_1)e^{i\varepsilon W_1} = D_0 + \varepsilon D_1 + \varepsilon^2\Delta_2, \tag{3.8}$$

where Δ_2 is a zero order pseudodifferential operator. The following lemma is a typical normal form type result, and is crucial for the following.

Lemma 3.2. *Let D_0 be a diagonal pseudodifferential matrix whose principal symbol $\mathcal{D}_0$ has a spectrum satisfying (3.1) and let Δ_1 be a pseudodifferential matrix.*

Then there exist two pseudodifferential matrices W and D_1, with D_1 diagonal, such that:

$$[D_0, W] + \Delta_1 = D_1 + \varepsilon\widetilde{\Delta}_2, \tag{3.9}$$

where $\widetilde{\Delta}_2$ is a pseudodifferential matrix of order 0.

Moreover the principal symbol of D_1 is the diagonal part of the principal symbol of Δ_1: we have $\sigma_p(D_1) = diag\, \sigma_p(\Delta_1)$.

Proof. By the non degeneracy condition of the spectrum of $\mathcal{D}_0$ we know, by standard arguments (see [13] for instance), that there exists a matrix $\mathcal{W}_0$ and a diagonal one $\mathcal{D}_1$ such that

$$[\mathcal{D}_0, \mathcal{W}_0] + \mathcal{D}_{1,0} = \mathcal{D}_1,$$

where $\mathcal{D}_{1,0}$ is the principal symbol of Δ_1.

Indeed it is enough to take $\mathcal{D}_1$ as the diagonal part of $\mathcal{D}_{1,0}$ and

$$(\mathcal{W}_0(x,\xi))_{i,j} = \frac{(\mathcal{D}_{1,0}(x,\xi))_{i,j}}{\delta_i(x,\xi) - \delta_j(x,\xi)} \tag{3.10}$$

and notice that the Weyl quantization of $\mathcal{W}_0$ satisfies (3.9). □

By Lemma 3.2 we know that there exists W_1 satisfying (3.7). Writing

$$e^{-i\varepsilon W_1}(D_0 + \varepsilon\Delta_1)e^{i\varepsilon W_1} = D_0 + \varepsilon(\Delta_1 + [D_0, W_1]) + \varepsilon^2(\Delta_2 - \tilde{\Delta}_2),$$

we get immediately (3.8).

It is easy to get convinced that all the W_k will satisfy recursively an equation of the form

$$[D_0, W_k] + \Delta_k = D_k + O(\varepsilon),$$

which can be solved by Lemma 3.2.

The expression for the principal symbol of D_1 follows by construction and the following well known lemma (see [9] for instance):

Lemma 3.3. *Let a and b two symbols. Then the principal symbol of $\mathrm{Op}^W_\varepsilon(a)\mathrm{Op}^W_\varepsilon(b)$ is ab and its subprincipal symbol is $\frac{1}{2i}\{a, b\}$.*

In order to derive (3.5) we have to compute the subprincipal symbol of the diagonal part of the right-hand side of (3.4), that is, for each $n = 1 \dots d$ and using Lemma 3.3,

$$\sum_{jk} \mathrm{Op}^W_\varepsilon(\overline{\mathcal{U}_{jn}})\mathrm{Op}^W_\varepsilon((\mathcal{A}_0 + \varepsilon\mathcal{A}_1)_{jk})\mathrm{Op}^W_\varepsilon(\mathcal{U}_{kn}) - \frac{1}{2i}\sum_{j=1}^{d} \delta_n\{\overline{u_{jn}}, u_{jn}\},$$

since $\mathcal{U}$ is unitary.

The term $\varepsilon\mathcal{A}_1$ is obviously responsible for the second term in the right-hand side of (3.5). Using Lemma 3.3 and the distributivity of the Poisson bracket, we get the following expression for the first one:

$$\sum_{jk} \frac{1}{2i}\left(\{\overline{\mathcal{U}_{jn}}, (\mathcal{A}_0)_{jk}\mathcal{U}_{kn}\} + \overline{\mathcal{U}_{jn}}\{(\mathcal{A}_0)_{jk}, \mathcal{U}_{kn}\}\right)$$

$$= \sum_{jk} \frac{1}{2i}\left(\overline{\mathcal{U}_{jn}}\{(\mathcal{A}_0)_{jk}, \mathcal{U}_{kn}\} + (\mathcal{A}_0)_{jk}\{\overline{\mathcal{U}_{jn}}, \mathcal{U}_{kn}\} + \mathcal{U}_{kn}\{\overline{\mathcal{U}_{jn}}, (\mathcal{A}_0)_{jk}\}\right)$$

Interverting j and k in half of the terms and noticing that, since $\mathcal{A}_0$ is Hermitian, $(\mathcal{A}_0)_{jk} = \overline{(\mathcal{A}_0)_{kj}}$, we get easily (3.5).

3.2. **Symbolic properties.**

With the hypothesis that both $\mathcal{D}$ and $\mathcal{U}$ are pseudodifferential matrices it is quite obvious that V_ε and D_ε are pseudodifferential matrices as well. Indeed the formal construction in the preceding section shows that the iterative process uses only three tools: multiplications of pseudodifferential operators, computation of subprincipal symbols and solving equation (3.9).

For (3.9), the formula (3.10) used in the proof of Lemma 3.2, together with the non-degeneracy condition (3.1) which shows clearly that $(\delta_i(x,\xi)-\delta_j(x,\xi))^{-1}$ is a symbol, implies that W_0 is a pseudodifferential operator.

Note that the microlocalization assumption is crucial in order that the expansions obtained by this iterative construction do define symbols. We have indeed no uniform control on the growth at infinity.

3.3. **The Rossby-Poincaré case.**

In the case of oceanic waves $A(x,\varepsilon D,\varepsilon)$ is given by (2.2):

$$A(x,\varepsilon D) := i\begin{pmatrix} \varepsilon\bar u\cdot\varepsilon\nabla & \varepsilon\partial_1 & \varepsilon\partial_2 \\ \varepsilon\partial_1 & \varepsilon\bar u\cdot\varepsilon\nabla+\varepsilon^2\partial_1\bar u_1 & -b+\varepsilon^2\partial_2\bar u_1 \\ \varepsilon\partial_2 & b+\varepsilon^2\partial_1\bar u_2 & \varepsilon\bar u\cdot\varepsilon\nabla+\varepsilon^2\partial_2\bar u_2\end{pmatrix}.$$

Therefore

$$\mathcal{A}_0(x,\xi) := \begin{pmatrix} 0 & \xi_1 & \xi_2 \\ \xi_1 & 0 & -ib \\ \xi_2 & ib & 0\end{pmatrix},$$

and

$$\mathcal{A}_1(x,\xi) := \begin{pmatrix} \bar u\cdot\xi & 0 & 0 \\ 0 & \bar u\cdot\xi & 0 \\ 0 & 0 & \bar u\cdot\xi\end{pmatrix} = \bar u\cdot\xi\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1\end{pmatrix}.$$

A straightforward computation shows that the spectrum of $\mathcal{A}_0$ is

$$\left\{0,\sqrt{\xi_1^2+\xi_2^2+b^2(x_2)},-\sqrt{\xi_1^2+\xi_2^2+b^2(x_2)}\right\}.$$

3.3.1. *Microlocalization.* The three eigenvalues of $\mathcal{A}_0$ are separated if and only if

$$\xi_1^2+\xi_2^2+b^2(x_2)\neq 0.$$

Therefore, considering a compact subset $\mathcal{K}$ of $\mathbf{R}^4$ such that

$$\mathcal{K}\cap\{(x_1,x_2,\xi_1,\xi_2)\,/\,\xi_1^2+\xi_2^2+b^2(x_2)=0\}=\emptyset$$

ensures that

- the eigenvalues do not cross, so that it is possible to get a unitary diagonalizing matrix with regular entries;
- the non degeneracy condition (3.1) is satisfied.

In other words, $A(x, \varepsilon D, \varepsilon)$ satisfies the assumptions of Theorem 2 provided that one considers only its action on vector fields which are suitably microlocalized.

We assume of course that this microlocalization condition is satisfied by the initial datum, which is the condition (2.4).

Furthermore, we shall prove in the next two sections that the propagation by the scalar operators $T_\pm$ and T_R (to be defined now) preserves this suitable microlocalization, thus justifying a posteriori the diagonalization procedure for all times.

3.3.2. *Computation of the Poincaré and Rossby Hamiltonians.* The above computations show that one can define the two Poincaré Hamiltonians as follows:

$$\tau_\pm := \pm\sqrt{\xi_1^2 + \xi_2^2 + b^2(x_2)}$$

and we shall denote the associate operator constructed via Theorem 2 by $T_\pm$.

Now let us consider the Rossby Hamiltonian. In all this paragraph, for the sake of readability, we shall denote

$$\langle \xi \rangle_b := \sqrt{\xi_1^2 + \xi_2^2 + b^2(x_2)}\,.$$

An easy computation shows that a (normalized) eigenvector of $\mathcal{A}_0(x,\xi)$ of zero eigenvalue is

$$u_0 = \frac{1}{\langle \xi \rangle_b} \begin{pmatrix} b \\ i\xi_2 \\ -i\xi_1 \end{pmatrix}.$$

By Theorem 2, the Rossby Hamiltonian is then given by the formula

$$\tau_R = \sum_{j,k=1\ldots 3} \left(\Im\left(\overline{u_{j0}}\{a_{jk}, u_{k0}\}\right) + \frac{a_{jk}\{\overline{u_{j0}}, u_{k0}\}}{2i} \right) + \sum_{j,k=1\ldots 3} (\mathcal{A}_1)_{jk} \overline{u_{j0}} u_{k0}. \tag{3.11}$$

In order to compute the different Lie brackets, we start with a couple of simple remarks :

$$\{\xi_j, f\} = \partial_{x_j} f \quad \text{ and } \quad \{b, f\} = -b' \partial_{\xi_2} f\,.$$

In particular, if f does not depend on x_1, then $\{\xi_1, f\} = 0$.

The contribution of the first term in the parenthesis in (3.11) is

$$\begin{aligned}
&\sum_{j,k=1\ldots 3} \left(\overline{u_{j0}}\{a_{jk}, u_{k0}\}\right) \\
&\quad = \frac{b}{\langle \xi \rangle_b}\left\{\xi_2, \frac{-i\xi_1}{\langle \xi \rangle_b}\right\} + \frac{i\xi_2}{\langle \xi \rangle_b}\left\{ib, \frac{-i\xi_1}{\langle \xi \rangle_b}\right\} + \frac{i\xi_1}{\langle \xi \rangle_b}\left(\left\{\xi_2, \frac{b}{\langle \xi \rangle_b}\right\} + \left\{ib, \frac{i\xi_2}{\langle \xi \rangle_b}\right\}\right) \\
&\quad = \frac{-ib\xi_1}{\langle \xi \rangle_b}\partial_{x_2}\frac{1}{\langle \xi \rangle_b} - \frac{i\xi_2\xi_1 b'}{\langle \xi \rangle_b}\partial_{\xi_2}\frac{1}{\langle \xi \rangle_b} + \frac{i\xi_1}{\langle \xi \rangle_b}\partial_{x_2}\frac{b}{\langle \xi \rangle_b} + \frac{i\xi_1 b'}{\langle \xi \rangle_b}\partial_{\xi_2}\frac{\xi_2}{\langle \xi \rangle_b} \\
&\quad = \frac{2i\xi_1 b'}{\langle \xi \rangle_b^2}.
\end{aligned}$$

Using the distributivity of the Poisson brackets, we get the contribution of the second term in a very similar way

$$\begin{aligned}
\sum_{j,k=1\dots 3} & \frac{a_{jk}\{\overline{u_{j0}}, u_{k0}\}}{2} \\
&= \xi_1 \left\{ \frac{b}{\langle\xi\rangle_b}, \frac{i\xi_2}{\langle\xi\rangle_b} \right\} - \xi_2 \left\{ \frac{b}{\langle\xi\rangle_b}, \frac{i\xi_1}{\langle\xi\rangle_b} \right\} + ib \left\{ \frac{i\xi_1}{\langle\xi\rangle_b}, \frac{i\xi_2}{\langle\xi\rangle_b} \right\} \\
&= \xi_1 \left(\frac{ib}{\langle\xi\rangle_b} \left\{ \frac{1}{\langle\xi\rangle_b}, \xi_2 \right\} + \frac{i\xi_2}{\langle\xi\rangle_b} \left\{ b, \frac{1}{\xi_b} \right\} + \frac{i}{\langle\xi\rangle_b^2} \{b, \xi_2\} \right) - \frac{i\xi_2\xi_1}{\langle\xi\rangle_b} \left\{ b, \frac{1}{\langle\xi\rangle_b} \right\} - \frac{ib\xi_1}{\langle\xi\rangle_b} \left\{ \frac{1}{\langle\xi\rangle_b}, \xi_2 \right\} \\
&= \frac{-ib'\xi_1}{\langle\xi\rangle_b^2} \cdot
\end{aligned}$$

The computation of the second term of the right hand side of (3.11) is trivial since $\mathcal{A}_1$ is a multiple of the identity. Adding the two previous expressions we get finally

$$\tau_R = \frac{\xi_1 b'}{\xi_1^2 + \xi_2^2 + b(x_2)^2} + \bar{u} \cdot \xi$$

and the associate operator will be denoted by T_R.

Remark 3.4. *Since the elementary steps of the diagonalization process use only multiplications, computations of subprincipal symbols and solving normal forms equations, all the subsymbols of T_R and $T_\pm$ depend on x_1 only through $\bar{u}$ and its derivatives.*

4. Study of the Poincaré waves

In Section 3 we constructed two linear operators, called $T_\pm$, whose principal symbols are

$$\tau_\pm = \pm\sqrt{\xi_1^2 + \xi_2^2 + b^2(x_2)}.$$

We now want to study the propagation equation associated to those operators, namely the linear equation in $\mathbf{R} \times \mathbf{R}^2$

$$i\varepsilon^2 \partial_t \varphi_\pm = T_\pm \varphi_\pm, \quad \varphi_{\pm|t=0} = \varphi_\pm^0 \tag{4.1}$$

where $\varphi_\pm^0$ are microlocalized in a compact set $\mathcal{C}$ satisfying Assumption (2.4). Before studying that equation we need to check that it makes sense, since a priori $T_\pm$ is only defined on vector fields microlocalized on such a compact set. This is achieved in the coming section, where we check that the separation of eigenvalues (3.1) required in the statement of Theorem 2 holds because $\sqrt{\xi_1^2 + \xi_2^2 + b^2(x_2)}$ remains bounded away from zero during the propagation.

Then we shall show that the solutions to these equations exit any compact set in finite time (Point (2) of Theorem 1).

4.1. Microlocalization.

Let us prove the following result, which provides the first part of Point (1) in Theorem 1 and allows to make sense of Equation (4.1) for all times.

Proposition 4.1. *Under the assumptions of Theorem 1, the operators $T_\pm$ are self-adjoint, and the function $\varphi(t) = e^{i\frac{t}{\varepsilon^2}T_\pm}\varphi_\pm^0$ are such that $\mu Supp \varphi_\pm(t)$ satisfies (2.4) for all times.*

Proof. The proof of that result relies on a spectral argument. Due to the form of the principal symbols of $T_\pm$ recalled above, the operators $T_\pm$ are self-adjoint (see [9]). We can therefore define two families $(\psi_n^\pm)_{n\in\mathbf{N}}$ of eigenvectors of $T_\pm$ in $L^2(\mathbf{R}^2)$ and two sequences of eigenvalues $\lambda_n^\pm$ such that if the initial data writes

$$\varphi_\pm^0(x) = \sum_n c_n^{\pm,0}\psi_n^\pm(x),$$

then

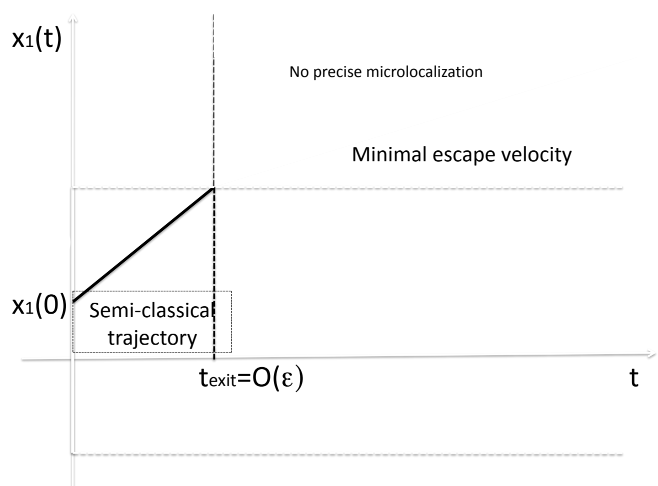
$$\varphi_\pm(t,x) = \sum_n e^{i\frac{\lambda_n^\pm t}{\varepsilon^2}} c_n^{\pm,0}\psi_n^\pm(x).$$

Since the eigenfunctions $\psi_n^\pm$ are microlocalized on the energy surfaces of the Poincaré Hamiltonians (see for instance [9]; Proposition 2.9.6), the result follows. □

4.2. **Dispersion.** In this paragraph we shall prove Point (2) of Theorem 1. The strategy is the following.

In Section 4.2.1 we prove using semi-classical analysis that for a very short time, the solutions to (4.1) remain microlocalized in a compact set satisfying assumption (2.5), and such that $\mu\mathrm{Supp}_{x_1}\varphi_\pm$ become disjoint from $\mathrm{Supp}_{x_1}\bar{u}$. Section 4.2.2 is then devoted to the long-time behaviour of the solution, and Mourre estimates allow to prove that the solution exits any compact set after some time, and that it remains microlocalized far from $\xi_1 = 0$.

The result of the analysis carried out in this paragraph is that the behaviour of $\mu\mathrm{Supp}_{x_1}\varphi_\pm$ is as depicted in the following figure.



4.2.1. *Short time behaviour.* The aim of this paragraph is to prove the following result. It shows that the solutions of (4.1) exit the support of $\bar{u}$ after a time $t_{\rm exit}\varepsilon$, for $|t_{\rm exit}|$ large enough (independent of ε). We only state the forward in time result: the backwards result is identical, up to changing the sign of time. We shall further restrict the analysis to T_+ since the argument for T_- is identical, up to some sign changes.

Proposition 4.2. *Let φ^0 be a function, microlocalized in a compact set $\mathcal{C}$ satisfying Assumption (2.5), and let φ be the associate solution of (4.1). Let $[u_-, u_+]$ be a closed interval of $\mathbf{R}$ containing $Supp_{x_1} \bar{u}$. There exists a constant $t_{\rm exit} > 0$ such that for any $\varepsilon \in]0,1[$, the function $\varphi(\varepsilon t_{\rm exit}, \cdot)$ is microlocalized in a compact set $\mathcal{K}$ such that the projection of $\mathcal{K}$ onto the x_1-axis does not interesect $[u_-, u_+]$. Moreover $\mu Supp_{\xi_1} \varphi$ is unchanged.*

More precisely, if $\mu Supp_{\xi_1} \varphi^0 \subset \mathbf{R}^+ \backslash \{0\}$, then $\mu Supp_{x_1} \varphi(\varepsilon t_{\rm exit}, \cdot) \subset]u_+, +\infty[$, and if $\mu Supp_{\xi_1} \varphi^0 \subset \mathbf{R}^- \setminus \{0\}$, then $\mu Supp_{x_1} \varphi(\varepsilon t_{\rm exit}, \cdot) \subset]-\infty, u_-[$.

Proof. Define the function $\psi(s) := \varphi(\varepsilon s)$. Then (4.1) reads

$$i\varepsilon \partial_s \psi = T_+ \psi, \quad \psi_{|s=0} = \varphi^0, \tag{4.2}$$

and any result proved on ψ on $[0, \mathcal{T}]$ will yield the same result for φ on $[0, \mathcal{T}\varepsilon]$. Notice that (4.2) is written in a semi-classical setting, so by the propagation of the microsupport theorem (see for instance [9], Theorem 4.3.7), the microsupport of ψ is propagated by the bicharacteristics, which are the integral curves of the principal symbol. Recall that the principal symbol of T_+ is

$$\tau_+(\xi_1, x_2, \xi_2) = \sqrt{\xi_1^2 + \xi_2^2 + b^2(x_2)}$$

and the bicharacteristics are given by the following set of ODEs:

$$\begin{cases} \dot{x}^t = \nabla_\xi \tau_+(\xi_1^t, x_2^t, \xi_2^t), & x^0 = (x_1^0, x_2^0) \\ \dot{\xi}^t = -\nabla_x \tau_+(\xi_1^t, x_2^t, \xi_2^t), & \xi^0 = (\xi_1^0, \xi_2^0). \end{cases}$$

Notice that τ_+ is independent of x_1, so $\dot{\xi}_1^t$ is identically zero and therefore $\xi_1^t \equiv \xi_1^0$. So for all $s \geq 0$, the microlocal support in ξ_1 of $\psi(s)$ remains unchanged, and in particular is far from $\xi_1 = 0$. Moreover one has

$$\dot{x}_1^t = \frac{\xi_1^0}{\sqrt{(\xi_1^0)^2 + (\xi_2^t)^2 + b^2(x_2^t)}} \cdot$$

Now we recall that the bicharacteristic curves lie on energy surfaces, meaning that on each bicharacteristic, $\tau_+(\xi_1^0, x_2^t, \xi_2^t)$ is a constant. That implies that $(\xi_2^t)^2 + b^2(x_2^t)$ is a constant on each bicharacteristic, so that for all times,

$$\dot{x}_1^t \equiv \frac{\xi_1^0}{\sqrt{(\xi_1^0)^2 + (\xi_2^0)^2 + b^2(x_2^0)}} \cdot$$

If $\xi_1^0 > 0$, then x_1 is propagated to the right and eventually escapes to the right of the support in x_1 of $\bar{u}$, whereas if $\xi_1^0 < 0$, the converse (to the left) occurs. Proposition 4.2 is proved. □

4.2.2. *Long time behaviour.* The aim of this paragraph is to prove the following result, which again is only proved for positive times for simplicity.

Proposition 4.3. *Under the assumptions of Proposition 4.2, let φ^+ be the solution of (4.1) associated with the data $\varphi(\varepsilon t_{\rm exit}, \cdot)$. Then $\mu Supp_{x_1} \varphi^+(t)$ does not intersect $Supp_{x_1} \bar{u}$ for $t \geq \varepsilon t_{\rm exit}$, and $\mu Supp_{\xi_1} \varphi^+(t)$ remains unchanged for $t \geq \varepsilon t_{\rm exit}$. Finally $\mu Supp_{x_1} \varphi^+(t)$ exits any compact set in x_1 as soon as $t > \varepsilon t_{\rm exit}$.*

Proof. Before going into the proof, we shall simplify the analysis by only studying the case of T_+ (the case T_- is obtained by identical arguments), and we shall only deal with the case when the support in ξ_1 of the data lies in the positive half space. The other case is obtained similarly.

The proof is based on Mourre's theory which we shall now briefly recall, and we refer to [7] and [3] for all details. Let us consider two self-adjoint operators H and A on a Hilbert space $\mathcal{H}$. We make the following assumptions:

(1) the intersection of the domains of A and H is dense in the domain $\mathcal{D}(H)$ of H .
(2) $t \mapsto e^{itA}$ maps $\mathcal{D}(H)$ to itself, and for all $\varphi^0 \in \mathcal{D}(H)$,

$$\sup_{t\in[0,1]} \|He^{itA}\varphi^0\| < \infty.$$

(3) The operator $i[H, A]$ is bounded from below and closable, and the domain $\mathcal{D}(B_1)$ where iB_1 is its closure, contains $\mathcal{D}(H)$. More generally for all $n \in \mathbf{N}$ the operator $i[iB_n, A]$ is bounded from below and closable and the domain $\mathcal{D}(B_{n+1})$ of its closure iB_{n+1} contains $\mathcal{D}(H)$, and finally B_{n+1} extends to a bounded operator from $\mathcal{D}(H)$ to its dual.
(4) There exists $\theta > 0$ and an open interval Δ of $\mathbf{R}$ such that if E_Δ is the corresponding spectral projection of H, then

$$E_\Delta i[H, A]E_\Delta \geq \theta E_\Delta. \tag{4.3}$$

Note that Assumptions (1 - 3) can be replaced by the fact that $[f(H), A]$ and all commutator iterates are bounded for any smooth, compactly supported function f (see [3]).

Under those assumptions, for any integer $m \in \mathbf{N}$ and for any $\theta' \in]0, \theta[$, there is a constant C such that

$$\|\chi_-(A - a - \theta' t)e^{-iHt}g(H)\chi_+(A - a)\| \leq Ct^{-m}$$

where $\chi_\pm$ is the characteristic function of $\mathbf{R}^\pm$, g is any smooth compactly supported function in Δ, and the above bound is uniform in $a \in \mathbf{R}$.

Let us apply this theory to our situation. We consider equation (4.1) with data $e^{i\frac{t_{\text{exit}}}{\varepsilon}T_+}\varphi_+^0$, and let us define the operator T_+^0 as the operator T_+ where $\bar{u}$ has been chosen identically zero. We shall start by studying the equation

$$i\varepsilon^2\partial_t\widetilde{\varphi} = T_+^0\widetilde{\varphi}, \quad \widetilde{\varphi}_{|t=\varepsilon t_{\text{exit}}} = e^{i\frac{t_{\text{exit}}}{\varepsilon}T_+}\varphi_+^0, \tag{4.4}$$

for which we shall prove Proposition 4.3. Then we shall prove that the solution $\widetilde{\varphi}$ actually solves the original equation (4.1) with the same data $e^{i\frac{t_{\text{exit}}}{\varepsilon}T_+}\varphi_+^0$ at $t = \varepsilon t_{\text{exit}}$ up to $O(\varepsilon^\infty)$, because its support in x_1 lies outside the support of $\bar{u}$ and because the symbolic expansion of T_+ depends on x_1 only through $\bar{u}$ and its derivatives (see Remark 3.4).

So let us start by applying Mourre's theory to (4.4). Let us write the projection of $\mathcal{K}$ onto the ξ_1-axis as included in $[d_0, d_1]$ with $0 < d_0 < d_1 < \infty$. We recall that on the support of $e^{i\frac{t_{\text{exit}}}{\varepsilon}T_+}\varphi_+^0$, x_1 remains to the right of the support of $\bar{u}$. Then we apply the theory to $H = T_+^0$ and to $A = x_1$ (the pointwise multiplication). Assumptions (1) to (3) are easy to check, in particular because this is a semiclassical setting, so only the principal symbols need to be considered. Similarly finding a lower bound for $E_\Delta i[T_+^0, x_1]E_\Delta$ boils down to computing the Poisson bracket $\{\tau_+, x_1\}$ where

$$\{f,g\} = \nabla_\xi f \cdot \nabla_x g - \nabla_x f \cdot \nabla_\xi g,$$

and one finds

$$\{\tau_+, x_1\} = \frac{\xi_1}{\sqrt{\xi_1^2+\xi_2^2+b^2(x_2)}}\cdot \tag{4.5}$$

Since T_+^0 has constant coefficients in x_1, ξ_1 is preserved, so in particular for all times one has $\mu\mathrm{Supp}_{\xi_1}\widetilde{\varphi}(t) \subset [d_0, d_1]$. One can furthermore choose for Δ an interval of $\mathbf{R}$ of the type $]D_0, D_1[$ where the constants D_0 and D_1 are chosen so that for any $(x,\xi) \in \mathcal{K}$, one has

$$D_0 < \sqrt{\xi_1^2+\xi_2^2+b^2(x_2)} < D_1. \tag{4.6}$$

As the microlocal supports of the eigenfunctions of T_+^0 lie on energy surfaces, we know that the solution to (4.4) will remain in E_Δ for all times. Now let us apply the results of [7] and [3]. By Lemma 3.3, (4.5), (4.6) and the assumption on ξ_1 written above, we have that

$$E_\Delta i[H,A]E_\Delta \geq \varepsilon \frac{d_0}{D_1} E_\Delta,$$

so (4.3) holds with $\theta = \varepsilon d_0/D_1$. It follows that the solution $e^{i\frac{(t-\varepsilon t_{\mathrm{exit}})}{\varepsilon^2}T_+^0}\left(e^{i\frac{t_{\mathrm{exit}}}{\varepsilon}T_+}\varphi_+^0\right)$ to (4.4) has a support in x_1 such that

$$x_1 > u_+ + \frac{d_0}{D_1}\frac{t}{\varepsilon}$$

which proves the result for (4.4).

Since $\mu\mathrm{Supp}_{x_1} e^{i\frac{(t-\varepsilon t_{\mathrm{exit}})}{\varepsilon^2}T_+^0}\left(e^{i\frac{t_{\mathrm{exit}}}{\varepsilon}T_+}\varphi_+^0\right)$ does not cross $\mathrm{Supp}_{x_1}\bar{u}$, one has actually

$$e^{i\frac{(t-\varepsilon t_{\mathrm{exit}})}{\varepsilon^2}T_+^0}\left(e^{i\frac{t_{\mathrm{exit}}}{\varepsilon}T_+}\varphi_+^0\right) = e^{i\frac{(t-\varepsilon t_{\mathrm{exit}})}{\varepsilon^2}T_+}\left(e^{i\frac{t_{\mathrm{exit}}}{\varepsilon}T_+}\varphi_+^0\right) \quad \text{in } L^2$$

locally uniformly in t (see Proposition 5.3 of the appendix). The proposition follows. □

5. Propagation of the Rossby waves

5.1. **Semiclassical transport equations and microlocalization.**

Because of the scaling of the Rossby hamiltonian (which is smaller than the Poincaré hamiltonians by one order of magnitude), on the times scales considered here the propagation of energy by Rossby waves is described by the hamiltonian dynamics

$$\frac{dx_i}{dt} = \frac{\partial \tau_R}{\partial \xi_i}, \quad \frac{d\xi_i}{dt} = -\frac{\partial \tau_R}{\partial x_i},$$

which can be written explicitly

$$
\begin{aligned}
\frac{dx_1}{dt} &= b'(x_2)\frac{\langle\xi\rangle_b^2 - 2\xi_1^2}{\langle\xi\rangle_b^4} + \bar{u}_1(x),\\
\frac{dx_2}{dt} &= -2b'(x_2)\frac{\xi_1\xi_2}{\langle\xi\rangle_b^4} + \bar{u}_2(x),\\
\frac{d\xi_1}{dt} &= -\partial_1\bar{u}_1(x)\xi_1 - \partial_1\bar{u}_2(x)\xi_2,\\
\frac{d\xi_2}{dt} &= \xi_1\frac{2b(b')^2 - b''\langle\xi\rangle_b^2}{\langle\xi\rangle_b^4} - \partial_2\bar{u}_1(x)\xi_1 - \partial_2\bar{u}_2(x)\xi_2
\end{aligned}
\tag{5.1}
$$

where we recall that $\langle\xi\rangle_b = \sqrt{\xi_1^2 + \xi_2^2 + b^2(x_2)}$. In order for the dynamics to be well defined and also in order to justify the diagonalization process, we need the quantity $\langle\xi\rangle_b$ to remain bounded from below for all times.

Proposition 5.1. *Let $\mathcal{C}$ be some compact subset of $\mathbf{R}^4$ such that*

$$\mathcal{C} \cap \{(x_1, x_2, \xi_1, \xi_2)\,/\,\xi_1^2 + \xi_2^2 + b^2(x_2) = 0\} = \emptyset\,.$$

Then the bicharacteristics $t \mapsto (x(t), \xi(t))$ of the Rossby Hamiltonian starting from any point $(x_1^0, x_2^0, \xi_1^0, \xi_2^0)$ of $\mathcal{C}$ are defined globally in time, and $\forall t \in \mathbf{R}$,

$$\inf_{(x_1^0,x_2^0,\xi_1^0,\xi_2^0)\in\mathcal{C}} (\xi_1(t)^2 + \xi_2(t)^2 + b^2(x_2(t))) > 0.$$

Proof. As b', b'', u and Du are Lipschitz, by the Cauchy-Lipschitz theorem the system of ODEs (5.1) has a unique maximal solution. In order to prove that this solution is defined globally, it is enough to prove that the time derivative of this solution is uniformly bounded. This comes from assumption (2.3) giving an upper bound on $b'/\big(1 + b^2(x_2)\big)^{\frac{1}{2}}$ and $b''/\big(1 + b^2(x_2)\big)^{\frac{1}{2}}$, and from the lower bound on $\langle\xi\rangle_b$ to be established now.

The crucial assumption here is the fact that b' and b do not vanish simultaneously.

So let us suppose that $\langle\xi\rangle_b$ vanishes, and consider the first time t^* at which $\langle\xi\rangle_b(t^*) = 0$. Assume to start with that $x(t^*)$ lies outside the support of $\bar{u}$. Then there is a small amount of time (t^-, t^*), $t^- < t^*$, on which $x(t)$ remains outside the support of $\bar{u}$. So on the interval $(t^-, t^*), \xi_1$ is a constant hence remains zero, and an inspection of the ODEs then shows that on $(t^-, t^*), x_2$ and ξ_2 are also constant, hence $\langle\xi\rangle_b(t) = 0$ which is impossible by definition of t^*.

Now let us assume that $x(t^*)$ does not lie outside the support of $\bar{u}$, where t^* is still the first time t^* at which $\langle\xi\rangle_b(t^*) = 0$, assuming such a time exists. We shall prove that

$$|\langle\xi\rangle_b(t)| \lesssim (t^* - t)^{\frac{1}{2}}, \quad t \to t^*. \tag{5.2}$$

Indeed we have clearly

$$
\begin{aligned}
\frac{1}{2}\frac{d}{dt}\langle\xi\rangle_b^2 &= b(x_2)b'(x_2)\frac{dx_2}{dt} + \xi_1\frac{d\xi_1}{dt} + \xi_2\frac{d\xi_2}{dt}\\
&= b(x_2)b'(x_2)\bar{u}_2(x) - \partial_1\bar{u}(x)\cdot\xi - \partial_2\bar{u}(x)\cdot\xi + \frac{b''(x_2)\xi_1\xi_2}{\langle\xi\rangle_b^2}
\end{aligned}
\tag{5.3}
$$

so in particular we find that $\frac{d}{dt}\langle\xi\rangle_b^2$ is bounded as t goes to t^*, hence (5.2) holds.

Moreover along a trajectory of the Rossby Hamiltonian, τ_R is conserved, and we have

$$\frac{dx_1}{dt} = \frac{b'(x_2)}{\langle\xi\rangle_b^2} - \frac{2(\tau_R - u(x)\cdot\xi)^2}{b'(x_2)} + u_1(x).$$

Since b' and b do not vanish simultaneously, this in turn implies that there is a constant C such that as t goes to t^*,

$$\Big|\frac{dx_1}{dt}\Big| \geq \frac{C}{t^*-t}\cdot$$

In particular there is a time $t < t^*$ at which the trajectory has escaped the support of $\bar u$, which is contrary to our assumption. This concludes the proof of the proposition. □

5.2. **Dynamics outside from the support of $\bar u$.**

Using the fact that $\bar u$ has compact support, and simple properties of the Rossby dynamics in the absence of zonal flow, we can prove the following

Proposition 5.2. *Let $\mathcal{C}$ be some compact subset of $\mathbf{R}^4$ such that*

$$\mathcal{C} \cap \{(x_1,x_2,\xi_1,\xi_2)\,/\,\xi_1^2+\xi_2^2+b^2(x_2)=0\} = \emptyset\,.$$

Then the bicharacteristics of the Rossby Hamiltonian starting from any point of $\mathcal{C}$ are bounded in x_2:

$$\forall t\in\mathbf{R},\qquad \sup_{(x_1^0,x_2^0,\xi_1^0,\xi_2^0)\in\mathcal{C}} |x_2(t)|<\infty.$$

Proof. • Let us start by describing the dynamics in the absence of zonal flow: ξ_1 is then an invariant of the motion, so that the dynamics in (x_2,ξ_2) can be decoupled. Furthermore, as the energy surfaces are compact

$$\tau = \frac{\xi_1 b'(x_2)}{\xi_1^2+\xi_2^2+b^2(x_2)}$$

the motion along x_2 is periodic (with infinite period for homoclinic and heteroclinic orbits).

The motion along x_1 is then determined by the equation

$$\frac{dx_1}{dt} = b'(x_2)\frac{\langle\xi\rangle_b^2-2\xi_1^2}{\langle\xi\rangle_b^4}\cdot$$

It is trapped if and only if the average of the right-hand side over one period is zero. Outside from saddle points, this quantity depends continuously on ξ_1, so that we expect the initial data leading to trapped trajectories to belong to a manifold of codimension 1. This can be proved rigorously if b^2 has only one non degenerate critical points (see [2]).

• Let us now turn to the influence of the zonal flow. We will first check that the only possible escape direction is again x_1. Indeed the energy surfaces corresponding to $\tau_R\neq 0$ are bounded in the x_2 direction : as $x_2\to\pm\infty$,

$$\frac{b'(x_2)\xi_1}{\langle\xi\rangle_b^2} + \bar u(x)\cdot\xi \to 0\,.$$

Consider now a trajectory on the energy level $\tau_R = 0$, and some point of this trajectory (y_1, y_2, ξ_1, ξ_2) such that $y_2 \notin \mathrm{Supp}_{x_2}\bar{u}$. One has

$$b'(y_2)\xi_1 = 0\,.$$

- If $b'(y_2) = 0$, then

$$\frac{dx_1}{dt} = \frac{dx_2}{dt} = \frac{d\xi_1}{dt} = \frac{d\xi_2}{dt} = 0\,.$$

The uniqueness in Cauchy-Lipschitz theorem implies then that the trajectory is nothing else than a fixed point, and therefore in particular is bounded.

- If $\xi_1 = 0$, then

$$\frac{dx_2}{dt} = \frac{d\xi_1}{dt} = \frac{d\xi_2}{dt} = 0 \quad \text{and} \quad \frac{dx_1}{dt} = b'(y_2)\frac{\xi_2^2 + b^2(y_2) - \xi_1^2}{\langle\xi\rangle_b^4}\,,$$

meaning that the trajectory is a uniform straight motion along x_1. In particular, it is bounded in the x_2-direction.

Finally, we conclude that trajectories on the energy level $\tau_R = 0$ are either trapped in the support $\mathrm{Supp}_{x_2}\bar{u}$, or trivial in the x_2-direction. □

5.3. **Perspectives.** As recalled in the introduction, it is generally believed that in the situation depicted in this paper (a flow around a large macroscopic current) Rossby waves are trapped. However due to the 2-dimensional setting (compared to the work in [2]) the trapping in the x_1 direction seems difficult to prove, outside some specific cases studied in the previous paragraph. One way to be convinced of the trapping phenomenon should be by implementing the dynamical system numerically. It should be pointed out however that actually in order to get physically relevant predictions for the oceanic eddies, one should consider 3D models, or at least 2D models involving the influence of stratification. The methods presented here seem to be robust and should be extended to such complex models, up to again the study of the hamiltonian system describing the Rossby dynamics.

Appendix : a comparison result

For the sake of completeness, we state here the result which shows the stability of the propagation under a $O(\varepsilon^\infty)$ error on the propagator. This result has been used in the proof of the diagonalization when comparing A and $T_\pm$, T_R, and in the proof of dispersion when comparing $T_\pm$ and $T_\pm^0$.

Proposition 5.3. *Let A_ε and $\tilde{A}_\varepsilon$ be two pseudo-differential operators such that*

- *iA_ε is hermitian in $L^2(\mathbf{R}^d)$,*
- *$A_\varepsilon - \tilde{A}_\varepsilon = O(\varepsilon^\infty)$ microlocally on $\Omega \subset \mathbf{R}^{2d}$.*

Let $\tilde{\varphi}$ be a solution to

$$i\partial_t\tilde{\varphi} + \tilde{A}_\varepsilon\tilde{\varphi} = 0$$

microlocalized in Ω, and φ be the solution to

$$i\partial_t\varphi + A_\varepsilon\varphi = 0$$

with the same initial data. Then, for all $N \in \mathbf{N}$,

$$\sup_{t \leq \varepsilon^{-N}} \|\varphi(t) - \tilde{\varphi}(t)\|_{L^2(\mathbf{R}^d)} = O(\varepsilon^\infty)\,.$$

Proof. The proof is based on a simple energy inequality and is completely straightforward. We have

$$\begin{aligned}\frac{d}{dt}\|\varphi - \tilde{\varphi}\|^2_{L^2(\mathbf{R}^d)} &= 2\langle iA_\varepsilon\varphi - i\tilde{A}_\varepsilon\tilde{\varphi}|\varphi - \tilde{\varphi}\rangle \\ &= 2\langle (iA_\varepsilon - i\tilde{A}_\varepsilon)\tilde{\varphi}|\varphi - \tilde{\varphi}\rangle \\ &\leq 2\|(A_\varepsilon - \tilde{A}_\varepsilon)\tilde{\varphi}\|_{L^2(\mathbf{R}^d)}\|\varphi - \tilde{\varphi}\|_{L^2(\mathbf{R}^d)}\,.\end{aligned}$$

This leads to

$$\|\varphi(t) - \tilde{\varphi}(t)\|^2_{L^2(\mathbf{R}^d)} = O(\varepsilon^\infty)t\,,$$

which concludes the proof. □

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(I. Gallagher) Institut de Mathématiques UMR 7586, Université Paris VII, 175, rue du Chevaleret, 75013 Paris, FRANCE

E-mail address: Isabelle.Gallagher@math.jussieu.fr

(T. Paul) CNRS and Centre de mathématiques Laurent Schwartz, École polytechnique, 91128 Palaiseau cedex, FRANCE

E-mail address: thierry.paul@math.polytechnique.fr

(L. Saint-Raymond) Université Paris VI and DMA École Normale Supérieure, 45 rue d'Ulm, 75230 Paris Cedex 05, FRANCE

E-mail address: Laure.Saint-Raymond@ens.fr