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Local Dispersive and Strichartz Estimates for the Schrödinger Operator on the Heisenberg Group

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Abstract. It was proved by Bahouri *et al.* [9] that the Schrödinger equation on the Heisenberg group $\mathbb{H}^d$, involving the sublaplacian, is an example of a totally non-dispersive evolution equation: for this reason global dispersive estimates cannot hold. This paper aims at establishing local dispersive estimates on $\mathbb{H}^d$ for the linear Schrödinger equation, by a refined study of the Schrödinger kernel S_t on $\mathbb{H}^d$. The sharpness of these estimates is discussed through several examples. Our approach, based on the explicit formula of the heat kernel on $\mathbb{H}^d$ derived by Gaveau [19], is achieved by combining complex analysis and Fourier-Heisenberg tools. As a by-product of our results we establish local Strichartz estimates and prove that the kernel S_t concentrates on quantized horizontal hyperplanes of $\mathbb{H}^d$.

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1 Introduction

1.1 Setting of the problem

It is well-known that the solution to the free Schrödinger equation on $\mathbb{R}^n$

$$(S) \quad \begin{cases} i\partial_t u - \Delta u = 0, \\ u_{|t=0} = u_0 \end{cases}$$

can be explicitly written with a convolution kernel for $t \neq 0$

$$u(t,\cdot) = u_0 \star \frac{\mathrm{e}^{-i\frac{|\cdot|^2}{4t}}}{(-4\pi i t)^{\frac{n}{2}}} \cdot \tag{1.1}$$

The proof of this explicit representation stems by a combination of Fourier and complex analysis arguments, from the expression of the heat kernel on $\mathbb{R}^n$. More precisely, taking the partial Fourier transform of (S) with respect to the variable x and integrating in time the resulting ODE, we get

$$\widehat{u}(t,\xi) = \mathrm{e}^{it|\xi|^2} \widehat{u}_0(\xi),$$

where for any function $g \in L^1(\mathbb{R}^n)$ we have defined

$$\widehat{g}(\xi) \stackrel{\text{def}}{=} \mathcal{F}(g)(\xi) \stackrel{\text{def}}{=} \int_{\mathbb{R}^n} \mathrm{e}^{-i\langle x,\xi\rangle} g(x)\, dx.$$

The heart of the matter to prove (1.1) then consists in computing in the sense of distributions the inverse Fourier transform of the complex Gaussian

$$\left(\mathcal{F}^{-1} \mathrm{e}^{it|\cdot|^2}\right)(x) = \frac{\mathrm{e}^{-i\frac{|x|^2}{4t}}}{(-4\pi i t)^{\frac{n}{2}}} \cdot \tag{1.2}$$

The proof of formula (1.2) is based on two observations: first, that for any x in $\mathbb{R}^n$, the two maps

$$\begin{aligned} z \in \mathbb{C} \quad &\longmapsto \quad H_1(z) \stackrel{\text{def}}{=} \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \mathrm{e}^{i\langle x,\xi\rangle} \mathrm{e}^{-z|\xi|^2} d\xi, \\ z \in \mathbb{C} \quad &\longmapsto \quad H_2(z) \stackrel{\text{def}}{=} \frac{1}{(4\pi z)^{\frac{n}{2}}} \mathrm{e}^{-\frac{|x|^2}{4z}} \end{aligned}$$

are holomorphic on the domain D of complex numbers with positive real part. Accordingly with the expression of the heat kernel, these two functions coincide on the intersection of the real line with D, and thus they coincide on the whole domain D. Second, if $(z_p)_{p\in\mathbb{N}}$ denotes a sequence of elements of D which converges to $-it$ for $t\neq 0$, the use of the Lebesgue dominated convergence theorem ensures that $H_1(z_p)$ and $H_2(z_p)$ converge in $\mathcal{S}'(\mathbb{R}^n)$, as p tends to infinity, which achieves the proof of (1.2).

Formula (1.1) implies by Young's inequality the following dispersive estimate:

$$\forall t\neq 0, \quad \|u(t,\cdot)\|_{L^\infty(\mathbb{R}^n)} \leq \frac{1}{(4\pi|t|)^{\frac{n}{2}}}\|u_0\|_{L^1(\mathbb{R}^n)}. \tag{1.3}$$

Such estimate plays a key role in the study of semilinear and quasilinear equations which appear in numerous physical applications. Combined with an abstract functional analysis argument known as the TT^*-argument, it yields a range of inequalities involving space-time Lebesgue norms, known as Strichartz estimates[†]. When u_0 is for instance in $L^2(\mathbb{R}^n)$, the above dispersive estimate (1.3) gives rise to the following Strichartz estimate for the solution to the free Schrödinger equation

$$\|u\|_{L^q(\mathbb{R};L^p(\mathbb{R}^n))} \leq C(p,q)\|u_0\|_{L^2(\mathbb{R}^n)}, \tag{1.4}$$

where (p,q) satisfies the scaling admissibility condition

$$\frac{2}{q}+\frac{n}{p}=\frac{n}{2} \quad \text{with} \quad q\geq 2 \quad \text{and} \quad (n,q,p)\neq(2,2,\infty). \tag{1.5}$$

The interest for this issue has soared in the last decades. We refer for instance to the monographs [4,30] and the references therein for an overview on this topic in the Euclidean framework.

In the present work, we aim at investigating this phenomenon for the Schrödinger equation on the Heisenberg group $\mathbb{H}^d$ involving the sublaplacian. Recall that in [9], the first author along with Gérard and Xu proved that no dispersion occurs for this equation, and in particular exhibited an example for which the Schrödinger operator on $\mathbb{H}^d$ behaves as a transport equation with respect to one direction, known as the vertical direction. More precisely they established the following result which shows that a global dispersive estimate of the type (1.3) cannot be expected on $\mathbb{H}^d$. We refer to the coming paragraph for the notation.

[†]For further details, one can consult the papers of Ginibre and Velo [22], Keel and Tao [24] and Strichartz [29].

Proposition 1.1 ([9])**.** *There exists a function u_0 in the Schwartz class $\mathcal{S}(\mathbb{H}^d)$ such that the solution to the free Schrödinger equation on $\mathbb{H}^d$ satisfies*

$$\forall t\in\mathbb{R}, \quad \forall (Y,s)\in\mathbb{H}^d, \quad u(t,Y,s)=u_0(Y,s+4td). \tag{1.6}$$

This result rules out an estimate of the type (1.3) in the setting of the Heisenberg group but does not exclude a degraded estimate: for instance in the case when u_0 is compactly supported, then the solution remains compactly supported, in a set transported along the vertical line, so a local L^∞ norm decays to zero with time. Inspired by the Euclidean strategy displayed above, we shall indeed be able to establish local decay in the spirit of (1.3). The precise result is stated in the next paragraph. As in the Euclidean case, such a local dispersive estimate stems from the explicit expression of the Schrödinger kernel S_t on $\mathbb{H}^d$, which turns out to be of type (1.2) in a horizontal strip of $\mathbb{H}^d$ (see Theorem 1.2).

Note also that a lack of dispersion was highlighted for the Schrödinger propagator (associated with the sublaplacian) in the framework of H-type groups [12] or more generally in the case of 2-step stratified Lie groups [8]. More precisely, if p denotes the dimension of the center of the H-type group, Del Hierro proved in [12] sharp dispersive inequalities for the Schrödinger equation solution (with a $|t|^{-\frac{(p-1)}{2}}$ decay). Concerning the more general case of 2-step stratified Lie groups, the authors along with Fermanian-Kammerer [8] emphasized the key role played by the canonical skew-symmetric form in determining the rate of decay of the solutions of the Schrödinger equation: they established that if p denotes the dimension of the center of a 2-step stratified Lie group G and k the dimension of the radical of its canonical skew-symmetric form, then the solutions of the Schrödinger equation on G satisfy dispersive estimates with a rate of decay at most of order $|t|^{-\frac{k+p-1}{2}}$.

1.2 Basic facts about the Heisenberg group

Recall that the d-dimensional Heisenberg group $\mathbb{H}^d$ can be defined as $T^\star\mathbb{R}^d\times\mathbb{R}$ where $T^\star\mathbb{R}^d$ is the cotangent bundle, endowed with the noncommutative product law[‡]

$$(Y,s)\cdot(Y',s')\stackrel{\text{def}}{=}\big(Y+Y',s+s'+2\langle\eta,y'\rangle-2\langle\eta',y\rangle\big), \tag{1.7}$$

where $w=(Y,s)=(y,\eta,s)$ and $w'=(Y',s')=(y',\eta',s')$ are elements of $\mathbb{H}^d$. The variable Y is called the horizontal variable, while the variable s is known as the vertical variable.

[‡]We refer to the monographs [7,17,18,28,31,32] and the references therein for further details.

The space $\mathbb{H}^d$ is provided with a smooth left invariant measure, the Haar measure, which in the coordinate system (Y,s) is simply the Lebesgue measure. In particular, one can define the following (noncommutative) convolution product for any two integrable functions f and g:

$$f\star g(w)\stackrel{\text{def}}{=}\int_{\mathbb{H}^d} f(w\cdot v^{-1})g(v)\,dv=\int_{\mathbb{H}^d} f(v)g(v^{-1}\cdot w)\,dv, \tag{1.8}$$

and the usual Young inequalities are valid

$$\|f\star g\|_{L^r(\mathbb{H}^d)}\le \|f\|_{L^p(\mathbb{H}^d)}\|g\|_{L^q(\mathbb{H}^d)},$$

whenever

$$1\le p,q,r\le\infty \quad \text{and} \quad \frac{1}{r}=\frac{1}{p}+\frac{1}{q}-1. \tag{1.9}$$

The dilation on $\mathbb{H}^d$ is defined for all $a>0$ by

$$\delta_a(Y,s)\stackrel{\text{def}}{=}\big(aY,a^2s\big). \tag{1.10}$$

Since for all $a>0$ and any $f\in L^1(\mathbb{H}^d)$,

$$\int_{\mathbb{H}^d} f\big(\delta_a(w)\big)\,dw=a^{-(2d+2)}\int_{\mathbb{H}^d} f(w)dw,$$

the homogeneous dimension of $\mathbb{H}^d$ is $Q\stackrel{\text{def}}{=}2d+2$.

The natural distance on $\mathbb{H}^d$ compatible with the product law (1.7) is called the Korányi distance and is defined by

$$d_{\mathbb{H}}(w,w')\stackrel{\text{def}}{=}\rho_{\mathbb{H}}(w^{-1}\cdot w') \tag{1.11}$$

for all w, w' in $\mathbb{H}^d$, where $\rho_{\mathbb{H}}$ stands for the distance to the origin

$$\rho_{\mathbb{H}}(w)=\rho_{\mathbb{H}}(Y,s)\stackrel{\text{def}}{=}\big(|Y|^4+s^2\big)^{\frac{1}{4}}. \tag{1.12}$$

In the following $B_{\mathbb{H}}(w_0,R)$ denotes the Heisenberg ball centered at w_0 and of radius R for the distance $d_{\mathbb{H}}$ defined by (1.11), namely

$$B_{\mathbb{H}}(w_0,R)\stackrel{\text{def}}{=}\Big\{w\in\mathbb{H}^d\,/\,d_{\mathbb{H}}(w,w_0)<R\Big\}.$$

Observing that the distance $d_{\mathbb{H}}$ is invariant by left translation, that is to say

$$\forall (w,w',w_0)\in(\mathbb{H}^d)^3, \quad d_{\mathbb{H}}\left(\tau_{w_0}(w),\tau_{w_0}(w')\right)=d_{\mathbb{H}}(w,w'),$$

where τ_{w_0} denotes the left translation defined by

$$\tau_{w_0}(w)\stackrel{\text{def}}{=}w_0\cdot w, \tag{1.13}$$

one can readily check that $\tau_{w_0}(B_{\mathbb{H}}(0,R))=B_{\mathbb{H}}(w_0,R)$.

Most classical analysis tools of $\mathbb{R}^n$ can be adapted to $\mathbb{H}^d$, resorting to the following left invariant vector fields:

$$\mathcal{X}_j\stackrel{\text{def}}{=}\partial_{y_j}+2\eta_j\partial_s \quad \text{and} \quad \Xi_j\stackrel{\text{def}}{=}\partial_{\eta_j}-2y_j\partial_s \quad \text{with} \quad j\in\{1,\dots,d\},$$

known as the horizontal left invariant vector fields. In particular, the sublaplacian is given by

$$\Delta_{\mathbb{H}}\stackrel{\text{def}}{=}\sum_{j=1}^{d}\left(\mathcal{X}_j^2+\Xi_j^2\right).$$

For instance the Schwartz space $\mathcal{S}(\mathbb{H}^d)$, which is nothing else than $\mathcal{S}(\mathbb{R}^{2d+1})$, can be characterized by means of $\Delta_{\mathbb{H}}$ and $\rho_{\mathbb{H}}$.

1.3 Main results

The main goal of this article is to establish local dispersive estimates for the free linear Schrödinger equation on $\mathbb{H}^d$ associated with the sublaplacian

$$(S_{\mathbb{H}}) \quad \begin{cases} i\partial_t u-\Delta_{\mathbb{H}}u=0, \\ u_{|t=0}=u_0. \end{cases}$$

As in the Euclidean case one can readily establish that the Cauchy problem $(S_{\mathbb{H}})$ admits a unique, global in time solution if $u_0\in L^2(\mathbb{H}^d)$, by resorting to Fourier-Heisenberg analysis tools or to functional calculus of the self-adjoint operator $-\Delta_{\mathbb{H}}$ (see Section 4 for further details). Denoting by $(\mathcal{U}(t))_{t\in\mathbb{R}}$ the solution operator, namely $\mathcal{U}(t)u_0$ is the solution of $(S_{\mathbb{H}})$ at time t associated with the data u_0, then similarly to the Euclidean case $(\mathcal{U}(t))_{t\in\mathbb{R}}$ is a one-parameter group of unitary operators on $L^2(\mathbb{H}^d)$.

The first result we establish states as follows.

Theorem 1.1. *Given $w_0 \in \mathbb{H}^d$, let u_0 be a function in $\mathcal{D}(B_{\mathbb{H}}(w_0,R_0))$. Then the solution to the Cauchy problem $(S_{\mathbb{H}})$ associated to u_0 disperses locally for large $|t|$, in the sense that, for any positive constant $\kappa < \sqrt{4d}$, the following estimate holds for all $2 \le p \le \infty$:*

$$\|u(t,\cdot)\|_{L^p(B_{\mathbb{H}}(w_0,\kappa|t|^{1/2}))} \le \left(\frac{M_\kappa}{|t|^{\frac{Q}{2}}}\right)^{1-\frac{2}{p}} \|u_0\|_{L^{p'}(\mathbb{H}^d)} \tag{1.14}$$

for all $|t| \ge T_{\kappa,R_0}$, where

$$T_{\kappa,R_0} \stackrel{def}{=} \left(\frac{R_0}{\sqrt{4d}-\kappa}\right)^2, \quad M_\kappa \stackrel{def}{=} \frac{1}{(4\pi)^{\frac{Q}{2}}} \int_{\mathbb{R}} \left(\frac{2\tau}{\sinh 2\tau}\right)^d \exp\left(\frac{\kappa^2\tau}{2}\right) d\tau \tag{1.15}$$

and p' is the conjugate exponent to p.

Remark 1.1. The counterexample (1.6) due to Bahouri *et al.* [9] is given by

$$u(t,Y,s) = \int_{\mathbb{R}} \mathrm{e}^{i(s+4dt)\lambda} \mathrm{e}^{-\lambda|Y|^2} g(\lambda) \lambda^d d\lambda \tag{1.16}$$

with g in $\mathcal{D}(]0,\infty[)$. Although u_0 does not belong to $\mathcal{D}(\mathbb{H}^d)$, one can easily check that for any $\delta > 0$, as soon as $|s+4dt| > \delta|t|$ then for any integer N there is a positive constant C depending on N and u_0 such that

$$|u(t,w)| \le \frac{C}{|s+4dt|^N} \le \frac{C}{(\delta|t|)^N}. \tag{1.17}$$

On the other hand estimate (1.17) fails, for any integer $N \ge 1$, in the case when $s = -4td$. This shows the sharpness of the bound on the constant κ appearing in Theorem 1.1. More generally it was established in [3] that for any integer ℓ, denoting by $L_\ell^{(d-1)}$ the Laguerre polynomial of order ℓ and type $d-1$ (see for instance [13, 23, 27]), then

$$u^{(\ell)}(t,Y,s) = \int_{\mathbb{R}} \mathrm{e}^{i(s+4t(2\ell+d))\lambda} \mathrm{e}^{-|\lambda||Y|^2} L_\ell^{(d-1)}(2|\lambda||Y|^2) g(\lambda) \lambda^d d\lambda$$

is a solution to $(S_{\mathbb{H}})$, and this solution satisfies (1.17) when $|s+4(2\ell+d)t| > \delta|t|$ and $|t|$ is large enough.

As in the Euclidean case outlined above, the (local) dispersive estimate (1.14) stems from Young inequalities (1.9) using an explicit formula of the type (1.1) for the Schrödinger kernel on $\mathbb{H}^d$. However, the study of the kernel S_t of the

Schrödinger operator on $\mathbb{H}^d$ is more involved than in the Euclidean case, because on the one hand the Fourier transform on $\mathbb{H}^d$ is an intricate tool and on the other hand S_t does not enjoy a formulation of type (1.2) globally on $\mathbb{H}^d$. In fact, as will be seen in Section 4 (see Proposition 4.1), one can compute S_t in the sense of distributions, using the Fourier-Heisenberg analysis tools developed in [5], and also it turns out (see Theorem 1.4) that S_t concentrates on quantized horizontal hyperplanes of $\mathbb{H}^d$. It follows that the explicit formula of the type (1.1) that we obtain here is only local. More precisely, our result states as follows. Its sharpness is discussed in Paragraph 1.4.

Theorem 1.2. *The kernel associated with the free Schrödinger equation* $(S_{\mathbb{H}})$ *reads for all* $t\neq 0$

$$S_t(Y,s)=\frac{1}{(-4i\pi t)^{\frac{Q}{2}}}\int_{\mathbb{R}}\left(\frac{2\tau}{\sinh 2\tau}\right)^d \exp\left(-\frac{\tau s}{2t}-i\frac{|Y|^2\tau}{2t\tanh 2\tau}\right)d\tau \tag{1.18}$$

provided that $|s|<4d|t|$.

Remark 1.2. Theorem 1.2 highlights the separate roles of the horizontal and vertical variables of $\mathbb{H}^d$. Note that in [26], Müller already emphasized the distinguished role of the horizontal variable in the study of the Fourier restriction theorem on $\mathbb{H}^d$.

Even though the dispersive estimate (1.14) we establish for the Schrödinger operator on $\mathbb{H}^d$ is only local, we are able to prove that the solutions of the Schrödinger equation $(S_{\mathbb{H}})$ enjoy locally Strichartz estimates in the spirit of (1.4). More precisely, we have the following result.

Theorem 1.3. *Under the notations of Theorem* 1.1, *given* $\kappa<\sqrt{4d}$ *and* (p,q) *belonging to the admissible set*

$$\mathcal{A}\stackrel{def}{=}\left\{(p,q)\,/\,\frac{2}{q}+\frac{Q}{p}=\frac{Q}{2}\quad with\quad 2\leq p\leq\infty\right\}, \tag{1.19}$$

there exists a positive constant $C(q,\kappa)$ *such that, for all* $u_0\in L^2(\mathbb{H}^d)$ *supported in the ball* $B_{\mathbb{H}}(w_0,R_0)$, *for some* $w_0\in\mathbb{H}^d$, *the solution to the Cauchy problem* $(S_{\mathbb{H}})$ *satisfies the following local Strichartz estimate:*

$$\|u\|_{L^q(]-\infty,-C_\kappa R_0^2]\cup[C_\kappa R_0^2,+\infty[;L^p(B_{\mathbb{H}}(0,\kappa\sqrt{|t|})))}\leq C(q,\kappa)\|u_0\|_{L^2(B_{\mathbb{H}}(w_0,R_0))}, \tag{1.20}$$

where $C_\kappa=(\sqrt{4d}-\kappa)^{-2}$.

Note that the Strichartz estimate (1.20) is invariant by scaling (through the scaling $u(t,w)\mapsto u(\lambda^2 t,\delta_\lambda w)$). Let us underline that there is a duality between the size of the support of u_0 and the time for which the Strichartz estimates holds. Indeed, letting R_0 go to zero, we find that for an initial data concentrated around some $w_0\in\mathbb{H}^d$, the Strichartz estimate is almost global in time. Conversely, letting R_0 go to infinity, the time from which (1.20) occurs is close to infinity. Let us also emphasize that the counterexamples introduced in Remark 1.1 show somehow the optimality of our result, since for these counterexamples a global integrability both with respect to t and s independently is excluded.

1.4 Refined study of the Schrödinger kernel on $\mathbb{H}^d$

Theorem 1.2 asserts that S_t, for $t\neq 0$, is a decaying smooth function on the strip $|s|<4d|t|$ (with a decay rate of order $|t|^{-\frac{Q}{2}}$). One may wonder if S_t ($t\neq 0$) which, according to Proposition 4.1, belongs to $\mathcal{S}'(\mathbb{H}^d)$ can be identified with a function on the horizontal hyperplanes $s=\pm 4d|t|$. The answer to this question is negative as asserted by the following result.

Theorem 1.4. *With the previous notations, for all* $\pm w_\ell \stackrel{def}{=} (0,\pm 4(2\ell+d)|t|)$, *where* $t\neq 0$ *and* $\ell\in\mathbb{N}$, *there exists an initial data* $u_0^{\pm,\ell}\in\mathcal{S}(\mathbb{H}^d)$ *such that* $u^{\pm,\ell}(t,\cdot)\stackrel{def}{=}\mathcal{U}(t)u_0^{\pm,\ell}$ *satisfies*

$$u^{\pm,\ell}(t,\pm w_\ell)=u_0^{\pm,\ell}(0)=\langle\delta_0,u_0^{\pm,\ell}\rangle_{\mathcal{S}'(\mathbb{H}^d)\times\mathcal{S}(\mathbb{H}^d)}. \tag{1.21}$$

Remark 1.3. Actually, the above theorem can be easily generalized to any element of the horizontal hyperplanes $s=\pm 4(2\ell+d)|t|$, $t\neq 0$ and $\ell\in\mathbb{N}$, namely $(Y_0,\pm 4(2\ell+d)|t|)$, where Y_0 is some fixed element of $T^\star\mathbb{R}^d$. The Cauchy data generating a solution which concentrates on hyperplanes $s=\pm 4(2\ell+d)|t|$ are linked to the counterexamples introduced in Remark 1.1.

The above result shows the optimality of the bound $4d|t|$ in Theorem 1.2. However, we are able to improve this bound when we restrict $(S_{\mathbb{H}})$ to some subspaces of Cauchy data as in the next statement.

Theorem 1.5. *There exists an orthogonal decomposition of* $L^2(\mathbb{H}^d)$

$$L^2(\mathbb{H}^d)=\oplus_{m\in\mathbb{N}^d}L^2_m(\mathbb{H}^d) \tag{1.22}$$

such that the restriction $S_t^{(\ell)}$ of S_t to the subspace $\mathcal{V}_\ell(\mathbb{H}^d)\stackrel{def}{=}\oplus_{|m|\geq\ell}L^2_m(\mathbb{H}^d)$ is well defined as soon as $|s|<4(2\ell+d)|t|$, and satisfies for any positive constant $\kappa<\sqrt{4(d+2\ell)}$,

$$\sup_{|s|\leq\kappa^2|t|}\sup_{Y\in T^*\mathbb{R}^d}\frac{1}{|t|^{\frac{Q}{2}}}\big|S_t^{(\ell)}(Y,s)\big|\leq C(\ell,\kappa).$$

Remark 1.4. Decomposition (1.22) is strongly tied to the spectral representation of the sublaplacian $-\Delta_{\mathbb{H}}$. In order to give a flavor of the above result, let us point out that, as we shall see in Section 2, the Fourier-Heisenberg transform exchanges $-\Delta_{\mathbb{H}}$ with the harmonic oscillator. Then in some sense, (1.22) consists in a decomposition of $L^2(\mathbb{H}^d)$ along Hermite-type functions, via the Fourier-Heisenberg transform.

1.5 Main steps of the proof of the main results and layout of the paper

Since the Schrödinger equation on $\mathbb{H}^d$ is invariant under left translations, one can assume without loss of generality in the proof of Theorem 1.1 that $w_0=0$. By Young inequalities (1.9), Theorem 1.1 readily follows from Theorem 1.2 (reducing the assumption $|s|<4d|t|$ to the fact that $\rho_{\mathbb{H}}(w)<\sqrt{4d|t|}$). To prove Theorem 1.1, we are thus reduced to establishing Theorem 1.2. Roughly speaking the proof of Theorem 1.2 is achieved in three steps. In the first step, using the Fourier-Heisenberg analysis on tempered distributions developed in [5] (see also Section 2.2 in this paper), we establish that the kernel S_t of the Schrödinger operator on $\mathbb{H}^d$ belongs to $\mathcal{S}'(\mathbb{H}^d)$ (Proposition 4.1). It is well-known since the paper of Gaveau [19], that the solution to the heat equation on $\mathbb{H}^d$ associated with the sublaplacian writes for all $t>0$

$$u(t,\cdot)=\frac{1}{t^{\frac{Q}{2}}}u_0\star\big(h\circ\delta_{\sqrt{t}}\big),$$

where $\delta_{\sqrt{t}}$ is the dilation operator defined in (1.10) and h is the function in the Schwartz class $\mathcal{S}(\mathbb{H}^d)$ given by

$$h(Y,s)\stackrel{\rm def}{=}\frac{1}{(4\pi)^{\frac{Q}{2}}}\int_{\mathbb{R}}\left(\frac{2\tau}{\sinh 2\tau}\right)^d\exp\left(i\frac{\tau s}{2}-\frac{|Y|^2\tau}{2\tanh 2\tau}\right)d\tau. \tag{1.23}$$

Then the second step is devoted to the proof of the fact that the fundamental solution of the heat equation on $\mathbb{H}^d$ coming from Fourier analysis on $\mathbb{H}^d$ coincides with the explicit formula (1.23) established by Gaveau [19] (see Proposition 3.1). This step uses Melher's formula, along with the Fourier approach developed in [5, 6]. The last step concludes the proof following the general method of the Euclidean case via complex analysis, described above (see Section 4.2). It is in this final step that the restriction $|s|<4d|t|$ appears.

As usual, the proof of the local Strichartz estimates stated in Theorem 1.3 is straightforward from the local dispersive estimate (1.14) thanks to standard functional analysis arguments.

Finally, the refined study of the Schrödinger kernel on $\mathbb{H}^d$ (through Theorems 1.4 and 1.5) is derived by a combination of Fourier-Heisenberg tools and the spectral analysis of the harmonic oscillator.

Let us describe the organization of the paper. Section 2 is dedicated to a brief description of the Fourier transform $\mathcal{F}_{\mathbb{H}}$ on $\mathbb{H}^d$ and the space of frequencies $\widehat{\mathbb{H}}^d$, as well as the extension of $\mathcal{F}_{\mathbb{H}}$ to tempered distributions – which is at the heart of the matter in this paper. In Section 3, we recover the explicit formula of the heat kernel on $\mathbb{H}^d$ established by Gaveau in [19], using Fourier analysis on $\mathbb{H}^d$. In Section 4, we investigate the kernel of the Schrödinger operator on $\mathbb{H}^d$ and prove Theorem 1.2, while Section 5 is devoted to the proof of Theorem 1.1 thanks to Theorem 1.2. Then, we establish the local Strichartz estimates (Theorem 1.3). In Section 6, we undertake a refined study of S_t and establish Theorems 1.4 and 1.5 making use of the Fourier-Heisenberg approach developed in [5, 6].

To avoid heaviness, all along this article C will denote a positive constant which may vary from line to line. We also use $A\lesssim B$ to denote an estimate of the form $A\leq CB$.

2 Fourier analysis on $\mathbb{H}^d$

2.1 The Fourier transform on $\mathbb{H}^d$

The Fourier transform on $\mathbb{H}^d$ is defined using irreducible unitary representations of $\mathbb{H}^d$. It is thus not a complex-valued function on some "frequency space" as in the Euclidean case, but a family of bounded operators on $L^2(\mathbb{R}^d)$ (see for instance [2, 11, 14, 15, 28, 31, 32] for further details). Recently, in [5, 6] the authors introduced an equivalent, intrinsic definition of the Fourier transform on $\mathbb{H}^d$ in terms of functions acting on a frequency set denoted $\widetilde{\mathbb{H}}^d \stackrel{\text{def}}{=} \mathbb{N}^{2d}\times\mathbb{R}\setminus\{0\}$. More precisely, denoting the elements of this set by $\widehat{w}\stackrel{\text{def}}{=}(n,m,\lambda)$, the Fourier transform of an integrable function on $\mathbb{H}^d$ is defined in the following way:

$$\forall \widehat{w} \in \widetilde{\mathbb{H}}^d, \quad \mathcal{F}_{\mathbb{H}} f(\widehat{w}) \stackrel{\text{def}}{=} \int_{\mathbb{H}^d} \overline{\mathrm{e}^{is\lambda} \mathcal{W}(\widehat{w},Y)} f(Y,s) dY ds \tag{2.1}$$

with $\mathcal{W}$ the Wigner transform of the (renormalized) Hermite functions

$$\begin{aligned} &\mathcal{W}(\widehat{w},Y) \stackrel{\text{def}}{=} \int_{\mathbb{R}^d} \mathrm{e}^{2i\lambda\langle \eta,z\rangle} H_{n,\lambda}(y+z) H_{m,\lambda}(-y+z) dz, \\ &H_{m,\lambda}(x) \stackrel{\text{def}}{=} |\lambda|^{\frac{d}{4}} H_m\big(|\lambda|^{\frac{1}{2}} x\big) \end{aligned} \tag{2.2}$$

with $(H_m)_{m\in\mathbb{N}^d}$ the Hermite orthonormal basis of $L^2(\mathbb{R}^d)$ given by the eigenfunctions of the harmonic oscillator

$$-\big(\Delta - |x|^2\big) H_m = (2|m|+d) H_m.$$

We recall that

$$H_m(x) \stackrel{\text{def}}{=} \left(\frac{1}{2^{|m|} m!}\right)^{\frac{1}{2}} \prod_{j=1}^{d} \big(-\partial_j H_0(x) + x_j H_0(x)\big)^{m_j} \tag{2.3}$$

with $H_0(x) \stackrel{\text{def}}{=} \pi^{-\frac{d}{4}} \mathrm{e}^{-\frac{|x|^2}{2}}$, $m! \stackrel{\text{def}}{=} m_1! \cdots m_d!$ and $|m| \stackrel{\text{def}}{=} m_1 + \cdots + m_d$.

In [6], the authors show that the completion of the set $\widetilde{\mathbb{H}}^d$ for the distance

$$\widehat{d}(\widehat{w},\widehat{w}') \stackrel{\text{def}}{=} \big|\lambda(n+m) - \lambda'(n'+m')\big|_{\ell^1(\mathbb{N}^d)} + \big|(n-m) - (n'-m')\big|_{\ell^1(\mathbb{N}^d)} + d|\lambda - \lambda'|$$

is the set

$$\widehat{\mathbb{H}}^d \stackrel{\text{def}}{=} \widetilde{\mathbb{H}}^d \cup \widehat{\mathbb{H}}^d_0 \quad \text{with} \quad \widehat{\mathbb{H}}^d_0 \stackrel{\text{def}}{=} \mathbb{R}^d_{\mp} \times \mathbb{Z}^d \quad \text{and} \quad \mathbb{R}^d_{\mp} \stackrel{\text{def}}{=} (\mathbb{R}_-)^d \cup (\mathbb{R}_+)^d.$$

In this setting, the classical statements of Fourier analysis hold in a similar way to the Euclidean case. In particular, the inversion and Fourier-Plancherel formulae read

$$f(w) = \frac{2^{d-1}}{\pi^{d+1}} \int_{\widetilde{\mathbb{H}}^d} \mathrm{e}^{is\lambda} \mathcal{W}(\widehat{w},Y) \mathcal{F}_{\mathbb{H}} f(\widehat{w}) d\widehat{w} \tag{2.4}$$

and

$$(\mathcal{F}_{\mathbb{H}} f | \mathcal{F}_{\mathbb{H}} g)_{L^2(\widetilde{\mathbb{H}}^d)} = \frac{\pi^{d+1}}{2^{d-1}} (f|g)_{L^2(\mathbb{H}^d)}, \tag{2.5}$$

where the measure $d\widehat{w}$ is defined in the following way[§]: for any function θ on $\widetilde{\mathbb{H}}^d$,

$$\int_{\widetilde{\mathbb{H}}^d} \theta(\widehat{w})\, d\widehat{w} \stackrel{\text{def}}{=} \int_{\mathbb{R}} \sum_{(n,m)\in\mathbb{N}^{2d}} \theta(n,m,\lambda)|\lambda|^d d\lambda.$$

Straightforward computations give

$$\mathcal{F}_{\mathbb{H}}(-\Delta_{\mathbb{H}} f)(\widehat{w}) = 4|\lambda|(2|m|+d)\mathcal{F}_{\mathbb{H}}(f)(\widehat{w}). \tag{2.6}$$

According to (2.4)-(2.5), one can easily check that

$$L^2(\mathbb{H}^d) = \oplus_{m\in\mathbb{N}^d} L^2_m(\mathbb{H}^d) \tag{2.7}$$

in the following way: any function $f\in L^2(\mathbb{H}^d)$ can be split as

$$\begin{aligned} &f = \sum_{m\in\mathbb{N}^d} f_m \quad \text{with} \\ &f_m(Y,s) = \frac{2^{d-1}}{\pi^{d+1}} \sum_{n\in\mathbb{N}^d} \int_{\mathbb{R}} e^{is\lambda} \mathcal{W}\big((n,m,\lambda),Y\big) \mathcal{F}_{\mathbb{H}} f(n,m,\lambda)|\lambda|^d d\lambda, \end{aligned} \tag{2.8}$$

and

$$\|f\|^2_{L^2(\mathbb{H}^d)} = \sum_{m\in\mathbb{N}^d} \|f_m\|^2_{L^2(\mathbb{H}^d)}.$$

Let us also note that if f and g are two functions of $L^1(\mathbb{H}^d)$ then for any $\widehat{w} = (n,m,\lambda)$ in $\widetilde{\mathbb{H}}^d$ there holds

$$\mathcal{F}_{\mathbb{H}}(f\star g)(\widehat{w}) = (\mathcal{F}_{\mathbb{H}} f \cdot \mathcal{F}_{\mathbb{H}} g)(\widehat{w}) \stackrel{\text{def}}{=} \sum_{p\in\mathbb{N}^d} \mathcal{F}_{\mathbb{H}} f(n,p,\lambda)\mathcal{F}_{\mathbb{H}} g(p,m,\lambda). \tag{2.9}$$

As we shall see, the heat and Schrödinger kernels on $\mathbb{H}^d$ are radial, in the sense that they are invariant under the action of the unitary group of $T^\star\mathbb{R}^d$. In addition, being functions of $-\Delta_{\mathbb{H}}$ they are even. In fact, the Fourier transform of radial functions turns out to be simpler than in the general case: if f is a radial function in $L^1(\mathbb{H}^d)$, then for any $(n,m,\lambda)\in\widetilde{\mathbb{H}}^d$,

$$\mathcal{F}_{\mathbb{H}}(f)(n,m,\lambda) = \mathcal{F}_{\mathbb{H}}(f)(n,m,\lambda)\delta_{n,m} = \mathcal{F}_{\mathbb{H}}(f)(|n|,|n|,\lambda)\delta_{n,m} \tag{2.10}$$

[§]As shown in [5], the measure $d\widehat{w}$ can be extended by 0 on $\widehat{\mathbb{H}}^d_0$.

with for all $\ell \in \mathbb{N}$

$$\begin{aligned}&\mathcal{F}_{\mathbb{H}}(f)(\ell,\ell,\lambda)\\&=\binom{\ell+d-1}{\ell}^{-1}\int_{\mathbb{H}^d}\mathrm{e}^{-is\lambda}\mathrm{e}^{-|\lambda||Y|^2}L_\ell^{(d-1)}\big(2|\lambda||Y|^2\big)f(Y,s)\,dYds,\end{aligned} \tag{2.11}$$

where $L_\ell^{(d-1)}$ stands for the Laguerre polynomial¶ of order ℓ and type $d-1$.

Obviously the inversion formula writes in that case

$$f(w)=\frac{2^{d-1}}{\pi^{d+1}}\sum_{\ell\in\mathbb{N}}\int_{\mathbb{R}}\mathrm{e}^{is\lambda}\widetilde{\mathcal{W}}(\ell,\lambda,Y)\mathcal{F}_{\mathbb{H}}(f)(\ell,\ell,\lambda)\,|\lambda|^d\,d\lambda, \tag{2.12}$$

where

$$\widetilde{\mathcal{W}}(\ell,\lambda,Y)\stackrel{\text{def}}{=}\sum_{\substack{n\in\mathbb{N}^d\\|n|=\ell}}\mathcal{W}(n,n,\lambda,Y)=\mathrm{e}^{-|\lambda||Y|^2}L_\ell^{(d-1)}\big(2|\lambda||Y|^2\big). \tag{2.13}$$

2.2 The Fourier transform on $\mathcal{S}'(\mathbb{H}^d)$

The new approach of the Fourier-Heisenberg transform developed in [6] enabled the authors in [5] to extend $\mathcal{F}_{\mathbb{H}}$ to $\mathcal{S}'(\mathbb{H}^d)$, the set of tempered distributions: note that since the Schwartz class $\mathcal{S}(\mathbb{H}^d)$ coincides with $\mathcal{S}(\mathbb{R}^{2d+1})$ then similarly $\mathcal{S}'(\mathbb{H}^d)$ is nothing else than $\mathcal{S}'(\mathbb{R}^{2d+1})$. Roughly speaking, the first step to achieve this extension consists in characterizing $\mathcal{S}(\widehat{\mathbb{H}}^d)$, the range of $\mathcal{S}(\mathbb{H}^d)$, by $\mathcal{F}_{\mathbb{H}}$. It will be useful to recall in the following that according to [5], the space $\mathcal{S}(\widehat{\mathbb{H}}^d)$ can be equipped with semi-norms $\|\cdot\|_{N,\mathcal{S}(\widehat{\mathbb{H}}^d)}$ and that in particular, for all $\theta\in\mathcal{S}(\widehat{\mathbb{H}}^d)$ and $N\in\mathbb{N}$, there exists C_N such that for all $\hat{w}=(n,m,\lambda)\in\widetilde{\mathbb{H}}^d$

$$|\theta(\hat{w})|\le C_N\big(1+4|\lambda|(2|m|+d)\big)^{-N}\|\theta\|_{N,\mathcal{S}(\widehat{\mathbb{H}}^d)}. \tag{2.14}$$

We refer to [5] for the definition of $\mathcal{S}(\widehat{\mathbb{H}}^d)$ and further details. Then the result follows by duality, as in the Euclidean case, once shown that the Fourier transform $\mathcal{F}_{\mathbb{H}}$ is a bicontinuous isomorphism between the spaces $\mathcal{S}(\mathbb{H}^d)$ and $\mathcal{S}(\widehat{\mathbb{H}}^d)$‖.

¶The interested reader can consult for instance [1,10,13] and the references therein.

‖Note that a first attempt in the description of the range of $\mathcal{S}(\mathbb{H}^d)$ by the Fourier-Heisenberg transform goes back to the pioneering works by Geller in [20,21], where asymptotic series are used. One can also consult the paper of Astengo *et al.* [2].

The map $\mathcal{F}_{\mathbb{H}}$ can thus be continuously extended from $\mathcal{S}'(\mathbb{H}^d)$ into $\mathcal{S}'(\widehat{\mathbb{H}}^d)$ in the following way:

$$\mathcal{F}_{\mathbb{H}}:\begin{cases}\mathcal{S}'(\mathbb{H}^d) & \longrightarrow \quad \mathcal{S}'(\widehat{\mathbb{H}}^d),\\ T & \longmapsto \quad \Big[\theta\mapsto \langle \mathcal{F}_{\mathbb{H}}T,\theta\rangle_{\mathcal{S}'(\widehat{\mathbb{H}}^d)\times\mathcal{S}(\widehat{\mathbb{H}}^d)}=\langle T,{}^t\mathcal{F}_{\mathbb{H}}\theta\rangle_{\mathcal{S}'(\mathbb{H}^d)\times\mathcal{S}(\mathbb{H}^d)}\Big],\end{cases} \tag{2.15}$$

where according to (2.4),

$$ {}^t\mathcal{F}_{\mathbb{H}}\theta(y,\eta,s)\stackrel{\rm def}{=}\frac{\pi^{d+1}}{2^{d-1}}\big(\mathcal{F}_{\mathbb{H}}^{-1}\theta\big)(y,-\eta,-s). \tag{2.16}$$

In particular one can compute the Fourier transform of the Dirac mass

$$\mathcal{F}_{\mathbb{H}}(\delta_0)=\mathbf{1}_{\{(n,m,\lambda)\,/\,n=m\}},$$

that is to say, for any θ in $\mathcal{S}(\widehat{\mathbb{H}}^d)$

$$\big\langle \mathcal{F}_{\mathbb{H}}(\delta_0),\theta\big\rangle_{\mathcal{S}'(\widehat{\mathbb{H}}^d)\times\mathcal{S}(\widehat{\mathbb{H}}^d)}=\sum_{n\in\mathbb{N}^d}\int_{\mathbb{R}}\theta(n,n,\lambda)|\lambda|^d d\lambda. \tag{2.17}$$

It will be useful later on to notice that $\mathcal{S}'(\widehat{\mathbb{H}}^d)$ contains (after suitable identification) all functions with moderate growth, that are defined as the locally integrable functions θ on $\widehat{\mathbb{H}}^d$ such that for some large enough integer N, the map

$$(n,m,\lambda)\ \longmapsto\ \big(1+|\lambda|(n+m|+d)+|n-m|\big)^{-N}\theta(n,m,\lambda) \tag{2.18}$$

belongs to $L^\infty(\widetilde{\mathbb{H}}^d)$. It is proved in [5] that any such function can be identified with a tempered distribution on $\widehat{\mathbb{H}}^d$.

Let us end this introduction on Fourier analysis on the Heisenberg group by recalling that if T is a tempered distribution on $\mathbb{H}^d$, then for all f in $\mathcal{S}(\mathbb{H}^d)$ and all w in $\mathbb{H}^d$,

$$(T\star f)(w)=\langle T,\check{f}\circ\tau_{w^{-1}}\rangle_{\mathcal{S}'(\mathbb{H}^d)\times\mathcal{S}(\mathbb{H}^d)},$$

and

$$(f\star T)(w)=\big\langle T,\check{f}\circ\tau^{\rm r}_w\big\rangle_{\mathcal{S}'(\mathbb{H}^d)\times\mathcal{S}(\mathbb{H}^d)}, \tag{2.19}$$

where

$$\check{f}(w)\stackrel{\rm def}{=}f(w^{-1}), \tag{2.20}$$

and τ_w denotes the left translation operator by w defined in (1.13) while $\tau^{\rm r}_w$ is the right translation operator by w defined by

$$\tau^{\rm r}_w(w')\stackrel{\rm def}{=}w'\cdot w. \tag{2.21}$$

3 On the kernel of the heat operator on $\mathbb{H}^d$

A striking consequence of Fourier analysis on $\mathbb{H}^d$ developed in [5] is that it provides another proof of the fact that the fundamental solution of the heat equation on $\mathbb{H}^d$ coincides with the explicit formula established by Gaveau in [19]. First note the following representation coming from Fourier-Heisenberg analysis.

Proposition 3.1. *If u denotes the solution to the free heat equation on $\mathbb{H}^d$*

$$(H_{\mathbb{H}})\quad \begin{cases} \partial_t u - \Delta_{\mathbb{H}} u = 0, \\ u_{|t=0} = u_0, \end{cases}$$

where u_0 is a given integrable function on $\mathbb{H}^d$, then for all $t>0$ there holds

$$u(t,\cdot) = u_0 \star h_t,$$

where h_t is defined by

$$h_t(Y,s) \stackrel{def}{=} \frac{2^{d-1}}{\pi^{d+1}} \sum_{n\in\mathbb{N}^d} \int_{\mathbb{R}} \mathrm{e}^{is\lambda} \mathcal{W}\big((n,n,\lambda),Y\big) \mathrm{e}^{-4t|\lambda|(2|n|+d)} |\lambda|^d d\lambda.$$

Proof. Applying $\mathcal{F}_{\mathbb{H}}$ to the heat equation and taking advantage of (2.6), we get for all (n,m,λ) in $\widetilde{\mathbb{H}}^d$,

$$\begin{cases} \dfrac{d\widehat{u}_{\mathbb{H}}}{dt}(t,n,m,\lambda) = -4|\lambda|(2|m|+d)\widehat{u}_{\mathbb{H}}(t,n,m,\lambda), \\ \widehat{u}_{\mathbb{H}|t=0} = \mathcal{F}_{\mathbb{H}} u_0. \end{cases}$$

By time integration, this implies that for all (n,m,λ) in $\widetilde{\mathbb{H}}^d$,

$$\widehat{u}_{\mathbb{H}}(t,n,m,\lambda) = \mathrm{e}^{-4t|\lambda|(2|m|+d)} \mathcal{F}_{\mathbb{H}} u_0(n,m,\lambda). \tag{3.1}$$

According to (2.9), we deduce that

$$\widehat{u}_{\mathbb{H}}(t,n,m,\lambda) = (\mathcal{F}_{\mathbb{H}} u_0 \cdot \theta_t)(n,m,\lambda) \quad \text{with} \quad \theta_t(n,m,\lambda) \stackrel{\text{def}}{=} \mathrm{e}^{-4t|\lambda|(2|n|+d)} \delta_{n,m},$$

where $\delta_{n,m}$ denotes the Kronecker symbol, which implies that

$$u(t,\cdot) = u_0 \star h_t \quad \text{with} \quad (\mathcal{F}_{\mathbb{H}} h_t)(n,m,\lambda) \stackrel{\text{def}}{=} \mathrm{e}^{-4t|\lambda|(2|n|+d)} \delta_{n,m}.$$

This concludes the proof thanks to the inversion formula (2.4). □

Remark 3.1. Performing the change of variable $t\lambda \mapsto \lambda$ in the heat kernel given by Proposition 3.1 readily implies that for all $t>0$

$$h_t(Y,s) = \frac{1}{t^{\frac{Q}{2}}} h\left(\frac{Y}{\sqrt{t}}, \frac{s}{t}\right)$$

with

$$h(Y,s) \stackrel{\text{def}}{=} \frac{2^{d-1}}{\pi^{d+1}} \sum_{m\in\mathbb{N}^d} \int_{\mathbb{R}} e^{is\lambda} \mathcal{W}\big((n,n,\lambda),Y\big) e^{-4|\lambda|(2|m|+d)} \delta_{n,m} |\lambda|^d d\lambda. \tag{3.2}$$

The following remarkable result due to Gaveau [19] asserts that the heat operator on $\mathbb{H}^d$ has a convolution kernel in $\mathcal{S}(\mathbb{H}^d)$.

Theorem 3.1 ([19])**.** *There exists a function h in $\mathcal{S}(\mathbb{H}^d)$ such that for any $u_0 \in L^1(\mathbb{H}^d)$, the solution to $(H_{\mathbb{H}})$ writes for all $t>0$*

$$u(t,\cdot) = u_0 \star h_t,$$

where h_t is defined by

$$h_t(Y,s) = \frac{1}{t^{\frac{Q}{2}}} h\left(\frac{Y}{\sqrt{t}}, \frac{s}{t}\right) \tag{3.3}$$

and the function h is given by the formula

$$h(Y,s) \stackrel{def}{=} \frac{1}{(4\pi)^{\frac{Q}{2}}} \int_{\mathbb{R}} \left(\frac{2\tau}{\sinh 2\tau}\right)^d \exp\left(i\frac{\tau s}{2} - \frac{|Y|^2 \tau}{2\tanh 2\tau}\right) d\tau. \tag{3.4}$$

Proof. Our purpose here is to establish that the formula coming from Fourier analysis given by (3.2), namely

$$\begin{aligned} h(y,\eta,s) = \frac{2^{d-1}}{\pi^{d+1}} \sum_{m\in\mathbb{N}^d} \int_{\mathbb{R}\times\mathbb{R}^d} & e^{is\lambda + 2i\lambda\langle\eta,z\rangle} e^{-4|\lambda|(2|m|+d)} \\ & \times H_{m,\lambda}(y+z) H_{m,\lambda}(-y+z) dz |\lambda|^d d\lambda \end{aligned}$$

coincides with the explicit expression of the heat kernel (3.4) given by Theorem 3.1. The proof relies on Melher's formula (see [16])

$$\sum_{m\in\mathbb{N}} P_m(x) P_m(\widetilde{x}) r^m = \frac{1}{\sqrt{1-r^2}} \exp\left(\frac{2x\widetilde{x}r - (x^2+\widetilde{x}^2)r^2}{1-r^2}\right), \tag{3.5}$$

that holds true for all $x,\widetilde{x}$ in $\mathbb{R}$ and r in $]-1,1[$, where P_m denotes the Hermite polynomial of order m defined by

$$P_m(x) \stackrel{\text{def}}{=} \pi^{\frac{1}{4}} H_m(x) \mathrm{e}^{\frac{|x|^2}{2}}$$

with H_m the Hermite function introduced in (2.3).

To this end, we shall use the following lemma.

Lemma 3.1. *Under the above notations, there holds for all $(y,z)\in\mathbb{R}^2$ and all positive real numbers λ and t,*

$$\begin{aligned}
&\sum_{m\in\mathbb{N}} \mathrm{e}^{-2mt\lambda} H_{m,\lambda}(z-y) H_{m,\lambda}(z+y) \\
&= \frac{1}{\pi^{\frac{1}{2}}} \left(\frac{\lambda}{1-\mathrm{e}^{-4t\lambda}}\right)^{\frac{1}{2}} \exp\left(-\lambda z^2 \tanh(t\lambda) - \frac{\lambda y^2}{\tanh(t\lambda)}\right).
\end{aligned}$$

Proof. Applying the Mehler formula (3.5) to the rescaled Hermite functions (2.3) yields

$$\begin{aligned}
&\sum_{m\in\mathbb{N}} \mathrm{e}^{-2tm\lambda} H_{m,\lambda}(z-y) H_{m,\lambda}(z+y) \\
&= \frac{1}{\pi^{\frac{1}{2}}} \mathrm{e}^{-\lambda(z^2+y^2)} \left(\frac{\lambda}{1-\mathrm{e}^{-4t\lambda}}\right)^{\frac{1}{2}} \\
&\quad \times \exp\left(\frac{1}{1-\mathrm{e}^{-4t\lambda}} \left(2\lambda(z^2-y^2)\mathrm{e}^{-2t\lambda} - 2\lambda(z^2+y^2)\mathrm{e}^{-4t\lambda}\right)\right).
\end{aligned}$$

The result follows from the following identities:

$$\begin{aligned}
&-\lambda(z^2+y^2) + \frac{1}{1-\mathrm{e}^{-4t\lambda}}\left(2\lambda(z^2-y^2)\mathrm{e}^{-2t\lambda} - 2\lambda(z^2+y^2)\mathrm{e}^{-4t\lambda}\right) \\
&= -\frac{\lambda}{1-\mathrm{e}^{-4t\lambda}} \left(z^2(1-\mathrm{e}^{-2t\lambda})^2 + y^2(1+\mathrm{e}^{-2t\lambda})^2\right), \\
&\frac{(1-\mathrm{e}^{-2t\lambda})^2}{1-\mathrm{e}^{-4t\lambda}} = \frac{(\mathrm{e}^{t\lambda}-\mathrm{e}^{-t\lambda})^2}{\mathrm{e}^{2t\lambda}-\mathrm{e}^{-2t\lambda}} = \frac{\mathrm{e}^{t\lambda}-\mathrm{e}^{-t\lambda}}{\mathrm{e}^{t\lambda}+\mathrm{e}^{-t\lambda}} = \tanh(t\lambda), \\
&\frac{(1+\mathrm{e}^{-2t\lambda})^2}{1-\mathrm{e}^{-4t\lambda}} = \frac{(\mathrm{e}^{t\lambda}+e^{-t\lambda})^2}{\mathrm{e}^{2t\lambda}-\mathrm{e}^{-2t\lambda}} = \frac{\mathrm{e}^{t\lambda}+\mathrm{e}^{-t\lambda}}{\mathrm{e}^{t\lambda}-\mathrm{e}^{-t\lambda}} = \frac{1}{\tanh(t\lambda)}.
\end{aligned}$$

The lemma is proved. ☐

Let us return to the proof of Theorem 3.1. In order to establish that the two formulae (3.2) and (3.4) coincide, it suffices to consider the case when $d=1$: (3.2) becomes

$$h(y,\eta,s)=\frac{1}{\pi^2}\sum_{m\in\mathbb{N}}\int_{\mathbb{R}^2}\mathrm{e}^{is\lambda+2i\lambda\eta z}\mathrm{e}^{-4|\lambda|(2m+1)} \times H_{m,\lambda}(y+z)H_{m,\lambda}(-y+z)\,dz|\lambda|d\lambda. \tag{3.6}$$

Then applying Lemma 3.1 with $t=4|\lambda|$, we find that

$$h(y,\eta,s)=\frac{1}{\pi^{\frac{5}{2}}}\int_{\mathbb{R}^2}\mathrm{e}^{is\lambda+2i\lambda\eta z-4|\lambda|}\left(\frac{|\lambda|}{1-\mathrm{e}^{-16|\lambda|}}\right)^{\frac{1}{2}} \times\exp\left(-|\lambda|z^2\tanh(4|\lambda|)-\frac{|\lambda|y^2}{\tanh(4|\lambda|)}\right)|\lambda|d\lambda dz.$$

Performing the change of variables $|\lambda|^{\frac{1}{2}}z\mapsto z$, this gives rise to

$$h(y,\eta,s)=\frac{1}{\pi^{\frac{5}{2}}}\int_{\mathbb{R}}\mathrm{e}^{is\lambda-4|\lambda|-\frac{|\lambda|y^2}{\tanh(4|\lambda|)}} \times\mathcal{F}\left(\mathrm{e}^{-\tanh(4|\lambda|)|\cdot|^2}\right)\left(2|\lambda|^{\frac{1}{2}}\eta\right)\left(\frac{|\lambda|}{1-\mathrm{e}^{-16|\lambda|}}\right)^{\frac{1}{2}}|\lambda|^{\frac{1}{2}}d\lambda.$$

Since

$$\mathcal{F}\left(\mathrm{e}^{-\tanh(4|\lambda|)|\cdot|^2}\right)\left(2|\lambda|^{\frac{1}{2}}\eta\right)=\sqrt{\frac{\pi}{\tanh(4|\lambda|)}}\,\mathrm{e}^{-\frac{|\lambda|\eta^2}{\tanh(4|\lambda|)}},$$

we discover that

$$h(y,\eta,s)=\frac{1}{\pi^2}\int_{\mathbb{R}}\mathrm{e}^{is\lambda-4|\lambda|-\frac{|\lambda|(y^2+\eta^2)}{\tanh(4|\lambda|)}}\left(\tanh(4|\lambda|)\left(1-\mathrm{e}^{-16|\lambda|}\right)\right)^{-\frac{1}{2}}|\lambda|d\lambda.$$

But

$$\tanh(4|\lambda|)\left(1-\mathrm{e}^{-16|\lambda|}\right)=4\mathrm{e}^{-8|\lambda|}\sinh^2(4|\lambda|),$$

which ends the proof of Theorem 3.1. □

4 On the kernel of the Schrödinger operator on $\mathbb{H}^d$

4.1 Representation of the free Schrödinger equation

Contrary to the heat equation (and as in the Euclidean case recalled in the introduction), the kernel of the Schrödinger operator does not belong to the Schwartz class $\mathcal{S}(\mathbb{H}^d)$. Nevertheless, one can solve explicitly the Schrödinger equation $(S_{\mathbb{H}})$ by means of the Fourier-Heisenberg transform introduced in Section 2, in the following way.

Proposition 4.1. *The solution to the free Schrödinger equation*

$$(S_{\mathbb{H}}) \quad \begin{cases} i\partial_t u - \Delta_{\mathbb{H}} u = 0, \\ u_{|t=0} = u_0 \end{cases}$$

reads for all $t \neq 0$ and all $u_0 \in \mathcal{S}(\mathbb{H}^d)$

$$u(t,\cdot) = u_0 \star S_t, \tag{4.1}$$

where S_t denotes the tempered distribution on $\mathbb{H}^d$ defined for all φ in $\mathcal{S}(\mathbb{H}^d)$ by

$$\langle S_t, \varphi \rangle_{\mathcal{S}'(\mathbb{H}^d)\times\mathcal{S}(\mathbb{H}^d)} = \langle e^{4it|\lambda|(2|n|+d)}\delta_{n,m}, \theta \rangle_{\mathcal{S}'(\widehat{\mathbb{H}}^d)\times\mathcal{S}(\widehat{\mathbb{H}}^d)}$$

with $\varphi = {}^t\mathcal{F}_{\mathbb{H}}\theta$, according to notation (2.16).

Proof. Arguing as for the proof of Proposition 3.1, we start by applying $\mathcal{F}_{\mathbb{H}}$ to $(S_{\mathbb{H}})$, which thanks to (2.6) implies that

$$\begin{cases} i\dfrac{d\widehat{u}_{\mathbb{H}}}{dt}(t,n,m,\lambda) = -4|\lambda|(2|m|+d)\widehat{u}_{\mathbb{H}}(t,n,m,\lambda), \\ \widehat{u}_{\mathbb{H}|t=0} = \mathcal{F}_{\mathbb{H}} u_0, \end{cases}$$

and leads by integration to

$$\widehat{u}_{\mathbb{H}}(t,n,m,\lambda) = e^{4it|\lambda|(2|m|+d)}\mathcal{F}_{\mathbb{H}} u_0(n,m,\lambda) \tag{4.2}$$

for all (n,m,λ) in $\widetilde{\mathbb{H}}^d$. Then taking advantage of (2.9), we find that

$$\widehat{u}_{\mathbb{H}}(t,n,m,\lambda) = (\mathcal{F}_{\mathbb{H}} u_0 \cdot \Theta_t)(n,m,\lambda) \quad \text{with} \quad \Theta_t(n,m,\lambda) \stackrel{\text{def}}{=} e^{4it|\lambda|(2|n|+d)}\delta_{n,m}.$$

One can easily check that Θ_t is a function with moderate growth in the sense of (2.18), and thus as it was proved in [5], it is a tempered distribution on $\widehat{\mathbb{H}}^d$. This ensures that the Schrödinger kernel S_t belongs to $\mathcal{S}'(\mathbb{H}^d)$.

Finally combining (2.16) together with (2.19), we readily gather that for all u_0 in $\mathcal{S}(\mathbb{H}^d)$ and all w in $\mathbb{H}^d$,

$$\begin{aligned} (u_0 \star S_t)(w) &= \langle S_t, \check{u}_0 \circ \tau_w^{\mathrm{r}} \rangle_{\mathcal{S}'(\mathbb{H}^d)\times\mathcal{S}(\mathbb{H}^d)} \\ &= \langle e^{4it|\lambda|(2|n|+d)}\delta_{n,m}, \theta \rangle_{\mathcal{S}'(\widehat{\mathbb{H}}^d)\times\mathcal{S}(\widehat{\mathbb{H}}^d)} \\ &= \sum_{n\in\mathbb{N}^d} \int_{\mathbb{R}} e^{4it|\lambda|(2|n|+d)}\theta(n,n,\lambda)|\lambda|^d d\lambda, \end{aligned} \tag{4.3}$$

where $\check{u}_0 \circ \tau_w^{\mathrm{r}} = {}^t\mathcal{F}_{\mathbb{H}}\theta$, which completes the proof of the proposition. □

4.2 Computation of the Schrödinger kernel on Heisenberg strips

Our goal now is to establish Theorem 1.2. As already mentioned, the proof of formula (1.18) goes along the same lines as the Euclidean proof, though more involved. Thanks to Theorem 3.1, the solution to the heat equation $(H_{\mathbb{H}})$ writes

$$f(t,\cdot)=f_0\star h_t,$$

where h_t is given for all $t>0$ by

$$\begin{aligned}h_t(Y,s)&=\frac{1}{(4\pi t)^{\frac{Q}{2}}}\int_{\mathbb{R}}\left(\frac{2\tau}{\sinh 2\tau}\right)^d\exp\left(i\frac{\tau s}{2t}-\frac{|Y|^2\tau}{2t\tanh 2\tau}\right)d\tau\\&=\frac{2^{d-1}}{\pi^{d+1}}\sum_{m\in\mathbb{N}^d}\int_{\mathbb{R}}\mathrm{e}^{is\lambda}\mathrm{e}^{-4t|\lambda|(2|m|+d)}\mathcal{W}\big((m,m,\lambda),Y\big)|\lambda|^d d\lambda.\end{aligned}$$

To achieve our goal, the first step consists in observing that the maps

$$z\longmapsto H^1_z(Y,s)\quad\text{and}\quad z\longmapsto H^2_z(Y,s)$$

with

$$H^1_z(Y,s)\stackrel{\text{def}}{=}\frac{2^{d-1}}{\pi^{d+1}}\sum_{m\in\mathbb{N}^d}\int_{\mathbb{R}}\mathrm{e}^{is\lambda}\mathrm{e}^{-4z|\lambda|(2|m|+d)}\mathcal{W}\big((m,m,\lambda),Y\big)|\lambda|^d d\lambda, \tag{4.4}$$

$$H^2_z(Y,s)\stackrel{\text{def}}{=}\frac{1}{(4\pi z)^{\frac{Q}{2}}}\int_{\mathbb{R}}\left(\frac{2\tau}{\sinh 2\tau}\right)^d\exp\left(i\frac{\tau s}{2z}-\frac{|Y|^2\tau}{2z\tanh 2\tau}\right)d\tau \tag{4.5}$$

are, for all (Y,s) in $\mathbb{H}^d$, holomorphic on a suitable domain of $\mathbb{C}$.

Actually on the one hand, performing the change of variables $\beta=\lambda(2|m|+d)$ in each integral of the right-hand side of (4.4), we get

$$\begin{aligned}H^1_z(Y,s)=&\frac{2^{d-1}}{\pi^{d+1}}\sum_{m\in\mathbb{N}^d}\frac{1}{(2|m|+d)^{d+1}}\\&\times\int_{\mathbb{R}}\mathrm{e}^{is\frac{\beta}{2|m|+d}}\mathrm{e}^{-4z|\beta|}\mathcal{W}\left(\left(m,m,\frac{\beta}{2|m|+d}\right),Y\right)|\beta|^d d\beta,\end{aligned}$$

where obviously in each term of the above identity the integrated function is holomorphic on $\mathbb{C}$. Moreover, using the fact that the modulus of the Wigner transform of the Hermite functions is bounded by 1, we obtain for all z in $\mathbb{C}$ satisfying $\mathrm{Re}(z)\geq a>0$

$$\int_{\mathbb{R}}\left|\mathrm{e}^{is\frac{\beta}{2|m|+d}}\mathrm{e}^{-4z|\beta|}\mathcal{W}\left(\left(m,m,\frac{\beta}{2|m|+d}\right),Y\right)\right||\beta|^d d\beta\leq\int_{\mathbb{R}}\mathrm{e}^{-4a|\beta|}|\beta|^d d\beta<\infty,$$

$$\int_{\mathbb{R}} \Big| \partial_z \Big(\mathrm{e}^{is\frac{\beta}{2|m|+d}} \mathrm{e}^{-4z|\beta|} \mathcal{W}\Big(\Big(m,m,\frac{\beta}{2|m|+d}\Big),Y\Big)\Big)\Big| |\beta|^d d\beta \leq 4\int_{\mathbb{R}} \mathrm{e}^{-4a|\beta|}|\beta|^{d+1} d\beta < \infty,$$

which by Lebesgue's derivation theorem ensures that the map $z \longmapsto H^1_z$ is holomorphic on the domain $D \stackrel{\text{def}}{=} \{z \in \mathbb{C}, \mathrm{Re}(z) > 0\}$.

On the other hand, we have by definition for all z in $\mathbb{C}^*$

$$H^2_z(Y,s) = \frac{1}{(4\pi z)^{\frac{Q}{2}}} \int_{\mathbb{R}} \Big(\frac{2\tau}{\sinh 2\tau}\Big)^d \exp\Big(i\frac{\tau s}{2z} - \frac{|Y|^2\tau}{2z\tanh 2\tau}\Big) d\tau,$$

where of course the integrated function is holomorphic on $\mathbb{C}^*$. Now our aim is to apply Lebesgue's derivation theorem to establish that $H^2_z(Y,s)$ is holomorphic on some domain of $\mathbb{C}$.

Writing

$$\frac{|Y|^2\tau}{2z\tanh 2\tau} = \frac{|Y|^2\tau}{2|z|^2\tanh 2\tau}\overline{z}$$

and setting $z = |z|\mathrm{e}^{i\arg(z)}$, we readily gather that

$$\Big|\exp\Big(-\frac{|Y|^2\tau}{2z\tanh 2\tau}\Big)\Big| = \exp\Big(\frac{-|Y|^2\tau}{2|z|\tanh 2\tau}\cos(\arg(z))\Big) \tag{4.6}$$

and

$$\Big|\partial_z \exp\Big(-\frac{|Y|^2|\tau|}{2z\tanh 2\tau}\Big)\Big| = \frac{|Y|^2\tau}{2|z|^2\tanh 2\tau}\exp\Big(\frac{-|Y|^2\tau}{2|z|\tanh 2\tau}\cos(\arg(z))\Big). \tag{4.7}$$

Along the same lines, one can easily check that

$$\Big|\exp\Big(\frac{i\tau s}{2z}\Big)\Big| = \exp\Big(\frac{\tau s}{2|z|}\sin(\arg(z))\Big),$$

and

$$\Big|\partial_z \exp\Big(\frac{i\tau s}{2z}\Big)\Big| = \frac{|\tau||s|}{2|z|^2}\exp\Big(\frac{\tau s}{2|z|}\sin(\arg(z))\Big).$$

We infer that for all $\tau \in \mathbb{R}$, $w = (Y,s) \in \mathbb{H}^d$ and all z in $\mathbb{C}$ satisfying $\mathrm{Re}(z) \geq a > 0$,

$$\Big|\exp\Big(i\frac{\tau s}{2z} - \frac{|Y|^2\tau}{2z\tanh 2\tau}\Big)\Big| \leq \exp\Big(\frac{|\tau||s|}{2|z|}\Big) \tag{4.8}$$

and

$$\Big|\partial_z\Big(\exp\Big(i\frac{\tau s}{2z} - \frac{|Y|^2\tau}{2z\tanh 2\tau}\Big)\Big)\Big| \leq \exp\Big(\frac{|\tau||s|}{2|z|}\Big)\Big(\frac{1}{a} + \frac{|\tau||s|}{2|z|^2}\Big). \tag{4.9}$$

Fix $0<C<4d$, then combining formula (4.5) together with the Lebesgue derivation theorem, we deduce that the map $z\longmapsto H^2_z$ is holomorphic on

$$\widetilde{D}_{|s|}\stackrel{\text{def}}{=}\left\{z\in D,|z|>\frac{|s|}{C}\right\}. \tag{4.10}$$

Since by Gaveau's result (see Section 3), the maps H^1_z and H^2_z coincide on the intersection of the real line with $\widetilde{D}_{|s|}$, we conclude that they also coincide on the whole domain $\widetilde{D}_{|s|}$.

Choose a sequence $(z_p)_{p\in\mathbb{N}}$ of elements of $\widetilde{D}_{|s|}$ which converges to $-it$, $t\in\mathbb{R}^*$, and consider $\lim_{p\to\infty}\langle H^1_{z_p},\varphi\rangle_{\mathcal{S}'(\mathbb{H}^d)\times\mathcal{S}(\mathbb{H}^d)}$ and $\lim_{p\to\infty}\langle H^2_{z_p},\varphi\rangle_{\mathcal{S}'(\mathbb{H}^d)\times\mathcal{S}(\mathbb{H}^d)}$ for φ in $\mathcal{S}(\mathbb{H}^d)$.

Let us start with $\lim_{p\to\infty}\langle H^1_{z_p},\varphi\rangle_{\mathcal{S}'(\mathbb{H}^d)\times\mathcal{S}(\mathbb{H}^d)}$. By (2.15), there holds

$$\langle H^1_{z_p},\varphi\rangle_{\mathcal{S}'(\mathbb{H}^d)\times\mathcal{S}(\mathbb{H}^d)}=\big\langle\mathcal{F}_{\mathbb{H}}H^1_{z_p},\theta\big\rangle_{\mathcal{S}'(\widehat{\mathbb{H}}^d)\times\mathcal{S}(\widehat{\mathbb{H}}^d)} \tag{4.11}$$

with $\varphi(y,\eta,s)=c_d(\mathcal{F}^{-1}_{\mathbb{H}}\theta)(y,-\eta,-s)$ for some constant c_d. But according to (4.4),

$$\begin{aligned}&\big\langle\mathcal{F}_{\mathbb{H}}H^1_{z_p},\theta\big\rangle_{\mathcal{S}'(\widehat{\mathbb{H}}^d)\times\mathcal{S}(\widehat{\mathbb{H}}^d)}\\&=\int_{\widetilde{\mathbb{H}}^d}\mathrm{e}^{-4z_p|\lambda|(2|m|+d)}\delta_{n,m}\theta(\hat{w})d\hat{w}\\&=\sum_{m\in\mathbb{N}^d}\int_{\mathbb{R}}\mathrm{e}^{-4z_p|\lambda|(2|m|+d)}\theta(m,m,\lambda)|\lambda|^d d\lambda.\end{aligned}$$

Besides since θ belongs to $\mathcal{S}(\widehat{\mathbb{H}}^d)$, it stems from (2.14) that for any integer N there exists C_N such that for all $\hat{w}=(m,m,\lambda)\in\widetilde{\mathbb{H}}^d$

$$|\theta(\hat{w})|\le C_N\big(1+4|\lambda|(2|m|+d)\big)^{-N}\|\theta\|_{N,\mathcal{S}(\widehat{\mathbb{H}}^d)}.$$

Then performing the change of variables $\beta=\lambda(2|m|+d)$ in each integral of the right-hand side of the above identity and choosing $N\ge d+2$, we get

$$\begin{aligned}&\sum_{m\in\mathbb{N}^d}\int_{\mathbb{R}}\big|\mathrm{e}^{-4z_p|\lambda|(2|m|+d)}\theta(m,m,\lambda)\big||\lambda|^d d\lambda\\&\le C_N\|\theta\|_{N,\mathcal{S}(\widehat{\mathbb{H}}^d)}\sum_{m\in\mathbb{N}^d}\frac{1}{(2|m|+d)^{d+1}}\int_{\mathbb{R}}(1+4|\beta|)^{-N}|\beta|^d d\beta<\infty.\end{aligned}$$

Applying Lebesgue's dominated convergence theorem, we infer that**

$$\lim_{p\to\infty}\langle H^1_{z_p},\varphi\rangle_{\mathcal{S}'(\mathbb{H}^d)\times\mathcal{S}(\mathbb{H}^d)}=\langle H^1_{-it},\varphi\rangle_{\mathcal{S}'(\mathbb{H}^d)\times\mathcal{S}(\mathbb{H}^d)}=\langle S_t,\varphi\rangle_{\mathcal{S}'(\mathbb{H}^d)\times\mathcal{S}(\mathbb{H}^d)}. \tag{4.12}$$

In order to deal with $H^2_{z_p}$, recall that by definition

$$H^2_{z_p}(Y,s)=\frac{1}{(4\pi z_p)^{\frac{Q}{2}}}\int_{\mathbb{R}}\left(\frac{2\tau}{\sinh 2\tau}\right)^d\exp\left(i\frac{\tau s}{2z_p}-\frac{|Y|^2\tau}{2z_p\tanh 2\tau}\right)d\tau.$$

By hypothesis, $(z_p)_{p\in\mathbb{N}}$ is a sequence of D satisfying $|z_p|>\frac{|s|}{C}$, with $0<C<4d$, and this implies that

$$\begin{aligned}&\int_{\mathbb{R}}\left|\left(\frac{2\tau}{\sinh 2\tau}\right)^d\exp\left(i\frac{\tau s}{2z_p}-\frac{|Y|^2\tau}{2z_p\tanh 2\tau}\right)\right|d\tau\\ &\leq\int_{\mathbb{R}}\left(\frac{2\tau}{\sinh 2\tau}\right)^d\exp\left(\frac{C|\tau|}{2}\right)d\tau<\infty.\end{aligned}$$

We deduce that for all $w=(Y,s)$ satisfying $|s|<4d|t|$, there holds

$$\begin{aligned}&\lim_{p\to\infty}H^2_{z_p}(Y,s)=H^2_{-it}(Y,s)\\ &=\frac{1}{(-4i\pi t)^{\frac{Q}{2}}}\int_{\mathbb{R}}\left(\frac{2\tau}{\sinh 2\tau}\right)^d\exp\left(-\frac{\tau s}{2t}-i\frac{|Y|^2\tau}{2t\tanh 2\tau}\right)d\tau,\end{aligned}$$

which of course ensures that, for all φ in $\mathcal{S}(\mathbb{H}^d)$, we have

$$\lim_{p\to\infty}\langle H^2_{z_p},\varphi\rangle_{\mathcal{S}'(\mathbb{H}^d)\times\mathcal{S}(\mathbb{H}^d)}=\langle H^2_{-it},\varphi\rangle_{\mathcal{S}'(\mathbb{H}^d)\times\mathcal{S}(\mathbb{H}^d)}.$$

This ends the proof of Theorem 1.2. □

5 Proof of the local dispersive and Strichartz estimates

5.1 Proof of Theorem 1.1

Since the linear Schrödinger equation on $\mathbb{H}^d$ is invariant by left translation, it suffices to prove the result for $w_0=0$. Let u_0 be a function in $\mathcal{D}(B_{\mathbb{H}}(0,R_0))$. Then

**Let us underline that formula (4.11) holds true for any sequence $(z_p)_{p\in\mathbb{N}}$ of D which converges to $-it$ with $t\in\mathbb{R}^*$.

invoking Theorem 1.2, we infer that the solution to the Cauchy problem $(S_{\mathbb{H}})$ assumes the form

$$u(t,\cdot)=u_0\star S_t, \tag{5.1}$$

where

$$S_t(Y,s)=\frac{1}{(-4i\pi t)^{\frac{Q}{2}}}\int_{\mathbb{R}}\left(\frac{2\tau}{\sinh 2\tau}\right)^d\exp\left(-\frac{\tau s}{2t}-i\frac{|Y|^2\tau}{2t\tanh 2\tau}\right)d\tau \tag{5.2}$$

on any Heisenberg ball $B_{\mathbb{H}}(0,\kappa\sqrt{|t|})$ with $\kappa<\sqrt{4d}$, since

$$\rho_{\mathbb{H}}(w)\leq\kappa\sqrt{|t|}\implies |s|\leq\kappa^2|t|<4d|t|.$$

Note that

$$\|S_t\|_{L^\infty(B_{\mathbb{H}}(0,\kappa\sqrt{|t|}))}\leq\frac{1}{(4\pi|t|)^{\frac{Q}{2}}}\int_{\mathbb{R}}\left(\frac{2\tau}{\sinh 2\tau}\right)^d\exp\left(\frac{\kappa^2|\tau|}{2}\right)d\tau\stackrel{\text{def}}{=}\frac{M_\kappa}{|t|^{\frac{Q}{2}}}. \tag{5.3}$$

But by definition of the convolution product on $\mathbb{H}^d$, we have

$$(u_0\star S_t)(w)=\int_{\mathbb{H}^d}u_0(v)S_t\big(v^{-1}\cdot w\big)\,dv. \tag{5.4}$$

Thanks to the triangle inequality, we have for any w in $B_{\mathbb{H}}(0,\kappa\sqrt{|t|})$ and any v in $B_{\mathbb{H}}(0,R_0)$

$$\rho(v^{-1}\cdot w)=d_{\mathbb{H}}(w,v)\leq\rho(w)+\rho(v)\leq\kappa\sqrt{|t|}+R_0<\sqrt{4d|t|}$$

provided that

$$|t|>T_{\kappa,R_0}=\left(\frac{R_0}{\sqrt{4d}-\kappa}\right)^2,$$

which by Young's inequality completes the proof of estimate (1.14) in the case $p=\infty$.

Furthermore by the conservation of the mass there holds

$$\|u(t,\cdot)\|_{L^2(B_{\mathbb{H}}(0,\kappa\sqrt{|t|}))}\leq\|u(t,\cdot)\|_{L^2(\mathbb{H}^d)}=\|u_0\|_{L^2(\mathbb{H}^d)}.$$

So resorting to a real interpolation argument, we get for all $2\leq p\leq\infty$ and any $|t|\geq T_{R_0,\kappa}$

$$\|u(t,\cdot)\|_{L^p(B_{\mathbb{H}}(0,\kappa\sqrt{|t|}))}\leq\left(\frac{M_\kappa}{|t|^{\frac{Q}{2}}}\right)^{1-\frac{2}{p}}\|u_0\|_{L^{p'}(\mathbb{H}^d)},$$

where p' denotes the conjugate exponent of p. This finishes the proof. □

5.2 Proof of Theorem 1.3

As already mentioned, the Strichartz estimates are straightforward from the dispersive estimates. Actually, Theorem 1.3 readily follows from the mass conservation and the following proposition, which can be seen as a corollary of Theorem 1.1.

Proposition 5.1. *Under the assumptions of Theorem* 1.1*, the solution to the Cauchy problem* $(S_{\mathbb{H}})$ *associated to* u_0 *satisfies, for all* $2\leq p\leq\infty$ *and all* q *such that* $\frac{1}{q}+\frac{Q}{p}<\frac{Q}{2}$*,*

$$\|u\|_{L^q(]-\infty,-C_\kappa R_0^2]\cup[C_\kappa R_0^2,\infty[;L^p(B_{\mathbb{H}}(w_0,\kappa\sqrt{|t|})))}\leq C(q,\kappa)R_0^{-Q(\frac{1}{2}-\frac{1}{p})+\frac{2}{q}}\|u_0\|_{L^2(\mathbb{H}^d)}.$$

Proof. Since u_0 is in supported in $B_{\mathbb{H}}(w_0,R_0)$, combining the Hölder inequality with (1.14), we infer that, for all $2\leq p\leq\infty$,

$$\|u(t,\cdot)\|_{L^p(B_{\mathbb{H}}(w_0,\kappa\sqrt{|t|}))}\leq C(\kappa)\frac{R_0^{Q(\frac{1}{2}-\frac{1}{p})}}{|t|^{\frac{Q}{2}-\frac{Q}{p}}}\|u_0\|_{L^2(\mathbb{H}^d)} \tag{5.5}$$

for all $|t|\geq T_{\kappa,R_0}$. The proposition follows after time integration, and Theorem 1.3 is a direct consequence. □

6 Concentration properties of the Schrödinger kernel on $\mathbb{H}^d$

6.1 Proof of Theorem 1.4

The proof relies on Fourier-Heisenberg analysis recalled in Section 2.1.

Denote $w_t=(0,-4dt)$. Then, making use of formula (4.3) and recalling that S_t is even, we get, for any $u_0\in\mathcal{S}(\mathbb{H}^d)$,

$$\begin{aligned}(u_0\star S_t)(w_t)&=\langle S_t,u_0\circ\tau_{w_t^{-1}}\rangle_{\mathcal{S}'(\mathbb{H}^d)\times\mathcal{S}(\mathbb{H}^d)}\\&=\sum_{n\in\mathbb{N}^d}\int_{\mathbb{R}}\mathrm{e}^{4it|\lambda|(2|n|+d)}\theta_t(n,n,\lambda)|\lambda|^d d\lambda,\end{aligned} \tag{6.1}$$

where $\tau_{w_t^{-1}}$ is the left translation operator by w_t^{-1} defined by (1.13) and

$${}^t\mathcal{F}_{\mathbb{H}}\theta_t(y,\eta,s)=\frac{\pi^{d+1}}{2^{d-1}}\left(\mathcal{F}_{\mathbb{H}}^{-1}\theta_t\right)(y,-\eta,-s)=u_0\circ\tau_{w_t^{-1}}. \tag{6.2}$$

Notice that, for all $f\in L^1(\mathbb{H}^d)$ and all $g_0=(0,s_0)$, we have

$$\mathcal{F}_{\mathbb{H}}\big(f\circ\tau_{g_0^{-1}}\big)(n,m,\lambda)=\mathrm{e}^{-is_0\lambda}(\mathcal{F}_{\mathbb{H}}f)(n,m,\lambda).$$

It then follows that

$$\mathcal{F}_{\mathbb{H}}\big(u_0\circ\tau_{w_t^{-1}}\big)(n,m,\lambda)=\mathrm{e}^{4idt\lambda}\mathcal{F}_{\mathbb{H}}(u_0)(n,m,\lambda).$$

Combining (2.1)-(2.2) together with (6.2), we deduce that

$$\theta_t(n,m,\lambda)=\frac{2^{d-1}}{\pi^{d+1}}\mathrm{e}^{-4idt\lambda}\mathcal{F}_{\mathbb{H}}(u_0)(m,n,-\lambda).$$

Consequently, we have

$$(u_0\star S_t)(w_t)=\frac{2^{d-1}}{\pi^{d+1}}\sum_{n\in\mathbb{N}^d}\int_{\mathbb{R}}\mathrm{e}^{4idt\lambda}\mathrm{e}^{4it|\lambda|(2|n|+d)}\mathcal{F}_{\mathbb{H}}(u_0)(n,n,\lambda)|\lambda|^d d\lambda.$$

In particular, if we consider u_0 so that

$$\mathcal{F}_{\mathbb{H}}(u_0)(n,n,\lambda)=\mathcal{F}_{\mathbb{H}}(u_0)(n,n,\lambda)\delta_{n,0}\mathbf{1}_{\lambda<0},$$

we obtain, thanks to (2.17),

$$(u_0\star S_t)(w_t)=\langle\delta_0,u_0\rangle_{\mathcal{S}'(\mathbb{H}^d)\times\mathcal{S}(\mathbb{H}^d)}.$$

Theorem 1.4 follows. $\square$

6.2 Proof of Theorem 1.5

Let $\ell\geq 1$ be a fixed integer. We revisit the proof of Theorem 1.2: recall that the restriction over s comes from the fact that the function $H_z^2(Y,s)$ given by (4.5) is holomorphic only on the set $|z|>|s|/(4d)$. Now let us define

$$\mathscr{V}_\ell(\mathbb{H}^d)\stackrel{\text{def}}{=}\oplus_{|m|\geq\ell}L^2_m(\mathbb{H}^d),$$

where L^2_m is defined in (2.7)-(2.8). In the case when the Cauchy data u_0 belongs to the set $\mathcal{S}(\mathbb{H}^d)\cap\mathscr{V}_\ell(\mathbb{H}^d)$, our goal is to write

$$u(t)=u_0\star S_t^{(\ell)},$$

where $S_t^{(\ell)}$ is a tempered distribution obtained, for $|s|<4(2\ell+d)|t|$, by the same complex analysis argument as in the proof of Theorem 1.2, where the function to be analyzed, arising from the heat equation, is now

$$h_t^{(\ell)}(Y,s)\stackrel{\text{def}}{=}\frac{2^{d-1}}{\pi^{d+1}}\sum_{|m|\geq\ell}\int_{\mathbb{R}\times\mathbb{R}^d}\mathrm{e}^{is\lambda+2i\lambda\langle\eta,z\rangle}\mathrm{e}^{-4|\lambda|(2|m|+d)t}\\ \times H_{m,\lambda}(y+z)H_{m,\lambda}(-y+z)dz|\lambda|^d d\lambda.$$

According to Gaveau's resulted recalled in Theorem 3.1, the function $h_t^{(\ell)}$ also reads

$$h_t^{(\ell)}(Y,s)=\frac{1}{(4\pi t)^{\frac{Q}{2}}}\int_{\mathbb{R}}\left(\frac{2\tau}{\sinh 2\tau}\right)^d\mathrm{e}^{i\frac{\tau s}{2t}-\frac{|Y|^2\tau}{2t\tanh 2\tau}}d\tau\\ -\frac{2^{d-1}}{\pi^{d+1}}\sum_{|m|\leq\ell-1}\int_{\mathbb{R}\times\mathbb{R}^d}\mathrm{e}^{is\lambda+2i\lambda\langle\eta,z\rangle}\mathrm{e}^{-4|\lambda|(2|m|+d)t}\\ \times H_{m,\lambda}(y+z)H_{m,\lambda}(-y+z)dz|\lambda|^d d\lambda.$$

But in view of (2.13), we have, for any integer k,

$$\frac{2^{d-1}}{\pi^{d+1}}\sum_{|m|=k}\int_{\mathbb{R}\times\mathbb{R}^d}\mathrm{e}^{is\lambda+2i\lambda\langle\eta,z\rangle}\mathrm{e}^{-4|\lambda|(2|m|+d)t}H_{m,\lambda}(y+z)H_{m,\lambda}(-y+z)dz|\lambda|^d d\lambda\\ =\frac{2^{d-1}}{\pi^{d+1}}\int_{\mathbb{R}}\mathrm{e}^{is\lambda}\mathrm{e}^{-4|\lambda|(2k+d)t}\mathrm{e}^{-|\lambda||Y|^2}L_k^{(d-1)}(2|\lambda||Y|^2)|\lambda|^d d\lambda,$$

where $L_k^{(d-1)}$ denotes the Laguerre polynomial of order k and type $d-1$. Then, performing the change of variables $\lambda=\frac{\tau}{2t}$, we readily gather that

$$h_t^{(\ell)}(Y,s)=\frac{1}{(4\pi t)^{\frac{Q}{2}}}\int_{\mathbb{R}}\left(\frac{2\tau}{\sinh 2\tau}\right)^d\mathrm{e}^{i\frac{\tau s}{2t}-\frac{|Y|^2\tau}{2t\tanh 2\tau}}d\tau\\ -\frac{1}{(4\pi t)^{\frac{Q}{2}}}\sum_{k\leq\ell-1}\int_{\mathbb{R}}\big(4|\tau|\big)^d\mathrm{e}^{-2|\tau|(2k+d)}L_k^{(d-1)}\left(\frac{|Y|^2|\tau|}{t}\right)\mathrm{e}^{i\frac{\tau s}{2t}-\frac{|Y|^2|\tau|}{2t}}d\tau.$$

Returning again to the strategy of the proof of Theorem 1.2, our first aim is therefore to prove that the maps

$$z\longmapsto H_z^{(\ell),1}(Y,s)\quad\text{and}\quad z\longmapsto H_z^{(\ell),2}(Y,s)$$

with

$$H_z^{(\ell),1}(Y,s) \stackrel{\text{def}}{=} \frac{2^{d-1}}{\pi^{d+1}} \sum_{|m|\geq \ell} \int_{\mathbb{R}} \mathrm{e}^{is\lambda} \mathrm{e}^{-4z|\lambda|(2|m|+d)} \mathcal{W}\big((m,m,\lambda),Y\big) |\lambda|^d d\lambda$$

and

$$\begin{aligned} H_z^{(\ell),2}(Y,s) \stackrel{\text{def}}{=} & \frac{1}{(4\pi z)^{\frac{Q}{2}}} \int_{\mathbb{R}} (2|\tau|)^d \mathrm{e}^{i\frac{\tau s}{2z}} \left(\frac{1}{(\sinh 2|\tau|)^d} \mathrm{e}^{-\frac{|Y|^2\tau}{2z\tanh 2\tau}} \right. \\ & \left. - \sum_{k\leq \ell-1} 2^d \mathrm{e}^{-2|\tau|(2k+d)} L_k^{(d-1)} \left(\frac{|Y|^2|\tau|}{z} \right) \mathrm{e}^{-\frac{|Y|^2|\tau|}{2z}} \right) d\tau \end{aligned} \tag{6.3}$$

are, for all (Y,s) in $\mathbb{H}^d$, holomorphic on a suitable domain of $\mathbb{C}^*$. The same reasoning as in the proof of Theorem 1.2 enables to check that the function $H_z^{(\ell),1}$ is holomorphic on the domain $D=\{z\in\mathbb{C},\mathrm{Re}(z)>0\}$ so now we concentrate on $H_z^{(\ell),2}$.

We shall prove that the function $H_z^{(\ell),2}$ is holomorphic on the domain

$$\widetilde{D}_{|s|}^{\ell} \stackrel{\text{def}}{=} \left\{ z\in D, |z| > \frac{|s|}{4(2\ell+d)} \right\}. \tag{6.4}$$

Let us start by re-writing (6.3) in the following form:

$$\begin{aligned} & H_z^{(\ell),2}(Y,s) = \frac{1}{(4\pi z)^{\frac{Q}{2}}} \int_{\mathbb{R}} (4|\tau|)^d \mathrm{e}^{i\frac{\tau s}{2z} - 2|\tau| d} \Phi_\ell \left(\frac{|Y|^2|\tau|}{z}, \mathrm{e}^{-4|\tau|} \right) d\tau \quad \text{with} \\ & \Phi_\ell(x,r) \stackrel{\text{def}}{=} \mathrm{e}^{-\frac{x}{2}} \left(\frac{\mathrm{e}^{-\frac{rx}{1-r}}}{(1-r)^d} - \sum_{k\leq \ell-1} r^k L_k^{(d-1)}(x) \right). \end{aligned} \tag{6.5}$$

From now on we set[††]

$$x \stackrel{\text{def}}{=} \frac{|Y|^2|\tau|}{z}, \quad r \stackrel{\text{def}}{=} \mathrm{e}^{-4|\tau|}. \tag{6.6}$$

Recalling that the generating function for the Laguerre polynomials is given by (see for instance [1,13,14,23,25,27])

$$\sum_{k\geq 0} r^k L_k^{(d-1)}(x) = \frac{\mathrm{e}^{-\frac{rx}{1-r}}}{(1-r)^d}, \quad |r|<1, \tag{6.7}$$

[††]It will be useful to point out that $|x| = \frac{|Y|^2|\tau|}{|z|}$ and $\mathrm{Re}(x) = \frac{|Y|^2|\tau|}{|z|}\cos(\arg(z))$.

we notice that the $e^{\frac{x}{2}}\Phi_\ell(x,r)$ is nothing else than the remainder of the Taylor expansion of the function $f(x,r)\stackrel{\text{def}}{=}e^{-\frac{rx}{1-r}}(1-r)^{-d}$ at order $\ell-1$, near $r=0$. We shall therefore argue differently depending on whether r is close to 1 or not. So let us fix $\tau_0>0$, and start by analyzing the case when $|\tau|\geq\tau_0$, since this implies that $r\leq r_0\stackrel{\text{def}}{=}e^{-4\tau_0}<1$. The case $|\tau|\leq\tau_0$ will be dealt with further down. Considering

$$H^{(\ell),2}_{z,\tau_0}(Y,s)\stackrel{\text{def}}{=}\frac{1}{(4\pi z)^{\frac{Q}{2}}}\int_{|\tau|\geq\tau_0}(4|\tau|)^d e^{i\frac{\tau s}{2z}-2|\tau|d}\Phi_\ell\left(\frac{|Y|^2|\tau|}{z},e^{-4|\tau|}\right)d\tau, \tag{6.8}$$

we are thus reduced to investigating, for $|\tau|\geq\tau_0$, the function

$$\big(4|\tau|\big)^d e^{-2|\tau|d}e^{i\frac{\tau s}{2z}}\Phi_\ell\left(\frac{|Y|^2|\tau|}{z},e^{-4|\tau|}\right) \tag{6.9}$$

and its derivative with respect to z. We shall actually restrict z to the domain

$$\begin{aligned}&D^\ell_{|s|,a,A}\stackrel{\text{def}}{=}\left\{z\in\mathbb{C},|z|>\frac{|s|}{\kappa(2\ell+d)}\right\}\cap\Omega_{a,A}\quad\text{with}\\&\Omega_{a,A}\stackrel{\text{def}}{=}\big\{z\in\mathbb{C},\mathrm{Re}(z)>a,|z|\leq A\big\},\end{aligned}$$

where $0<\kappa<4$ and a, $A>0$ are arbitrary fixed constants.

Recalling

$$f(x,r)\stackrel{\text{def}}{=}\frac{e^{-\frac{rx}{1-r}}}{(1-r)^d}$$

and applying Taylor's formula, it readily follows from (6.7) that

$$\Phi_\ell(x,r)=e^{-\frac{x}{2}}\frac{r^\ell}{(\ell-1)!}\int_0^1(1-s)^{\ell-1}\big(\partial_r^\ell f\big)(x,rs)ds.$$

After some computations we infer that there is a positive constant $C(\ell,\tau_0)$ such that, for all $r\leq r_0=e^{-4\tau_0}$, we have

$$\begin{aligned}&\big|\Phi_\ell(x,r)\big|\leq C(\ell,\tau_0)r^\ell e^{-\frac{\mathrm{Re}(x)}{2}}(1+|x|)^\ell,\\&\big|\partial_x\Phi_\ell(x,r)\big|\leq C(\ell,\tau_0)r^\ell e^{-\frac{\mathrm{Re}(x)}{2}}(1+|x|)^\ell.\end{aligned} \tag{6.10}$$

Observing that

$$\left|e^{-\frac{|Y|^2|\tau|}{2z}}\right|=e^{-\frac{|Y|^2|\tau|}{2|z|}\cos(\arg(z))} \tag{6.11}$$

with
$$\inf_{z\in\Omega_{a,A}} \cos\big(\arg(z)\big) \geq \alpha(a,A) > 0,$$
we deduce that there is a positive constant $C(\ell,\tau_0,a,A)$ such that, for all (Y,s) in $\mathbb{H}^d$, $z\in D^\ell_{|s|,a,A}$ and $|\tau|\geq\tau_0$, there holds
$$\left|(4|\tau|)^d \mathrm{e}^{i\frac{\tau s}{2z}-2|\tau|d}\Phi_\ell\left(\frac{|Y|^2|\tau|}{z}, \mathrm{e}^{-4|\tau|}\right)\right| \leq C(\ell,\tau_0,a,A)|\tau|^d \mathrm{e}^{\frac{-(4-\kappa)(2\ell+d)|\tau|}{2}}$$
and similarly
$$\left|\partial_z(4|\tau|)^d \mathrm{e}^{i\frac{\tau s}{2z}-2|\tau|d}\Phi_\ell\left(\frac{|Y|^2|\tau|}{z}, \mathrm{e}^{-4|\tau|}\right)\right| \leq C(\ell,\tau_0,a,A)|\tau|^d \mathrm{e}^{\frac{-(4-\kappa)(2\ell+d)|\tau|}{2}}.$$
This readily ensures that the function $H^{(\ell),2}_{z,\tau_0}$ is holomorphic on the domain $\widetilde{D}^\ell_{|s|}$ defined in (6.4).

To deal with $H^{(\ell),2}_z - H^{(\ell),2}_{z,\tau_0}$, let us first observe that according to (6.3), we have
$$\begin{aligned}H^{(\ell),2}_{z,\tau_0}(Y,s) = \frac{1}{(4\pi z)^{\frac{Q}{2}}}\int_{|\tau|\leq\tau_0}(2|\tau|)^d \mathrm{e}^{i\frac{\tau s}{2z}}\Bigg(&\frac{1}{(\sinh 2|\tau|)^d}\mathrm{e}^{-\frac{|Y|^2\tau}{2z\tanh 2\tau}}\\ &-\sum_{k\leq\ell-1}2^d \mathrm{e}^{-2|\tau|(2k+d)}L_k^{(d-1)}\left(\frac{|Y|^2|\tau|}{z}\right)\mathrm{e}^{-\frac{|Y|^2|\tau|}{2z}}\Bigg)d\tau.\end{aligned}$$
Then invoking (6.11), we infer that for any integer ℓ and all positive real numbers a and A, there exists a positive constant $C(\ell,a,A)$ such that
$$\begin{aligned}&\sup_{\substack{z\in\Omega_{a,A}\\ k\leq\ell-1}}\left|L_k^{(d-1)}\left(\frac{|Y|^2|\tau|}{z}\right)\mathrm{e}^{-\frac{|Y|^2|\tau|}{2z}}\right| \leq C(\ell,a,A),\\ &\sup_{\substack{z\in\Omega_{a,A}\\ k\leq\ell-1}}\left|\frac{d}{dz}\left(L_k^{(d-1)}\left(\frac{|Y|^2|\tau|}{z}\right)\mathrm{e}^{-\frac{|Y|^2|\tau|}{2z}}\right)\right| \leq C(\ell,a,A).\end{aligned}$$
We deduce that for all (Y,s) in $\mathbb{H}^d$, $z\in D^\ell_{|s|,a,A}$ and $|\tau|\leq\tau_0$, there holds
$$\left|\sum_{k\leq\ell-1}(4|\tau|)^d \mathrm{e}^{-2|\tau|(2k+d)}L_k^{(d-1)}\left(\frac{|Y|^2|\tau|}{z}\right)\mathrm{e}^{i\frac{\tau s}{2z}-\frac{|Y|^2|\tau|}{2z}}\right| \leq C(\ell,a,A)\tau_0^d \mathrm{e}^{\frac{\tau_0\kappa(2\ell+d)}{2}} \tag{6.12}$$

and

$$\begin{aligned}&\left|\partial_z\left(\sum_{k\leq\ell-1}(4|\tau|)^d \mathrm{e}^{-2|\tau|(2k+d)}L_k^{(d-1)}\left(\frac{|Y|^2|\tau|}{z}\right)\mathrm{e}^{i\frac{\tau s}{2z}-\frac{|Y|^2|\tau|}{2z}}\right)\right|\\&\leq C(\ell,a,A)\mathrm{e}^{\frac{\tau_0\kappa(2\ell+d)}{2}}\left(1+\frac{\tau_0\kappa(2\ell+d)}{2}\right).\end{aligned}\tag{6.13}$$

This obviously implies that the function

$$\frac{1}{(4\pi z)^{\frac{Q}{2}}}\sum_{k\leq\ell-1}\int_{|\tau|\leq\tau_0}(4|\tau|)^d \mathrm{e}^{i\frac{\tau s}{2z}}\mathrm{e}^{-2|\tau|(2k+d)}L_k^{(d-1)}\left(\frac{|Y|^2|\tau|}{z}\right)\mathrm{e}^{-\frac{|Y|^2|\tau|}{2z}}d\tau$$

is holomorphic on the domain $D^\ell_{|s|,a,A}$. The part

$$\frac{1}{(4\pi z)^{\frac{Q}{2}}}\int_{|\tau|\leq\tau_0}\left(\frac{2\tau}{\sinh 2\tau}\right)^d \mathrm{e}^{i\frac{\tau s}{2z}}\mathrm{e}^{-\frac{|Y|^2\tau}{2z\tanh 2\tau}}d\tau$$

can easily be dealt with, which achieves the proof of the fact that the function $H_z^{(\ell),2}$ is holomorphic on $\widetilde{D}^\ell_{|s|}$.

Finally, let $(z_p)_{p\in\mathbb{N}}$ be a sequence in $\widetilde{D}^\ell_{|s|}$ which converges to $-it$ with $t\in\mathbb{R}^*$. Then arguing as in the proof of Theorem 1.2, one can readily gather that for any function φ in $\mathcal{S}(\mathbb{H}^d)\cap\mathscr{V}_\ell(\mathbb{H}^d)$, we have

$$\begin{aligned}&\lim_{p\to\infty}\langle H_{z_p}^{(\ell),1},\varphi\rangle_{\mathcal{S}'(\mathbb{H}^d)\times\mathcal{S}(\mathbb{H}^d)}=\langle H_{-it}^{(\ell),1},\varphi\rangle_{\mathcal{S}'(\mathbb{H}^d)\times\mathcal{S}(\mathbb{H}^d)}\\&=\langle S_t^{(\ell)},\varphi\rangle_{\mathcal{S}'(\mathbb{H}^d)\times\mathcal{S}(\mathbb{H}^d)}=\langle S_t,\varphi\rangle_{\mathcal{S}'(\mathbb{H}^d)\times\mathcal{S}(\mathbb{H}^d)}.\end{aligned}$$

To show that

$$\lim_{p\to\infty}\langle H_{z_p}^{(\ell),2},\varphi\rangle_{\mathcal{S}'(\mathbb{H}^d)\times\mathcal{S}(\mathbb{H}^d)}=\langle H_{-it}^{(\ell),2},\varphi\rangle_{\mathcal{S}'(\mathbb{H}^d)\times\mathcal{S}(\mathbb{H}^d)}$$

we shall as above investigate separately $H_{z,\tau_0}^{(\ell),2}$ and $H_z^{(\ell),2}-H_{z,\tau_0}^{(\ell),2}$. Let $(z_p)_{p\in\mathbb{N}}$ be a sequence in $\widetilde{D}^\ell_{|s|}$ which converges to $-it$ with $t\in\mathbb{R}^*$ and let us start by studying the part corresponding to $H_{z,\tau_0}^{(\ell),2}$. One can assume without loss of generality that $\frac{|s|}{|z_p|}\leq\kappa(2\ell+d)$ with $0<\kappa<4$, and also that $(1-\delta)|t|\leq|z_p|\leq(1+\delta)|t|$ for some small δ. Then taking advantage of estimate (6.10), we readily gather that

$$|\varphi(Y,s)|\left|(4|\tau|)^d\mathrm{e}^{-2|\tau|d}\mathrm{e}^{i\frac{\tau s}{2z_p}}\Phi_\ell\left(\frac{|Y|^2|\tau|}{z_p},\mathrm{e}^{-4|\tau|}\right)\right|$$

$$\leq C(\ell,\delta,\tau_0)|\varphi(Y,s)|\left(1+\frac{|Y|^2}{(1-\delta)|t|}\right)^{\ell}|\tau|^{d+\ell}\,\mathrm{e}^{\frac{(\kappa-4)(2\ell+d)|\tau|}{2}},$$

which implies that

$$\lim_{p\to\infty}\langle H^{(\ell),2}_{z_p,\tau_0},\varphi\rangle_{\mathcal{S}'(\mathbb{H}^d)\times\mathcal{S}(\mathbb{H}^d)}=\langle H^{(\ell),2}_{-it,\tau_0},\varphi\rangle_{\mathcal{S}'(\mathbb{H}^d)\times\mathcal{S}(\mathbb{H}^d)}.$$

Now to study the part corresponding to $H^{(\ell),2}_{z}-H^{(\ell),2}_{z,\tau_0}$, let us first observe that it stems from (6.11) that $|\exp(-\frac{|Y|^2|\tau|}{2z_p})|\leq 1$. Consequently, there exists a positive constant $C(\ell,\delta,\tau_0)$ such that, for all (Y,s) in $\mathbb{H}^d$ and $|\tau|\leq\tau_0$, there holds

$$\begin{aligned}&|\varphi(Y,s)|\left|\sum_{k\leq\ell-1}(4|\tau|)^d\mathrm{e}^{-2|\tau|(2k+d)}L_k^{(d-1)}\left(\frac{|Y|^2|\tau|}{z_p}\right)\mathrm{e}^{i\frac{\tau s}{2z_p}-\frac{|Y|^2|\tau|}{2z_p}}\right|\\&\leq C(\ell,\delta,\tau_0)|\varphi(Y,s)|\left(1+\frac{|Y|^2|\tau_0|}{(1-\delta)|t|}\right)^{\ell}\mathrm{e}^{\frac{\kappa(2\ell+d)\tau_0}{2}}.\end{aligned}$$

Since the first part can be easily dealt, we readily gather that

$$\lim_{p\to\infty}\left\langle H^{(\ell),2}_{z_p}-H^{(\ell),2}_{z_p,\tau_0},\varphi\right\rangle_{\mathcal{S}'(\mathbb{H}^d)\times\mathcal{S}(\mathbb{H}^d)}=\left\langle H^{(\ell),2}_{-it}-H^{(\ell),2}_{-it,\tau_0},\varphi\right\rangle_{\mathcal{S}'(\mathbb{H}^d)\times\mathcal{S}(\mathbb{H}^d)}.$$

This ends the proof of Theorem 1.5. □

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References

[1] R. Askey and S. Waigner, *Mean convergence of expansions in Laguerre and Hermite series*, Amer. J. Math. 87 (1965), 695–708.

[2] F. Astengo, B. Di Blasio and F. Ricci, *Fourier transform of Schwartz functions on the Heisenberg group*, Studia Math. 214 (2013), 201–222.

[3] H. Bahouri, D. Barilari, and I. Gallagher, *Strichartz estimates and Fourier Restriction Theorems on the Heisenberg group*, arXiv:1911.03729.

[4] H. Bahouri, J.-Y. Chemin and R. Danchin, *Fourier Analysis and Nonlinear Partial Differential Equations*, Grundlehren der Mathematischen Wisserchaften, Vol. 343, Springer-Verlag, 2011.

[5] H. Bahouri, J.-Y. Chemin and R. Danchin, *Fourier transform of tempered distributions on the Heisenberg group*, Annales Henri Lebesgue 1 (2018), 1–45.

[6] H. Bahouri, J.-Y. Chemin and R. Danchin, *A frequency space for the Heisenberg group*, Annales de l'Institut de Fourier 69 (2019), 365–407.

[7] H. Bahouri, C. Fermanian-Kammerer and I. Gallagher, *Phase-space analysis and pseudo-differential calculus on the Heisenberg group*, Astérisque 340, 2012.

[8] H. Bahouri, C. Fermanian-Kammerer and I. Gallagher, *Dispersive estimates for the Schrödinger operator on step 2 stratified Lie groups*, Analysis and PDE 9 (2016), 545–574.

[9] H. Bahouri, P. Gérard, and C.-J. Xu, *Espaces de Besov et estimations de Strichartz généralisées sur le groupe de Heisenberg*, Journal d'Analyse Mathématique 82 (2000), 93–118.

[10] R. Beals and R. Wong, *Special Functions and Orthogonal Polynomials*, Cambridge Stud. Adv. Math. 153, 2016.

[11] L.-J. Corwin and F.-P. Greenleaf, *Representations of Nilpotent Lie Groups and Their Applications, Part 1: Basic Theory and Examples*, Cambridge Stud. Adv. Math. 18, 1990.

[12] M. Del Hierro, *Dispersive and Strichartz estimates on H-type groups*, Studia Math. 169 (2005), 1–20.

[13] A. Erdélyi, W. Magnus, F. Oberhettinger, and F.-G. Tricomi, *Higher Transcendental Functions*, McGraw-Hill 2, 1953.

[14] J. Faraut and K. Harzallah, *Deux cours d'analyse harmonique*, École d'Été d'analyse harmonique de Tunis, Progress in Mathematics, Birkhäuser, 1984.

[15] C. Fermanian-Kammerer and V. Fischer, *Defect measures on graded lie groups*, to appear in Annali della Scuola Normale Superiore di Pisa.

[16] R.-P. Feynman and A.-R. Hibbs, *Quantum mechanics and path integrals*, in: *International Series in Pure and Applied Physics*, Maidenhead, Berksh: McGraw-Hill Publishing Company, 1965.

[17] V. Fischer and M.-V. Ruzhansky, *A pseudo-differential calculus on graded nilpotent Lie groups*, Fourier analysis, Trends Math Birkhäuser (2014), 107–132.

[18] G. B. Folland, *Harmonic Analysis in Phase Space*, Ann. of Math. Stud., Princeton University Press, 1989.

[19] B. Gaveau, *Principe de moindre action, propagation de la chaleur et estimées sous-elliptiques sur certains groupes nilpotents*, Acta Math. 139 (1977), 95–153.

[20] D. Geller, *Fourier analysis on the Heisenberg groups*, in: *Proceedings of the National Academy of Sciences of the United States of America*, 74 (1977), 1328–1331.

[21] D. Geller, *Fourier analysis on the Heisenberg group I, the Schwartz space*, J. Funct. Anal. 36 (1980), 205–254.

[22] J. Ginibre and G. Velo, *Generalized Strichartz inequalities for the wave equations*, J. Funct. Anal. 133 (1995), 50–68.

[23] I.-S. Gradshteyn and I.-M. Ryzhik, *Table of Integrals, Series and Products*, Elsevier, 2007.

[24] M. Keel and T. Tao, *Endpoint Strichartz estimates*, Amer. J. Math. 120 (1998), 955–980.

[25] N. Lebedev, *Special Functions and Their Applications*, Prentice Hall, 1965.

[26] D. Müller, *A restriction theorem for the Heisenberg group*, Annales of Mathematics 131 (1990), 567–587.

[27] F. W. J. Olver, *Asymptotics and Special Functions*, Academic Press, 1974.

[28] E. M. Stein, *Harnomic Analysis*, Princeton University Press, 1993.

[29] R. Strichartz, *Restriction Fourier transform of quadratic surfaces and decay of solutions of the wave equations*, Duke Math. J. 44 (1977), 705–714.

[30] T. Tao, *Nonlinear Dispersive Equations. Local and Global Analysis*, CBMS Regional Conference Series in Mathematics, Vol. 106, AMS, 2006.

[31] M. E. Taylor, *Noncommutative Harmonic Analysis*, Mathematical Survey and Monographs, Vol. 22, AMS, 1986.

[32] S. Thangavelu, *Harmonic Analysis on the Heisenberg group*, Progress in Mathematics, Vol. 159, Birkhaüser, 1998.