Isabelle Gallagher

Laure Saint-Raymond

Benjamin Texier

FROM NEWTON TO BOLTZMANN: HARD SPHERES AND SHORT-RANGE POTENTIALS

I. Gallagher

Institut de Mathématiques de Jussieu UMR CNRS 7586, Université Paris-Diderot (Paris 7),
175, rue du Chevaleret, 75013 Paris, FRANCE.

E-mail : `gallagher@math.univ-paris-diderot.fr`

L. Saint-Raymond

Université Paris VI and DMA École Normale Supérieure, 45 rue d'Ulm, 75230 Paris Cedex 05,
FRANCE.

E-mail : `Laure.Saint-Raymond@ens.fr`

B. Texier

Institut de Mathématiques de Jussieu UMR CNRS 7586, Université Paris-Diderot (Paris 7),
175, rue du Chevaleret, 75013 Paris, FRANCE,
and DMA École Normale Supérieure, 45 rue d'Ulm, 75230 Paris Cedex 05, FRANCE.

E-mail : `Benjamin.Texier@math.jussieu.fr`

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Abstract. — We provide a rigorous derivation of the Boltzmann equation as the mesoscopic limit of systems of hard spheres, or Newtonian particles interacting via a short-range potential, as the number of particles N goes to infinity and the characteristic length of interaction ε simultaneously goes to 0, in the Boltzmann-Grad scaling $N\varepsilon^{d-1} \equiv 1$.

The time of validity of the convergence is a fraction of the average time of first collision, due to a limitation of the time on which one can prove uniform estimates for the BBGKY and Boltzmann hierarchies.

Our proof relies on the fundamental ideas of Lanford, and the important contributions of King, Cercignani, Illner and Pulvirenti, and Cercignani, Gerasimenko and Petrina. The main novelty here is the detailed study of pathological trajectories involving recollisions, which proves the term-by-term convergence for the correlation series expansion.

CONTENTS

PART I

INTRODUCTION

CHAPTER 1

THE LOW DENSITY LIMIT

We are interested in this monograph in the qualitative behavior of systems of particles with short-range interactions. We study the qualitative behaviour of particle systems with short-range binary interactions, in two cases: hard spheres, that move in uniform rectilinear motion until they undergo elastic collisions, and smooth, monotonic, compactly supported potentials.

• For hard spheres, the equations of motion are

$$(1.0.1) \qquad \frac{dx_i}{dt} = v_i\,, \qquad \frac{dv_i}{dt} = 0\,,$$

for $1 \le i \le N$, where $(x_i, v_i) \in \mathbf{R}^d \times \mathbf{R}^d$ denote the position and velocity of particle i, provided that the exclusion condition $|x_i(t) - x_j(t)| > \sigma$ is satisfied, where σ denotes the diameter of the particles. We further have to prescribe a reflection condition at the boundary

$$(1.0.2) \qquad \begin{aligned} v_i^{in} &= v_i^{out} - \nu^{i,j} \cdot (v_i^{out} - v_j^{out})\,\nu^{i,j} \\ v_j^{in} &= v_j^{out} + \nu^{i,j} \cdot (v_i^{out} - v_j^{out})\,\nu^{i,j} \ \text{ if } \ \exists j \ne i\,, |x_i - x_j| = \sigma\,, \end{aligned}$$

where $\nu^{i,j} := (x_i - x_j)/|x_i - x_j|$. Note that it is not obvious to check that (1.0.1)-(1.0.2) defines global dynamics. This question is addressed in Chapter 4.

• In the case of smooth interactions, the Hamiltonian equations of motion are

$$(1.0.3) \qquad \frac{dx_i}{dt} = v_i\,, \qquad m_i\frac{dv_i}{dt} = -\sum_{j \ne i} \nabla\Phi(x_i - x_j)\,,$$

where m_i is the mass of particle i (which we shall assume equal to 1 to simplify) and the force exerted by particle j on particle i is $-\nabla\Phi(x_i - x_j)$.

When the system is constituted of two elementary particles, in the reference frame attached to the center of mass, the dynamics is two-dimensional. The deflection of the particle trajectories from straight lines can then be described through explicit formulas (which are given in Chapter 8).

When the system is constituted of three particles or more, the integrability is lost, and in general the problem becomes very complicated, as already noted by Poincaré [**37**].

Remark 1.0.1. — *Note that the dynamics of hard spheres is in some sense a limit of the smooth-forces case with*

$$\Phi(x) = +\infty \ \textit{if } |x| < \sigma\,, \qquad \Phi(x) = 0 \ \textit{if } |x| > \sigma\,.$$

Nevertheless, to our knowledge, there does not exist any mathematical statement concerning these asymptotics.

We will however see in the sequel that the two types of systems exhibit very similar qualitative behaviours in the low density limit. Once the dynamics is defined (i.e. provided that we can discard multiple collisions), the case of hard spheres is actually simpler and we will discuss it in Part II to explain the main ideas and conceptual difficulties. We will then explain, in Part III, how to extend the arguments to the smoother case of Hamiltonian systems.

1.1. The Liouville equation

In the large N limit, individual trajectories become irrelevant, and our goal is to describe an average behaviour.

This average will be of course over particles which are indistiguishable, meaning that we will be only interested in some distribution related to the empirical measure

$$\mu_N\big(t, X_N(0), V_N(0)\big) := \frac{1}{N} \sum_{i=1}^{N} \delta_{x_i(t), v_i(t)} \,,$$

with $X_N(0) := (x_1(0), \dots, x_N(0)) \in \mathbf{R}^{dN}$ and $V_N(0) := (v_1(0), \dots, v_N(0)) \in \mathbf{R}^{dN}$, and $(x_i(t), v_i(t))$ is the state at time t of particle i in the system with initial configuration $\big(X_N(0), V_N(0)\big)$.

But, because we have only a vague knowledge of the state of the system at initial time, we will further average over initial configurations. At time 0, we thus start with a distribution $f_N^0(Z_N)$, where we use the following notation: for any set of s particles with positions $X_s := (x_1, \dots, x_s) \in \mathbf{R}^{ds}$ and velocities $V_s := (v_1, \dots, v_s) \in \mathbf{R}^{ds}$, we write $Z_s := (z_1, \dots, z_s) \in \mathbf{R}^{2ds}$ with $z_i := (x_i, v_i) \in \mathbf{R}^{2d}$.

We then aim at describing the evolution of the distribution

$$\int \frac{1}{N} \sum_{i=1}^{N} \delta_{z_i(t)} f_N^0(Z_N) dZ_N \,.$$

We thus define the probability $f_N = f_N(t, Z_N)$, referred to as the *N-particle distribution function*, and we assume that it satisfies for all permutations σ of $\{1, \dots, N\}$,

(1.1.1) $$f_N(t, Z_{\sigma(N)}) = f_N(t, Z_N) \,,$$

with $Z_{\sigma(N)} = (x_{\sigma(1)}, v_{\sigma(1)}, \dots, x_{\sigma(N)}, v_{\sigma(N)})$. This corresponds to the property that the particles are indistinguishable.

The distribution we are interested in is therefore nothing else than the first marginal $f_N^{(1)}$ of the distribution function f_N, defined by

$$f_N^{(1)}(t, Z_1) := \int f_N(t, Z_N) \, dz_2 \dots dz_N \,.$$

Since f_N is an invariant of the particle system, the *Liouville equation* relative to the particle system (1.0.3) is

$$(1.1.2) \qquad \partial_t f_N + \sum_{i=1}^N v_i \cdot \nabla_{x_i} f_N - \sum_{i=1}^N \sum_{\substack{j=1 \\ j \neq i}}^N \nabla_x \Phi \left(x_i - x_j \right) \cdot \nabla_{v_i} f_N = 0 \,.$$

For hard spheres, provided that we can prove that the dynamics is well defined for almost all initial configurations, we find the Liouville equation

$$(1.1.3) \qquad \partial_t f_N + \sum_{i=1}^N v_i \cdot \nabla_{x_i} f_N = 0$$

on the domain

$$\mathcal{D}_N := \left\{ Z_N \in \mathbf{R}^{2dN} \,/\, \forall i \neq j, \, |x_i - x_j| > \sigma \right\}$$

with the boundary condition $f_N(t, Z_N^{in}) = f_N(t, Z_N^{out})$.

1.2. Mean field versus collisional dynamics

In this framework, in order for the average energy per particle to remain bounded, one has to assume that the energy of each pairwise interaction is small. In other words, one has to consider a rescaled potential Φ_ε obtained
– either by scaling the strength of the force,
– or by scaling the range of potential.
According to the scaling chosen, we expect to obtain different asymptotics.

• In the case of a weak coupling, i.e. when the strength of the individual interaction becomes small (of order $1/N$) but the range remains macroscopic, the convenient scaling in order for the macroscopic dynamics to be sensitive to the coupling is:

$$\partial_t f_N + \sum_{i=1}^N v_i \cdot \nabla_{x_i} f_N - \frac{1}{N} \sum_{i=1}^N \sum_{\substack{j=1 \\ j \neq i}}^N \nabla_x \Phi \left(x_i - x_j \right) \cdot \nabla_{v_i} f_N = 0 \,.$$

Then each particle feels the effect of the force field created by all the (other) particles

$$F_N(x) = -\frac{1}{N} \sum_{j=1}^N \nabla_x \Phi \left(x - x_j \right) \sim - \iint \nabla_x \Phi(x - y) f_N^{(1)}(t, y, v) dy dv \,.$$

In particular, the dynamics seems to be stable under small perturbations of the positions or velocities of the particles.

In the limit $N \to \infty$, we thus get a *mean field approximation*, that is an equation of the form

$$\partial_t f + v \cdot \nabla_x f + F \cdot \nabla_v f = 0$$

for the first marginal, where the coupling arises only through some average

$$F := -\nabla_x \Phi * \int f dv \,.$$

An important amount of literature is devoted to such asymptotics, but this is not our purpose here. We refer to [**11, 41**] for pioneering results, to [**25**] for a recent study and to [**21**] for a review on that topic.

• The scaling we shall deal with in the present work corresponds to a strong coupling, i.e. to the case when the amplitude of the potential remains of size $O(1)$, but its range becomes small.

Introduce a small parameter $\varepsilon > 0$ corresponding to the typical interaction length of the particles. For hard spheres, ε is simply the diameter of particles. In the case of Hamiltonian systems, ε will be the range of the interaction potential. We shall indeed assume throughout this text the following properties for Φ (a *short-range* potential).

Assumption 1.2.1. — *The potential $\Phi : \mathbf{R}^d \to \mathbf{R}$ is a radial, nonnegative, nonincreasing function supported in the unit ball of $\mathbf{R}^d$, of class C^2 in $\{x \in \mathbf{R}^d, 0 < |x| < 1\}$. Moreover it is assumed that Φ is unbounded near zero, goes to zero at $|x| = 1$ with bounded derivatives, and that $\nabla\Phi$ vanishes only on $|x| = 1$.*

Then in the macroscopic spatial and temporal scales, the Hamiltonian system becomes

$$(1.2.1) \qquad \frac{dx_i}{dt} = v_i\,, \qquad \frac{dv_i}{dt} = -\frac{1}{\varepsilon}\sum_{j \neq i}\nabla\Phi\left(\frac{x_i - x_j}{\varepsilon}\right)\,,$$

and the Liouville equation takes the form

$$(1.2.2) \qquad \partial_t f_N + \sum_{i=1}^{N} v_i \cdot \nabla_{x_i} f_N - \sum_{i=1}^{N}\sum_{\substack{j=1\\j\neq i}}^{N}\frac{1}{\varepsilon}\nabla_x\Phi\left(\frac{x_i - x_j}{\varepsilon}\right)\cdot\nabla_{v_i} f_N = 0\,.$$

With such a scaling, the dynamics is very sensitive to the positions of the particles.
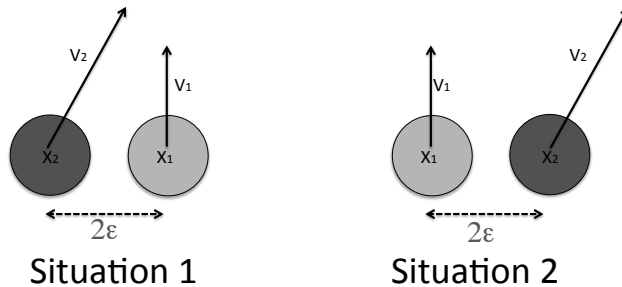
FIGURE 1. Instability

Situations 1 and 2 on Figure 1 are differ by a spatial translation of $O(\varepsilon)$ only. However in Situation 1, particles will interact and be deviated from their free motion, while in Situation 2, they will evolve under free flow.

1.3. The Boltzmann-Grad limit

Particles move with uniform rectilinear motion as long as they remain at a distance greater than ε to other particles. In the limit $\varepsilon \to 0$, we thus expect trajectories to be almost polylines.

Deflections are due to elementary interactions
– which occur when two particles are at a distance smaller than ε (exactly ε in the case of hard spheres),
– during a time interval of order ε (if the relative velocity is not too small) or even instantaneously in the case of hard spheres,
– which involve generally only two particles : the probability that a third particle enters a security ball of radius ε should indeed tend to 0 as $\varepsilon \to 0$ in the convenient scaling. We are therefore brought back to the case of the two-body system, which is completely integrable (see Chapter 8).

In order for the interactions to have a macroscopic effect on the dynamics, each particle should undergo a finite number of collisions per unit of time. A scaling argument, giving the mean free path in terms of N and ε, then shows that $N\varepsilon^{d-1} = O(1)$: indeed a particle travelling at speed bounded by R covers in unit time an area of size $R\varepsilon^{d-1}$, and there are N such particles. This is the Boltzmann-Grad scaling (see [**24**]).

The Boltzmann equation, which is the master equation in collisional kinetic theory [**15, 46**], is expected to describe such a dynamics.

CHAPTER 2

THE BOLTZMANN EQUATION

2.1. Transport and collisions

As mentioned in the previous chapter, the state of the system in the low density limit should be described (at the statistical level) by the kinetic density, i.e. by the probability $f \equiv f(t, x, v)$ of finding a particle with position x and velocity v at time t.

This density is expected to evolve under both the effects of transport and binary elastic collisions, which is expressed in the Boltzmann equation (introduced by Boltzmann in [**8**]-[**9**]) :

$$(2.1.1) \qquad \underbrace{\partial_t f + v \cdot \nabla_x f}_{\text{free transport}} \quad = \quad \underbrace{Q(f,f)}_{\text{localized binary collisions}}$$

The Boltzmann collision operator, present in the right-hand side of (2.1.1), is the quadratic form, acting on the velocity variable, associated with the bilinear operator

$$(2.1.2) \qquad Q(f,f) = \iint [f'f_1' - ff_1] \, b(v - v_1, \omega) \, dv_1 d\omega$$

where we have used the standard abbreviations

$$f = f(v), \quad f' = f(v'), \quad f_1' = f(v_1'), \quad f_1 = f(v_1),$$

with (v', v_1') given by

$$v' = v + \omega \cdot (v_1 - v) \, \omega, \quad v_1' = v_1 - \omega \cdot (v_1 - v) \, \omega.$$

One can easily show that the quadruple (v, v_1, v', v_1') parametrized by $\omega \in \mathbf{S}_1^{d-1}$ (where $\mathbf{S}_\rho^{d-1}$ denotes the sphere of radius ρ in $\mathbf{R}^d$) provides the family of all solutions to the system of $d + 1$ equations

$$(2.1.3) \qquad \begin{aligned} v + v_1 &= v' + v_1', \\ |v|^2 + |v_1|^2 &= |v'|^2 + |v_1'|^2, \end{aligned}$$

which, at the kinetic level, express the fact that collisions are elastic and thus conserve momentum and energy. Notice that the transformation $(v, v_1, \omega) \mapsto \left(v', v_1', \frac{v - v_1}{|v - v_1|}\right)$ is an involution.

The Boltzmann collision operator can therefore be split, at least formally, into a gain term and a loss term (see [**13, 46**])

$$Q(f,f) = Q^+(f,f) - Q^-(f,f).$$

The loss term counts all collisions in which a given particle of velocity v will encounter another particle, of velocity v_1, and thus will change its velocity leading to a loss of particles of velocity v, whereas the

gain term measures the number of particles of velocity v which are created due to a collision between particles of velocities v' and v'_1.

Denote by $(w, \omega) \in \mathbf{R}^d \times \mathbf{S}_1^{d-1}$ the integrand of (2.1.2). The collision cross-section $b = b(w, \omega)$, or collision kernel, is a measurable function positive almost everywhere, which measures the statistical repartition of post-collisional velocities (v, v_1) given the pre-collisional velocities (v', v'_1). Its precise form depends crucially on the nature of the microscopic interactions, and will be discussed in more details in the sequel. Note that, due to the Galilean invariance of collisions, it only depends on the magnitude of the relative velocity $|w|$ and on the deviation angle θ, or deflection (scattering) angle, defined by $\cos \theta = k \cdot \omega$ where $k = w/|w|$.

2.2. Boltzmann's H theorem and irreversibility

From (2.1.3) and using the well-known facts (see [**13**]) that transforming $(v, v_1) \mapsto (v_1, v)$ and $(v, v_1, \omega) \mapsto (v', v'_1, \omega)$ merely induces mappings with unit Jacobian determinants, known as the pre-post-collisional changes of variables or simply collisional symmetries, one can show that

$$(2.2.1) \qquad \int Q(f, f)\varphi dv = 0$$

for all f regular enough, if and only if $\varphi(v)$ is a collision invariant, i.e. $\varphi(v)$ is a linear combination of $\{1, v_1, \ldots, v_d, |v|^2\}$. Thus, successively multiplying the Boltzmann equation (2.1.1) by the collision invariants and then integrating in velocity yields formally the local conservation laws

$$(2.2.2) \qquad \partial_t \int_{\mathbf{R}^d} f \begin{pmatrix} 1 \\ v \\ \frac{|v|^2}{2} \end{pmatrix} dv + \nabla_x \cdot \int_{\mathbf{R}^d} f \begin{pmatrix} v \\ v \otimes v \\ \frac{|v|^2}{2} v \end{pmatrix} dv = 0 \,,$$

which provides the link to a macroscopic description of the gas.

The other very important feature of the Boltzmann equation comes also from the symmetries of the collision operator. Disregarding integrability issues, we choose $\varphi = \log f$ and use the properties of the logarithm, to find

$$(2.2.3) \qquad \begin{aligned} D(f) &\equiv -\int Q(f, f) \log f \, dv \\ &= \frac{1}{4} \int_{\mathbf{R}^d \times \mathbf{R}^d \times \mathbf{S}_1^{d-1}} b(v - v_1, \omega)(f' f'_1 - f f_1) \log \frac{f' f'_1}{f f_1} \, dv dv_1 d\omega \geq 0 \,. \end{aligned}$$

The so-defined entropy dissipation is therefore a nonnegative functional.

This leads to Boltzmann's H theorem, also known as second principle of thermodynamics, stating that the entropy is (at least formally) a Lyapunov functional for the Boltzmann equation.

$$(2.2.4) \qquad \partial_t \int_{\mathbf{R}^d} f \log f \, dv + \nabla_x \cdot \int_{\mathbf{R}^d} f \log f v \, dv \leq 0 \,.$$

As to the equation $Q(f, f) = 0$, it is possible to show that it is only satisfied by the so-called Maxwellian distributions $M_{\rho, u, \theta}$, which are defined by

$$(2.2.5) \qquad M_{\rho, u, \theta}(v) := \frac{\rho}{(2\pi\theta)^{\frac{d}{2}}} e^{-\frac{|v-u|^2}{2\theta}} \,,$$

where $\rho \in \mathbf{R}_+$, $u \in \mathbf{R}^d$ and $\theta \in \mathbf{R}_+$ are respectively the macroscopic density, bulk velocity and temperature, under some appropriate choice of units. The relation $Q(f, f) = 0$ expresses the fact that collisions are no longer responsible for any variation in the density and so, that the gas has reached statistical equilibrium. In fact, it is possible to show that if the density f is a Maxwellian distribution for some $\rho(t, x)$, $u(t, x)$ and $\theta(t, x)$, then the macroscopic conservation laws (2.2.2) turn out to constitute the compressible Euler system.

More generally, the H-theorem (2.2.4) together with the conservation laws (2.2.2) constitute the key elements of the study of hydrodynamic limits.

Remark 2.2.1. — *Note that the irreversibility inherent to the Boltzmann dynamics seems at first sight to contradict the possible existence of a connection with the microscopic dynamics which is reversible and satisfies the Poincaré recurrence theorem (while the Boltzmann dynamics predict some relaxation towards equilibrium).*

That irreversibility will actually appear in the limiting process as an arbitrary choice of the time direction (encoded in the distinction between pre-collisional and post-collisional particles), and more precisely as an arbitrary choice of the initial time, which is the only time for which one has a complete information on the correlations. The point is that the joint probability of having particles of velocity (v', v_1') (respectively of velocities (v, v_1)) before the collision is assumed to be equal to $f(t, x, v')f(t, x, v_1')$ (resp. to $f(t, x, v)f(t, x, v_1)$), meaning that particles should be independent before collision.

2.3. The Cauchy problem

Let us first describe briefly the most apparent problems in trying to construct a general, good Cauchy theory for the Boltzmann equation. In the full, general situation, known a priori estimates for the Boltzmann equation are only those which are associated with the basic physical laws, namely the formal conservation of mass and energy, and the bounds on entropy and entropy dissipation. Note that, when the physical space is unbounded, the dispersive properties of the free transport operator allow to further expect some control on the moments with respect to x-variables. Yet the Boltzmann collision integral is a quadratic operator that is purely local in the position and time variables, meaning that it acts as a convolution in the v variable, but as a pointwise multiplication in the t and x variables : thus, with the only a priori estimates which seem to hold in full generality, the collision integral is even not a well-defined distribution with respect to x-variables. This major obstruction is one of the reasons why the Cauchy problem for the Boltzmann equation is so tricky, another reason being the intricate nature of the Boltzmann operator.

For the sake of simplicity, we shall consider here only bounded collision cross-sections b. A huge literature is devoted to the study of more singular cross-sections insofar as the presence of long range interactions always creates singularities associated to grazing collisions. However, at the present time, there is no extension of Lanford's convergence result in this framework.

2.3.1. Short time existence of continuous solutions. — The easiest way to construct local solutions to the Boltzmann equation is to use a fixed point argument in the space of continuous functions.

Remarking that the free transport operator preserves weighted L^∞ norms

$$\left\|f_{in}(x - vt, v) \exp\left(\frac{\beta}{2}|v|^2\right)\right\|_{L^\infty} = \left\|f_{in}(x, v) \exp(\frac{\beta}{2}|v|^2)\right\|_{L^\infty},$$

and that the following continuity property holds for the collision operator

$$\left\|Q(f, f)(v) \exp(\frac{\beta}{2}|v|^2)\right\|_{L^\infty} \leq C_\beta \left\|f(v) \exp(\frac{\beta}{2}|v|^2)\right\|_{L^\infty}^2,$$

we get the existence of continuous solutions, the lifespan of which is inversely proportional to the norm of the initial data.

Theorem 1. — *Let $f_{in} \in C^0(\mathbf{R}^d \times \mathbf{R}^d)$ such that*

(2.3.1)
$$\left\|f_{in} \exp(\frac{\beta}{2}|v|^2)\right\|_{L^\infty} < +\infty$$

for some $\beta > 0$.

Then, there exists $C_\beta > 0$ (depending only β) such that the Boltzmann equation (2.1.1) with initial data f_{in} has a unique continuous solution on $[0, T]$ with

$$T = \frac{C_\beta}{\left\|f_{in} \exp(\frac{\beta}{2}|v|^2)\right\|_{L^\infty}}.$$

Note that the weigthed L^∞ norm controls in particular the macroscopic density

$$\rho(t, x) := \int f(t, x, v) dv \leq C_\beta \|f(t, x, v) \exp(\frac{\beta}{2}|v|^2)\|_\infty,$$

therefore the possible concentrations for which the collision process can become very pathological. This restriction, even coming from a very rough analysis, has therefore a physical meaning.

2.3.2. Fluctuations around some global equilibrium. — Historically the first global existence result for the spatially inhomogeneous Boltzmann equation is due to S. Ukai [**43, 44**], who considered initial data that are fluctuations around a global equilibrium, for instance around the reduced centered Gaussian $M := M_{1,0,1}$ with notation (2.2.5):

$$f_{in} = M(1 + g_{in}).$$

He proved the global existence of a solution to the Cauchy poblem for (2.1.1) under the assumption that the initial perturbation g_{in} is smooth and small enough in a norm that involves derivatives and weights so as to ensure decay for large v.

The convenient functional space to be considered is indeed

$$H_{\ell,k} = \{g \equiv g(x, v) / \|g\|_{\ell,k} := \sup_v (1 + |v|^k)\|M^{1/2} g(\cdot, v)\|_{H_x^\ell} < +\infty\}.$$

Theorem 2 ([**43, 44**]). — *Let $g_{in} \in H_{\ell,k}$ for $\ell > d/2$ and $k > d/2 + 1$ such that*

(2.3.2)
$$\|g_{in}\|_{\ell,k} \leq a_0$$

for some a_0 sufficiently small.

Then, there exists a unique global solution $f = M(1 + g)$ with $g \in L^\infty(\mathbf{R}^+, H_{\ell,k}) \cap C(\mathbf{R}^+, H_{\ell,k})$ to the Boltzmann equation (2.1.1) with initial data

$$g_{|t=0} = g_{in}.$$

Such a global existence result is based on Duhamel's formula and on Picard fixed point theorem. It requires a very precise study of the linearized collision operator $\mathcal{L}_M$ defined by

$$\mathcal{L}_M g := -\frac{2}{M} Q(M, Mg),$$

and more precisely of the semi-group U generated by

$$v \cdot \nabla_x + \mathcal{L}_M.$$

The main disadvantage inherent to that strategy is that one cannot expect to extend such a result to classes of initial data with less regularity.

2.3.3. Renormalized solutions. — The theory of renormalized solutions goes back to the late 80s and is due to R. DiPerna and P.-L. Lions [18]. It holds for physically admissible initial data of arbitrary sizes, but does not yield solutions that are known to solve the Boltzmann equation in the usual weak sense.

Rather, it gives the existence of a global weak solution to a class of formally equivalent initial-value problems.

Definition 2.3.1. — *A renormalized solution of the Boltzmann equation* (2.1.1) *relatively to the global equilibrium M is a function*

$$f \in C(\mathbf{R}^+, L^1_{loc}(\mathbf{R}^d \times \mathbf{R}^d))$$

which satisfies in the sense of distributions

(2.3.3)
$$M\big(\partial_t + v \cdot \nabla_x\big)\Gamma\left(\frac{f}{M}\right) = \Gamma'\left(\frac{f}{M}\right) Q(f, f) \quad on \ \mathbf{R}^+ \times \mathbf{R}^d \times \mathbf{R}^d,$$
$$f_{|t=0} = f_{in} \geq 0 \quad on \ \mathbf{R}^d \times \mathbf{R}^d.$$

for any $\Gamma \in C^1(\mathbf{R}^+)$ such that $|\Gamma'(z)| \leq C/\sqrt{1+z}$.

With the above definition of renormalized solution relatively to M, the following existence result holds :

Theorem 3 ([18]). — *Given any initial data f_{in} satisfying*

(2.3.4)
$$H(f_{in}|M) \equiv \int\int \left(f_{in} \log \frac{f_{in}}{M} - f_{in} + M\right)(x, v) \, dv \, dx < +\infty,$$

there exists a renormalized solution $f \in C(\mathbf{R}^+, L^1_{loc}(\mathbf{R}^d \times \mathbf{R}^d))$ relatively to M to the Boltzmann equation (2.1.1) *with initial data f_{in}.*

Moreover, f satisfies

- the continuity equation

(2.3.5)
$$\partial_t \int f dv + \nabla_x \cdot \int f v dv = 0;$$

- the momentum equation with defect measure

(2.3.6)
$$\partial_t \int f v dv + \nabla_x \cdot \int f v \otimes v dv + \nabla_x \cdot m = 0$$

where m is a Radon measure on $\mathbf{R}^+ \times \mathbf{R}^d$ with values in the nonnegative symmetric matrices;

- the entropy inequality

$$(2.3.7) \qquad H(f|M)(t) + \int \operatorname{trace}(m)(t) + \int_0^t \int D(f)(s,x)dsdx$$
$$\leq H(f_{in}|M)$$

where $\operatorname{trace}(m)$ *is the trace of the nonnegative symmetric matrix* m, *and the entropy dissipation* $D(f)$ *is defined by* (2.2.3).

The weak stability of approximate solutions is inherited from the entropy inequality. In order to take limits in the renormalized Boltzmann equation, we have further to obtain some strong compactness. The crucial idea here is to use the velocity averaging lemma due to F. Golse, P.-L. Lions, B. Perthame and R. Sentis [**22**], stating that the moments in v of the solution to some transport equation are more regular than the function itself.

Remark 2.3.2. — *As we will see, the major weakness of the convergence theorem describing the Boltzmann equation as the low density limit of large systems of particles is the very short time on which it holds. However, the present state of the art regarding the Cauchy theory for the Boltzmann equation makes it very difficult to improve.*

*Because of the scaling of the microscopic interactions, the conditioning on energy surfaces (see Chapter 6) introduces strong spatial oscillations in the initial data. We therefore do not expect to get regularity so that we could take advantage of the perturbative theory of S. Ukai [**43**, **44**]. A coarse graining argument would be necessary to retrieve spatial regularity on the kinetic distribution, but we are not aware of any breakthrough in this direction.*

*As for using the DiPerna-Lions theory [**18**], the first step would be to understand the counterpart of renormalization at the level of the microscopic dynamics, which seems to be also a very challenging problem.*

CHAPTER 3

MAIN RESULTS

3.1. Lanford and King's theorems

The main goal of this monograph is to prove the two following statements. We give here compact, and somewhat informal, statements of our two main results. Precise statements are given in Chapters 6 and 11 (see Theorem 8 page 47 for the hard-spheres case, and 11 page 85 for the potential case).

The following statement concerns the case of hard spheres dynamics, and the main ideas behind its proof go back to the fundamental work of Lanford [**34**].

Theorem 4. — *Let $f_0 : \mathbf{R}^{2d} \mapsto \mathbf{R}^+$ be a continuous density of probability such that*

$$\left\| f_0(x,v) \exp(\frac{\beta}{2}|v|^2) \right\|_{L^\infty(\mathbf{R}^{2d})} < +\infty$$

for some $\beta > 0$.

Consider the system of N hard spheres of diameter ε, initially distributed according to f_0 and "independent", governed by the system (1.0.1)-(1.0.2). Then, in the Boltzmann-Grad limit $N \to \infty$, $N\varepsilon^{d-1} \sim 1$, its distribution function converges to the solution to the Boltzmann equation (2.1.1) with the cross-section $b(w,\omega) := (\omega \cdot w)_+$ and with initial data f_0, in the sense of observables.

The next theorem concerns the Hamiltonian case (with a repulsive potential), and important steps of the proof can be found in the thesis of King [**30**].

Theorem 5. — *Assume that the repulsive potential Φ satisfies Assumption 1.2.1 as well as the technical assumption (8.3.1). Let $f_0 : \mathbf{R}^{2d} \mapsto \mathbf{R}^+$ be a continuous density of probability such that*

$$\left\| f_0 \exp(\frac{\beta}{2}|v|^2) \right\|_{L^\infty} < +\infty$$

for some $\beta > 0$.

Consider the system of N particles, initially distributed according to f_0 and "independent", governed by the system (1.2.1). Then, in the Boltzmann-Grad limit $N \to \infty$, $N\varepsilon^{d-1} \sim 1$, its distribution function converges to the solution to the Boltzmann equation (2.1.1) with a bounded cross-section, depending on Φ implicitly, and with initial data f_0, in the sense of observables.

Remark 3.1.1. — *Convergence in the sense of observables means that, for any test function φ in $C_c^0(\mathbf{R}_v^d)$, the corresponding observable*

$$\phi_\varepsilon(t,x) := \int f_\varepsilon(t,x,v)\varphi(v)dv \longrightarrow \phi(t,x) := \int f(t,x,v)\varphi(v)dv$$

uniformly in t and x. We indeed recall that the kinetic distribution cannot be measured, only averages can be reached by physical experiments : this accounts for the terminology "observables".

In mathematical terms, this means that we establish only weak convergence with respect to the v-variable. Such a convergence result does not exclude the existence of pathological behaviors, in particular dynamics obtained by reversing the arrow of time and which are predicted by the (reversible) microscopic system. We shall only prove that these behaviors have negligible probability in the limit $\varepsilon \to 0$.

Remark 3.1.2. — *The initial independence assumption has to be understood also asymptotically. It will be discussed with much details in Chapter 6 (see Chapter 11 in the case of a potential): it is actually related to some coarse-graining arguments which are rather not intuitive at first sight.*

For hard spheres, the exclusion obviously prevents independence for fixed ε, but we expect to retrieve this independence as $\varepsilon \to 0$ if we consider a fixed number s of spheres. The question is to deal with an infinite number of such spheres.

The case of the smooth Hamiltonian system could seem to be simpler insofar as particles can occupy the whole space. Nevertheless, in order to control the decay at large energies, we need to introduce some conditioning on energy surfaces, which is very similar to exclusion.

Remark 3.1.3. — *The technical assumption (8.3.1) will be made explicit in Chapter 8 : it ensures that the deviation angle is a suitable parametrization of the collision, and more precisely that we can retrieve the impact parameter from both the ingoing velocity and the deviation angle. What we will use is the fact that the jacobian of this change of variables is bounded at least locally.*

Such an assumption is not completely compulsory for the proof. We can imagine of splitting the integration domain in many subdomains where the deviation angle is a good parametrization of the collision, but then we have to extend the usual definition of the cross-section. The important point is that the deviation angle cannot be a piecewise constant function of the impact parameter.

3.2. Background and references

The problem of asking for a rigorous derivation of the Boltzmann equation from the Hamiltonian dynamics goes back to Hilbert [**27**], who suggested to use the Boltzmann equation as an intermediate step between the Hamiltonian dynamics and fluid mechanics, and who described this axiomatization of physics as a major challenge for mathematicians of the twentieth century.

We shall not give an exhaustive presentation of the studies that have been carried out on this question but indicate some of the fundamental landmarks, concerning for most of them the case of hard spheres. First one should mention N. Bogoliubov [**6**], M. Born, and H. S. Green [**10**], J. G. Kirkwood [**31**] and J. Yvan [**47**], who gave their names to the BBGKY hierarchy on the successive marginals, which we shall be using extensively in this study. H. Grad was able to obtain in [**23**] a differential equation on the first marginal which after some manipulations converges towards the Boltzmann equation.

The first mathematical result on this problem goes back to C. Cercignani [12] and O. Lanford [34] who proved that the propagation of chaos should be established by a careful study of trajectories of a hard spheres system, and who exhibited – for the first time – the origin of irreversibility. The proof, even though incomplete, is therefore an important breakthrough. The limits of their methods, on which we will comment later on – especially regarding the short time of convergence – are still challenging questions.

The argument of O. Lanford was then revisited and completed in several works. Let us mention especially the contributions of K. Uchiyama [42], C. Cercignani, R. Illner and M. Pulvirenti [15] and H. Spohn [40] who introduced a mathematical formalism, in particular for the existence of solutions to the BBGKY hierarchy which turns out to be a theory in the spirit of the Cauchy-Kowalewskaya theorem.

The term-by-term convergence of the hierarchy in the Boltzmann-Grad scaling was studied in more details by C. Cercignani, V. I. Gerasimenko and D. I. Petrina [14] : they provide for the first time quantative estimates on the set of "pathological trajectories", i.e. trajectories for which the Boltzmann equation does not provide a good approximation of the dynamics. What is not completely clear in this approach is the stability of the estimates under microscopic spatial translations.

The method of proof was then extended
- to the case when the initial distribution is close to vacuum, in which case global in time results may be proved [15, 28, 29];
- to the case when interactions are localized but not pointwise [30]. Because multiple collisions are no longer negligible, this requires a careful study of clusters of particles.

Many review papers deal with those different results, see [19, 38, 46] for instance.

Let us summarize the strategy of the proofs. Their are two main steps:

(i) a short time bound for the series expansion expressing the correlations of the system of N particles and the corresponding quantities of the Boltzmann equation;

(ii) the term by term convergence.

In the case of hard spheres, point (i) is just a matter of explicit estimates, while point (ii) is usually considered as almost obvious (but deep). Among experts in the field the hard sphere case is therefore considered to be completely solved. However, we could not find a proof for the measure zero estimates (i.e. the control of recollisions) in the litterature. It might be that to experts in the field such an estimate is easy, but from our point of view it turned out to be quite delicate.

- For the Boltzmann dynamics, it seems to be correct that a zero measure argument allows to control recollisions inasmuch as particles are pointwise.
- For fixed ε, we will see that the set of velocities leading to recollisions (even in the case of three particles) is small but not zero : this cannot be obtained by a straightforward thickening argument without any **geometrical information** on the limiting zero measure set.
- For the microscopic system of N particles, collisional particles are at a distance ε from each other, we thus expect that even "good trajectories" deviate from trajectories associated to the Boltzmann dynamics. We shall therefore need some **stability of "pathological sets"** of velocities with respect to microscopic spatial translations, to be able to iterate the process.

3.3. New contributions

Our goal here is to provide a self-contained presentation, which includes all the details of the proofs, especially concerning term-by-term convergence which to our knowledge is not completely written anywhere, even in the hard-spheres case.

Part II is a review of known results in the case of hard spheres. Following Lanford's strategy, we shall establish the starting hierarchy of equations, providing a short time, uniform estimate.

We focus especially on the **definition of functional spaces**: we shall see that the short time estimate is obtained as an analytical type result, meaning that we control all correlation functions together. The functional spaces we consider are in some sense natural from the point of view of statistical physics, since they involve two parameters β and μ related to the inverse temperature and chemical potential to control the growth of energy and of the number of particles. Nevertheless, instead of usual L^1 norms, we use L^∞ norms, which are needed to control collision integrals (see Remark 2.3.2).

The second point we discuss in details is the **notion of independence**. As noted in Remark 3.1.2, for any fixed $\varepsilon > 0$, because of the exclusion, particles cannot be independent. In the $2Nd$-dimensional phase-space, we shall see actually that the Gibbs measure has support on only a very small set. Careful estimates on the partition function show however that the marginal of order s (for any fixed s) converges to some tensorized distribution, meaning that independence is recovered at the limit $\varepsilon \to 0$.

Part III deals with the case of the Hamiltonian system, with a repulsive potential. It basically follows King's thesis [**30**], filling in some gaps.

In the limit $\varepsilon \to 0$ with $N\varepsilon^{d-1} \equiv 1$, we would like to obtain a kind of homogeneization result : we want to average the motion over the small scales in t and x, and replace the localized interactions by pointwise collisions as in the case of hard spheres. We therefore introduce an **artificial boundary** (following [**30**]) so that
– on the exterior domain, the dynamics reduces to free transport,
– on the interior domain, the dynamics can be integrated in order to compute outwards boundary conditions in terms of the incoming flux. Note that such a scattering operator is relevant only if we can guarantee that there is no other particle involved in the interaction.
An important point is therefore to control multiple collisions, which - contrary to the case of hard spheres - could happen for a non zero set of initial data. We however expect that they become negligible in the Boltzmann-Grad limit (as the probability of finding three particles having approximately the same position tends to zero). **Cluster estimates**, based on suitable partitions of the $2Nd$-dimensional phase-space and symmetry arguments, give the required asymptotic bound on multiple collisions.

Part IV is the heart of our contribution, where we establish the term-by-term convergence. Note that the arguments work in the same way in both situations (hard spheres and potential case), up to some minor technical points due to the fact that, for the N-particle Hamiltonian system, pre-collisional and post-collisional configurations differ by their velocities but also by their microscopic positions and by some microscopic shift in time.

However the two main difficulties are exactly the same:

– describing geometrically the set of "pathological" velocities and deflection angles leading to possible recollisions, in order to get a **quantitative estimate** of its measure;
– proving that this set is **stable under small translations** of positions.

Note that the estimates we establish depend only on the scattering operator, so that we have a rate of convergence which can be made explicit for instance in the case of hard spheres.

To control the set of recolliding trajectories by means of explicit estimates, we make use of properties of the cross-section which are not guaranteed a priori for a generic repulsive potential. Assumption (8.3.1) guarantees that these conditions are satisfied.

PART II

THE CASE OF HARD SPHERES

CHAPTER 4

MICROSCOPIC DYNAMICS AND BBGKY HIERARCHY

In this chapter we define the N-particle flow for hard spheres (introduced in Chapter 1), and write down the associated BBGKY hierarchy. Finally we present a formal derivation of the Boltzmann hierarchy, and the Boltzmann equation of hard spheres. This chapter follows the classical approaches of [**1**], [**14**], [**15**], [**34**], among others.

4.1. The N-particle flow

We consider N particles in the space $\mathbf{R}^d$, the motion of which is described by N positions $(x_1, \ldots, x_N)$ and N velocities $(v_1, \ldots, v_N)$, each in $\mathbf{R}^d$. Denoting by $Z_N := (z_1, \ldots, z_N)$ the set of particles, each particle $z_i := (x_i, v_i) \in \mathbf{R}^{2d}$ is submitted to free flow on the domain

$$\mathcal{D}_N := \left\{ Z_N \in \mathbf{R}^{2dN} \,/\, \forall i \neq j, \, |x_i - x_j| > \varepsilon \right\}$$

and bounces off the boundary $\partial \mathcal{D}_N$ according to the laws of elastic reflection: if

$$\forall 1 \leq i \leq N, \quad \frac{dx_i}{dt} = v_i, \quad \frac{dv_i}{dt} = 0 \text{ on } \mathcal{D}_N$$

(4.1.1)
$$v_i^{in} = v_i^{out} - \nu^{i,j} \cdot (v_i^{out} - v_j^{out}) \, \nu^{i,j}$$

$$v_j^{in} = v_j^{out} + \nu^{i,j} \cdot (v_i^{out} - v_j^{out}) \, \nu^{i,j} \text{ if } \exists j \neq i, \, |x_i - x_j| = \varepsilon,$$

where $\nu^{i,j} := (x_i - x_j)/|x_i - x_j|$, and in the case when $\nu^{i,j} \cdot (v_i^{in} - v_j^{in}) < 0$ (meaning that the ingoing velocities are precollisional).

Contrary to the potential case studied in Part III, it is not obvious to check that (4.1.1) defines a global dynamics, at least for almost all initial data. Note indeed that this is not a simple consequence of the Cauchy-Lipschitz theorem since the boundary condition is not smooth, and even not defined for all configurations. We call *pathological* a trajectory such that

- either there exists a collision involving more than two particles, or the collision is grazing (meaning that $\nu^{i,j} \cdot (v_i^{in} - v_j^{in}) = 0$) hence the boundary condition is not well defined;

- or there are an infinite number of collisions in finite time so the dynamics cannot be globally defined.

In [**2**, Proposition 4.3], it is stated that outside a negligible set of initial data there are no pathological trajectories; the complete proof is provided in [**1**]. Actually the setting of [**1**] is more complicated than ours since an infinite number of particles is considered. The arguments of [**1**] can however be easily

adapted to our case to yield the following result, whose proof we detail for the convenience of the reader.

Proposition 4.1.1. — *Let N, ε be fixed. The set of initial configurations leading to a pathological trajectory is of measure zero in $\mathbf{R}^{2dN}$.*

We first prove the following elementary lemma, in which we have used the following notation: for any $s \in \mathbf{N}^*$ and $R > 0$, we denote $B_R^s := \{V_s \in \mathbf{R}^{ds}, |V_s| \leq R\}$ where $|\cdot|$ is the euclidean norm; we often write $B_R := B_R^1$.

Lemma 4.1.1. — *Let $\rho, R > 0$ be given, and $\delta < \varepsilon/2$. Define*

$$I := \left\{ Z_N \in B_\rho^N \times B_R^N \;/\; \text{one particle will collide with two others on the time interval } [0, \delta] \right\}.$$

Then $|I| \leq C(N, \varepsilon, R)\, \rho^{d(N-2)} \delta^2$.

Proof. — We notice that I is a subset of

$$\left\{ Z_N \in B_\rho^N \times B_R^N \;/\; \exists \{i, j, k\} \text{ distinct}, \quad |x_i - x_j| \in [\varepsilon, \varepsilon + 2R\delta] \quad \text{and} \quad |x_i - x_k| \in [\varepsilon, \varepsilon + 2R\delta] \right\},$$

and the lemma follows directly. □

Proof of Proposition 4.1.1. — Let $R > 0$ be given and fix some time $t > 0$. Let $\delta < \varepsilon/2$ be a parameter such that t/δ is an integer.

Lemma 4.1.1 implies that there is a subset $I_0(\delta, R)$ of $B_R^N \times B_R^N$ of measure at most $C(N, \varepsilon, R)R^{d(N-2)}\delta^2$ such that any initial configuration belonging to $(B_R^N \times B_R^N) \setminus I_0(\delta, R)$ generates a solution on $[0, \delta]$ such that each particle encounters at most one other particle on $[0, \delta]$. Moreover up to removing a measure zero set of initial data each collision is non-grazing.

Now let us start again at time δ. We recall that in the velocity variables, the ball of radius R in $\mathbf{R}^{dN}$ is stable by the flow, whereas the positions at time δ lie in the ball $B_{R+R\delta}^N$. Let us apply Lemma 4.1.1 again to that new initial configuration space. Since the measure is invariant by the flow, we can construct a subset $I_1(\delta, R)$ of the initial positions $B_R^N \times B_R^N$, of size $C(N, \varepsilon, R)R^{d(N-2)}(1 + \delta)^{d(N-2)}\delta^2$ such that outside $I_0 \cup I_1(\delta, R)$, the flow starting from any initial point in $B_R^N \times B_R^N$ is such that each particle encounters at most one other particle on $[0, \delta]$, and then at most one other particle on $[\delta, 2\delta]$, again in a non-grazing collision. We repeat the procedure t/δ times: we construct a subset

$$I_\delta(t, R) := \bigcup_{j=0}^{t/\delta - 1} I_j(\delta, R)$$

of $B_R^N \times B_R^N$, of measure

$$|I_\delta(t, R)| \leq C(N, \varepsilon, R)R^{d(N-2)}\delta^2 \sum_{j=0}^{t/\delta - 1} (1 + j\delta)^{d(N-2)}$$

$$\leq C(N, R, t, \varepsilon)\delta,$$

such that for any initial configuration in $B_R^N \times B_R^N$ outside that set, the flow is well-defined up to time t. The intersection $I(t, R) := \bigcap_{\delta > 0} I_\delta(t, R)$ is of measure zero, and any initial configuration in $B_R^N \times B_R^N$ outside $I(t, R)$ generates a well-defined flow until time t. Finally we consider the countable union of

those zero measure sets $I := \bigcup_n I(t_n, R_n)$ where t_n and R_n go to infinity, and any initial configuration in $\mathbf{R}^{2dN}$ outside I generates a globally defined flow. The proposition is proved. $\quad\square$

4.2. The Liouville equation and the BBGKY hierarchy

According to Part I, Paragraph 1.1, the Liouville equation relative to the particle system (4.1.1) is

$$(4.2.1) \qquad \partial_t f_N + \sum_{i=1}^{N} v_i \cdot \nabla_{x_i} f_N = 0 \quad \text{in} \quad \mathbf{R}_+ \times \mathcal{D}_N$$

with the boundary condition $f_N(t, Z_N^{in}) = f_N(t, Z_N^{out})$. We recall the assumption that f_N is invariant by permutation in the sense of (1.1.1), meaning that the particles are indistinguishable.

The classical strategy to obtain asymptotically a kinetic equation such as (2.1.1) is to write the evolution equation for the first marginal of the distribution function f_N, namely

$$f_N^{(1)}(t, z_1) := \int_{\mathbf{R}^{2d(N-1)}} f_N(t, z_1, z_2, \ldots, z_N) \mathbb{1}_{Z_N \in \mathcal{D}_N} \, dz_2 \ldots dz_N \,.$$

The point to be noted is that the evolution of $f_N^{(1)}$ depends actually on $f_N^{(2)}$ because of the quadratic interaction imposed by the boundary condition. And in the same way, the equation on $f_N^{(2)}$ depends on $f_N^{(3)}$. Instead of a kinetic equation, we therefore obtain a hierarchy of equations involving all the marginals of f_N

$$(4.2.2) \qquad f_N^{(s)}(t, Z_s) := \int_{\mathbf{R}^{2d(N-s)}} f_N(t, Z_s, z_{s+1}, \ldots, z_N) \mathbb{1}_{Z_N \in \mathcal{D}_N} \, dz_{s+1} \cdots dz_N \,.$$

Notice that $f_N^{(s)}(t, Z_s)$ is defined on $\mathcal{D}_s$ only, and that

$$(4.2.3) \qquad f_N^{(s)}(t, Z_s) = \int_{\mathbf{R}^{2d}} f_N^{(s+1)}(t, Z_s, z_{s+1}) \, dz_{s+1} \,.$$

Finally by integration of the boundary condition on f_N we find that $f_N^{(s)}(t, Z_s^{in}) = f_N^{(s)}(t, Z_s^{out})$. An equation for the marginals is derived in weak form in Section 4.3, and from that equation we derive formally the Boltzmann hierarchy in the Boltzmann-Grad limit (see Section 4.4).

4.3. Weak formulation of Liouville's equation

Our goal in this section is to find the weak formulation of the system of equations satisfied by the family of marginals $\left(f_N^{(s)}\right)_{1 \le s \le N}$ defined above in (4.2.2). From now on we assume that f_N decays at infinity in the velocity variable (the functional setting will be made precise in Chapter 5).

Given a smooth, compactly supported function ϕ defined on $\mathbf{R}_+ \times \mathcal{D}_s$ and satisfying the symmetry assumption (1.1.1) as well as the boundary condition $\phi(t, Z_s^{in}) = \phi(t, Z_s^{out})$, we have

$$(4.3.1) \qquad \int_{\mathbf{R}_+ \times \mathbf{R}^{2dN}} \left(\partial_t f_N + \sum_{i=1}^{N} v_i \cdot \nabla_{x_i} f_N \right) \phi(t, Z_s) \mathbb{1}_{Z_N \in \mathcal{D}_N} \, dZ_N dt = 0 \,.$$

We now use integrations by parts to derive from (4.3.1) the weak form of the equation in the marginals $f_N^{(s)}$. On the one hand an integration by parts in the time variable gives

$$
\int_{\mathbf{R}_+ \times \mathbf{R}^{2dN}} \partial_t f_N(t, Z_N) \phi(t, Z_s) \mathbb{1}_{Z_N \in \mathcal{D}_N}\, dZ_N dt = -\int_{\mathbf{R}^{2dN}} f_N(0, Z_N)\phi(0, Z_s)\mathbb{1}_{Z_N \in \mathcal{D}_N}\, dZ_N
$$
$$
-\int_{\mathbf{R}_+ \times \mathbf{R}^{2dN}} f_N(t, Z_N)\partial_t \phi(t, Z_s)\mathbb{1}_{Z_N \in \mathcal{D}_N}\, dZ_N dt\,,
$$

hence, by definition of $f_N^{(s)}$ in (4.2.2),

$$
\int_{\mathbf{R}_+ \times \mathbf{R}^{2dN}} \partial_t f_N(t, Z_N) \phi(t, Z_s) \mathbb{1}_{Z_N \in \mathcal{D}_N}\, dZ_N dt = -\int_{\mathbf{R}^{2ds}} f_N^{(s)}(0, Z_s)\phi(0, Z_s)\, dZ_s
$$
$$
-\int_{\mathbf{R}_+ \times \mathbf{R}^{2ds}} f_N^{(s)}(t, Z_s)\partial_t \phi(t, Z_s)\, dZ_s dt\,.
$$

Now let us compute

$$
\sum_{i=1}^N \int_{\mathbf{R}^{2dN}} v_i \cdot \nabla_{x_i} f_N(t, Z_N) \phi(t, Z_s) \mathbb{1}_{Z_N \in \mathcal{D}_N}\, dZ_N = \int_{\mathbf{R}^{2dN}} \mathrm{div}_{X_N}\big(V_N\, f_N(t, Z_N)\big)\phi(t, Z_s)\mathbb{1}_{Z_N \in \mathcal{D}_N}\, dZ_N
$$

using Green's formula. The boundary terms involve configurations with at least one pair (i, j) satisfying $|x_i - x_j| = \varepsilon$. According to Paragraph 4.1 we may neglect configurations where more than two particles collide at the same time, so the boundary condition is well defined. For any i and j in $\{1, \dots, N\}$ we denote

$$
\Sigma_N(i, j) := \Big\{ X_N \in \mathbf{R}^{2dN},\ |x_i - x_j| = \varepsilon \Big\}\,,
$$

and $n^{i,j}$ is the outward normal to $\Sigma_N(i, j)$ in $\mathbf{R}^{dN}$. We obtain by Green's formula:

$$
\sum_{i=1}^N \int_{\mathbf{R}_+ \times \mathbf{R}^{2dN}} v_i \cdot \nabla_{x_i} f_N(t, Z_N) \phi(t, Z_s) \mathbb{1}_{Z_N \in \mathcal{D}_N}\, dZ_N\, dt
$$
$$
= -\sum_{i=1}^s \int_{\mathbf{R}_+ \times \mathbf{R}^{2dN}} f_N(t, Z_N) v_i \cdot \nabla_{x_i} \phi(t, Z_s) \mathbb{1}_{Z_N \in \mathcal{D}_N}\, dZ_N dt
$$
$$
+ \sum_{1 \le i \ne j \le N} \int_{\mathbf{R}_+ \times \mathbf{R}^{dN} \times \Sigma_N(i,j)} n^{i,j} \cdot V_N\, f_N(t, Z_N)\phi(t, Z_s)\, d\sigma_N^{i,j} dV_N dt\,,
$$

with $d\sigma_N^{i,j}$ the surface measure on $\Sigma_N(i, j)$, induced by the Lebesgue measure. Now we split the last term into four parts:

$$
\sum_{1 \le i \ne j \le N} \int_{\mathbf{R}_+ \times \mathbf{R}^{dN} \times \Sigma_N(i,j)} n^{i,j} \cdot V_N\, f_N(t, Z_N)\phi(t, Z_s)\, d\sigma_N^{i,j} dV_N dt
$$
$$
= \sum_{i=1}^s \sum_{j=s+1}^N \int_{\mathbf{R}_+ \times \mathbf{R}^{dN} \times \Sigma_N(i,j)} n^{i,j} \cdot V_N\, f_N(t, Z_N)\phi(t, Z_s)\, d\sigma_N^{i,j} dV_N dt
$$
$$
+ \sum_{i=s+1}^N \sum_{j=1}^s \int_{\mathbf{R}_+ \times \mathbf{R}^{dN} \times \Sigma_N(i,j)} n^{i,j} \cdot V_N\, f_N(t, Z_N)\phi(t, Z_s)\, d\sigma_N^{i,j} dV_N dt
$$
$$
+ \sum_{1 \le i \ne j \le s} \int_{\mathbf{R}_+ \times \mathbf{R}^{dN} \times \Sigma_N(i,j)} n^{i,j} \cdot V_N\, f_N(t, Z_N)\phi(t, Z_s)\, d\sigma_N^{i,j} dV_N dt
$$
$$
+ \sum_{s+1 \le i \ne j \le N} \int_{\mathbf{R}_+ \times \mathbf{R}^{dN} \times \Sigma_N(i,j)} n^{i,j} \cdot V_N\, f_N(t, Z_N)\phi(t, Z_s)\, d\sigma_N^{i,j} dV_N dt\,.
$$

The boundary condition on f_N and ϕ imply that the two last terms of on the right-hand side are zero. By symmetry (1.1.1) and by definition of $f_N^{(s)}$, we can write

$$\sum_{i=1}^{s} \sum_{j=s+1}^{N} \int_{\mathbf{R}_+ \times \mathbf{R}^{dN} \times \Sigma_N(i,j)} n^{i,j} \cdot V_N \, f_N(t, Z_N) \phi(t, Z_s) \, d\sigma_N^{i,j} dV_N dt$$

$$+ \sum_{i=s+1}^{N} \sum_{j=1}^{s} \int_{\mathbf{R}_+ \times \mathbf{R}^{dN} \times \Sigma_N(i,j)} n^{i,j} \cdot V_N \, f_N(t, Z_N) \phi(t, Z_s) \, d\sigma_N^{i,j} dV_N dt$$

$$= -(N-s)\varepsilon^{d-1} \sum_{i=1}^{s} \int_{\mathbf{R}_+ \times \mathbf{S}_1^{d-1} \times \mathbf{R}^d \times \mathbf{R}^{2ds}} \omega \cdot (v_{s+1} - v_i) \, f_N^{(s+1)}(t, Z_s, x_i + \varepsilon\omega, v_{s+1}) \phi(t, Z_s) \, dZ_s d\omega dv_{s+1} dt.$$

Finally we obtain

$$\sum_{i=1}^{N} \int_{\mathbf{R}_+ \times \mathbf{R}^{2dN}} v_i \cdot \nabla_{x_i} f_N(t, Z_N) \phi(t, Z_s) \mathbb{1}_{Z_N \in \mathcal{D}_N} \, dZ_N \, dt$$

$$= -\sum_{i=1}^{s} \int_{\mathbf{R}_+ \times \mathbf{R}^{2ds}} f_N^{(s)}(t, Z_s) v_i \cdot \nabla_{x_i} \phi(t, Z_s) \, dZ_s dt$$

$$- (N-s)\varepsilon^{d-1} \sum_{i=1}^{s} \int_{\mathbf{R}_+ \times \mathbf{S}_1^{d-1} \times \mathbf{R}^d \times \mathbf{R}^{2ds}} \omega \cdot (v_{s+1} - v_i) \, f_N^{(s+1)}(t, Z_s, x_i + \varepsilon\omega, v_{s+1}) \phi(t, Z_s) \, dZ_s d\omega dv_{s+1} dt.$$

It remains to define the *collision operator*

$$(4.3.2) \qquad \begin{aligned} \left(\mathcal{C}_{s,s+1} f_N^{(s+1)} \right)(t, Z_s) := & -(N-s)\varepsilon^{d-1} \sum_{i=1}^{s} \int_{\mathbf{S}_1^{d-1} \times \mathbf{R}^d} \omega \cdot (v_{s+1} - v_i) \\ & \times f_N^{(s+1)}(t, Z_s, x_i + \varepsilon\omega, v_{s+1}) d\omega dv_{s+1}, \end{aligned}$$

where recall that $\mathbf{S}_1^{d-1}$ is the unit sphere of $\mathbf{R}^d$, and in the end we obtain the weak formulation of the BBGKY hierarchy

$$(4.3.3) \qquad \partial_t f_N^{(s)} + \sum_{1 \le i \le s} v_i \cdot \nabla_{x_i} f_N^{(s)} = \mathcal{C}_{s,s+1} f_N^{(s+1)} \quad \text{in} \quad \mathbf{R}_+ \times \mathcal{D}_s,$$

with the boundary conditions $f_N^{(s)}(t, Z_s^{in}) = f_N^{(s)}(t, Z_s^{out})$.

In the integrand of the collision operators $\mathcal{C}_{s,s+1}$ defined in (4.3.2), we now distinguish between pre- and post-collisional configurations, as we decompose

$$\mathcal{C}_{s,s+1} = \mathcal{C}_{s,s+1}^+ - \mathcal{C}_{s,s+1}^-$$

where

$$(4.3.4) \qquad \mathcal{C}_{s,s+1}^{\pm} f^{(s+1)} = \sum_{i=1}^{s} \mathcal{C}_{s,s+1}^{\pm,i} f^{(s+1)}$$

the index i referring to the index of the interaction particle among the s "fixed" particles, with the notation

$$\left(\mathcal{C}_{s,s+1}^{\pm,i} f^{(s+1)} \right)(Z_s) := (N-s)\varepsilon^{d-1} \int_{\mathbf{S}_1^{d-1} \times \mathbf{R}^d} (\omega \cdot (v_{s+1} - v_i))_{\pm} f^{(s+1)}(Z_s, x_i + \varepsilon\omega, v_{s+1}) \, d\omega dv_{s+1},$$

the index $+$ corresponding to post-collisional configurations and the index $-$ to pre-collisional configurations.

Denote by $\boldsymbol{\Psi}_s(t)$ the s-particle flow associated with the hard-spheres system, and by $\mathbf{T}_s$ the associated solution operator:

$$(4.3.5) \qquad \mathbf{T}_s(t): \qquad f \in C^0(\mathcal{D}_s; \mathbf{R}) \mapsto f(\boldsymbol{\Psi}_s(-t, \cdot)) \in C^0(\mathcal{D}_s; \mathbf{R}).$$

The time-integrated form of equation (4.3.3) is

$$(4.3.6) \qquad f_N^{(s)}(t, Z_s) = \mathbf{T}_s(t) f_N^{(s)}(0, Z_s) + \int_0^t \mathbf{T}_s(t-\tau) \mathcal{C}_{s,s+1} f_N^{(s+1)}(\tau, Z_s) \, d\tau.$$

The *total flow* and *total collision* operators $\mathbf{T}$ and $\mathbf{C}_N$ are defined on finite sequences $G_N = (g_s)_{1 \le s \le N}$ as follows:

$$(4.3.7) \qquad \begin{cases} \forall s \le N, \ (\mathbf{T}(t) G_N)_s := \mathbf{T}_s(t) g_s, \\ \forall s \le N-1, \ (\mathbf{C}_N G_N)_s := \mathcal{C}_{s,s+1} g_{s+1}, \quad (\mathbf{C}_N G_N)_N := 0. \end{cases}$$

We finally define *mild solutions* to the BBGKY hierarchy (4.3.6) to be solutions of

$$(4.3.8) \qquad F_N(t) = \mathbf{T}(t) F_N(0) + \int_0^t \mathbf{T}(t-\tau) \mathbf{C}_N F_N(\tau) \, d\tau, \qquad F_N = (f_N^{(s)})_{1 \le s \le N}.$$

4.4. The Boltzmann hierarchy and the Boltzmann equation

Starting from (4.3.8) we now consider the limit $N \to \infty$ under the Boltzmann-Grad scaling $N\varepsilon^{d-1} \equiv 1$, in order to derive formally the expected form of the Boltzmann hierarchy.

Because of the scaling assumption $N\varepsilon^{d-1} \equiv 1$, the collision term $\mathcal{C}_{s,s+1} f^{(s+1)}(Z_s)$ is approximately equal to

$$\sum_{i=1}^{s} \int_{\mathbf{S}_1^{d-1} \times \mathbf{R}^d} \omega \cdot (v_{s+1} - v_i) f_N^{(s+1)}(Z_s, x_i + \varepsilon\omega, v_{s+1}) \, d\omega dv_{s+1}$$

which we may split into two terms, depending on the sign of $\omega \cdot (v_{s+1} - v_i)$, as in (4.3.4):

$$\sum_{i=1}^{s} \int_{\mathbf{S}_1^{d-1} \times \mathbf{R}^d} \left(\omega \cdot (v_{s+1} - v_i) \right)_+ f_N^{(s+1)}(Z_s, x_i + \varepsilon\omega, v_{s+1}) \, d\omega dv_{s+1}$$

$$- \sum_{i=1}^{s} \int_{\mathbf{S}_1^{d-1} \times \mathbf{R}^d} \left(\omega \cdot (v_{s+1} - v_i) \right)_- f_N^{(s+1)}(Z_s, x_i + \varepsilon\omega, v_{s+1}) \, d\omega dv_{s+1}.$$

Recall that pre-collisional particles are particles (x_i, v_i) and (x_{s+1}, v_{s+1}) whose distance is decreasing up to collision time, meaning that for which

$$(x_{s+1} - x_i) \cdot (v_{s+1} - v_i) < 0.$$

With the above notation this means that

$$\omega \cdot (v_{s+1} - v_i) < 0.$$

On the contrary the case when $\omega \cdot (v_{s+1} - v_i) > 0$ is called the post-collisional case; we recall that grazing collisions, satisyfing $\omega \cdot (v_{s+1} - v_i) = 0$ can be neglected (see Paragraph 4.1 above).

Consider a set of particles $Z_{s+1} = (Z_s, x_i + \varepsilon\omega, v_{s+1})$ such that (x_i, v_i) and $(x_i + \varepsilon\omega, v_{s+1})$ are post-collisional. We recall the boundary condition

$$f_N^{(s+1)}(t, Z_s, x_i + \varepsilon\omega, v_{s+1}) = f_N^{(s+1)}(t, Z_s^*, x_i + \varepsilon\omega, v_{s+1}^*)$$

where $Z_s^* = (z_1, \ldots, z_i^*, \ldots z_s)$ and (v_i^*, v_{s+1}^*) is the pre-image of (v_i, v_{s+1}) by (4.1.1):

(4.4.1)
$$v_i^* := v_i - \omega \cdot (v_i - v_{s+1}) \, \omega$$
$$v_{s+1}^* := v_{s+1} + \omega \cdot (v_i - v_{s+1}) \, \omega \,,$$

while $x_i^* := x_i$. In the following writing also $x_{s+1}^* := x_{s+1}$ we shall use the notation

(4.4.2)
$$\sigma(z_i^*, z_{s+1}^*) := (z_i, z_{s+1}) \,.$$

Then neglecting the small spatial translations in the arguments of $f_N^{(s+1)}$ and using the fact that $f_N^{(s+1)}$ is right-continuous for all s we obtain the following asymptotic expression for the collision operator at the limit:

(4.4.3)
$$\mathcal{C}_{s,s+1}^0 f^{(s+1)}(t, Z_s) := \sum_{i=1}^{s} \int \left(\omega \cdot (v_{s+1} - v_i) \right)_+$$
$$\times \left(f^{(s+1)}(t, x_1, v_1, \ldots, x_i, v_i^*, \ldots, x_s, v_s, x_i, v_{s+1}^*) - f^{(s+1)}(t, Z_s, x_i, v_{s+1}) \right) d\omega \, dv_{s+1} \,.$$

The asymptotic dynamics are therefore governed by the following integral form of the Boltzmann hierarchy:

(4.4.4)
$$f^{(s)}(t) = \mathbf{S}_s(t) f_0^{(s)} + \int_0^t \mathbf{S}_s(t - \tau) \mathcal{C}_{s,s+1}^0 f^{(s+1)}(\tau) \, d\tau \,,$$

where $\mathbf{S}_s(t)$ denotes the s-particle free-flow.

Similarly to (4.3.7), we can define the total Boltzmann flow and collision operators $\mathbf{S}$ and $\mathbf{C}$ as follows:

(4.4.5)
$$\begin{cases} \forall s \geq 1, \ (\mathbf{S}(t)G)_s := \mathbf{S}_s(t) g_s \,, \\ \forall s \geq 1, \ (\mathbf{C^0}G)_s := \mathcal{C}_{s,s+1}^0 g_{s+1} \,, \end{cases}$$

so that *mild solutions* to the Boltzmann hierarchy (4.4.4) are solutions of

(4.4.6)
$$F(t) = \mathbf{S}(t) F(0) + \int_0^t \mathbf{S}(t - \tau) \mathbf{C^0} F(\tau) \, d\tau \,, \qquad F = (f^{(s)})_{s \geq 1} \,.$$

Note that if $f^{(s)}(t, Z_s) = \prod_{i=1}^{s} f(t, z_i)$ (meaning $f^{(s)}(t)$ is *tensorized*) then f satisfies the Boltzmann equation (2.1.1)-(2.1.2), where the cross-section is $b(w, \omega) := (\omega \cdot w)_+$.

CHAPTER 5

UNIFORM A PRIORI ESTIMATES FOR THE BBGKY AND BOLTZMANN HIERARCHIES

This chapter is devoted to the statement and proof of uniform a priori estimates for mild solutions to the BBGKY hierarchy, defined in (4.3.8), which we reproduce here:

$$(5.0.1) \qquad F_N(t) = \mathbf{T}(t)F_N(0) + \int_0^t \mathbf{T}(t-\tau)\mathbf{C}_N F_N(\tau)\, d\tau\,, \qquad F_N = (f_N^{(s)})_{1 \le s \le N}\,,$$

as well as for the limit Boltzmann hierarchy defined in (4.4.6)

$$(5.0.2) \qquad F(t) = \mathbf{S}(t)F(0) + \int_0^t \mathbf{S}(t-\tau)\mathbf{C^0} F(\tau)\, d\tau\,, \qquad F = (f^{(s)})_{s \ge 1}\,.$$

Those results are obtained in Paragraphs 5.2 and 5.3 by use of a Cauchy-Kowalevskaya type argument in some adequate function spaces defined in Paragraph 5.1.

5.1. Functional spaces and statement of the results

In order to obtain uniform a priori bounds for mild solutions to the BBGKY hierarchy, we need to introduce some norms on the space of sequences $(g^{(s)})_{s \ge 1}$. Given $\varepsilon > 0$, $\beta > 0$, an integer $s \ge 1$, and a continuous function $g_s : \mathcal{D}_s \to \mathbf{R}$, we let

$$(5.1.1) \qquad\qquad |g_s|_{\varepsilon,s,\beta} := \sup_{Z_s \in \mathcal{D}_s} \left(|g_s(Z_s)| \exp\left(\beta E_0(Z_s)\right)\right)$$

where E_0 is the free Hamiltonian:

$$(5.1.2) \qquad\qquad E_0(Z_s) := \sum_{1 \le i \le s} \frac{|v_i|^2}{2}\,.$$

Note that the dependence on ε of the norm is through the constraint $Z_s \in \mathcal{D}_s$. We also define, for a continuous function $g_s : \mathbf{R}^{2ds} \to \mathbf{R}$,

$$(5.1.3) \qquad\qquad |g_s|_{0,s,\beta} := \sup_{Z_s \in \mathbf{R}^{2ds}} \left(|g_s(Z_s)| \exp\left(\beta E_0(Z_s)\right)\right)\,.$$

Definition 5.1.1. — *For $\varepsilon > 0$ and $\beta > 0$, we denote $X_{\varepsilon,s,\beta}$ the Banach space of continuous functions $\mathcal{D}_s \to \mathbf{R}$ with finite $|\cdot|_{\varepsilon,s,\beta}$ norm, and similarly $X_{0,s,\beta}$ is the Banach space of continuous functions $\mathbf{R}^{2ds} \to \mathbf{R}$ with finite $|\cdot|_{0,s,\beta}$ norm.*

For sequences of continuous functions $G = (g_s)_{s \geq 1}$, with $g_s : \mathcal{D}_s \to \mathbf{R}$, we let for $\varepsilon > 0$, $\beta > 0$, and $\mu \in \mathbf{R}$,

$$\|G\|_{\varepsilon,\beta,\mu} := \sup_{s \geq 1} \left(|g_s|_{\varepsilon,s,\beta} \exp(\mu s) \right).$$

We define similarly for $G = (g_s)_{s \geq 1}$, with $g_s : \mathbf{R}^{2ds} \to \mathbf{R}$,

$$\|G\|_{0,\beta,\mu} := \sup_{s \geq 1} \left(|g_s|_{0,s,\beta} \exp(\mu s) \right).$$

Definition 5.1.2. — *For $\varepsilon \geq 0$, $\beta > 0$, and $\mu \in \mathbf{R}$, we denote $\mathbf{X}_{\varepsilon,\beta,\mu}$ the Banach space of sequences of functions $G = (g_s)_{s \geq 1}$, with $g_s \in X_{\varepsilon,s,\beta}$ and $\|G\|_{\varepsilon,\beta,\mu} < \infty$.*

The following inclusions hold:

(5.1.4)   if $\beta' \leq \beta$ and $\mu' \leq \mu$, then $X_{\varepsilon,s,\beta'} \subset X_{\varepsilon,s,\beta}$, $\mathbf{X}_{\varepsilon,\beta',\mu'} \subset \mathbf{X}_{\varepsilon,\beta,\mu}$.

Remark 5.1.3. — *These norms are rather classical in statistical physics (up to replacing the L^∞ norm by an L^1 norm) , where probability measures are called "ensembles".*

At the canonical level, the ensemble $\mathbb{1}_{Z_s \in \mathcal{D}_s} e^{-\beta E_0(Z_s)} dZ_s$ is a normalization of the Lebesgue measure, where $\beta \sim \theta^{-1}$ (and θ is the absolute temperature) specifies fluctuations of energy. The Boltzmann-Gibbs principle states that the average value of any quantity in the canonical ensemble is its equilibrium value at temperature θ.

The micro-canonical level consists in restrictions of the ensemble to energy surfaces.

At the grand-canonical level the number of particles may vary, with variations indexed by chemical potential $\mu \in \mathbf{R}$.

Existence and uniqueness for (5.0.1) comes from the theory of linear transport equations which provides a unique, global solution to the Liouville equation (4.2.1). Nevertheless, in order to obtain a similar result for the limiting hierarchy (5.0.2), we need to obtain uniform a priori estimates with respect to N, on the marginals $f_N^{(s)}$ for any fixed s. We shall thus deal with both systems (5.0.1) and (5.0.2) simultaneously, using analytical-type techniques which will provide short-time existence in the spaces of $\mathbf{X}_{\varepsilon,\beta,\mu}$-valued functions of time (resp. $\mathbf{X}_{0,\beta,\mu}$). Actually the parameters β and μ will themselves depend on time: in the sequel we choose for simplicity a linear dependence in time, though other, decreasing functions of time could be chosen just as well. Such a time dependence on the parameters of the function spaces is a situation which occurs whenever continuity estimates involve a loss, which is the case here since the continuity estimates on the collision operators lead to a deterioration in the parameters β and μ. We refer to Section 5.4 for some comments.

Definition 5.1.4. — *Given $T > 0$, a positive function $\boldsymbol{\beta}$ and a real valued function $\boldsymbol{\mu}$ defined on $[0,T]$ we denote $\mathbf{X}_{\varepsilon,\boldsymbol{\beta},\boldsymbol{\mu}}$ the space of functions $G : t \in [0,T] \mapsto G(t) = (g_s(t))_{1 \leq s} \in \mathbf{X}_{\varepsilon,\beta(t),\mu(t)}$, such that for all $Z_s \in \mathbf{R}^{2ds}$, the map $t \in [0,T] \mapsto g_s(t, Z_s)$ is measurable, and*

(5.1.5) $\|G\|_{\varepsilon,\boldsymbol{\beta},\boldsymbol{\mu}} := \sup_{0 \leq t \leq T} \|G(t)\|_{\varepsilon,\beta(t),\mu(t)} < \infty.$

We define similarly

$$\|G\|_{0,\boldsymbol{\beta},\boldsymbol{\mu}} := \sup_{0 \leq t \leq T} \|G(t)\|_{0,\beta(t),\mu(t)}.$$

We shall prove the following uniform bounds for the BBGKY hierarchy.

Theorem 6 (**Uniform estimates for the BBGKY hierarchy**). — *Let $\beta_0 > 0$ and $\mu_0 \in \mathbf{R}$ be given. There is a time $T > 0$ as well as two nonincreasing functions $\boldsymbol{\beta} > 0$ and $\boldsymbol{\mu}$ defined on $[0, T]$, satisfying $\boldsymbol{\beta}(0) = \beta_0$ and $\boldsymbol{\mu}(0) = \mu_0$, such that in the Boltzmann-Grad scaling $N\varepsilon^{d-1} \equiv 1$, any family of initial marginals $F_N(0) = \big(f_N^{(s)}(0)\big)_{1 \le s \le N}$ in $\mathbf{X}_{\varepsilon,\beta_0,\mu_0}$ gives rise to a unique solution $F_N(t) = (f_N^{(s)}(t))_{1 \le s \le N}$ in $\mathbf{X}_{\varepsilon,\boldsymbol{\beta},\boldsymbol{\mu}}$ to the BBGKY hierarchy (5.0.1) satisfying the following bound:*

$$\|F_N\|_{\varepsilon,\boldsymbol{\beta},\boldsymbol{\mu}} \le 2\|F_N(0)\|_{\varepsilon,\beta_0,\mu_0} \,.$$

This is a *uniform* existence result, in the sense that the existence time T does not depend on the number of particles N, which of course is crucial in the perspective of the limit $N \to \infty$. Note that actually the only assumption made is on *bounds* on the initial family of marginals.

Remark 5.1.5. — *The proof of Theorem 6 provides a lower bound of the time T on which one has a uniform bound, in terms of the initial parameters β_0, μ_0 and the dimension d: one finds*

$$(5.1.6) \qquad T \ge C_d e^{\mu_0}(1 + \beta_0^{\frac{1}{2}})^{-1} \max_{\beta \in [0,\beta_0]} \beta e^{-\beta}(\beta_0 - \beta)^{\frac{d+1}{2}} \,,$$

where C_d is a constant depending only on d.

In particular if $d \ll \beta_0$, there holds $\max_{\beta \in [0,\beta_0]} \beta e^{-\beta}(\beta_0 - \beta)^{\frac{d+1}{2}} = \beta_0^{\frac{d+1}{2}}\big(1 + o(1)\big)$, hence an existence time of the order of $e^{\mu_0}\beta_0^{d/2}$.

The proof of Theorem 6 uses neither the fact that the BBGKY hierarchy is closed by the transport equation satisfied by f_N, nor possible cancellations of the collision operators. It only relies on crude estimates and in particular the limiting hierarchy satisfies the same result, proved similarly.

Theorem 7 (**Existence for the Boltzmann hierarchy**). — *Let $\beta_0 > 0$ and $\mu_0 \in \mathbf{R}$ be given. There is a time $T > 0$ as well as two nonincreasing functions $\boldsymbol{\beta} > 0$ and $\boldsymbol{\mu}$ defined on $[0, T]$, satisfying $\boldsymbol{\beta}(0) = \beta_0$ and $\boldsymbol{\mu}(0) = \mu_0$, such that any family of initial marginals $F(0) = \big(f^{(s)}(0)\big)_{s \ge 1}$ in $\mathbf{X}_{0,\beta_0,\mu_0}$ gives rise to a unique solution $F(t) = (f^{(s)}(t))_{s \ge 1}$ in $\mathbf{X}_{0,\boldsymbol{\beta},\boldsymbol{\mu}}$ to the Boltzmann hierarchy (5.0.2), satisfying the following bound:*

$$\|F\|_{0,\boldsymbol{\beta},\boldsymbol{\mu}} \le 2\|F(0)\|_{0,\beta_0,\mu_0} \,.$$

5.2. Main steps of the proofs

The proofs of Theorems 6 and 7 are typical of analytical-type results, such as the classical Cauchy-Kowalevskaya theorem. We follow here Ukai's approach [**45**], which turns out to be remarkably short and self-contained.

Let us give the main steps of the proof: we start by noting that the conservation of energy for the s-particle flow is reflected in identities

$$(5.2.1) \qquad \begin{aligned} |\mathbf{T}_s(t)g_s|_{\varepsilon,s,\beta} &= |g_s|_{\varepsilon,s,\beta} \quad \text{and} \quad \|\mathbf{T}(t)G_N\|_{\varepsilon,\beta,\mu} = \|G_N\|_{\varepsilon,\beta,\mu} \,, \\ |\mathbf{S}_s(t)g_{0,s}|_{0,s,\beta} &= |g_s|_{0,s,\beta} \quad \text{and} \quad \|\mathbf{S}(t)G_0\|_{0,\beta,\mu} = \|G_0\|_{0,\beta,\mu} \,, \end{aligned}$$

for all parameters $\beta > 0$, $\mu \in \mathbf{R}$, and for all $g_s \in X_{\varepsilon,s,\beta}$, $g_{0,s} \in X_{0,s,\beta}$, $G_N = (g_s)_{1 \le s \le N} \in \mathbf{X}_{\varepsilon,\beta,\mu}$, $G_0 = (g_{0,s})_{s \ge 1} \in \mathbf{X}_{0,\beta,\mu}$, and all $t \ge 0$. Next assume that in the Boltzmann-Grad scaling $N\varepsilon^{d-1} \equiv 1$,

there holds the bound

$$(5.2.2) \qquad \forall\, 0 < \varepsilon \le \varepsilon_0 \,, \quad \left\| \int_0^t \mathbf{T}(t-\tau)\mathbf{C}_N G_N(\tau)\, d\tau \right\|_{\varepsilon,\boldsymbol{\beta},\boldsymbol{\mu}} \le \frac{1}{2}\, \|G_N\|_{\varepsilon,\boldsymbol{\beta},\boldsymbol{\mu}} \,,$$

for some functions $\boldsymbol{\beta}$ and $\boldsymbol{\mu}$ as in the statement of Theorem 6. Under (5.2.2), the linear operator

$$\mathfrak{L}: \quad G_N \in \mathbf{X}_{\varepsilon,\boldsymbol{\beta},\boldsymbol{\mu}} \mapsto \left(t \mapsto \int_0^t \mathbf{T}(t-\tau)\mathbf{C}_N G_N(\tau)\, d\tau \right) \in \mathbf{X}_{\varepsilon,\boldsymbol{\beta},\boldsymbol{\mu}}$$

is linear continuous from $\mathbf{X}_{\varepsilon,\boldsymbol{\beta},\boldsymbol{\mu}}$ to itself with norm strictly smaller than one. In particular, the operator $\mathrm{Id} - \mathfrak{L}$ is invertible in the Banach algebra $\mathcal{L}(\mathbf{X}_{\varepsilon,\boldsymbol{\beta},\boldsymbol{\mu}})$. Next given $F_N(0) \in \mathbf{X}_{\varepsilon,\beta_0,\mu_0}$, by conservation of energy (5.2.1), inclusions (5.1.4) and decay of $\boldsymbol{\beta}$ and $\boldsymbol{\mu}$, there holds

$$\big(t \mapsto \mathbf{T}(t)F_N(0) \big) \in \mathbf{X}_{\varepsilon,\boldsymbol{\beta},\boldsymbol{\mu}} \,.$$

Hence, there exists a unique solution $F_N \in \mathbf{X}_{\varepsilon,\boldsymbol{\beta},\boldsymbol{\mu}}$ to $(\mathrm{Id} - \mathfrak{L})F_N = \mathbf{T}(\cdot)F_N(0)$, an equation which is equivalent to (5.0.1).

The reasoning is identical for Theorem 7, replacing (5.2.2) by

$$(5.2.3) \qquad \left\| \int_0^t \mathbf{S}(t-\tau)\mathbf{C}^0 G(\tau)\, d\tau \right\|_{0,\boldsymbol{\beta},\boldsymbol{\mu}} \le \frac{1}{2}\, \|G\|_{0,\boldsymbol{\beta},\boldsymbol{\mu}} \,.$$

The next section is devoted to the proofs of (5.2.2) and (5.2.3).

5.3. Continuity estimates

In order to prove (5.2.2) and (5.2.3), we first establish bounds, in the above defined functional spaces, for the collision operators defined in (4.3.2) and (4.4.3), and for the total collision operators. In $\mathcal{C}_{s,s+1}$, the sum in i over $[1,s]$ will imply a loss in μ, while the linear velocity factor will imply a loss in β. The losses are materialized in (5.3.2) below by inequalities $\beta' < \beta$, $\mu' < \mu$.

The next statement concerns the BBGKY collision operator.

Proposition 5.3.1. — *Given $\beta > 0$ and $\mu \in \mathbf{R}$, for $1 \le s \le N-1$, the collision operator $\mathcal{C}_{s,s+1}$ satisfies the bound, for all $G_N = (g_s)_{1 \le s \le N} \in \mathbf{X}_{\varepsilon,\beta,\mu}$ in the Boltzmann-Grad scaling $N\varepsilon^{d-1} \equiv 1$,*

$$(5.3.1) \qquad \big| \mathcal{C}_{s,s+1} g_{s+1}(Z_s) \big| \le C_d\, \beta^{-\frac{d}{2}} \Big(s\beta^{-\frac{1}{2}} + \sum_{1 \le i \le s} |v_i| \Big) e^{-\beta E_0(Z_s)} |g_{s+1}|_{\varepsilon,s+1,\beta} \,,$$

for some $C_d > 0$ depending only on d.

Moreover for all $0 < \beta' < \beta$ and $\mu' < \mu$, the total collision operator $\mathbf{C}_N$ satisfies the bound

$$(5.3.2) \qquad \|\mathbf{C}_N G_N\|_{\varepsilon,\beta',\mu'} \le C_d(1 + \beta^{-\frac{1}{2}}) \Big(\frac{1}{\beta - \beta'} + \frac{1}{\mu - \mu'} \Big) \|G_N\|_{\varepsilon,\beta,\mu}$$

in the Boltzmann-Grad scaling $N\varepsilon^{d-1} \equiv 1$.

Estimate (5.3.2), a continuity estimate with loss for the total collision operator $\mathbf{C}_N$, is not directly used in the following. In the existence proof, we use instead the pointwise bound (5.3.1). Note that the more abstract (and therefore more complicated in our particular setting) approach of L. Nirenberg [35] and T. Nishida [36] would require the loss estimate (5.3.2).

Proof. — Recall that as in (4.3.2),

$$\left(\mathcal{C}_{s,s+1} f_N^{(s+1)}\right)(t, Z_s) := -(N-s) \sum_{i=1}^{s} \int_{\mathbf{S}_1^{d-1} \times \mathbf{R}^d} \omega \cdot (v_{s+1} - v_i) \, f_N^{(s+1)}(t, Z_s, x_i + \varepsilon\omega, v_{s+1}) d\omega dv_{s+1} \,.$$

Estimating each term in the sum separately, regardless of possible cancellations between "gain" and "loss" terms, it is obvious that

$$|\mathcal{C}_{s,s+1} g_{s+1}(Z_s)| \le \kappa_d \varepsilon^{d-1} (N-s) |g_{s+1}|_{\varepsilon,s+1,\beta} \sum_{1 \le i \le s} I_i(V_s) \,,$$

where κ_d is the volume of the unit ball of $\mathbf{R}^d$, and where

$$I_i(V_s) := \int_{\mathbf{R}^d} \left(|v_{s+1}| + |v_i|\right) \exp\left(-\frac{\beta}{2} \sum_{j=1}^{s+1} |v_j|^2\right) dv_{s+1} \,.$$

Since a direct calculation gives

$$I_i(V_s) \le C_d \beta^{-\frac{d}{2}} \left(\beta^{-\frac{1}{2}} + |v_i|\right) \exp\left(-\frac{\beta}{2} \sum_{1 \le j \le s} |v_j|^2\right),$$

the result (5.3.1) is deduced directly in the Boltzmann-Grad scaling $N\varepsilon^{d-1} \equiv 1$.

We turn to the proof of (5.3.2). From the pointwise inequality (due to Cauchy-Schwarz)

$$(5.3.3) \qquad \sum_{1 \le i \le s} |v_i| \exp\left(-(\gamma/2) \sum_{1 \le j \le s} |v_j|^2\right) \le s^{\frac{1}{2}} (e\gamma)^{-1/2}, \qquad \gamma > 0,$$

we deduce for the above velocity integral $I_i(V_s)$ the bound, for $0 < \beta' < \beta$,

$$\sum_{1 \le i \le s} \exp\left((\beta'/2) \sum_{1 \le j \le s} |v_j|^2\right) I_i(V_s) \le C_d \beta^{-\frac{d}{2}} \left(s\beta^{-\frac{1}{2}} + s^{\frac{1}{2}} (\beta - \beta')^{-\frac{1}{2}}\right).$$

Since

$$\sup_{1 \le s \le N} \left(\left(s\beta^{-\frac{1}{2}} + s^{1/2}(\beta - \beta')^{-\frac{1}{2}}\right) e^{-(\mu - \mu')s}\right) \le e^{-1}(\mu - \mu')^{-1}\left(\beta^{-\frac{1}{2}} + 1\right) + e^{-1}(\beta - \beta')^{-1},$$

we find (5.3.2). Proposition 5.3.1 is proved. $\square$

A similar result holds for the limiting collision operator.

Proposition 5.3.2. — *Given $\beta > 0$, $\mu \in \mathbf{R}$, the collision operator $\mathcal{C}_{s,s+1}^0$ satisfies the following bound, for all $g_{s+1} \in X_{0,s+1,\beta}$:*

$$(5.3.4) \qquad \left|\mathcal{C}_{s,s+1}^0 g_{s+1}(Z_s)\right| \le C_d \beta^{-\frac{d}{2}} \left(s\beta^{-\frac{1}{2}} + \sum_{1 \le i \le s} |v_i|\right) e^{-\beta E_0(Z_s)} |g_{s+1}|_{0,s+1,\beta} \,,$$

for some $C_d > 0$ depending only on d.

Proof. — There holds

$$\left|\mathcal{C}_{s,s+1}^0 g_{s+1}(Z_s)\right| \le \sum_{1 \le i \le s} \int_{\mathbf{S}^{d-1} \times \mathbf{R}^d} \left(|v_{s+1}| + |v_i|\right)\left(|g_{s+1}(v_i^*, v_{s+1}^*)| + |g_{s+1}(v_i, v_{s+1})|\right) d\omega dv_{s+1},$$

omitting most of the arguments of g_{s+1} in the integrand. By definition of $|\cdot|_{0,s,\beta}$ norms and conservation of energy (5.2.1), there holds

$$|g_{s+1}(v_i^*, v_{s+1}^*)| + |g_{s+1}(v_i, v_{s+1})| \le \left(e^{-\beta E_0(Z_s^*)} + e^{-\beta E_0(Z_s)}\right)|g_{s+1}|_{0,\beta}$$
$$= 2e^{-\beta E_0(Z_s)}|g_{s+1}|_{0,s+1,\beta},$$

where Z_s^* is identical to Z_s except for v_i and v_{s+1} changed to v_i^* and v_{s+1}^*. This gives

$$\left| \mathcal{C}_{s,s+1}^0 g_{s+1}(Z_s) \right| \le C_d |g_{s+1}|_{0,s+1,\beta} e^{-\beta E_0(Z_s)} \sum_{1 \le i \le s} I_i(V_s) \,,$$

borrowing notation from the proof of Proposition 5.3.1, and we conclude as above. $\qquad\square$

Propositions 5.3.1 and 5.3.2 are the key to the proof of (5.2.2) and (5.2.3). Let us first prove a continuity estimate based on Proposition 5.3.1, which implies directly (5.2.2).

Lemma 5.3.1. — *Let $\beta_0 > 0$ and $\mu_0 \in \mathbf{R}$ be given. For all $\lambda > 0$ and $t > 0$ such that $\lambda t < \beta_0$, there holds the bound*

$$(5.3.5) \qquad e^{s(\mu_0 - \lambda t)} \left| \int_0^t \mathbf{T}_s(t - \tau) \mathcal{C}_{s,s+1} g_{s+1}(\tau) \, d\tau \right|_{\varepsilon,s,\beta_0 - \lambda t} \le \bar{c}(\beta_0, \mu_0, \lambda, T) \|G_N\|_{\varepsilon,\boldsymbol{\beta},\boldsymbol{\mu}} \,,$$

for all $G_N = (g_{s+1})_{1 \le s \le N} \in \mathbf{X}_{\varepsilon,\boldsymbol{\beta},\boldsymbol{\mu}}$, with $\bar{c}(\beta_0, \mu_0, \lambda, T)$ computed explicitly in (5.3.11) below. In particular there is $T > 0$ depending only on β_0 and μ_0 such that for an appropriate choice of λ in $(0, \beta_0/T)$, there holds for all $t \in [0, T]$

$$(5.3.6) \qquad e^{s(\mu_0 - \lambda t)} \left| \int_0^t \mathbf{T}_s(t - \tau) \mathcal{C}_{s,s+1} g_{s+1}(\tau) \, d\tau \right|_{\varepsilon,s,\beta_0 - \lambda t} \le \frac{1}{2} \|G_N\|_{\varepsilon,\boldsymbol{\beta},\boldsymbol{\mu}} \,.$$

Proof. — Let us define, for all $\lambda > 0$ and $t > 0$ such that $\lambda t < \beta_0$, the functions

$$(5.3.7) \qquad\qquad \beta_0^\lambda(t) := \beta_0 - \lambda t \quad \text{and} \quad \mu_0^\lambda(t) := \mu_0 - \lambda t \,.$$

By conservation of energy (5.2.1), there holds the bound

$$\left| \int_0^t \mathbf{T}(t - \tau) \mathcal{C}_{s,s+1} g_{s+1}(\tau) \, d\tau \right|_{\varepsilon,s,\beta_0^\lambda(t)} \le \sup_{Z_s \in \mathbf{R}^{2ds}} \int_0^t e^{\beta_0^\lambda(t) E_0(Z_s)} \left| \mathcal{C}_{s,s+1} g_{s+1}(\tau, Z_s) \right| d\tau \,.$$

Estimate (5.3.1) from Proposition 5.3.1 gives

$$e^{\beta_0^\lambda(t) E_0(Z_s)} \left| \mathcal{C}_{s,s+1} g_{s+1}(\tau, Z_s) \right|$$

$$\le C_d \left(\beta_0^\lambda(\tau) \right)^{-\frac{d}{2}} |g_{s+1}(\tau)|_{\varepsilon,s+1,\beta_0^\lambda(\tau)} \left(s(\beta_0^\lambda(\tau))^{-\frac{1}{2}} + \sum_{1 \le i \le s} |v_i| \right) e^{\lambda(\tau - t) E_0(Z_s)} \,.$$

By definition of norms $\| \cdot \|_{\varepsilon,\beta,\mu}$ and $\|\!| \cdot |\!\|_{\varepsilon,\boldsymbol{\beta},\boldsymbol{\mu}}$ we have

$$(5.3.8) \qquad \begin{aligned} |g_{s+1}(\tau)|_{\varepsilon,s+1,\beta_0^\lambda(\tau)} &\le e^{-(s+1)\mu_0^\lambda(\tau)} \|G_N(\tau)\|_{\varepsilon,\beta_0^\lambda(\tau),\mu_0^\lambda(\tau)} \\ &\le e^{-(s+1)\mu_0^\lambda(\tau)} \|\!|G_N|\!\|_{\varepsilon,\boldsymbol{\beta},\boldsymbol{\mu}} \,. \end{aligned}$$

The above bounds yield, since β_0^λ and μ_0^λ are nonincreasing,

$$e^{s\mu_0^\lambda(t)} \left| \int_0^t \mathbf{T}(t - \tau) \mathcal{C}_{s,s+1} g_{s+1}(\tau) \, d\tau \right|_{\varepsilon,s,\beta_0^\lambda(t)}$$

$$\le C_d \|\!|G_N|\!\|_{\varepsilon,\boldsymbol{\beta},\boldsymbol{\mu}} e^{-\mu_0^\lambda(T)} \left(\beta_0^\lambda(T) \right)^{-\frac{d}{2}} \sup_{Z_s \in \mathbf{R}^{2ds}} \int_0^t \overline{C}(\tau, t, Z_s) \, d\tau \,,$$

where, for $\tau \le t$,

$$(5.3.9) \qquad \overline{C}(\tau, t, Z_s) := \left(s(\beta_0^\lambda(\tau))^{-\frac{1}{2}} + \sum_{1 \le i \le s} |v_i| \right) e^{\lambda(\tau - t)(s + E_0(Z_s))} \,.$$

Since

$$(5.3.10) \qquad\qquad \sup_{Z_s \in \mathbf{R}^{2ds}} \int_0^t \overline{C}(\tau, t, Z_s) \, d\tau \le \frac{C_d}{\lambda} \left(1 + \left(\beta_0^\lambda(T) \right)^{-\frac{1}{2}} \right),$$

there holds finally

$$e^{s\mu_0^\lambda(t)}\Big|\int_0^t \mathbf{T}(t-\tau)\mathcal{C}_{s,s+1}g_{s+1}(\tau)\,d\tau\Big|_{\varepsilon,s,\beta_0^\lambda(t)} \le \bar{c}(\beta_0,\mu_0,\lambda,T)\|\!|G_N\|\!|_{\varepsilon,\boldsymbol{\beta},\boldsymbol{\mu}}\,,$$

where, with a possible change of the constant C_d,

$$(5.3.11) \qquad \bar{c}(\beta_0,\mu_0,\lambda,T) := C_d\,e^{-\mu_0^\lambda(T)}\lambda^{-1}\big(\beta_0^\lambda(T)\big)^{-\frac{d}{2}}\Big(1+\big(\beta_0^\lambda(T)\big)^{-\frac{1}{2}}\Big).$$

The result (5.3.5) follows. To deduce (5.3.6) we need to find $T>0$ and $\lambda>0$ such that $\lambda T<\beta_0$ and

$$(5.3.12) \qquad C_d(1+(\beta_0-\lambda T)^{-\frac{1}{2}})e^{-\mu_0+\lambda T}(\beta_0-\lambda T)^{-\frac{d}{2}}=\frac{\lambda}{2}\,.$$

With $\beta := \lambda T \in (0,\beta_0)$, condition (5.3.12) becomes

$$T = C_d e^{\mu_0}\beta e^{-\beta}\frac{(\beta_0-\beta)^{\frac{d+1}{2}}}{1+(\beta_0-\beta)^{\frac{1}{2}}}$$

$$\ge C_d e^{\mu_0}(1+\beta_0^{\frac{1}{2}})^{-1}\beta e^{-\beta}(\beta_0-\beta)^{\frac{d+1}{2}}\,,$$

up to changing the constant C_d and (5.3.6) follows. Notice that (5.1.6) is a consequence of this computation. $\qquad\square$

The proof of the corresponding result (5.2.3) for the Boltzmann hierarchy is identical, since the estimates for $\mathcal{C}^0_{s,s+1}$ and $\mathcal{C}_{s,s+1}$ are essentially identical (compare estimate (5.3.1) from Proposition 5.3.1 with estimate (5.3.4) from Proposition 5.3.2).

5.4. Some remarks on the strategy of proof

The key in the proof of (5.2.2) is *not* to apply Minkowski's integral inequality, which would indeed lead here to

$$\Big\|\int_0^t \mathbf{T}(t-\tau)\mathbf{C}_N G_N(\tau)\,d\tau\Big\|_{\varepsilon,\beta_0^\lambda(t),\mu_0^\lambda(t)} \le \int_0^t \big\|\mathbf{C}_N G_N(\tau)\big\|_{\varepsilon,\beta_0^\lambda(t),\mu_0^\lambda(t)}\,d\tau\,,$$

by (5.2.1), and then to a divergent integral of the type

$$\Big\|\int_0^t \mathbf{T}(t-\tau)\mathbf{C}_N G_N(\tau)\,d\tau\Big\|_{\varepsilon,\beta_0^\lambda(t),\mu_0^\lambda(t)} \le C\big(\beta_0^\lambda(T),\mu_0^\lambda(T)\big)\int_0^t\Big(\frac{1}{\beta_0^\lambda(\tau)-\beta_0^\lambda(t)}+\frac{1}{\mu_0^\lambda(\tau)-\mu_0^\lambda(t)}\Big)\,d\tau\,.$$

The difference is that by Minkowski the upper bound appears as the time integral of a supremum in s, while in the proof of (5.2.2), the upper bound is a supremum in s of a time integral.

As pointed out in Section 5.1, other proofs of Theorems 6 and 7 can be devised, using tools inspired by the proof of the Cauchy-Kowalevskaya theorem: we recall for instance the approaches of [35] and [36], as well as [34] and [30].

CHAPTER 6

STATEMENT OF THE CONVERGENCE RESULT

We state here our first main result, describing convergence of mild solutions to the BBGKY hierarchy (4.3.6) to mild solutions of the Boltzmann hierarchy (4.4.4). This result implies in particular Theorem 5 stated in the Introduction page 17.

The first part of this chapter is devoted to a precise description of Boltzmann initial data which are *admissible*, i.e., which can be obtained as the limit of BBGKY initial data satisfying the required uniform bounds. This involves discussing the notion of "quasi-independence" mentioned in the Introduction, via a conditioning of the initial data. Then we state the main convergence result (Theorem 8 page 47) and sketch the main steps of its proof.

6.1. Quasi-independence

In this paragraph we discuss the notion of "quasi-independent" initial data. We first define admissible Boltzmann initial data, meaning data which can be reached from BBGKY initial data (which are bounded families of marginals) by a limiting procedure, and then show how to "condition" the initial BBGKY initial data so as to converge towards admissible Boltzmann initial data. Finally we characterize admissible Boltzmann initial data.

6.1.1. Admissible Boltzmann data. — In the following we denote

$$\Omega_s := \{Z_s \in \mathbf{R}^{2ds} , \ \forall i \neq j , \ x_i \neq x_j\} .$$

Definition 6.1.1 (**Admissible Boltzmann data**). — *Admissible Boltzmann data are defined as families $F_0 = (f_0^{(s)})_{s \geq 1}$, with each $f_0^{(s)}$ nonnegative, integrable and continuous over Ω_s, such that*

$$(6.1.1) \qquad \int_{\mathbf{R}^{2d}} f_0^{(s+1)}(Z_s, z_{s+1}) \, dz_{s+1} = f_0^{(s)}(Z_s) ,$$

and which are limits of BBGKY initial data $F_{0,N} = (f_{0,N}^{(s)})_{1 \leq s \leq N} \in \mathbf{X}_{\varepsilon,\beta_0,\mu_0}$ in the following sense: for some $F_{0,N}$ satisfying

$$(6.1.2) \qquad \sup_{N \geq 1} \|F_{0,N}\|_{\varepsilon,\beta_0,\mu_0} < \infty , \quad \text{for some } \beta_0 > 0 , \ \mu_0 \in \mathbf{R} , \ \text{as } N\varepsilon^{d-1} \equiv 1 ,$$

for each given $s \in [1, N]$, the marginal of order s defined by

$$(6.1.3) \qquad f_{0,N}^{(s)}(Z_s) = \int_{\mathbf{R}^{2d(N-s)}} \mathbb{1}_{Z_N \in \mathcal{D}_N} f_{0,N}^{(N)}(Z_N) \, dz_{s+1} \ldots dz_N \,, \quad 1 \le s < N \,,$$

converges in the Boltzmann-Grad limit:

$$(6.1.4) \qquad f_{0,N}^{(s)} \longrightarrow f_0^{(s)} \quad as \ N \to \infty \ with \ N\varepsilon^{d-1} \equiv 1 \,, \ locally \ uniformly \ in \ \Omega_s \,.$$

In this section we shall prove the following result.

Proposition 6.1.1. — *The set of admissible Boltzmann data, in the sense of Definition* 6.1.1, *is the set of families of marginals F_0 as in* (6.1.1) *satisfying a uniform bound $\|F_0\|_{0,\beta_0,\mu_0} < \infty$ for some $\beta_0 > 0$ and $\mu_0 \in \mathbf{R}$.*

6.1.2. Conditioning. — We first consider "chaotic" configurations, corresponding to tensorized initial measures, or initial densities which are products of one-particle distributions:

$$(6.1.5) \qquad f_0^{\otimes s}(Z_s) = \prod_{1 \le i \le s} f_0(z_i) \,, \qquad 1 \le s \le N \,,$$

where f_0 is nonnegative, normalized, and belongs to some X_{0,μ_0,β_0} space (see Definition 5.1.1 page 33):

$$(6.1.6) \qquad f_0 \ge 0 \,, \quad \int_{\mathbf{R}^{2d}} f_0(z)dz = 1 \,, \quad f_0 \in X_{0,\mu_0,\beta_0} \quad \text{for some } \beta_0 > 0 \,, \mu_0 \in \mathbf{R} \,.$$

Such initial data are particularly meaningful insofar as they will produce the Boltzmann equation (2.1.1), and we shall show in Proposition 6.1.2 that they are admissible.

We then consider the initial data $\mathbb{1}_{Z_N \in \mathcal{D}_N} f_0^{\otimes N}(Z_N)$, and the property of normalization is preserved by introduction of the partition function

$$(6.1.7) \qquad \mathcal{Z}_N := \int_{\mathbf{R}^{2dN}} \mathbb{1}_{Z_N \in \mathcal{D}_N} f_0^{\otimes N}(Z_N) \, dZ_N \,.$$

Conditioned datum built on f_0 is then defined as $\mathcal{Z}_N^{-1} \mathbb{1}_{Z_N \in \mathcal{D}_N} f_0^{\otimes N}(Z_N)$. This operation is called conditioning on energy surfaces, and is a classical tool in statistical mechanics (see [**20, 32, 33**] for instance).

The partition function defined in (6.1.7) satisfies the next result, which will be useful in the following.

Lemma 6.1.2. — *Given f_0 satisfying* (6.1.6), *there holds for $1 \le s \le N$ the bound*

$$1 \le \mathcal{Z}_N^{-1} \mathcal{Z}_{N-s} \le \left(1 - \varepsilon \kappa_d |f_0|_{L^\infty L^1}\right)^{-s} \,,$$

in the scaling $N\varepsilon^{d-1} \equiv 1$, where $|f_0|_{L^\infty L^1}$ denotes the $L^\infty(\mathbf{R}_x^d, L^1(\mathbf{R}_v^d))$ norm of f_0, and κ_d denotes the volume of the unit ball in $\mathbf{R}^d$.

Proof. — We have by definition

$$\mathcal{Z}_{s+1} = \int_{\mathbf{R}^{2d(s+1)}} \mathbb{1}_{Z_s \in \mathcal{D}_s} \left(\prod_{i=1}^s \mathbb{1}_{|x_i - x_{s+1}| > \varepsilon}\right) f_0^{\otimes(s+1)}(Z_{s+1}) \, dZ_{s+1} \,.$$

By Fubini, we deduce

$$\mathcal{Z}_{s+1} = \int_{\mathbf{R}^{2ds}} \left(\int_{\mathbf{R}^{2d}} \left(\prod_{1 \le i \le s} \mathbb{1}_{|x_i - x_{s+1}| > \varepsilon} \right) f_0(z_{s+1}) dz_{s+1} \right) \mathbb{1}_{Z_s \in \mathcal{D}_s} f_0^{\otimes s}(Z_s) dZ_s \,.$$

Since

$$\int_{\mathbf{R}^{2d}} \Big(\prod_{1 \leq i \leq s} \mathbb{1}_{|x_i - x_{s+1}| > \varepsilon} \Big) f_0(z_{s+1}) dz_{s+1} \geq \|f_0\|_{L^1} - \kappa_d s \varepsilon^d |f_0|_{L^\infty L^1} \,,$$

we deduce from the above, by nonnegativity of $f_0^{\otimes s}$ and the fact that $\|f_0\|_{L^1} = 1$ the lower bound

$$\mathcal{Z}_{s+1} \geq \mathcal{Z}_s \big(1 - \kappa_d s \varepsilon^d |f_0|_{L^\infty L^1} \big) \,,$$

implying by induction

$$\mathcal{Z}_N \geq \mathcal{Z}_{N-s} \prod_{j=N-s}^{N-1} (1 - j \varepsilon^d \kappa_d |f_0|_{L^\infty L^1}) \geq \mathcal{Z}_{N-s} \big(1 - \varepsilon \kappa_d |f_0|_{L^\infty L^1} \big)^s \,,$$

where we used $s \leq N$ and the scaling $N \varepsilon^{d-1} \equiv 1$. That proves the lemma. □

6.1.3. Characterization of admissible Boltzmann initial data. — The aim of this paragaph is to prove Proposition 6.1.1.

Let us start by proving the following statement, which provides examples of admissible Boltzmann initial data, in terms of tensor products.

Proposition 6.1.2. — *Given f_0 satisfying (6.1.6), the data $F_0 = (f_0^{\otimes s})_{s \geq 1}$ is admissible in the sense of Definition 6.1.1.*

Proof. — Let us define, with notation (6.1.7),

$$f_{0,N}^{(N)} := \mathcal{Z}_N^{-1} \mathbb{1}_{Z_N \in \mathcal{D}_N} f_0^{\otimes N}(Z_N)$$

and let $F_{0,N} := \big(f_{0,N}^{(s)} \big)_{s \leq N}$ be the set of its marginals. In a first step we prove they satisfy uniform bounds as in (6.1.2). In a second step, we prove the local uniform convergence to zero of $f_{0,N}^{(s)} - f_0^{\otimes s}$ in Ω_s, as in (6.1.3).

First step. We have clearly

$$f_{0,N}^{(s)}(Z_s) \leq \mathcal{Z}_N^{-1} \mathbb{1}_{Z_s \in \mathcal{D}_s} f_0^{\otimes s}(Z_s) \int_{\mathbf{R}^{2d(N-s)}} \prod_{s+1 \leq i < j \leq N} \mathbb{1}_{|x_i - x_j| > \varepsilon} \prod_{s+1 \leq i \leq N} f_0(z_i) \, dZ_{(s+1,N)} \,,$$

where we have used the notation

$$dZ_{(s+1,N)} := dz_{s+1} \dots dz_N \,.$$

This gives

$$f_{0,N}^{(s)}(Z_s) \leq \mathcal{Z}_N^{-1} \mathcal{Z}_{N-s} \mathbb{1}_{Z_s \in \mathcal{D}_s} f_0^{\otimes s}(Z_s)$$
$$\leq \big(1 - \varepsilon \kappa_d |f_0|_{L^\infty L^1} \big)^{-s} \mathbb{1}_{Z_s \in \mathcal{D}_s} f_0^{\otimes s}(Z_s) \,,$$

the second inequality by Lemma 6.1.2.

By $2x + \ln(1-x) \geq 0$ for $x \in [0, 1/2]$, there holds

$$(6.1.8) \qquad (1 - \varepsilon \kappa_d |f_0|_{L^\infty L^1})^{-s} \leq e^{2s\varepsilon \kappa_d |f_0|_{L^\infty L^1}} \,, \qquad \text{if } 2\varepsilon \kappa_d |f_0|_{L^\infty L^1} < 1 \,,$$

so that for N larger than some N_0 (equivalently, for ε small enough),

$$e^{s\mu_0} \big| f_{0,N}^{(s)} \big|_{\varepsilon,s,\beta_0} \leq e^{s(\mu_0 + 2\varepsilon \kappa_d |f_0|_{L^\infty L^1})} \big| \mathbb{1}_{Z_s \in \mathcal{D}_s} f_0^{\otimes s}(Z_s) \big|_{\varepsilon,s,\beta_0} \leq \Big(e^{2\varepsilon \kappa_d |f_0|_{L^\infty L^1}} |f_0|_{0,\beta_0,\mu_0} \Big)^s \,.$$

Given μ_0 such that $|f_0|_{0,\beta_0,\mu_0} < 1$, for N larger than some N_1, which we may assume to be larger than N_0, there holds $e^{2\varepsilon\kappa_d|f_0|_{L^\infty L^1}}|f_0|_{0,\beta_0,\mu_0} < 1$. The above then implies

$$\sup_{N\geq N_1} \|F_{0,N}\|_{\varepsilon,\beta_0,\mu_0} \leq \sup_{N\geq N_1}\sup_{1\leq s\leq N}\left(e^{2\varepsilon\kappa_d|f_0|_{L^\infty L^1}}|f_0|_{0,\beta_0,\mu_0}\right)^s < \infty,$$

which of course implies the uniform bound $\displaystyle\sup_{N\geq 1}\|F_{0,N}\|_{\varepsilon,\beta_0,\mu_0} < \infty$.

Second step. We compute for $s \leq N$:

$$f_{0,N}^{(s)} = \mathcal{Z}_N^{-1}\mathbb{1}_{Z_s\in\mathcal{D}_s}f_0^{\otimes s}\int_{\mathbf{R}^{2d(N-s)}}\prod_{s+1\leq i<j\leq N}\mathbb{1}_{|x_i-x_j|>\varepsilon}\prod_{i\leq s<j}\mathbb{1}_{|x_i-x_j|>\varepsilon}\prod_{s+1\leq i\leq N}f_0(z_i)\,dZ_{(s+1,N)}.$$

We deduce, by symmetry,

$$(6.1.9) \qquad f_{0,N}^{(s)} = \mathcal{Z}_N^{-1}\mathbb{1}_{Z_s\in\mathcal{D}_s}f_0^{\otimes s}\left(\mathcal{Z}_{N-s} - \mathcal{Z}_{(s+1,N)}^\flat\right)$$

with the notation

$$\mathcal{Z}_{(s+1,N)}^\flat = \int_{\mathbf{R}^{2d(N-s)}}\left(1 - \prod_{i\leq s<j}\mathbb{1}_{|x_i-x_j|>\varepsilon}\right)\prod_{s+1\leq i<j\leq N}\mathbb{1}_{|x_i-x_j|>\varepsilon}\prod_{s+1\leq i\leq N}f_0(z_i)\,dZ_{(s+1,N)},$$

so that $\mathcal{Z}_{(s+1,N)}^\flat$ is a function of X_s.

From there, the difference $\mathbb{1}_{Z_s\in\mathcal{D}_s}f_0^{\otimes s} - f_{0,N}^{(s)}$ decomposes as a sum:

$$(6.1.10) \qquad \mathbb{1}_{Z_s\in\mathcal{D}_s}f_0^{\otimes s} - f_{0,N}^{(s)} = \left(1 - \mathcal{Z}_N^{-1}\mathcal{Z}_{N-s}\right)\mathbb{1}_{Z_s\in\mathcal{D}_s}f_0^{\otimes s} + \mathcal{Z}_N^{-1}\mathcal{Z}_{(s+1,N)}^\flat\mathbb{1}_{Z_s\in\mathcal{D}_s}f_0^{\otimes s}.$$

By Lemma 6.1.2, there holds $1 - \mathcal{Z}_N^{-1}\mathcal{Z}_{N-s} \to 0$ as $N \to \infty$, for fixed s. Since $f_0^{\otimes s}$ is uniformly bounded in Ω_s, this implies that the first term in the right-hand side of (6.1.10) tends to 0 as $N \to \infty$, uniformly in Ω_s. Besides, by

$$0 \leq 1 - \prod_{i\leq s<j}\mathbb{1}_{|x_i-x_j|>\varepsilon} \leq \sum_{\substack{1\leq i\leq s\\ s+1\leq j\leq N}}\mathbb{1}_{|x_i-x_j|<\varepsilon},$$

we bound

$$\mathcal{Z}_{(s+1,N)}^\flat \leq \sum_{1\leq i\leq s}\int_{\mathbf{R}^{2d(N-s)}}\left(\sum_{s+1\leq j\leq N}\mathbb{1}_{|x_i-x_j|<\varepsilon}\right)\prod_{s+1\leq i<j\leq N}\mathbb{1}_{|x_i-x_j|>\varepsilon}\prod_{s+1\leq i\leq N}f_0(z_i)\,dZ_{(s+1,N)}.$$

Given $1 \leq i \leq s$, there holds by symmetry and Fubini,

$$\int_{\mathbf{R}^{2d(N-s)}}\left(\sum_{s+1\leq j\leq N}\mathbb{1}_{|x_i-x_j|<\varepsilon}\right)\prod_{s+1\leq i<j\leq N}\mathbb{1}_{|x_i-x_j|>\varepsilon}\prod_{s+1\leq i\leq N}f_0(z_i)\,dZ_{(s+1,N)}$$

$$\leq (N-s)\int_{\mathbf{R}^{2d}}\mathbb{1}_{|x_i-x_{s+1}|<\varepsilon}f_0(z_{s+1})dz_{s+1}$$

$$\times \int_{\mathbf{R}^{2d(N-s-1)}}\prod_{s+2\leq i<j\leq N}\mathbb{1}_{|x_i-x_j|>\varepsilon}\prod_{s+2\leq i\leq N}f_0(z_i)\,dZ_{(s+2,N)}$$

$$= (N-s)\int_{\mathbf{R}^{2d}}\mathbb{1}_{|x_i-x_{s+1}|<\varepsilon}f_0(z_{s+1})dz_{s+1}\times\mathcal{Z}_{N-s-1},$$

so that

$$(6.1.11) \qquad \mathcal{Z}_{(s+1,N)}^\flat \leq s(N-s)\varepsilon^d\kappa_d|f_0|_{L^\infty L^1}\mathcal{Z}_{N-s-1},$$

where $|f_0|_{L^\infty L^1}$ denotes the $L^\infty(\mathbf{R}_x^d, L^1(\mathbf{R}_v^d))$ norm of f_0. By Lemma 6.1.2, we obtain

$$\mathcal{Z}_N^{-1}\mathcal{Z}_{(s+1,N)}^\flat \leq \varepsilon s\kappa_d|f_0|_{L^\infty L^1}\left(1 - \varepsilon\kappa_d|f_0|_{L^\infty L^1}\right)^{-(s+1)},$$

and the upper bound tends to 0 as $N \to \infty$, for fixed s. This implies convergence to 0, uniformly in Ω_s, of the second term in the right-hand side of (6.1.10).

We thus proved the uniform convergence $f_{0,N}^{(s)} - \mathbb{1}_{Z_s \in \mathcal{D}_s} f_0^{\otimes s} \to 0$ in Ω_s, and hence $f_{0,N}^{\otimes s} \to f_0^{\otimes s}$ holds locally uniformly in Ω_s. We conclude that $f_{0,N}^{(s)}$ converges locally uniformly to tensor products $f_0^{\otimes s}$ in Ω_s.

The proposition is proved. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

By Proposition 6.1.2, tensor products $(f_0^{\otimes s})_{s \geq 1}$, with f_0 satisfying (6.1.6), are admissible Boltzmann data. It is easy to generalize that result (see Proposition 6.1.4 below) to the convex hull of the set of tensor products. We shall actually also show the converse: all admissible Boltzmann data belong to the convex hull of tensor products, and that will enable us to deduce Proposition 6.1.1.

We first remark that given a Boltzmann datum F_0, and an associated BBGKY datum $F_{0,N}$, there holds

$$(6.1.12) \qquad\qquad\qquad\qquad \|F_0\|_{0,\beta_0,\mu_0} < \infty \,,$$

with β_0 and μ_0 as in (6.1.2). Indeed, let $C_0 = \sup_{N \geq 1} \|F_{0,N}\|_{\varepsilon,\beta_0,\mu_0} < \infty$. Given s and $Z_s \in \Omega_s$, for ε small enough, $\mathbb{1}_{Z_s \in \mathcal{D}_s} = 1$. Besides, by (6.1.4) there holds the pointwise convergence $f_{0,N}^{(s)}(Z_s) \to f_0^{(s)}(Z_s)$. Hence taking the limit $\varepsilon \to 0$ in the left-hand side of the inequality $e^{s\mu_0 + \beta_0 E_\varepsilon(Z_s)}|f_{0,N}^{(s)}(Z_s)| \leq C_0$, we find (6.1.12).

The Hewitt-Savage theorem reveals the specific role played by tensor products: the set of families $F_0 = (f_0^{(s)})_{s \geq 1}$ of marginals (6.1.1) satisfying the uniform bound (6.1.12) is the convex hull of tensorized initial data, as described in the following statement. We define $\mathcal{P} = \mathcal{P}(\mathbf{R}^{2d})$ be the set of continuous densities of probability in $\mathbf{R}^{2d}$:

$$(6.1.13) \qquad\qquad \mathcal{P} := \left\{ h \in C^0(\mathbf{R}^{2d}; \mathbf{R}) \,, \quad h \geq 0 \,, \quad \int_{\mathbf{R}^{2d}} h(z) dz = 1 \right\}.$$

Proposition 6.1.3. — *Given $F_0 = (f_0^{(s)})_{s \geq 1}$ a family of marginals (6.1.1) satisfying the uniform bound (6.1.12) with constants $\beta_0 > 0$ and $\mu_0 \in \mathbf{R}$, there exists a probability measure π over the set $\mathcal{P}$, with*

$$(6.1.14) \qquad\qquad\qquad \operatorname{supp} \pi \subset \left\{ g \in \mathcal{P}, \, |g|_{0,\beta_0} \leq e^{-\mu_0} \right\},$$

such that the following representation holds:

$$(6.1.15) \qquad\qquad\qquad f_0^{(s)} = \int_{\mathcal{P}} g^{\otimes s} d\pi(g) \,, \qquad s \geq 1 \,.$$

Proof. — Given a family F_0 satisfying (6.1.1) and (6.1.12), the existence of π satisfying (6.1.15) is granted by the Hewitt-Savage theorem [**26**]. The goal is then to prove the inclusion (6.1.14). Assume by contradiction that, for some $\alpha > 0$,

$$(6.1.16) \qquad\qquad \pi(A_\alpha) = \kappa_\alpha > 0 \,, \quad \text{where } A_\alpha := \left\{ g \in \mathcal{P}(\mathbf{R}^{2d}), \quad |g|_{0,1,\beta_0} \geq e^{-\mu_0} + \alpha \right\}.$$

We then have by (6.1.15)

$$f_0^{(s)} \geq \int_{A_\alpha} g^{\otimes s} d\pi(g),$$

hence by $f_0^{(s)} \le e^{-s\mu_0} \|F_0\|_{0,\beta_0,\mu_0}$, we infer that $\|F_0\|_{0,\beta_0,\mu_0} \ge \kappa_\alpha(1 + \alpha e^{\mu_0})^s$, which cannot hold for some $\alpha > 0$ and all s, since $1 + \alpha e^{\mu_0} > 1$. Hence (6.1.16) does not hold, which proves the result. $\square$

We now give the generalization of Proposition 6.1.2 that will be useful in the proof of Proposition 6.1.1. Let π be a probability measure on $\mathcal{P}$ satisfying (6.1.14) for some $\beta_0 > 0$ and some $\mu_0 \in \mathbf{R}$. Next we define

$$(6.1.17) \qquad\qquad \pi^{(s)} := \int_{\mathcal{P}} h^{\otimes s} d\pi(h) \,.$$

In the case that $\pi = \delta_{f_0}$, then (6.1.17) reduces to the tensor product (6.1.5)-(6.1.6). We let

$$(6.1.18) \qquad\qquad \mathcal{Z}_N := \int_{\mathbf{R}^{2dN}} \mathbb{1}_{Z_N \in \mathcal{D}_N} h^{\otimes N}(Z_N) \, dZ_N \,, \quad h \in \mathcal{P} \,,$$

generalizing (6.1.7).

The following result is an obvious generalization of Lemma 6.1.2.

Lemma 6.1.3. — *Given π satisfying (6.1.14) and $h \in \operatorname{supp} \pi$, the family of partition functions $\mathcal{Z}_s$ defined in (6.1.18) satisfies for $1 \le s \le N$ the bound*

$$1 \le \mathcal{Z}_N^{-1} \mathcal{Z}_{N-s} \le \left(1 - \varepsilon C_d e^{-\mu_0} \beta_0^{-1/2}\right)^{-s} ,$$

where C_d depends only on d.

The next statement generalizes Proposition 6.1.2. Its proof is an immediate extension of the proof of Proposition 6.1.2 thanks to the dominated convergence theorem, using the obvious bound $\mathbb{1}_{Z_s \in \mathcal{D}_s} h^{\otimes s}(Z_s) \le e^{-s\mu_0}$.

Proposition 6.1.4. — *Given π satisfying (6.1.14), the data $(\pi^{(s)})_{s \ge 1}$, with $\pi^{(s)}$ defined in (6.1.17), is admissible in the sense of Definition 6.1.1.*

Proof of Proposition 6.1.1. — We already observed in (6.1.12) that admissible Boltzmann data are bounded families of marginals. Conversely, given a bounded family of marginals F_0, by Proposition 6.1.3 representation (6.1.15) holds. Then, by Proposition 6.1.4, F_0 is an admissible Boltzmann datum. This proves Proposition 6.1.1. $\square$

Combining Propositions 6.1.1 and 6.1.3, we see that all admissible Boltzmann data are built on tensor products, in the sense that given an admissible Boltzmann datum, representation (6.1.15) holds for some π satisfying (6.1.14).

6.2. Main result: Convergence of the BBGKY hierarchy to the Boltzmann hierarchy

6.2.1. Statement of the result. —

Our main result is a *weak convergence* result, in the sense of convergence of observables, or averages with respect to the momentum variables. Moreover, since the marginals are defined in $\mathcal{D}_s$, we must also eliminate, in the convergence, the diagonals in physical space. Let us give a precise definition of the convergence we shall be considering.

Definition 6.2.1 (**Convergence**). — *Given a sequence* $(h_N^s)_{1 \leq s \leq N}$ *of functions* $h_N^s \in C^0(\mathcal{D}_s; \mathbf{R})$, *a sequence* $(h^s)_{s \geq 1}$ *of functions* $h^s \in C^0(\Omega_s; \mathbf{R})$, *we say that* (h_N^s) *converges on average and locally uniformly outside the diagonals to* (h^s), *and we denote*

$$(h_N^s)_{1 \leq s \leq N} \xrightarrow{\sim} (h^s)_{1 \leq s},$$

when for any fixed s, any test function $\varphi_s \in \mathcal{C}_c^\infty(\mathbf{R}^{ds}; \mathbf{R})$, *there holds*

$$I_{\varphi_s}(h_N^s - h^s)(X_s) := \int_{\mathbf{R}^{ds}} \varphi_s(V_s)(h_N^s - h^s)(Z_s) dV_s \longrightarrow 0, \quad as \ N \to \infty,$$

locally uniformly in $\{X_s \in \mathbf{R}^{ds}, \ x_i \neq x_j \text{ for } i \neq j\}$.

With regard to spatial variables, this notion of convergence is similar to the convergence in the sense of Chacon.

We remark that local uniform convergence in Ω_s implies convergence in the sense of Definition 6.2.1:

Lemma 6.2.2. — *Given* $(f_N^{(s)})_{1 \leq s \leq N}$ *a bounded sequence in* $\mathbf{X}_{\varepsilon,\boldsymbol{\beta},\boldsymbol{\mu}}$ *with the notation of Definition 10.2.3, if* $f_N^{(s)} \to f^{(s)}$ *for fixed s, uniformly in* $t \in [0,T]$ *and locally uniformly in* Ω_s, *then there holds* $f_N^{(s)} \xrightarrow{\sim} f^{(s)}$, *uniformly in* $t \in [0,T]$.

Proof. — Let K_s be compact in $\{X_s \in \mathbf{R}^{ds}, \ x_i \neq x_j \text{ for } i \neq j\}$. There holds

$$\left| I_{\varphi_s}(f_N^{(s)} - f^{(s)})(X_s) \right| \leq \|\varphi_s\|_{L^1(\mathbf{R}^d)} \sup_{V_s \in \text{supp } \varphi_s} \left| (f_N^{(s)} - f^{(s)})(X_s, V_s) \right|.$$

The set $K_s \times \text{supp}\, \varphi_s$ is compact in Ω_s. Hence the above upper bound converges to 0 as $N \to \infty$, uniformly in $[0,T] \times K_s$. $\qquad \square$

We can now state our main result.

Theorem 8 (**Convergence**). — *Let* $\beta_0 > 0$ *and* $\mu_0 \in \mathbf{R}$ *be given. There is a time* $T > 0$ *such that the following holds. Let* F_0 *in* $\mathbf{X}_{0,\beta_0,\mu_0}$ *be an admissible Boltzmann datum, with associated family of BBGKY datum* $(F_{0,N})_{N \geq 1}$, *in* $\mathbf{X}_{\varepsilon,\beta_0,\mu_0}$. *Let* F *and* F_N *be the solutions to the Boltzmann and BBGKY hierarchy produced by* F_0 *and* $F_{0,N}$ *respectively. There holds*

(6.2.1)
$$F_N \xrightarrow{\sim} F,$$

uniformly on $[0,T]$,.

In particular, if $F_0 = (f_0^{\otimes s})_{s \geq 1}$, *then the first marginal* $f_N^{(1)}$ *converges to the solution* f *of the Boltzmann equation* (2.1.1) *with initial data* f_0.

Finally in the case when $F_0 = (f_0^{\otimes s})_{s \geq 1}$ *with* f_0 *Lipschitz, then the convergence* (6.2.1) *holds at a rate* $O(\varepsilon^\alpha)$ *for any* $\alpha < (d-1)/(d+1)$.

Solutions to the Boltzmann hierarchy issued from tensorized initial data are themselves tensorized. For such data, the Boltzmann hierarchy then reduces to the nonlinear Boltzmann equation (2.1.1), and Theorem 8 describes an asymptotic form of propagation of chaos, in the sense that an initial property of independence is propagated in time, in the limit. This corresponds to Theorem 5 stated in the Introduction page 17.

6.2.2. About the proof of Theorem 8: outline of Chapter 7 and Part IV. —

The formal derivation presented in Chapter 4, Section 4.4, fails because of a number of incorrect arguments:

– Since mild solutions to the BBGKY hierarchy are defined by the Duhamel formula (4.3.6) where the solution itself occurs in the source term, we need some precise information on the convergence to take limits directly in (4.3.6).

– The irreversibility inherent to the Boltzmann hierarchy appears in the limiting process as an arbitrary choice of the time direction (encoded in the distinction between pre-collisional and post-collisional particles), and more precisely as an arbitrary choice of the initial time, which is the only time for which one has a complete information on the family of marginals $F_{0,N}$. This specificity of the initial time does not appear clearly in (4.3.6).

– The heuristic argument which allows to neglect pathological trajectories, meaning trajectories for which the reduced dynamics with s-particles does not coincide with the free transport ($\mathbf{T}_s \neq \mathbf{S}_s$), requires to be quantified. Indeed the question of the stability with respect to micro-translations in space must be analyzed; moreover we have more or less to repeat the operation infinitely many times, since mild solutions are defined by a loop process.

– Because of the conditioning, the initial data are not so smooth. The operations such as infinitesimal translations on the arguments require therefore a careful treatment.

To overcome the two first difficulties, the idea is to start from the iterated Duhamel formula, which allows to express any marginal $f_N^{(s)}(t, Z_s)$ in terms of the initial data $F_{0,N}$. By successive integrations in time, we have indeed the following representation of $f_N^{(s)}$:

$$(6.2.1) \qquad f_N^{(s)}(t) = \sum_{n=0}^{\infty} \int_0^t \int_0^{t_1} \dots \int_0^{t_{n-1}} \mathbf{T}_s(t-t_1)\mathcal{C}_{s,s+1}\mathbf{T}_{s+1}(t_1-t_2)\mathcal{C}_{s+1,s+2}\dots$$
$$\dots \mathbf{T}_{s+n}(t_n)f_N^{(s+n)}(0)\,dt_n\dots dt_1$$

where by convention $f_N^{(j)}(0) \equiv 0$ for $j > N$.

Using a **dominated convergence argument**, we shall first reduce (in Chapter 7) to the study of a functional

– defined as a finite sum of terms (independent of N),

– where the energies of the particles are assumed to be bounded (namely $E_0(Z_N) \leq R^2$),

– and where the collision times are supposed to be well separated (namely $|t_j - t_{j+1}| \geq \delta$).

The reason for the two last assumptions is essentially technical, and will appear more clearly in the next step.

The heart of the proof, in Part IV, is then to prove the term by term convergence, dealing with pathological trajectories. Let us recall that each collision term is defined as an integral with respect to positions and velocities. The main idea consists then in proving that we cannot build pathological trajectories if we exclude at each step a small domain of integration. The explicit construction of this "bad set" lies on

– a very simple geometrical lemma which ensures that two particles of size ε will not collide in the future provided that their relative velocity does not belong to a small subset of $\mathbf{R}^d$ (see Lemma 12.2.1),

– scattering estimates which tell us how these properties of the transport are modified when a particle is deviated by a collision (see Lemma 12.2.3).

This construction, which is the technical part of the proof, will be detailed in Chapter 12. The conclusion of the convergence proof is presented in Chapter 14.

CHAPTER 7

STRATEGY OF THE CONVERGENCE PROOF

The goal of this chapter is to use dominated convergence arguments to reduce the proof of Theorem 8 stated page 47 to the term-by-term convergence of some functionals involving a finite (uniformly bounded) number of marginals (Section 7.1). In order to further simplify the convergence analysis, we shall modify these functionals by eliminating some small domains of integration in the time and velocity variables corresponding to pathological dynamics, namely by removing large energies in Section 7.2 and clusters of collision times in Section 7.3.

We consider therefore families of initial data: Boltzmann initial data $F_0 = (f_0^{(s)})_{s \in \mathbf{N}}$ such that

$$\|F_0\|_{0,\beta_0,\mu_0} = \sup_{s \in \mathbf{N}} \sup_{Z_s} \left(\exp(\beta_0 E_0(Z_s) + \mu_0 s) f_0^{(s)}(Z_s) \right) < +\infty$$

and for each N, BBGKY initial data $F_{N,0} = (f_{N,0}^{(s)})_{1 \le s \le N}$ such that

$$\sup_N \|F_{N,0}\|_{\varepsilon,\beta_0,\mu_0} = \sup_N \sup_{s \le N} \sup_{Z_s \in \mathcal{D}_s} \left(\exp(\beta_0 E_0(Z_s) + \mu_0 s) f_{N,0}^{(s)}(Z_s) \right) < +\infty \,.$$

We then associate the respective unique mild solutions (constructed in Theorems 6 and 7 in Chapter 5) of the hierarchies

$$f^{(s)}(t) = \mathbf{S}_s(t) f_0^{(s)} + \int_0^t \mathbf{S}_s(t-\tau) \mathcal{C}_{s,s+1}^0 f^{(s+1)}(\tau) \, d\tau$$

and

$$f_N^{(s)}(t) = \mathbf{T}_s(t) f_{N,0}^{(s)} + \int_0^t \mathbf{T}_s(t-\tau) \mathcal{C}_{s,s+1} f_N^{(s+1)}(\tau) \, d\tau \,.$$

In terms of the initial datum, they can be rewritten

$$f^{(s)}(t, Z_s) = \sum_{n=0}^{\infty} \int_0^t \int_0^{t_1} \cdots \int_0^{t_{n-1}} \mathbf{S}_s(t-t_1) \mathcal{C}_{s,s+1}^0 \mathbf{S}_{s+1}(t_1-t_2) \mathcal{C}_{s+1,s+2}^0 \cdots$$
$$\cdots \mathbf{S}_{s+n}(t_n) f_0^{(s+n)} \, dt_n \ldots dt_1$$

and

$$f_N^{(s)}(t, Z_s) = \sum_{n=0}^{\infty} \int_0^t \int_0^{t_1} \cdots \int_0^{t_{n-1}} \mathbf{T}_s(t-t_1) \mathcal{C}_{s,s+1} \mathbf{T}_{s+1}(t_1-t_2) \mathcal{C}_{s+1,s+2} \cdots$$
$$\cdots \mathbf{T}_{s+n}(t_n) f_{N,0}^{(s+n)} \, dt_n \ldots dt_1 \,.$$

The observables we are interested in (recall the definition of convergence provided in Definition 6.2.1) are the following:

$$I_s(t)(X_s) := \int \varphi_s(V_s) f_N^{(s)}(t, Z_s) dV_s \quad \text{and} \quad I_s^0(t)(X_s) := \int \varphi_s(V_s) f^{(s)}(t, Z_s) dV_s,$$

and they therefore involve infinite sums, as there may be infinitely many particles involved (the sum over n is unbounded).

7.1. Reduction to a finite number of collision times

Due to the uniform bounds derived in Chapter 5, the dominated convergence theorem implies that it is enough to consider finite sums of elementary functions

$$f_N^{(s,k)}(t) := \int_0^t \int_0^{t_1} \ldots \int_0^{t_{k-1}} \mathbf{T}_s(t - t_1) \mathcal{C}_{s,s+1} \mathbf{T}_{s+1}(t_1 - t_2) \mathcal{C}_{s+1,s+2} \ldots$$

$$\ldots \mathbf{T}_{s+k}(t_k) f_{N,0}^{(s+k)} \, dt_k \ldots dt_1$$

(7.1.1)

$$f^{(s,k)}(t) := \int_0^t \int_0^{t_1} \ldots \int_0^{t_{n-1}} \mathbf{S}_s(t - t_1) \mathcal{C}_{s,s+1} \mathbf{S}_{s+1}(t_1 - t_2) \mathcal{C}_{s+1,s+2} \ldots$$

$$\ldots \mathbf{S}_{s+k}(t_k) f_0^{(s+k)} \, dt_k \ldots dt_1,$$

and the associate elementary observables :

$$(7.1.2) \quad I_{s,k}(t)(X_s) := \int \varphi_s(V_s) f_N^{(s,k)}(t, Z_s) dV_s, \quad \text{and} \quad I_{s,k}^0(t)(X_s) := \int \varphi_s(V_s) f^{(s,k)}(t, Z_s) dV_s.$$

and therefore to study the term-by-term convergence (for any fixed k), as expressed by the following statement.

Proposition 7.1.1. — *With the notation of Theorems 6 and 7 page 35, for each given $s \in \mathbf{N}^*$ and $t \in [0, T]$ there is a constant $C > 0$ such that for each $n \in \mathbf{N}^*$,*

$$\left\| I_s(t) - \sum_{k=0}^n I_{s,k}(t) \right\|_{L^\infty(\mathbf{R}^{ds})} \leq C \|\varphi_s\|_{L^\infty(\mathbf{R}^{ds})} \left(\frac{1}{2} \right)^n \|F_{N,0}\|_{\varepsilon,\beta_0,\mu_0}$$

and

$$\left\| I_s^0(t) - \sum_{k=0}^n I_{s,k}^0(t) \right\|_{L^\infty(\mathbf{R}^{ds})} \leq C \|\varphi_s\|_{L^\infty(\mathbf{R}^{ds})} \left(\frac{1}{2} \right)^n \|F_0\|_{0,\beta_0,\mu_0},$$

uniformly in N and $t \leq T$, in the Boltzmann-Grad scaling $N\varepsilon^{d-1} \equiv 1$.

Proof. — We use the notation of Chapter 5. Using the continuity estimate (5.2.2) we have

$$(7.1.3) \qquad \sup_{t \in [0,T]} \left\| \int_0^t \mathbf{T}(t - t') \mathbf{C}_N F_N(t') \, dt' \right\|_{\varepsilon,\beta(t),\mu(t)} \leq \frac{1}{2} \|F_N\|_{\varepsilon,\boldsymbol{\beta},\boldsymbol{\mu}}.$$

Recalling the definition of the Hamiltonian

$$E_0(Z_s) := \sum_{1 \leq i \leq s} \frac{|v_i|^2}{2}$$

we then deduce that

$$(7.1.4) \quad e^{\beta(t)E_0(Z_s)+s\mu(t)}\Big\| \sum_{k=n+1}^{\infty} \int_0^t \int_0^{t_1} \dots \int_0^{t_{k-1}} \mathbf{T}_s(t-t_1)\mathcal{C}_{s,s+1}\mathbf{T}_{s+1}(t_1-t_2)\mathcal{C}_{s+1,s+2}\dots$$

$$\dots \mathbf{T}_{s+k}(t_k)f_{N,0}^{(s+k)}\,dt_k\dots dt_1\Big\|_{L^\infty} \leq C\left(\frac{1}{2}\right)^n \|F_N\|_{\varepsilon,\boldsymbol{\beta},\boldsymbol{\mu}}\,.$$

Combining this estimate together with the uniform bound on $\|F_N\|_{\varepsilon,\boldsymbol{\beta},\boldsymbol{\mu}}$ given in Theorem 6 leads to the first statement in Proposition 7.1.1. The second statement is established exactly in an analogous way, using estimate (5.2.3) together with the uniform bound obtained in Theorem 7. $\qquad\square$

7.2. Energy truncation

We introduce a parameter $R > 0$ and define

$$f_{N,R}^{(s,k)}(t) := \sum_{k=0}^n \int_0^t \int_0^{t_1} \dots \int_0^{t_{k-1}} \mathbf{T}_s(t-t_1)\mathcal{C}_{s,s+1}\mathbf{T}_{s+1}(t_1-t_2)\mathcal{C}_{s+1,s+2}\dots$$

$$\dots \mathbf{T}_{s+k}(t_k)\mathbb{1}_{E_0(Z_{s+k})\leq R^2}f_{N,0}^{(s+k)}\,dt_k\dots dt_1\,,$$

$$(7.2.1)$$

$$f_R^{(s,k)}(t) := \sum_{k=0}^n \int_0^t \int_0^{t_1} \dots \int_0^{t_{k-1}} \mathbf{S}_s(t-t_1)\mathcal{C}_{s,s+1}^0\mathbf{S}_{s+1}(t_1-t_2)\mathcal{C}_{s+1,s+2}^0\dots$$

$$\dots \mathbf{S}_{s+k}(t_k)\mathbb{1}_{E_0(Z_{s+k})\leq R^2}f_0^{(s+k)}\,dt_k\dots dt_1$$

and the corresponding observables

$$(7.2.2) \qquad I_{s,k}^R(t)(X_s) := \int \varphi_s(V_s)f_{N,R}^{(s,k)}(t,Z_s)dV_s \quad \text{and} \quad I_{s,k}^{0,R}(t)(X_s) := \int \varphi_s(V_s)f_R^{(s,k)}dV_s\,.$$

Using the bounds derived in Chapter 5 we find easily that $\sum_k (I_{s,k} - I_{s,k}^R)(t)$ and $\sum_k (I_{s,k}^0 - I_{s,k}^{0,R})(t)$ can be made arbitrarily small when R is large. More precisely the following result holds.

Proposition 7.2.1. — *Let $s \in \mathbf{N}^*$ and $t \in [0,T]$ be given. There are two nonnegative constants C, C' such that for each n,*

$$\Big\| \sum_{k=0}^n (I_{s,k} - I_{s,k}^R)(t)\Big\|_{L^\infty(\mathbf{R}^{ds})} \leq C\|\varphi_s\|_{L^\infty(\mathbf{R}^{ds})}e^{-C'R^2}\|F_{N,0}\|_{\varepsilon,\beta_0,\mu_0}\,,$$

and

$$\Big\| \sum_{k=0}^n (I_{s,k}^0 - I_{s,k}^{0,R})(t)\Big\|_{L^\infty(\mathbf{R}^{ds})} \leq C\|\varphi_s\|_{L^\infty(\mathbf{R}^{ds})}e^{-C'R^2}\|F_0\|_{0,\beta_0,\mu_0}\,.$$

Proof. — Let $0 < \beta_0' < \beta_0$ be given, and define the associate functions $\boldsymbol{\beta}'$ and $\boldsymbol{\beta}$ as in Theorem 6 stated in Chapter 5. Assume that $\beta_0 - \beta_0'$ is small enough so that on $[0,T]$, one has $\boldsymbol{\beta}' > 0$. Then according to the results of Chapter 5 and similarly to (7.1.4) we know that

$$\int_0^t \int_0^{t_1} \dots \int_0^{t_{k-1}} \mathbf{T}_s(t-t_1)\mathcal{C}_{s,s+1}\mathbf{T}_{s+1}(t_1-t_2)\dots\mathbf{T}_{s+k}(t_k)\mathbb{1}_{E_0(Z_{s+k})\geq R^2}f_{N,0}^{(s+k)}\,dt_k\dots dt_1$$

$$\leq C\,2^{-k}e^{-\boldsymbol{\beta}'(T)E_0(Z_s)-s\mu_0(T)}\|G_{N,0,s}\|_{\varepsilon,\beta_0',\mu_0}\,,$$

where we have defined

$$G_{N,0,s} := (g_{N,0}^{s+k})_{0 \leq k \leq N-s}, \quad \text{with} \quad g_{N,0}^{s+k}(Z_{s+k}) := \mathbb{1}_{E_0(Z_{s+k}) \geq R^2} f_{N,0}^{(s+k)}(Z_{s+k}).$$

The result then follows from the fact that

$$\|G_{N,0,s}\|_{\varepsilon,\beta_0',\mu_0} \leq C e^{(\beta_0'-\beta_0)R^2} \|F_{N,0}\|_{\varepsilon,\beta_0,\mu_0}.$$

The argument is identical for $I_{s,n}^0(t) - I_{s,n}^{0,R}(t)$. $\qquad \square$

Remark 7.2.1. — *It is useful to notice that the collision operators preserve the bound on high energies, in the sense that*

$$\mathcal{C}_{s,s+1} \mathbb{1}_{E_0(Z_{s+1}) \leq R^2} \equiv \mathbb{1}_{E_0(Z_s) \leq R^2} \mathcal{C}_{s,s+1} \mathbb{1}_{E_0(Z_{s+1}) \leq R^2}$$

$$\mathcal{C}_{s,s+1}^0 \mathbb{1}_{E(Z_{s+1}) \leq R^2} \equiv \mathbb{1}_{E(Z_s) \leq R^2} \mathcal{C}_{s,s+1}^0 \mathbb{1}_{E(Z_{s+1}) \leq R^2}.$$

7.3. Time separation

We choose another small parameter $\delta > 0$ and further restrict the study to the case when $t_i - t_{i+1} \geq \delta$. That is, we define

$$\mathcal{T}_k(t) := \Big\{ T_k = (t_1, \ldots, t_k) \, / \, t_i < t_{i-1} \ \text{ with } \ t_{k+1} = 0 \ \text{ and } \ t_0 = t \Big\},$$

$$\mathcal{T}_{k,\delta}(t) := \Big\{ T_k \in \mathcal{T}_k(t) \, / \, t_i - t_{i+1} \geq \delta \Big\},$$

and

$$I_{s,k}^{R,\delta}(t)(X_s) := \int \varphi_s(V_s) \int_{\mathcal{T}_{k,\delta}(t)} \mathbf{T}_s(t-t_1) \mathcal{C}_{s,s+1} \mathbf{T}_{s+1}(t_1-t_2) \mathcal{C}_{s+1,s+2}$$

$$\ldots \mathcal{C}_{s+k-1,s+k} \mathbf{T}_{s+k}(t_k - t_{k+1}) \mathbb{1}_{E_0(Z_{s+k}) \leq R^2} f_{N,0}^{(s+k)} dT_k dV_s,$$

(7.3.1)

$$I_{s,k}^{0,R,\delta}(t)(X_s) := \int \varphi_s(V_s) \int_{\mathcal{T}_{k,\delta}(t)} \mathbf{S}_s(t-t_1) \mathcal{C}_{s,s+1}^0 \mathbf{S}_{s+1}(t_1-t_2) \mathcal{C}_{s+1,s+2}^0$$

$$\ldots \mathcal{C}_{s+k-1,s+k}^0 \mathbf{S}_{s+k}(t_k - t_{k+1}) \mathbb{1}_{E_0(Z_{s+k}) \leq R^2} f_0^{(s+k)} dT_k dV_s.$$

Again applying the continuity bounds for the transport and collision operators, the error on the functions $\sum_k (I_{s,k}^R - I_{s,k}^{R,\delta})(t)$ and $\sum_k (I_{s,k}^{0,R} - I_{s,k}^{0,R,\delta})(t)$ can be estimated as follows.

Proposition 7.3.1. — *Let $s \in \mathbf{N}^*$ and $t \in [0,T]$ be given. There is a constant C such that for each n and R,*

$$\Big\| \sum_{k=0}^n (I_{s,k}^R - I_{s,k}^{R,\delta})(t) \Big\|_{L^\infty(\mathbf{R}^{ds})} \leq C\delta \|\varphi\|_{L^\infty(\mathbf{R}^{ds})} \|F_{N,0}\|_{\varepsilon,\beta_0,\mu_0}$$

and

$$\Big\| \sum_{k=0}^n (I_{s,k}^{0,R} - I_{s,k}^{0,R,\delta})(t) \Big\|_{L^\infty(\mathbf{R}^{ds})} \leq C\delta \|\varphi\|_{L^\infty(\mathbf{R}^{ds})} \|F_0\|_{0,\beta_0,\mu_0}.$$

7.4. Reformulation in terms of pseudo-trajectories

Putting together Propositions 7.1.1, 7.2.1 and 7.3.1 we obtain the following result.

Corollary 7.4.1. — *With the notation of Theorem 9, given $s \in \mathbf{N}^*$ and $t \in [0, T]$, there are two positive constants C and C' such that for each $n \in \mathbf{N}^*$,*

$$\Big\| I_s(t) - \sum_{k=0}^{n} I_{s,k}^{R,\delta}(t) \Big\|_{L^\infty(\mathbf{R}^{ds})} \le C(2^{-n} + e^{-C'R^2} + \delta) \|\varphi\|_{L^\infty(\mathbf{R}^{ds})} \|F_{N,0}\|_{\varepsilon,\beta_0,\mu_0} .$$

In the same way as in (4.3.4) we now decompose the Boltzmann collision operators (4.4.3) into

$$\mathcal{C}_{s,s+1}^0 = \mathcal{C}_{s,s+1}^{0,+} - \mathcal{C}_{s,s+1}^{0,-} ,$$

where the index $+$ corresponding to post-collisional configurations and the index $-$ to pre-collisional configurations. By definition of the collision cross-section, we have

$$\big(\mathcal{C}_{s,s+1}^{0,-,m} f^{(s+1)}\big)(Z_s) := \int_{\mathbf{S}_1^{d-1} \times \mathbf{R}^d} b(v_{s+1} - v_m, \omega) f^{(s+1)}(Z_s, x_m, v_{s+1}) \, d\omega dv_{s+1}$$

$$= \int_{\mathbf{S}_1^{d-1} \times \mathbf{R}^d} ((v_{s+1} - v_m) \cdot \omega)_- f^{(s+1)}(Z_s, x_m, v_{s+1}) \, d\omega dv_{s+1} \quad \text{and}$$

$$\big(\mathcal{C}_{s,s+1}^{0,+,m} f^{(s+1)}\big)(Z_s) := \int_{\mathbf{S}_1^{d-1} \times \mathbf{R}^d} b(v_{s+1} - v_m, \omega) f^{(s+1)}(z_1, \ldots, x_m, v_m^*, \ldots, z_s, x_m, v_{s+1}^*) \, d\omega dv_{s+1}$$

$$= \int_{\mathbf{S}_1^{d-1} \times \mathbf{R}^d} ((v_{s+1} - v_m) \cdot \omega)_+ f^{(s+1)}(z_1, \ldots, x_m, v_m^*, \ldots, z_s, x_m, v_{s+1}^*) \, d\omega dv_{s+1} .$$

Performing the change of variables $\omega \mapsto -\omega$ in the integral defining $\mathcal{C}_{s,s+1}^{0,-,m}$, we get similar formulas as for the BBGKY collision operators.

The elementary BBGKY and Boltzmann observables we are interested in can therefore be decomposed as

$$\text{(7.4.1)} \qquad \begin{aligned} I_{s,k}^{R,\delta}(t)(X_s) &= \sum_{J,M} \Big(\prod_{i=1}^{k} j_i\Big) I_{s,k}^{R,\delta}(t, J, M)(X_s) \quad \text{and} \\ I_{s,k}^{0,R,\delta}(t)(X_s) &= \sum_{J,M} I_{s,k}^{0,R,\delta}(t, J, M)(X_s) \end{aligned}$$

where the *elementary functionals* $I_{s,k}^{R,\delta}(t, J, M)$ are defined by

$$\text{(7.4.2)} \qquad \begin{aligned} I_{s,k}^{R,\delta}(t, J, M)(X_s) &:= \int \varphi_s(V_s) \int_{\mathcal{T}_{k,\delta}(t)} \mathbf{T}_s(t - t_1) \mathcal{C}_{s,s+1}^{j_1,m_1} \mathbf{T}_{s+1}(t_1 - t_2) \mathcal{C}_{s+1,s+2}^{j_2,m_2} \\ &\qquad \qquad \ldots \mathbf{T}_{s+k}(t_k - t_{k+1}) \mathbb{1}_{E_0(Z_{s+k}) \le R^2} f_{N,0}^{(s+k)} dT_k dV_s , \\ I_{s,k}^{0,R,\delta}(t, J, M)(X_s) &:= \int \varphi_s(V_s) \int_{\mathcal{T}_{k,\delta}(t)} \mathbf{S}_s(t - t_1) \mathcal{C}_{s,s+1}^{0,j_1,m_1} \mathbf{S}_{s+1}(t_1 - t_2) \mathcal{C}_{s+1,s+2}^{0,j_2,m_2} \\ &\qquad \qquad \ldots \mathbf{S}_{s+k}(t_k - t_{k+1}) \mathbb{1}_{E_0(Z_{s+k}) \le R^2} f_0^{(s+k)} dT_k dV_s , \end{aligned}$$

with

$$J := (j_1, \ldots, j_k) \in \{+, -\}^k \quad \text{and} \quad M := (m_1, \ldots, m_k) \quad \text{with} \quad m_i \in \{1, \ldots, s + i - 1\} .$$

Each one of the previous functionals $I_{s,k}^{R,\delta}(t,J,M)$ and $I_{s,k}^{0,R,\delta}(t,J,M)$ defined in (7.4.2) can be viewed as the observable associated with some dynamics, which of course is not the actual dynamics in physical space since
– the total number of particles is not conserved;
– the distribution does even not remain nonnegative because of the sign of loss collision operators.
This explains the terminology of "pseudo-trajectories" we choose to describe the process.

In this formulation, the characteristics associated with the operators $\mathbf{T}_{s+i}(t_i - t_{i+1})$ and $\mathbf{S}_{s+i}(t_i - t_{i+1})$ are followed *backwards* in time between two consecutive times t_{i+1} and t_i, and collision terms (associated with $\mathcal{C}_{s+i,s+i+1}$ and $\mathcal{C}_{s+i,s+i+1}^0$) are seen as *source terms*, in which, in the words of Lanford [**34**], "additional particles" are "adjoined" to the marginal.

The main heuristic idea is that for the BBGKY hierarchy, in the time interval $[t_{i+1}, t_i]$ between two collisions $\mathcal{C}_{s+i-1,s+i}$ and $\mathcal{C}_{s+i,s+i+1}$, the particles should not interact in general so trajectories should correspond to the free flow $\mathbf{S}_{s+i}$. On the other hand at a collision time t_i, the velocities of the two particles in interaction are liable to change. This is depicted in Figure 2.
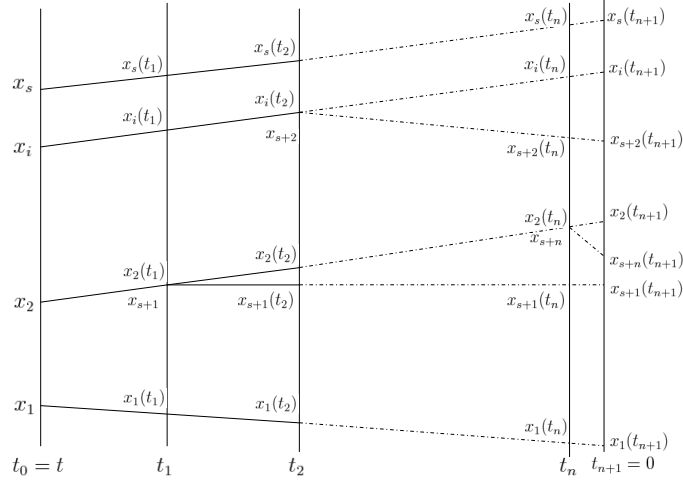

FIGURE 2. Pseudo-trajectories

At this stage however, we still cannot study directly the convergence of $I_{s,k}^{R,\delta}(t,J,M) - I_{s,k}^{0,R,\delta}(t,J,M)$ since the transport operators $\mathbf{T}_k$ do not coincide everywhere with the free transport operators $\mathbf{S}_k$, which means – in terms of pseudo-trajectories – that there are recollisions. We shall thus prove that these recollisions arise only for a few pathological pseudo-trajectories, which can be eliminated by additional truncations of the domains of integration. This is the goal of Part IV.

PART III

THE CASE OF SHORT RANGE POTENTIALS

CHAPTER 8

TWO-PARTICLE INTERACTIONS

In the case when the microscopic interaction between particles is governed by a short-range repulsive potential, collisions are no more instantaneous and pointwise, and they possibly involve more than two particles. Our analysis in Chapter 11 shows however that the low density limit $N\varepsilon^{d-1} \to 0$ requires only a description of two-particle interactions, at the exclusion of more complicated interactions.

In this chapter we therefore study precisely, following the lines of [**13**], the Hamiltonian system (1.2.1) for $N = 2$. The study of the reduced motion is carried out in Section 8.1, while the scattering map is introduced in Section 8.2, and the cross-section, which will play in important role in the Boltzmann hiearchy, is described in Section 8.3.

8.1. Reduced motion

We first define a notion of pre- and post-collisional particles, by analogy with the dynamics of hard spheres.

Definition 8.1.1. — *Two particles z_1, z_2 are said to be pre-collisional if their distance is ε and decreasing:*

$$|x_1 - x_2| = \varepsilon\,, \qquad (v_1 - v_2) \cdot (x_1 - x_2) < 0\,.$$

Two particles z_1, z_2 are said to be post-collisional if their distance is ε and increasing:

$$|x_1 - x_2| = \varepsilon\,, \qquad (v_1 - v_2) \cdot (x_1 - x_2) > 0\,.$$

We consider here only two-particle systems, and show in Lemma 8.1.2 that, if z_1 and z_2 are pre-collisional at time t_-, then there exists a post-collisional configuration z_1', z_2', attained at $t_+ > t_-$. Since $\nabla\Phi(x/\varepsilon)$ vanishes on $\{|x| \geq \varepsilon\}$, the particles z_1 and z_2 travel at constant velocities v_1' and v_2' for ulterior $(t > t_+)$ times.

Momentarily changing back the macroscopic scales of (1.2.1) to the microscopic scales of (1.0.3) by defining $\tau := (t - t_-)/\varepsilon$ and $y(\tau) := x/\varepsilon(\tau)$, $w(\tau) = v(\tau)$, we find that the two-particle dynamics is governed by the equations

$$
\text{(8.1.1)} \qquad
\begin{cases}
\dfrac{dy_1}{d\tau} = w_1\,, \quad \dfrac{dy_2}{d\tau} = w_2\,, \\[2mm]
\dfrac{dw_1}{d\tau} = -\nabla\Phi\left(y_1 - y_2\right) = -\dfrac{dw_2}{d\tau}\,,
\end{cases}
$$

whence the conservations

$$(8.1.2) \qquad \frac{d}{d\tau}(w_1 + w_2) = 0, \qquad \frac{d}{d\tau}\left(\frac{1}{4}(w_1 + w_2)^2 + \frac{1}{4}(w_1 - w_2)^2 + \Phi(y_1 - y_2)\right) = 0.$$

From (8.1.2) we also deduce that the center of mass has a uniform, rectilinear motion:

$$(8.1.3) \qquad (y_1 + y_2)(\tau) = (y_1 + y_2)(0) + \tau(w_1 + w_2),$$

and that pre- and post-collisional velocities are linked by the classical relations

$$(8.1.4) \qquad w_1' + w_2' = w_1 + w_2, \quad |w_1'|^2 + |w_2'|^2 = |w_1|^2 + |w_2|^2.$$

A consequence of (8.1.1) is that $(\delta y, \delta w) := (y_1 - y_2, w_1 - w_2)$ solves

$$(8.1.5) \qquad \frac{d}{d\tau}\delta y = \delta w, \qquad \frac{d}{d\tau}\delta w = -2\nabla\Phi(\delta y).$$

We notice that, Φ being radial, there holds

$$\frac{d}{d\tau}(\delta y \wedge \delta w) = \delta w \wedge \delta w - 2\delta y \wedge \nabla\Phi(\delta y) = 0,$$

implying that, if the initial angular momentum $\delta y_0 \wedge \delta w_0$ is non-zero, then δy remains for all times in the hyperplane orthogonal to $\delta y_0 \wedge \delta w_0$. In this hyperplane, introducing spherical coordinates (ρ, φ) in $\mathbf{R}_+ \times \mathbf{S}_1^{d-2}$, such that

$$\delta y = \rho e_\rho \quad \text{and} \quad \delta w = \dot\rho e_\rho + \rho\dot\varphi e_\varphi$$

the conservations of energy and angular momentum take the form

$$\frac{1}{2}(\dot\rho^2 + (\rho\dot\varphi)^2) + 2\Phi(\rho) = \frac{1}{2}|\delta w_0|^2,$$

$$\rho^2|\dot\varphi| = |\delta y_0 \wedge \delta w_0|,$$

implying $\rho > 0$ for all times, and

$$(8.1.6) \qquad \dot\rho^2 + \Psi(\rho, \mathcal{E}_0, \mathcal{J}_0) = \mathcal{E}_0, \qquad \Psi := \frac{\mathcal{E}_0 \mathcal{J}_0^2}{\rho^2} + 4\Phi(\rho),$$

where we have defined

$$(8.1.7) \qquad \mathcal{E}_0 := |\delta w_0|^2 \quad \text{and} \quad \mathcal{J}_0 := |\delta y_0 \wedge \delta w_0|/|\delta w_0| =: \sin\alpha,$$

which are respectively (twice) the energy and the impact parameter, $\pi - \alpha$ being the angle between δw_0 and δy_0 (notice that $\alpha \geq \pi/2$ for pre-collisional situations). In the limit case when $\alpha = 0$, the movement is confined to a line since $\dot\varphi \equiv 0$.

We consider the sets corresponding to pre- and post-collisional configurations:

$$(8.1.8) \qquad \mathcal{S}^\pm := \left\{(\delta y, \delta w) \in \mathbf{S}_1^{d-1} \times \mathbf{R}^d, \quad \pm\delta y \cdot \delta w > 0\right\},$$

where $\mathbf{S}_1^{d-1}$ is the unit sphere centered at the origin in $\mathbf{R}^d$; in spherical coordinates pre-collisional configurations correspond to $\rho = 1$ and $\dot\rho < 0$ while post-collisional configurations correspond to $\rho = 1$ and $\dot\rho > 0$.

Lemma 8.1.2 (Description of the reduced motion). — *For the differential equation* (8.1.5) *with pre-collisional datum* $(\delta y_0, \delta w_0) \in \mathcal{S}^-$, *there holds* $|\delta y(\tau)| \geq \rho_*$ *for all* $\tau \geq 0$, *with the notation*

$$(8.1.9) \qquad \rho_* = \rho_*(\mathcal{E}_0, \mathcal{J}_0) := \max\left\{\rho \in (0, 1), \quad \Psi(\rho, \mathcal{E}_0, \mathcal{J}_0) = \mathcal{E}_0\right\},$$

and for τ_* *defined by*

$$(8.1.10) \qquad \tau_* := 2\int_{\rho_*}^1 (\mathcal{E}_0 - \Psi(\rho, \mathcal{E}_0, \mathcal{J}_0))^{-1/2}\, d\rho,$$

the configuration is post-collisional ($\rho = 1$, $\dot{\rho} > 0$) at $\tau = \tau_*$.

Proof. — Solutions to (8.1.6) satisfy $\dot{\rho} = \iota(\rho)\big(\mathcal{E}_0 - \Psi(\rho)\big)^{1/2}$, with $\iota(\rho) = \pm 1$, possibly changing values only on $\{\Psi = \mathcal{E}_0\}$, by Darboux's theorem (a derivative function satisfies the intermediate value theorem). The initial configuration being pre-collisional, there holds initially $\iota = -1$, corresponding to a decreasing radius. The existence of ρ_* satisfying (8.1.9) is then easily checked: we have $|\delta y_0| = 1$ and $\delta y_0 \cdot \delta w_0 \neq 0$, so there holds $\Psi(1, \mathcal{E}_0, \mathcal{J}_0) < \mathcal{E}_0$, and Ψ is increasing as ρ is decreasing. The set $\{\tau \geq 0, \rho(\tau) \geq \rho_*\}$ is closed by continuity. It is also open: since Φ is nonincreasing, then $\partial_\rho \Psi \neq 0$ everywhere and in particular at $(\rho_*, \mathcal{E}_0, \mathcal{J}_0)$. So $\mathcal{E}_0 - \Psi$ changes sign at ρ_*, which forces, by (8.1.6), the sign function ι to jump from $-$ to $+$ as ρ reaches the value ρ_* from above. This proves $\rho \geq \rho_*$ by connexity. The minimal radius $\rho = \rho_*$ is attained at $\tau_*/2$, where τ_* is defined by (8.1.10), the integral being finite since $\partial_\rho \Psi$ does not vanish. Assume finally that for all $\tau > 0$, there holds $\rho(\tau) < 1$. Then on $[\tau_*/2, +\infty)$, ρ is increasing and bounded, hence converges to a limit radius, which contradicts the definition of ρ_*. This proves $\rho = 1$ at $\tau = \tau_*$, a time at which $\dot{\rho} > 0$, since ι has jumped exactly once, by definition of ρ_*. $\qquad\square$

Remark 8.1.3. — *Denoting $A : (y, w) \to (y, -w)$, and $\phi_t : \mathbf{R}^{2d} \to \mathbf{R}^{2d}$ the flow of (8.1.5), we find that $\phi_{-t} = A \circ \phi_t \circ A$, implying $A \circ \phi_t \circ A \circ \phi_t \equiv \mathrm{Id}$, and time-reversibility of the two-particle dynamics.*
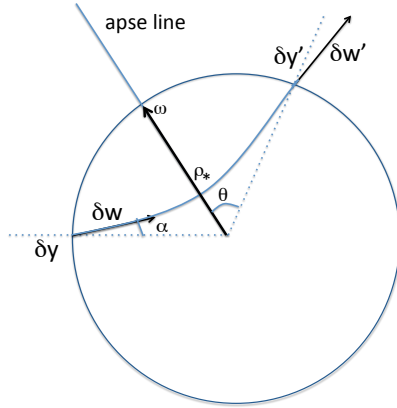

FIGURE 3. Reduced dynamics

The reduced dynamics is pictured on Figure 3, where the half-deflection angle θ is the integral of the angle φ as a function of ρ over $[\rho_*, 1]$:

$$(8.1.11) \qquad \theta = \int_{\rho_*}^1 \frac{\mathcal{E}_0^{1/2} \mathcal{I}_0}{\rho^2} \left(\mathcal{E}_0 - \Psi(\rho, \mathcal{E}_0, \mathcal{I}_0)\right)^{-1/2} d\rho\,,$$

With the initialization choice $\varphi_0 = 0$, the post-collisional configuration is $(\rho, \varphi)(\tau_*) = (1, 2\theta)$; it can be deduced from the pre-collisional configuration by symmetry with respect to the apse line, which by definition is the line through the origin and the point of closest approach $(\delta y(\tau_*/2), \delta w(\tau_*/2))$. The direction of this line is denoted $\omega \in \mathbf{S}_1^{d-1}$.

8.2. Scattering map

We shall now define a microscopic scattering map $\tilde{\sigma}_0$ that sends pre- to post-collisional configurations:

$$\tilde{\sigma}_0: \quad (\delta y_0, \delta w_0) \in \mathcal{S}^- \longrightarrow (\delta y(\tau_*), \delta w(\tau_*)) = \phi_{\tau_*}(\delta y_0, \delta w_0) \in \mathcal{S}^+\,.$$

By uniqueness of the trajectory of (8.1.5) issued from $(\delta y_0, \delta w_0)$ (a consequence of the regularity assumption on the potential, via the Cauchy-Lipschitz theorem), the scattering is one-to-one. It is also onto, by Remark 8.1.3: the pre-image of $(\delta y, \delta w) \in \mathcal{S}^+$ by the scattering is $A \circ \phi_{\tau_*}(\delta y, -\delta w) \in \mathcal{S}^-$.

Back in the macroscopic variables, we now define a corresponding scattering operator for the two-particle dynamics. In this view, we introduce the sets

$$\mathcal{S}_\varepsilon^\pm := \left\{ (z_1, z_2) \in \mathbf{R}^{4d},\ |x_1 - x_2| = \varepsilon,\ \pm(x_1 - x_2) \cdot (v_1 - v_2) > 0 \right\}.$$

We define, as in (8.1.7),

$$(8.2.1) \qquad \mathcal{E}_0 = |v_1 - v_2|^2 \quad \text{and} \quad \mathcal{J}_0 := \frac{|(x_1 - x_2) \wedge (v_1 - v_2)|}{\varepsilon |v_1 - v_2|} =: \sin\alpha.$$

Definition 8.2.1 (Scattering operator). — *The* scattering operator *is defined as*

$$\sigma_\varepsilon: (x_1, v_1, x_2, v_2) \in \mathcal{S}_\varepsilon^- \longrightarrow (x_1', v_1', x_2', v_2') \in \mathcal{S}_\varepsilon^+\,,$$

where

$$(8.2.2) \quad \begin{aligned} x_1' &:= \frac{1}{2}(x_1 + x_2) + \frac{\varepsilon\tau_*}{2}(v_1 + v_2) + \frac{\varepsilon}{2}\delta y(\tau_*) = -x_1 + \omega \cdot (x_1 - x_2)\omega + \frac{\varepsilon\tau_*}{2}(v_1 + v_2)\,, \\ x_2' &:= \frac{1}{2}(x_1 + x_2) + \frac{\varepsilon\tau_*}{2}(v_1 + v_2) - \frac{\varepsilon}{2}\delta y(\tau_*) = -x_2 - \omega \cdot (x_1 - x_2)\omega + \frac{\varepsilon\tau_*}{2}(v_1 + v_2)\,, \\ v_1' &:= \frac{1}{2}(v_1 + v_2) + \frac{1}{2}\delta w(\tau_*) = v_1 - \omega \cdot (v_1 - v_2)\,\omega\,, \\ v_2' &:= \frac{1}{2}(v_1 + v_2) - \frac{1}{2}\delta w(\tau_*) = v_2 + \omega \cdot (v_1 - v_2)\,\omega\,, \end{aligned}$$

where τ_* is the microscopic interaction time, as defined in Lemma 8.1.2, $(\delta y(\tau_*), \delta w(\tau_*))$ is the microscopic post-collisional configuration: $(\delta y(\tau_*), \delta w(\tau_*)) = \tilde{\sigma}_0((x_1 - x_2)/\varepsilon, v_1 - v_2)$, and ω is the direction of the apse line. Denoting by $\nu := (x_1 - x_2)/|x_1 - x_2|$ we also define

$$\sigma_0(\nu, v_1, v_2) := (\nu', v_1', v_2')\,.$$

The above description of (x_1', v_1') and (x_2', v_2') in terms of ω is deduced from the identities

$$\delta v(\tau_*) = \delta v_0 - 2\omega \cdot \delta v_0\,\omega \quad \text{and} \quad \delta y(\tau_*) = -\delta y_0 + 2\omega \cdot \delta y_0\,\omega$$

in the reduced microscopic coordinates.

By $\partial_\rho \Psi \neq 0$ in $(0, 1)$ and the implicit function theorem, the map $(\mathcal{E}, \mathcal{J}) \to \rho_*(\mathcal{E}, \mathcal{J})$ is C^2 just like Ψ. Similarly, $\tau_* \in C^2$. By Definition 8.2.1 and C^1 regularity of $\nabla\Phi$ (Assumption 1.2.1), this implies that the scattering operator σ_ε is C^1, just like the flow map of the two-particle scattering (denoted ϕ in Remark 8.1.3). The scattering σ_ε is also bijective, for the same reason that the microsopic scattering $\tilde{\sigma}_0$ is bijective; the inverse map is $\sigma_\varepsilon^{-1} := A \circ \sigma_\varepsilon \circ A$, with notation introduced in Remark 8.1.3.

Proposition 8.2.1. — *Let $R > 0$ be given and consider*

$$\mathcal{S}_{\varepsilon,R}^\pm := \left\{ (z_1, z_2) \in (\mathbf{R}^d \times B_R)^2,\ |x_1 - x_2| = \varepsilon,\ \pm(v_1 - v_2) \cdot (x_1 - x_2) > 0 \right\}.$$

The scattering operator σ_ε is a bijection from $\mathcal{S}_{\varepsilon,R}^-$ to $\mathcal{S}_{\varepsilon,R}^+$.

The macroscopic time of interaction $\mathcal{T}_* = \mathcal{T}_*(\mathcal{E}_0, \mathcal{J}_0) := \varepsilon\tau_*$, where τ_* is defined in (8.1.10), is uniformly bounded on compact sets of $\mathbf{R}^+ \setminus \{0\} \times [0, 1]$.

Proof. — We already know that σ_ε is a bijection from $\mathcal{S}_\varepsilon^-$ to $\mathcal{S}_\varepsilon^+$. By (8.1.4), it also preserves the velocity bound. Hence σ_ε is bijective $\mathcal{S}_{\varepsilon,R}^- \to \mathcal{S}_{\varepsilon,R}^+$.

Now given $\mathcal{E}_0 > 0$ and $\mathcal{J}_0 \in [0, 1]$, we shall show that τ_* can be bounded by a constant depending only on $\mathcal{E}_0$. Since $\Phi(\rho_*) \leq \mathcal{E}_0/4$, then $\rho_* \geq \Phi^{-1}(\mathcal{E}_0/4)$. Let us then define $i_0 \in (0, 1)$ by

$$i_0 := \frac{1}{2\sqrt{2}}\Phi^{-1}\left(\frac{\mathcal{E}_0}{4}\right),$$

so that $\rho_*^2 \geq 8i_0^2$.

On the one hand it is easy to see, after a change of variable in the integral, using

$$\frac{d}{d\rho}(\mathcal{E}_0 - \Psi(\mathcal{E}_0, \mathcal{J}_0, \rho)) = \frac{2\mathcal{E}_0\mathcal{J}_0^2}{\rho^3} - 4\Phi'(\rho) \geq \frac{2\mathcal{E}_0\mathcal{J}_0^2}{\rho^3} \geq 2\mathcal{E}_0\mathcal{J}_0^2,$$

that there holds the bound

$$\tau_* \leq \frac{1}{\mathcal{E}_0\mathcal{J}_0^2}\int_0^{\mathcal{E}_0(1-\mathcal{J}_0^2)}\frac{dy}{\sqrt{y}} \leq \frac{2\sqrt{1-\mathcal{J}_0^2}}{\mathcal{J}_0^2\sqrt{\mathcal{E}_0}}.$$

So if $\mathcal{J}_0 \geq i_0$, we find that

$$\tau_* \leq \frac{2}{\sqrt{\mathcal{E}_0}i_0^2} = \frac{16}{\sqrt{\mathcal{E}_0}\left(\Phi^{-1}\left(\frac{\mathcal{E}_0}{4}\right)\right)^2}.$$

On the other hand for $\mathcal{J}_0 \leq i_0$ we define $\gamma := \Phi^{-1}(\mathcal{E}_0/8)$ and we cut the integral defining τ_* into two parts:

$$\tau_* = \tau_*^{(1)} + \tau_*^{(2)} \quad \text{with} \quad \tau_*^{(1)} = 2\int_{\rho_*}^\gamma (\mathcal{E}_0 - \Psi(\mathcal{E}_0, \mathcal{J}_0, \rho))^{-1/2}\,d\rho.$$

Notice that since $\rho_*^2 \geq 8i_0^2$ and $\mathcal{J}_0 \leq i_0$, then $\mathcal{E}_0/4 - \mathcal{E}_0\mathcal{J}_0^2/4\rho_*^2 \geq 7\mathcal{E}_0/32 \geq \mathcal{E}_0/8$ so

$$\rho_* = \Phi^{-1}\left(\frac{\mathcal{E}_0}{4} - \frac{\mathcal{E}_0\mathcal{J}_0^2}{4\rho_*^2}\right) \leq \Phi^{-1}\left(\frac{\mathcal{E}_0}{8}\right) = \gamma.$$

The first integral $\tau_*^{(1)}$ is estimated using the fact that Φ' does not vanish outside 1 as stated in Assumption 1.2.1: defining

$$M(\Phi) := \inf_{i_0 \leq \rho \leq \gamma}|\Phi'(\rho)| > 0,$$

we find that on $[i_0, \gamma]$,

$$\frac{d}{d\rho}(\mathcal{E}_0 - \Psi(\mathcal{E}_0, \mathcal{J}_0, \rho)) = \frac{2\mathcal{E}_0\mathcal{J}_0^2}{\rho^3} - 4\Phi'(\rho) \geq 4M(\Phi)$$

so

$$\tau_*^{(1)} \leq \frac{\left(\mathcal{E}_0/2 - \mathcal{E}_0\mathcal{J}_0^2/\gamma^2\right)^{\frac{1}{2}}}{M(\Phi)} \leq \frac{\sqrt{\mathcal{E}_0}}{\sqrt{2}M(\Phi)}.$$

For the second integral we estimate simply

$$\tau_*^{(2)} \leq \frac{2}{\left(\mathcal{E}_0/2 - \mathcal{E}_0\mathcal{J}_0^2/\gamma^2\right)^{\frac{1}{2}}} \leq \frac{2}{\left(\mathcal{E}_0/2 - \mathcal{E}_0/8\right)^{\frac{1}{2}}} = \frac{4\sqrt{2}}{\sqrt{3\mathcal{E}_0}}.$$

The result follows. $\qquad\square$

Remark 8.2.2. — *If Φ is convex then $M(\Phi) = |\Phi'(\gamma)|$. Moreover if Φ is of the type $\dfrac{1}{\rho^s} \exp(-\dfrac{1}{1-\rho^2})$ then the proof of Proposition 8.2.1 shows that τ_* may be bounded from above by a constant of the order of $C/\sqrt{e_0}(1 + \log e_0)$ if $\mathcal{E}_0 \geq e_0$.*

8.3. Scattering cross-section and the Boltzmann collision operator

The scattering operator in Definition 8.2.1 is parametrized by the impact parameter and the two ingoing (or outgoing) velocities. However in the Boltzmann limit the impact parameter makes no longer sense: the observed quantity is the *deflection angle* or *scattering angle*, defined as the angle between ingoing and outgoing relative velocities. The next paragraph defines that angle as well as the scattering cross-section, and the following paragraph defines the Boltzmann collision operators using that formulation.

8.3.1. Scattering cross-section. — With notation from the previous paragraphs, the deflection angle is equal to $\pi - 2\Theta$ where $\Theta := \alpha + \theta$, the angle α being defined in (8.2.1) and θ being defined in (8.1.11), so that

$$\Theta = \Theta(\mathcal{E}_0, \mathcal{J}_0) := \arcsin \mathcal{J}_0 + \mathcal{J}_0 \int_{\rho_*}^{1} \frac{d\rho}{\sqrt{1 - \frac{4\Phi(\rho)}{\mathcal{E}_0} - \frac{\mathcal{J}_0^2}{\rho^2}}} \, .$$

The following result, and its proof, are due to [**39**]:

Lemma 8.3.1. — *Under Assumption 1.2.1, assume moreover that for all $\rho \in (0,1)$,*

$$(8.3.1) \qquad\qquad \rho\Phi''(\rho) + 2\Phi'(\rho) \geq 0 \, .$$

Then for all $\mathcal{E}_0 > 0$, the function $\mathcal{J}_0 \mapsto \Theta(\mathcal{E}_0, \mathcal{J}_0) \in [0, \pi/2]$ satisfies $\Theta(\mathcal{E}_0, 0) = 0$ and is strictly monotonic: $\partial_{\mathcal{J}_0}\Theta > 0$ for all $\mathcal{J}_0 \in (0,1)$. Moreover, it satisfies

$$\lim_{\mathcal{J}_0 \to 0} \partial_{\mathcal{J}_0}\Theta \in (0, \infty] \quad and \quad \lim_{\mathcal{J}_0 \to 1} \partial_{\mathcal{J}_0}\Theta = 0 \, .$$

Proof. — An energy $\mathcal{E}_0 > 0$ being fixed, the limiting values $\Theta(\mathcal{E}_0, 0) = 0$ and $\Theta(\mathcal{E}_0, 1) = \pi/2$ are found by direct computation. To prove monotonicity, the main idea of Saffirio and Simonella is to use the change of variable

$$\sin^2 \varphi := \frac{4\Phi(\rho)}{\mathcal{E}_0} + \frac{\mathcal{J}_0^2}{\rho^2}$$

which yields

$$\Theta(\mathcal{E}_0, \mathcal{J}_0) = \arcsin \mathcal{J}_0 + \int_{\arcsin \mathcal{J}_0}^{\frac{\pi}{2}} \frac{\sin \varphi}{\frac{\mathcal{J}_0}{\rho} - \frac{2\rho\Phi'(\rho)}{\mathcal{E}_0\mathcal{J}_0}} \, d\varphi \, .$$

Computing the derivative of this expression with respect to $\mathcal{J}_0$ gives

$$\frac{\partial\Theta}{\partial\mathcal{J}_0}(\mathcal{E}_0, \mathcal{J}_0) = \frac{1}{\sqrt{1 - \mathcal{J}_0^2}} \left(1 - \frac{\mathcal{E}_0\mathcal{J}_0^2}{\mathcal{E}_0\mathcal{J}_0^2 - \Phi'(1)} \right)$$

$$+ \int_{\arcsin \mathcal{J}_0}^{\frac{\pi}{2}} \frac{\mathcal{E}_0^2\mathcal{J}_0^2\rho^4 \sin \varphi}{(\mathcal{J}_0^2\mathcal{E}_0 - \rho^3\Phi'(\rho))^3} \left(\rho\Phi''(\rho) + 2\Phi'(\rho) + \frac{\rho^3}{\mathcal{E}_0\mathcal{J}_0^2}(\Phi'(\rho))^2 \right) d\varphi$$

where φ is defined by

$$\sin^2 \varphi = \frac{\mathcal{J}_0^2}{\rho^2} + \frac{2\Phi(\rho)}{\mathcal{E}_0} \, .$$

In view of the formula giving $\partial_{\mathcal{J}_0}\Theta$, it turns out assumption (8.3.1) implies $\partial_{\mathcal{J}_0}\Theta > 0$ for all $\mathcal{J}_0 \in (0,1)$, and also the limits

$$\lim_{\mathcal{J}_0 \to 0} \partial_{\mathcal{J}_0}\Theta \in (0,\infty] \quad \text{and} \quad \lim_{\mathcal{J}_0 \to 1} \partial_{\mathcal{J}_0}\Theta = 0$$

as soon as $\Phi'(1) = 0$ (if not then $\lim_{\mathcal{J}_0 \to 1} \partial_{\mathcal{J}_0}\Theta = \infty$). The result follows. $\qquad\square$

Remark 8.3.2. — *Note that one can construct examples that violate assumption* (8.3.1) *and for which monotonicity fails, regardless of convexity properties of the potential* Φ ([**39**]).

By Lemma 8.3.1, for each $\mathcal{E}_0$ we can locally invert the map $\Theta(\mathcal{E}_0, \cdot)$, and thus define $\mathcal{J}_0$ as a smooth function of $\mathcal{E}_0$ and Θ. This enables us to define a scattering cross-section (or *collision kernel*), as follows.
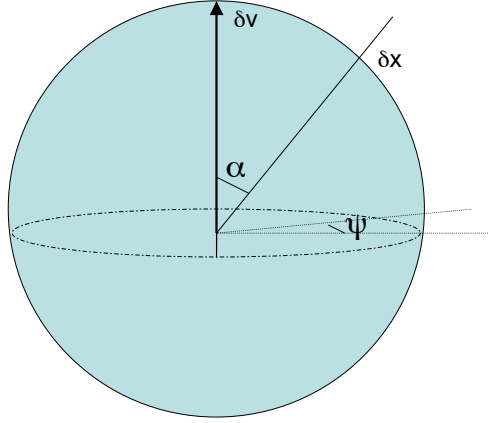

FIGURE 4. Spherical coordinates

For fixed x_1, we denote $d\sigma_1$ the surface measure on the sphere $\{y \in \mathbf{R}^d, |y - x_1| = \varepsilon\}$, to which x_2 belongs. We can parametrize the sphere by (α, ψ), with $\psi \in \mathbf{S}_1^{d-2}$, where α is the angle defined in (8.2.1). There holds

$$d\sigma_1 = \varepsilon^{d-1}(\sin\alpha)^{d-2}d\alpha d\psi.$$

The direction of the apse line is $\omega = (\Theta, \psi)$, so that, denoting $d\omega$ the surface measure on the unit sphere, there holds

(8.3.2) $$d\omega = (\sin\Theta)^{d-2}d\Theta d\psi.$$

By definition of α in (8.2.1), there holds

$$(x_1 - x_2) \cdot (v_1 - v_2) = \varepsilon|v_1 - v_2|\cos\alpha,$$

so that

$$\frac{1}{\varepsilon}(x_1 - x_2) \cdot (v_1 - v_2)\, d\sigma_1 = \varepsilon^{d-1}|v_1 - v_2|\cos\alpha\,(\sin\alpha)^{d-2}\,d\alpha d\psi$$

$$= \varepsilon^{d-1}|v_1 - v_2|\,\mathcal{J}_0^{d-2}d\mathcal{J}_0 d\psi,$$

where in the second equality we used the definition of $\mathcal{J}_0$ in (8.2.1). This gives

(8.3.3) $$\frac{1}{\varepsilon}(x_1 - x_2) \cdot (v_1 - v_2)\, d\sigma_1 = \varepsilon^{d-1}|v_1 - v_2|\mathcal{J}_0^{d-2}\partial_\Theta\mathcal{J}_0\, d\Theta d\psi,$$

wherever $\partial_\Theta\mathcal{J}_0$ is defined, that is, according to Lemma 8.3.1, for $\mathcal{J}_0 \in [0,1)$.

Definition 8.3.3. — *The scattering cross-section is defined for $|v_1 - v_2| > 0$ and $\Theta \in (0, \pi/2]$ by $b(w, \Theta)/|w|$, where*

$$(8.3.4) \qquad b(v_1 - v_2, \Theta) := |v_1 - v_2| \mathcal{J}_0^{d-2} \partial_\Theta \mathcal{J}_0 (\sin \Theta)^{2-d}.$$

Abusing notation we shall write $b(v_1 - v_2, \Theta) = b(v_1 - v_2, \omega)$.

By Lemma 8.3.1, the cross-section b is a locally bounded function of the relative velocities and scattering angle.

8.3.2. Scattering cross-section. — The relevance of b is made clear in the derivation of the Boltzmann hierarchy, where we shall use the identity

$$(8.3.5) \qquad \frac{1}{\varepsilon} (x_1 - x_2) \cdot (v_1 - v_2) \, d\sigma_1 = \varepsilon^{d-1} b(v_1 - v_2, \omega) d\omega \,,$$

derived from (8.3.2), (8.3.3) and Definition 8.3.3. As in Chapter 4 (see in particular Paragraph 4.4), we can formally derive the Boltzmann collision operators using this formulation: we thus define

$$(8.3.6) \qquad \mathcal{C}_{s,s+1}^0 f^{(s+1)}(t, Z_s) := \sum_{i=1}^s \int \mathbb{1}_{\nu \cdot (v_{s+1} - v_i) > 0} \, \nu \cdot (v_{s+1} - v_i)$$
$$\times \left(f^{(s+1)}(t, x_1, v_1, \ldots, x_i, v_i^*, \ldots, x_s, v_s, x_i, v_{s+1}^*) - f^{(s+1)}(t, Z_s, x_i, v_{s+1}) \right) d\nu dv_{s+1} \,,$$

where (v_i^*, v_{s+1}^*) is obtained from (v_i, v_{s+1}) by applying the inverse scattering operator σ_0^{-1}, using

$$\sigma_0^{-1} \left(\frac{x_i - x_{s+1}}{|x_i - x_{s+1}|}, v_i, v_{s+1} \right) = \left(\frac{x_i - x_{s+1}}{|x_i - x_{s+1}|}, v_i^*, v_{s+1}^* \right).$$

This can also be written using the cross-section:

$$(8.3.7) \qquad \mathcal{C}_{s,s+1}^0 f^{(s+1)}(t, Z_s) := \sum_{i=1}^s \int b(v_1 - v_2, \omega)$$
$$\times \left(f^{(s+1)}(t, x_1, v_1, \ldots, x_i, v_i^*, \ldots, x_s, v_s, x_i, v_{s+1}^*) - f^{(s+1)}(t, Z_s, x_i, v_{s+1}) \right) d\omega dv_{s+1} \,.$$

Remark 8.3.4. — *It is not possible to define an integrable cross-section if the potential is not compactly supported, no matter how fast it might be decaying. This issue is related to the occurrence of grazing collisions and discussed in particular in [46], Chapter 1, Section 1.4. However it is still possible to study the limit towards the Boltzmann equation, if one is ready to change the formulation of the Boltzmann equation by renouncing to the cross-section formulation ([39]).*

The question of the convergence to Boltzmann in the case of long-range potentials is a challenging open problem; it was considered by L. Desvillettes and M. Pulvirenti in [16] in the linear case, while L. Desvillettes and V. Ricci studied grazing collisions in [17].

TRUNCATED MARGINALS AND THE BBGKY HIERARCHY

Our starting point is the Liouville equation (1.2.2) satisfied by the N-particle distribution function f_N. We reproduce here equation (1.2.2):

$$(9.0.1) \qquad \partial_t f_N + \sum_{1 \le i \le N} v_i \cdot \nabla_{x_i} f_N - \sum_{1 \le i \ne j \le N} \frac{1}{\varepsilon} \nabla \Phi \left(\frac{x_i - x_j}{\varepsilon} \right) \cdot \nabla_{v_i} f_N = 0 \,.$$

The arguments of f_N in (9.0.1) are $(t, Z_N) \in \mathbf{R}_+ \times \Omega_N$, where we recall that

$$\Omega_N := \left\{ Z_N \in \mathbf{R}^{2dN} \,, \, \forall i \ne j \,, \, x_i \ne x_j \right\} \,.$$

As recalled in Part II, Chapter 4, the classical strategy to obtain a kinetic equation is to write the evolution equation for the first marginal of the distribution function f_N, namely

$$f_N^{(1)}(t, z_1) := \int_{\mathbf{R}^{2d(N-1)}} f_N(t, z_1, z_2, \ldots, z_N) \, dz_2 \ldots dz_N \,,$$

which leads to the study of the hierarchy of equations involving all the marginals of f_N

$$(9.0.2) \qquad f_N^{(s)}(t, Z_s) := \int_{\mathbf{R}^{2d(N-s)}} f_N(t, Z_s, z_{s+1}, \ldots, z_N) \, dz_{s+1} \cdots dz_N \,.$$

In Section 9.1 it is shown that due to the presence of the potential, and contrary to the hard spheres case described in Paragraph 4.2, it is necessary to truncate those marginals away from the set Ω_N. An equation for the *truncated marginals* is derived in weak form in Section 9.2. In order to introduce adequate collision operators, the notion of cluster is introduced and described in Section 9.3. Then collision operators are introduced in Section 9.4, and finally the integral formulation of the equation is written in Section 9.5.

9.1. Truncated marginals

From (9.0.1), we deduce by integration that the *untruncated marginals* defined in (9.0.2) solve

$$
\begin{aligned}
(9.1.1) \qquad & \partial_t f_N^{(s)}(t, Z_s) + \sum_{i=1}^{s} v_i \cdot \nabla_{x_i} f_N^{(s)}(t, Z_s) - \frac{1}{\varepsilon} \sum_{\substack{i,j=1 \\ i \ne j}}^{s} \nabla \Phi \left(\frac{x_i - x_j}{\varepsilon} \right) \cdot \nabla_{v_i} f_N^{(s)}(t, Z_s) \\
& = \frac{N-s}{\varepsilon} \sum_{i=1}^{s} \int \nabla \Phi \left(\frac{x_i - x_{s+1}}{\varepsilon} \right) \cdot \nabla_{v_i} f_N^{(s+1)}(t, Z_s, z_{s+1}) \, dz_{s+1} \,.
\end{aligned}
$$

There are several differences between (9.1.1) and the BBGKY hierarchy for hard spheres (4.3.2)-(4.3.3). One is that the transport operator in the left-hand side of (9.1.1) involves a force term. Another is that the integral term in the right-hand side of (9.1.1) involves velocity derivatives. Also, that integral term is a linear integral operator acting on higher-order marginals, just like (4.3.2), but, contrary to (4.3.2), is *not* spatially localized, in the sense that the integral in x_{s+1} is over the whole ball $B(x_i, \varepsilon)$, as opposed to an integral over a sphere in (4.3.2).

This leads us to distinguish spatial configurations in which interactions do take place from spatial configurations in which particles are pairwise at a distance greater than ε, by truncating off the interaction domain $\{Z_N, |x_i - x_j| \leq \varepsilon \text{ for some } i \neq j\}$ in the integrals defining the marginals. For the resulting truncated marginals, collision operators will appear as integrals over a piece of the boundary of the interaction domain, just like in the case of hard spheres. The scattering operator of Chapter 8 (Section 8.2) will then play the role that the boundary condition plays in the case of hard spheres in Chapter 4.

Suitable quantities to be studied are therefore not the marginals defined in (9.0.2) but rather the *truncated marginals*

$$(9.1.2) \qquad \widetilde{f}_N^{(s)}(t, Z_s) := \int_{\mathbf{R}^{2d(N-s)}} f_N(t, Z_s, z_{s+1}, \ldots, z_N) \prod_{\substack{i \in \{1,\ldots,s\} \\ j \in \{s+1,\ldots,N\}}} \mathbb{1}_{|x_i - x_j| > \varepsilon} \, dz_{s+1} \cdots dz_N \,,$$

where $|\cdot|$ denotes the euclidean norm. Notice that

$$(\widetilde{f}_N^{(1)} - f_N^{(1)})(t, z_1) = \int_{\mathbf{R}^{2d(N-1)}} f_N(t, z_1, z_2, \ldots, z_N) \prod_{j \in \{2,\ldots,N\}} (1 - \mathbb{1}_{|x_1 - x_j| > \varepsilon}) \, dz_2 \cdots dz_N$$

so that

$$(9.1.3) \qquad \|(\widetilde{f}_N^{(1)} - f_N^{(1)})(t)\|_{L^\infty(\mathbf{R}^{2d})} \leq C(N-1)\varepsilon^d \|f_N^{(2)}(t)\|_{L^\infty(\Omega_2)} \,.$$

We therefore expect both functions to have the same asymptotic behaviour in the Boltzmann-Grad limit $N\varepsilon^{d-1} = 1$. This is indeed proved in Lemma 11.1.2.

Given $1 \leq i < j \leq N$, we recall that $dZ_{(i,j)}$ denotes the $2d(j - i + 1)$-dimensional Lebesgue measure $dz_i dz_{i+1} \ldots dz_j$, and $dX_{(i,j)}$ the $d(j - i + 1)$-dimensional Lebesgue measure $dx_i dx_{i+1} \ldots dx_j$. We also define

$$(9.1.4) \qquad \mathcal{D}_N^s := \left\{ X_N \in \mathbf{R}^{dN}, \, \forall(i,j) \in [1, s] \times [s+1, N], \, |x_i - x_j| > \varepsilon \right\},$$

where $[1, s]$ is short for $[1, s] \cap \mathbf{N} = \{k \in \mathbf{N}, \, 1 \leq k \leq s\}$. Then the truncated marginals (9.1.2) may be formulated as follows:

$$(9.1.5) \qquad \widetilde{f}_N^{(s)}(t, Z_s) = \int_{\mathbf{R}^{2d(N-s)}} f_N(t, Z_s, z_{s+1}, \ldots, z_N) \prod_{\substack{i \in [1,s] \\ j \in [s+1,N]}} \mathbb{1}_{|x_i - x_j| > \varepsilon} \mathbb{1}_{X_N \in \mathcal{D}_N^s} \, dZ_{s+1,N} \,.$$

The key in introducing the truncated marginals (9.1.5), following King [**30**], is that it allows for a derivation of a hierarchy that is similar to the case of hard spheres. The main drawback is that contrary to the hard-spheres case in (4.2.3), truncated marginals are not actual *marginals*, in the sense that

$$(9.1.6) \qquad \widetilde{f}_N^{(s)}(Z_s) \neq \int_{\mathbf{R}^{2d}} \mathbb{1}_{X_{s+1} \in B} \widetilde{f}_N^{(s+1)}(Z_s, z_{s+1}) \, dz_{s+1} \,,$$

for any $B \subset \mathbf{R}^{d(s+1)}$, in particular if $B = \mathbf{R}^{d(s+1)}$, simply because $\mathcal{D}_N^s$ is *not* included in $\mathcal{D}_N^{s+1}$. Indeed, conditions $|x_j - x_{s+1}| > \varepsilon$, for $j \leq s$, hold for $X_N \in \mathcal{D}_N^s$, but not necessarily for $X_N \in \mathcal{D}_N^{s+1}$.

Furthermore, $\mathcal{D}_N^s$ intersects all the $\mathcal{D}_N^{s+m}$, for $m \in [1, N-s]$. A consequence is the existence of higher-order interactions between truncated marginals, as seen below in (9.4.8). Proposition 10.3.1 in Chapter 10 states however that these higher-order interactions are negligible in the Boltzmann-Grad limit.

9.2. Weak formulation of Liouville's equation

Our goal in this section is to find the weak formulation of the system of equations satisfied by the family of truncated marginals $\big(\widetilde{f}_N^{(s)}\big)_{s \in [1,N]}$ defined above in (9.1.5). The strategy will be similar to that followed in Chapter 4 in the hard spheres case. From now on we assume that f_N decays at infinity in the velocity variable.

Given a smooth, compactly supported function ϕ defined on $\mathbf{R}_+ \times \mathbf{R}^{2ds}$ and satisfying the symmetry assumption (1.1.1), we have

$$(9.2.1) \qquad \int_{\mathbf{R}_+ \times \mathbf{R}^{2dN}} \Big(\partial_t f_N + \sum_{i=1}^N v_i \cdot \nabla_{x_i} f_N - \frac{1}{\varepsilon} \sum_{i=1}^N \sum_{j \neq i} \nabla \Phi \Big(\frac{x_i - x_j}{\varepsilon} \Big) \cdot \nabla_{v_i} f_N \Big)(t, Z_N)$$
$$\times \phi(t, Z_s) \mathbb{1}_{X_N \in \mathcal{D}_N^s} \, dZ_N dt = 0 \,.$$

Note that in the above double sum in i and j, all the terms vanish except when $(i, j) \in [1, s]^2$ and when $(i, j) \in [s+1, N]^2$, by assumption on the support of F.

We now use integrations by parts to derive from (9.2.1) the weak form of the equation in the marginals $\widetilde{f}_N^{(s)}$. On the one hand an integration by parts in the time variable gives

$$\int_{\mathbf{R}_+ \times \mathbf{R}^{2dN}} \partial_t f_N(t, Z_N) \phi(t, Z_s) \mathbb{1}_{X_N \in \mathcal{D}_N^s} \, dZ_N dt = -\int_{\mathbf{R}^{2dN}} f_N(0, Z_N) \phi(0, Z_s) \mathbb{1}_{X_N \in \mathcal{D}_N^s} \, dZ_N$$
$$-\int_{\mathbf{R}_+ \times \mathbf{R}^{2dN}} f_N(t, Z_N) \partial_t \phi(t, Z_s) \mathbb{1}_{X_N \in \mathcal{D}_N^s} \, dZ_N dt \,,$$

hence, by definition of $\widetilde{f}_N^{(s)}$,

$$\int_{\mathbf{R}_+ \times \mathbf{R}^{2dN}} \partial_t f_N(t, Z_N) \phi(t, Z_s) \mathbb{1}_{X_N \in \mathcal{D}_N^s} \, dZ_N dt = -\int_{\mathbf{R}^{2ds}} \widetilde{f}_N^{(s)}(0, Z_s) \phi(0, Z_s) \, dZ_s$$
$$-\int_{\mathbf{R}_+ \times \mathbf{R}^{2ds}} \widetilde{f}_N^{(s)}(t, Z_s) \partial_t \phi(t, Z_s) \, dZ_s dt \,.$$

Now let us compute

$$\sum_{i=1}^N \int_{\mathbf{R}^{2dN}} v_i \cdot \nabla_{x_i} f_N(t, Z_N) \phi(t, Z_s) \mathbb{1}_{X_N \in \mathcal{D}_N^s} \, dZ_N = \int_{\mathbf{R}^{2dN}} \mathrm{div}_{X_N} \big(V_N \, f_N(t, Z_N) \big) \phi(t, Z_s) \mathbb{1}_{X_N \in \mathcal{D}_N^s} \, dZ_N$$

using Green's formula. The boundary of $\mathcal{D}_N^s$ is made of configurations with at least one pair (i, j), satisfying $1 \leq i \leq s$ and $s+1 \leq j \leq N$, with $|x_i - x_j| = \varepsilon$.

Let us define, for any couple $(i, j) \in [1, N]^2$,

$$(9.2.2) \qquad \Sigma_N^s(i, j) := \Big\{ X_N \in \mathbf{R}^{dN}, \quad |x_i - x_j| = \varepsilon$$
$$\text{and} \quad \forall (k, \ell) \in [1, s] \times [s+1, N] \setminus \{i, j\}, \, |x_k - x_\ell| > \varepsilon \Big\} \,.$$

We notice that $\Sigma_N^s(i,j)$ is a submanifold of $\left\{ X_N \in \mathbf{R}^{dN}, |x_i - x_j| = \varepsilon \right\}$, which is a smooth, codimension 1 manifold of $\mathbf{R}^{dN}$ (locally isomorphic to the space $\mathbf{S}_\varepsilon^d \times \mathbf{R}^{d(N-1)}$), and we denote by $d\sigma_N^{i,j}$ its surface measure, induced by the Lebesgue measure. Configurations with more than one *collisional* pair, i.e., (i,j) and (i',j') with $1 \leq i, i' \leq s$, $s+1 \leq j, j' \leq N$, with $|x_i - x_j| = |x_{i'} - x_{j'}| = \varepsilon$, and $\{i,j\} \neq \{i',j'\}$, are subsets of submanifols of $\mathbf{R}^{dN}$ of dimension at least two, and therefore contribute nothing to the boundary terms. Denoting $n^{i,j}$ the outward normal to $\Sigma_N^s(i,j)$ we therefore obtain by Green's formula:

$$\sum_{i=1}^N \int_{\mathbf{R}_+ \times \mathbf{R}^{2dN}} v_i \cdot \nabla_{x_i} f_N(t, Z_N) \phi(t, Z_s) \mathbb{1}_{X_N \in \mathcal{D}_N^s}\, dZ_N\, dt$$

$$= -\sum_{i=1}^s \int_{\mathbf{R}_+ \times \mathbf{R}^{2dN}} f_N(t, Z_N) v_i \cdot \nabla_{x_i} \phi(t, Z_s) \mathbb{1}_{X_N \in \mathcal{D}_N^s}\, dZ_N dt$$

$$+ \sum_{1 \leq i \neq j \leq N} \int_{\mathbf{R}_+ \times \mathbf{R}^{dN} \times \Sigma_N^s(i,j)} n^{i,j} \cdot V_N\, f_N(t, Z_N) \phi(t, Z_s)\, d\sigma_N^{i,j} dV_N dt\,.$$

By symmetry (1.1.1) and recalling that $\nu^{i,j} = (x_i - x_j)/|x_i - x_j|$ this gives

$$\sum_{i=1}^N \int_{\mathbf{R}_+ \times \mathbf{R}^{2dN}} v_i \cdot \nabla_{x_i} f_N(t, Z_N) \phi(t, Z_s) \mathbb{1}_{X_N \in \mathcal{D}_N^s}\, dZ_N\, dt$$

$$= -\sum_{i=1}^s \int_{\mathbf{R}_+ \times \mathbf{R}^{2dN}} f_N(t, Z_N) v_i \cdot \nabla_{x_i} \phi(t, Z_s) \mathbb{1}_{X_N \in \mathcal{D}_N^s}\, dZ_N dt$$

$$+ \frac{N-s}{\sqrt{2}} \sum_{i=1}^s \int_{\mathbf{R}_+ \times \mathbf{R}^{dN} \times \Sigma_N^s(i,j)} \nu^{i,s+1} \cdot (v_{s+1} - v_i)\, f_N(t, Z_N) \phi(t, Z_s)\, d\sigma_N^{i,j} dV_N dt\,,$$

so finally by definition of $\widetilde{f}_N^{(s)}$, we obtain

$$\sum_{i=1}^N \int_{\mathbf{R}_+ \times \mathbf{R}^{2dN}} v_i \cdot \nabla_{x_i} f_N(t, Z_N) \phi(t, Z_s) \mathbb{1}_{X_N \in \mathcal{D}_N^s}\, dZ_N\, dt$$

(9.2.3) $$= -\sum_{i=1}^s \int_{\mathbf{R}_+ \times \mathbf{R}^{2ds}} \widetilde{f}_N^{(s)}(t, Z_s) v_i \cdot \nabla_{x_i} \phi(t, Z_s)\, dZ_s dt$$

$$+ \frac{N-s}{\sqrt{2}} \sum_{i=1}^s \int_{\mathbf{R}_+ \times \mathbf{R}^{dN} \times \Sigma_N^s(i,j)} \nu^{i,s+1} \cdot (v_{s+1} - v_i)\, f_N(t, Z_N) \phi(t, Z_s)\, d\sigma_N^{i,j} dV_N dt\,.$$

Now let us consider the contribution of the potential in (9.2.1). We split the sum as follows:

$$\frac{1}{\varepsilon} \sum_i \sum_{j \neq i} \int_{\mathbf{R}_+ \times \mathbf{R}^{2dN}} \nabla \Phi\left(\frac{x_i - x_j}{\varepsilon} \right) \cdot \nabla_{v_i} f_N(t, Z_N) \phi(t, Z_s) \mathbb{1}_{X_N \in \mathcal{D}_N^s}\, dZ_N dt$$

$$= \frac{1}{\varepsilon} \sum_{\substack{i,j=1 \\ j \neq i}}^s \int_{\mathbf{R}_+ \times \mathbf{R}^{2dN}} \nabla \Phi\left(\frac{x_i - x_j}{\varepsilon} \right) \cdot \nabla_{v_i} f_N(t, Z_N) \phi(t, Z_s) \mathbb{1}_{X_N \in \mathcal{D}_N^s}\, dZ_N dt$$

$$+ \frac{1}{\varepsilon} \sum_{\substack{i,j=s+1 \\ j \neq i}}^N \int_{\mathbf{R}_+ \times \mathbf{R}^{2dN}} \nabla \Phi\left(\frac{x_i - x_j}{\varepsilon} \right) \cdot \nabla_{v_i} f_N(t, Z_N) \phi(t, Z_s) \mathbb{1}_{X_N \in \mathcal{D}_N^s}\, dZ_N dt\,.$$

We notice that the second term in the right-hand side vanishes identically. It follows that

$$\frac{1}{\varepsilon} \sum_i \sum_{j \neq i} \int_{\mathbf{R}_+ \times \mathbf{R}^{2dN}} \nabla \Phi \left(\frac{x_i - x_j}{\varepsilon} \right) \cdot \nabla_{v_i} f_N(t, Z_N) \phi(t, Z_s) \mathbb{1}_{X_N \in \mathcal{D}_N^s} \, dZ_N dt$$

$$= -\frac{1}{\varepsilon} \sum_{\substack{i,j=1 \\ j \neq i}}^s \int_{\mathbf{R}_+ \times \mathbf{R}^{2ds}} \nabla \Phi \left(\frac{x_i - x_j}{\varepsilon} \right) \cdot \nabla_{v_i} \phi(t, Z_s) \widetilde{f}_N^{(s)}(t, Z_s) \, dZ_s dt$$

so in the end we obtain

$$\int_{\mathbf{R}_+ \times \mathbf{R}^{2ds}} \widetilde{f}_N^{(s)}(t, Z_s) \Big(\partial_t \phi + \operatorname{div}_{X_s} (V_s \phi) - \frac{1}{\varepsilon} \sum_{\substack{i,j=1 \\ j \neq i}}^s \nabla \Phi \left(\frac{x_i - x_j}{\varepsilon} \right) \cdot \nabla_{v_i} \phi \Big)(t, Z_s) \, dZ_s dt$$

$$(9.2.4) \qquad = - \int_{\mathbf{R}^{2ds}} \widetilde{f}_N^{(s)}(0, Z_s) \phi(0, Z_s) \, dZ_s$$

$$- \frac{N-s}{\sqrt{2}} \sum_{i=1}^s \int_{\mathbf{R}_+ \times \Sigma_N^s(i,s+1)} \nu^{i,s+1} \cdot (v_{s+1} - v_i) \, f_N(t, Z_N) \phi(t, Z_s) \, d\sigma_N^{i,s+1} dt \,.$$

Remark 9.2.1. — *Using the weak form of Liouville's equation, we see that multiple collisions (which occur as a boundary integral on a zero measure subset of $\partial \mathcal{D}_N^s$) can be neglected.*

9.3. Clusters

We want to analyze the second term on the right-hand side of (9.2.4). We notice that in the space-velocity integration the variables $x_{s+2}, \ldots, x_N$ are integrated over $\mathbf{R}^{d(N-s-1)}$ (with the restriction that they must be at a distance at least ε from X_s) whereas x_{s+1} must lie in the sphere centered at x_i and of radius ε. It is therefore natural to try to express that contribution in terms of the marginal $\widetilde{f}_N^{(s+1)}(Z_{s+1})$. However as pointed out in (9.1.6),

$$\int \widetilde{f}_N^{(s+1)}(Z_{s+1}) \, dz_{s+1} \neq \widetilde{f}_N^{(s)}(Z_s) \,.$$

The difference between those two terms is that on the one hand

$$\forall X_N \in \mathcal{D}_N^{s+1}, \qquad \text{one has } |x_j - x_{s+1}| > \varepsilon \text{ for all } j \geq s+2 \,,$$

which is not the case for $X_N \in \mathcal{D}_N^s$, and on the other hand

$$\forall X_N \in \mathcal{D}_N^s, \qquad \text{one has } |x_j - x_{s+1}| > \varepsilon \text{ for all } j \leq s \,,$$

a condition which does not appear in the definition of $\mathcal{D}_N^{s+1}$.

This leads to the following definition.

Definition 9.3.1 (ε-**closure**). — *Given a subset $X_N = \{x_1, \ldots, x_N\}$ of $\mathbf{R}^{dN}$ and an integer s in $[1, N]$, the ε-closure $E(X_s, X_N)$ of X_s in X_N is defined as the intersection of all subsets Y of X_N which contain X_s and satisfy the separation condition*

$$(9.3.1) \qquad \forall y \in Y, \quad \forall x \in X_N \setminus Y, \quad |x - y| > \varepsilon \,.$$

We denote $|E(X_s, X_N)|$ the cardinal of $E(X_s, X_N)$.

Now let us introduce the following notation, useful in situations where X_N belongs to $\Sigma_N^s(i, s+1)$, defined in (9.2.2).

Notation 9.3.2. — *If $X_{s+m} = E(X_s, X_{s+m})$ and if for some integers $j_0 \leq s < k_0 \leq s + m$, there holds $|x_j - x_k| > \varepsilon$ for all $(j, k) \in [1, s] \times [s + 1, s + m] \setminus \{(j_0, k_0)\}$, then we say that $E(X_s, X_{s+m})$ has a weak link at (j_0, k_0), and we denote $X_{s+m} = E_{\langle j_0, k_0 \rangle}(X_s, X_{s+m})$.*

Moreover the following notion, following King [**30**], will turn out to be very useful.

Definition 9.3.3 (**Cluster**). — *A cluster of base $X_s = \{x_1, \ldots, x_s\}$ and length m is any point $\{x_{s+1}, \ldots, x_{s+m}\}$ in $\mathbf{R}^{dm}$ such that $E(X_s, X_{s+m}) = X_{s+m}$. We denote $\Delta_m(X_s)$ the set of all such clusters.*

The proof of the following lemma is completely elementary.

Lemma 9.3.4. — *The following equivalences hold, for $m \geq 1$:*

$$(9.3.2) \qquad \Big(E(X_s, X_N) = X_{s+m} \Big) \iff \Big(E(X_s, X_{s+m}) = X_{s+m} \text{ and } X_N \in \mathcal{D}_N^{s+m} \Big),$$

$$(9.3.3) \qquad \begin{pmatrix} E(X_s, X_N) = X_{s+m} \\ X_N \in \Sigma_N^s(i, s+1) \end{pmatrix} \iff \begin{pmatrix} E_{\langle i, s+1 \rangle}(X_s, X_{s+m}) = X_{s+m} \\ X_N \in \mathcal{D}_N^{s+m} \\ |x_i - x_{s+1}| = \varepsilon \end{pmatrix},$$

as well as the implication, for $m \geq 2$,

$$(9.3.4) \qquad \Big(E_{\langle i, s+1 \rangle}(X_s, X_{s+m}) = X_{s+m} \Big) \implies \Big(\{x_{s+2}, \ldots, x_{s+m}\} \in \Delta_{m-1}(x_{s+1}) \Big).$$
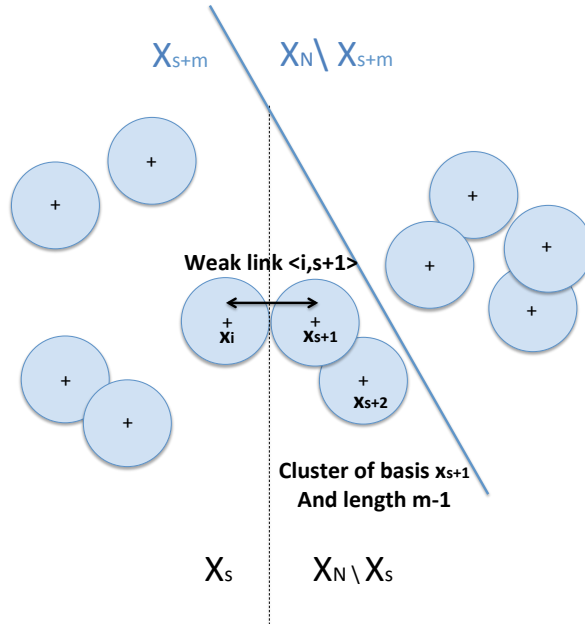

FIGURE 5. Clusters with weak links

9.4. Collision operators

With the help of the notions introduced in Section 9.3, we now can reformulate the boundary integral in (9.2.4).

Given $1 \le s \le N-1$ and X_N in $\Sigma_N^s(i, s+1)$, there holds $|x_{s+1} - x_i| = \varepsilon$, so that x_{s+1} belongs to $E(X_s, X_N)$, implying $|E(X_s, X_N)| \ge s+1$. We decompose $\Sigma_N^s(i, s+1)$ into a disjoint union over the possible cardinals of the ε-closure of X_s in X_N :

$$(9.4.1) \qquad \Sigma_N^s(i, s+1) = \bigcup_{1 \le m \le N-s} \left(\Sigma_N^s(i, s+1) \bigcap \left\{ Y_N, \ |E(Y_s, Y_N)| = s+m \right\} \right),$$

implying

$$\int_{\mathbf{R}^{dN} \times \Sigma_N^s(i, s+1)} \nu^{i, s+1} \cdot (v_{s+1} - v_i) \, f_N(Z_N) \phi(Z_s) \, d\sigma_N^{i, s+1} dV_N$$

$$= \sum_{1 \le m \le N-s} \int_{\mathbf{R}^{dN} \times \Sigma_N^s(i, s+1)} \mathbb{1}_{|E(X_s, X_N)| = s+m} \, \nu^{i, s+1} \cdot (v_{s+1} - v_i) \, f_N(Z_N) \phi(Z_s) \, d\sigma_N^{i, s+1} dV_N \,.$$

By assumption of symmetry (1.1.1) for f_N and ϕ, if $|E(X_s, X_N)| = s+m$, we can index the particles so that $E(X_s, X_N) = X_{s+m}$: we obtain

$$(9.4.2) \quad \begin{aligned} &\int_{\mathbf{R}^{dN} \times \Sigma_N^s(i, s+1)} \mathbb{1}_{|E(X_s, X_N)| = s+m} \, \nu^{i, s+1} \cdot (v_{s+1} - v_i) \, f_N(Z_N) \phi(Z_s) \, d\sigma_N^{i, s+1} dV_N \\ &= C_{N-s-1}^{m-1} \int_{\mathbf{R}^{dN} \times \Sigma_N^s(i, s+1)} \mathbb{1}_{E(X_s, X_N) = X_{s+m}} \, \nu^{i, s+1} \cdot (v_{s+1} - v_i) \, f_N(Z_N) \phi(Z_s) \, d\sigma_N^{i, s+1} dV_N \,. \end{aligned}$$

We use equivalence (9.3.3) from Lemma 9.3.4 and Fubini's theorem to write

$$\int_{\mathbf{R}^{dN} \times \Sigma_N^s(i, s+1)} \mathbb{1}_{E(X_s, X_N) = X_{s+m}} \nu^{i, s+1} \cdot (v_{s+1} - v_i) f_N(Z_N) \phi(Z_s) d\sigma_N^{i, s+1} dV_N$$

$$= \sqrt{2} \int_{S_\varepsilon(x_i) \times \mathbf{R}^d} \nu^{i, s+1} \cdot (v_{s+1} - v_i) \phi(Z_s)$$

$$\times \left(\int_{\mathbf{R}^{2d(m-1)}} \mathbb{1}_{E_{\langle i, s+1 \rangle}(X_s, X_{s+m}) = X_{s+m}} f_N^{(s+m)}(Z_{s+m}) dZ_{(s+1, s+m)} \right) d\sigma_i(x_{s+1}) \,,$$

with $d\sigma_i$ the surface measure on $S_\varepsilon(x_i) := \left\{ x \in \mathbf{R}^d, \ |x - x_i| = \varepsilon \right\}$. With (9.3.4), if $m \ge 2$, then the above integral over $\mathbf{R}^{2d(m-1)}$ appears as an integral over $\Delta_{m-1}(x_{s+1})$. We also remark that in the case $m = 1$, we have a simple description of $E_{\langle i, s+1 \rangle}(X_s, X_{s+1}) = X_{s+1}$:

$$(9.4.3) \qquad \left(\mathbb{1}_{E_{\langle i, s+1 \rangle}(X_s, X_{s+1}) = X_{s+1}} \neq 0 \right) \iff \begin{pmatrix} |x_i - x_{s+1}| \le \varepsilon \\ |x_j - x_{s+1}| > \varepsilon \quad \text{for } j \in [1, s] \setminus \{i\} \end{pmatrix} \,.$$

This leads to the following definition of the collision term of order $m \ge 1$, for $s + m \le N$: we define

$$(9.4.4) \quad \begin{aligned} \mathcal{C}_{s, s+m} \widetilde{f}_N^{(s+m)}(Z_s) := m C_{N-s}^m \sum_{i=1}^s \int_{S_\varepsilon(x_i) \times \mathbf{R}^d} \nu^{s+1, i} \cdot (v_{s+1} - v_i) \\ \times G_{\langle i, s+1 \rangle}^{(m-1)}(f_N^{(s+m)})(Z_{s+1}) \, d\sigma_i(x_{s+1}) dv_{s+1} \,, \end{aligned}$$

where for $m = 1$, by (9.4.3):

$$(9.4.5) \qquad G_{\langle i, s+1 \rangle}^{(0)}(\widetilde{f}_N^{(s+1)})(Z_{s+1}) := \left(\prod_{\substack{1 \le j \le s \\ j \ne i}} \mathbb{1}_{|x_{s+1} - x_j| > \varepsilon} \right) \widetilde{f}_N^{(s+1)}(Z_{s+1}) \,,$$

and for $m \geq 2$:

$$
\begin{aligned}
(9.4.6) \quad & G^{(m-1)}_{\langle i, s+1 \rangle}(\widetilde{f}^{(s+m)}_N)(Z_{s+1}) \\
& := \int_{\Delta_{m-1}(x_{s+1}) \times \mathbf{R}^{d(m-1)}} \mathbb{1}_{E_{\langle i, s+1 \rangle}(X_s, X_{s+m}) = X_{s+m}} \widetilde{f}^{(s+m)}_N(Z_{s+m}) dZ_{(s+2, s+m)} \, .
\end{aligned}
$$

The complex-looking indicator function $\mathbb{1}_{E_{\langle i, s+1 \rangle}(X_s, X_{s+m}) = X_{s+m}}$ will, in the estimates of the next chapters, be simply bounded from above by one. This will be the case for instance in an estimate showing that higher-order collision operators (9.4.6) are negligible in the thermodynamical limit; this estimate is (10.3.2) in Proposition 10.3.1. One should notice on the other hand that the operator $\mathcal{C}_{s,s+1}$ is very similar to the corresponding collision operator (4.3.2) in the hard-spheres situation.

With $(N-s)C^{m-1}_{N-s-1} = mC^m_{N-s}$, we can now reformulate (9.2.4) into

$$
\begin{aligned}
(9.4.7) \quad & \int_{\mathbf{R}_+ \times \mathbf{R}^{2ds}} \widetilde{f}^{(s)}_N(t, Z_s) \Big(\partial_t \phi + \mathrm{div}_{X_s}(V_s \phi) - \frac{1}{\varepsilon} \sum_{\substack{i,j=1 \\ j \neq i}}^{s} \nabla \Phi \left(\frac{x_i - x_j}{\varepsilon} \right) \cdot \nabla_{v_i} \phi \Big)(t, Z_s) \, dZ_s dt \\
& + \int_{\mathbf{R}^{2ds}} \widetilde{f}^{(s)}_N(0, Z_s) \phi(0, Z_s) \, dZ_s = \sum_{m=1}^{N-s} \int_{\mathbf{R}^+ \times \mathbf{R}^{2ds}} \phi(t, Z_s) \mathcal{C}_{s,s+m} \widetilde{f}^{(s+m)}_N(t, Z_s) \, dt dZ_s \, ,
\end{aligned}
$$

so that $\widetilde{f}^{(s)}_N$ appears as a (formal) weak solution to

$$
(9.4.8) \quad \partial_t \widetilde{f}^{(s)}_N + \sum_{1 \leq i \leq s} v_i \cdot \nabla_{x_i} \widetilde{f}^{(s)}_N - \frac{1}{\varepsilon} \sum_{1 \leq i \neq j \leq s} \nabla \Phi \left(\frac{x_i - x_j}{\varepsilon} \right) \cdot \nabla_{v_i} \widetilde{f}^{(s)}_N = \sum_{m=1}^{N-s} \mathcal{C}_{s,s+m} \widetilde{f}^{(s+m)}_N \, .
$$

9.5. Mild solutions

We now define the integral formulation of (9.4.8). Denote by $\mathbf{\Phi}_s(t)$ the s-particle Hamiltonian flow, and by $\mathbf{H}_s$ the associated solution operator:

$$
(9.5.1) \quad \mathbf{H}_s(t) : \qquad f \in C^0(\Omega_s; \mathbf{R}) \mapsto f(\mathbf{\Phi}_s(-t, \cdot)) \in C^0(\Omega_s; \mathbf{R}) \, .
$$

The time-integrated form of equation (9.4.8) is

$$
(9.5.2) \quad \widetilde{f}^{(s)}_N(t, Z_s) = \mathbf{H}_s(t) \widetilde{f}^{(s)}_N(0, Z_s) + \sum_{m=1}^{N-s} \int_0^t \mathbf{H}_s(t-\tau) \mathcal{C}_{s,s+m} \widetilde{f}^{(s+m)}_N(\tau, Z_s) \, d\tau \, .
$$

The *total flow* and *total collision* operators $\mathbf{H}$ and $\mathbf{C}_N$ are defined on finite sequences $G_N = (g_s)_{1 \leq s \leq N}$ as follows:

$$
(9.5.3) \quad
\begin{cases}
\forall s \leq N, \ (\mathbf{H}(t) G_N)_s := \mathbf{H}_s(t) g_s \, , \\
\forall s \leq N-1, \ (\mathbf{C}_N G_N)_s := \displaystyle\sum_{m=1}^{N-s} \mathcal{C}_{s,s+m} g_{s+m} \, , \quad (\mathbf{C}_N G_N)_N := 0 \, .
\end{cases}
$$

We define *mild solutions* to the BBGKY hierarchy (9.5.2) to be solutions of

$$
(9.5.4) \quad \widetilde{F}_N(t) = \mathbf{H}(t) \widetilde{F}_N(0) + \int_0^t \mathbf{H}(t-\tau) \mathbf{C}_N \widetilde{F}_N(\tau) \, d\tau \, , \qquad \widetilde{F}_N = (\widetilde{f}^{(s)}_N)_{1 \leq s \leq N} \, .
$$

Remark 9.5.1. — *At this stage, the use of weak formulations could seem a little bit suspicious since they are used essentially as a technical artifice to go from the Liouville equation (1.2.2) to the mild form of the BBGKY hierarchy (9.5.2). In particular, this allows to ignore pathological trajectories involving multiple collisions. Nevertheless, the existence of mild solutions to the BBGKY hierarchy (to be proved*

in the next two chapters) provides the existence of weak solutions to the BBGKY hierarchy, and in particular to the Liouville equation (which is nothing else than the last equation of the hierarchy). The classical uniqueness result for kinetic transport equations then implies that the object we consider, that is the family of truncated marginals, is uniquely determined (almost everywhere).

9.6. The limiting Boltzmann hierarchy

This limit of the BBGKY collision operators (9.4.4) was obtained formally in Section 8.3.2, following the formal derivation of the hard-spheres case in Paragraph 4.4, assuming higher-order interactions can be neglected. We recall the form of the collision operator as given in (8.3.7):

$$\mathcal{C}^0_{s,s+1} f^{(s+1)}(t, Z_s) := \sum_{i=1}^{s} \int b(v_1 - v_2, \omega)$$

$$\times \Big(f^{(s+1)}(t, x_1, v_1, \ldots, x_i, v_i^*, \ldots, x_s, v_s, x_i, v_{s+1}^*) - f^{(s+1)}(t, Z_s, x_i, v_{s+1}) \Big) d\omega dv_{s+1}.$$

where (v_i^*, v_{s+1}^*) is obtained from (v_i, v_{s+1}) by applying the inverse scattering operator σ_0^{-1} defined in Definition 8.2.1 and $b(w, \omega)$ is the cross-section given by Definition 8.3.3.

The asymptotic dynamics are therefore governed by the following integral form of the Boltzmann hierarchy:

$$(9.6.1) \qquad f^{(s)}(t) = \mathbf{S}_s(t) f_0^{(s)} + \int_0^t \mathbf{S}_s(t - \tau) \mathcal{C}^0_{s,s+1} f^{(s+1)}(\tau) \, d\tau,$$

where $\mathbf{S}_s(t)$ denotes the s-particle free-flow.

Similarly to (4.3.7), we can define the total Boltzmann flow and collision operators $\mathbf{S}$ and $\mathbf{C}$ as follows:

$$(9.6.2) \qquad \begin{cases} \forall s \geq 1, \ (\mathbf{S}(t)G)_s := \mathbf{S}_s(t) g_s, \\ \forall s \geq 1, \ (\mathbf{C^0}G)_s := \mathcal{C}^0_{s,s+1} g_{s+1}, \end{cases}$$

so that *mild solutions* to the Boltzmann hierarchy (9.6.1) are solutions of

$$(9.6.3) \qquad F(t) = \mathbf{S}(t) F(0) + \int_0^t \mathbf{S}(t - \tau) \mathbf{C^0} F(\tau) \, d\tau, \qquad F = (f^{(s)})_{s \geq 1}.$$

Note that if $f^{(s)}(t, Z_s) = \prod_{i=1}^{s} f(t, z_i)$ (meaning $f^{(s)}(t)$ is *tensorized*) then f satisfies the Boltzmann equation (2.1.1)-(2.1.2), with the cross-section $b(w, \omega)$ given by Definition 8.3.3.

CHAPTER 10

CLUSTER ESTIMATES AND UNIFORM A PRIORI ESTIMATES

In view of proving the existence of mild solutions to the BBGKY hierarchy (9.5.2), we need continuity estimates on the linear collision operators $\mathcal{C}_{s,s+m}$ defined in (9.4.4)-(9.4.5)-(9.4.6), and the total collision operator $\mathbf{C}_N$ defined in (9.5.3).

We first note that, by definition, the operator $\mathcal{C}_{s,s+m}$ involves only configurations with clusters of length m. Classical computations of statistical mechanics, presented in Section 10.1, show that the probability of finding such clusters is exponentially decreasing with m.

It is then natural to introduce functional spaces encoding the decay with respect to energy and the growth with respect to the order of the marginal (see Section 10.2, where norms are introduced, generalizing the norms introduced in Chapter 5 for the hard spheres case). In these appropriate functional spaces, we can establish uniform continuity estimates for the BBGKY collision operators (Section 10.3). These will enable us in Section 10.4 to obtain directly uniform bounds for the hierarchy as in Chapter 5.

10.1. Cluster estimates

A point $X_s \in \mathbf{R}^{ds}$ being given, we recall that $\Delta_m(X_s)$ is the set of all clusters of base X_s and length m (this notation is introduced in Definition 9.3.3 page 70).

Lemma 10.1.1. — *For any symmetric function φ on $\mathbf{R}^{Nd}$, any $s \in [1, N-1]$, any $X_s \in \mathbf{R}^{ds}$, the following identity holds:*

(10.1.1)
$$\int_{\mathbf{R}^{(N-s)d}} \varphi(X_N) dX_{(s+1,N)} = \int_{\mathbf{R}^{d(N-s)}} \mathbb{1}_{X_N \in \mathcal{D}_N^s} \, \varphi(X_N) \, dX_{(s+1,N)}$$
$$+ \sum_{m=1}^{N-s} C_{N-s}^m \int_{\Delta_m(X_s)} \left(\int_{\mathbf{R}^{d(N-s-m)}} \mathbb{1}_{X_N \in \mathcal{D}_N^{s+m}} \, \varphi(X_N) \, dX_{(s+m+1,N)} \right) dX_{(s+1,s+m)} \,,$$

implying, for $\zeta > 0$,

(10.1.2)
$$\frac{1}{m!} \int_{\Delta_m(X_s)} dX_{(s+1,s+m)} \leq \zeta^{-m} \exp\left(\zeta \kappa_d (s+m) \varepsilon^d \right)$$

and

(10.1.3) $$\sum_{m \geq 1} \frac{\zeta^{m+1} \exp\left(-\zeta \kappa_d (m+1)\varepsilon^d\right)}{m!} \int_{\Delta_m(x_1)} dX_{(2,m+1)} \leq \zeta\left(1 - \exp\left(-\zeta \kappa_d \varepsilon^d\right)\right),$$

where κ_d is the volume of the unit ball in $\mathbf{R}^d$.

Proof. — The first identity (10.1.1) is obtained by a simple partitioning argument, which extends the splitting used to define $\mathcal{C}_{s,s+m}$ in (9.4.4) in the previous chapter. We recall that, given any $X_s \in \mathbf{R}^{ds}$, the family

$$\left\{(x_{s+1}, \ldots, x_N), |E(X_s, X_N)| = s + m\right\} \quad \text{for } 0 \leq m \leq N - s,$$

is a partition of $\mathbf{R}^{(N-s)d}$. Then we use the symmetry assumption, as we did in (9.4.2), to find

$$\int_{\mathbf{R}^{(N-s)d}} \varphi(X_N) dX_{(s+1,N)} = \sum_{0 \leq m \leq N-s} C_{N-s}^m \int_{\mathbf{R}^{(N-s)d}} \mathbb{1}_{E(X_s, X_N) = X_{s+m}} \varphi(X_N) dX_{(s+1,N)}.$$

It then suffices to use equivalence (9.3.2) from Lemma 9.3.4, noting that the set of all $(x_{s+1}, \ldots, x_{s+m})$ in $\mathbf{R}^{md}$ such that $E(X_s, X_{s+m}) = X_{s+m}$ coincides with $\Delta_m(X_s)$. This proves (10.1.1).

Estimates (10.1.2) and (10.1.3) come from the counterpart of (10.1.1) at the grand canonical level, i.e. when the activity ζ^{-1} is fixed, rather than the total number N of particles; Remark 5.1.3 expands on this terminology.

For any bounded $\Lambda \subset \mathbf{R}^d$, the associated grand-canonical ensemble for n non-interacting particles is defined as the probability measure with density

$$\varphi_n(X_n) := \frac{\zeta^n \exp(-\zeta|\Lambda|)}{n!} \prod_{1 \leq i \leq n} \mathbb{1}_{x_i \in \Lambda}.$$

The s-point correlation function g_s and the truncated s-point correlation function $\widetilde{g}_s$ are defined by

$$g_s(X_s) := \sum_{n=s}^{\infty} \frac{n!}{(n-s)!} \int_{\mathbf{R}^{(n-s)d}} \varphi_n(X_n) dX_{(s+1,n)},$$

$$\widetilde{g}_s(X_s) := \sum_{n=s}^{\infty} \frac{n!}{(n-s)!} \int_{\mathbf{R}^{(n-s)d}} \mathbb{1}_{X_n \in \mathcal{D}_n^s} \varphi_n(X_n) dX_{(s+1,n)}.$$

We compute

$$\int_{\mathbf{R}^{(n-s)d}} \varphi_n(X_n) dX_{(s+1,n)} = \zeta^s \exp\left(-\zeta|\Lambda|\right) \frac{(\zeta|\Lambda|)^{n-s}}{n!} \prod_{1 \leq i \leq s} \mathbb{1}_{x_i \in \Lambda},$$

so that

(10.1.4) $$g_s(X_s) = \zeta^s \exp\left(-\zeta|\Lambda|\right) \sum_{k=0}^{\infty} \frac{(\zeta|\Lambda|)^k}{k!} \prod_{1 \leq i \leq s} \mathbb{1}_{\Lambda}(x_i) = \zeta^s \prod_{1 \leq i \leq s} \mathbb{1}_{x_i \in \Lambda}.$$

Similarly, by definition of $\mathcal{D}_n^s$ in (9.1.4),

$$\int_{\mathbf{R}^{(n-s)d}} \mathbb{1}_{X_n \in \mathcal{D}_n^s} \prod_{s+1 \leq j \leq n} \mathbb{1}_{x_i \in \Lambda} dX_{(s+1,n)} = \left|\Lambda \cap \left(\bigcap_{1 \leq i \leq s} {}^c B_\varepsilon(x_i)\right)\right| = \left|\Lambda \cap {}^c B_\varepsilon(X_s)\right|,$$

where we denote $B_\varepsilon(X_s) := \bigcup_{1 \leq i \leq s} B_\varepsilon(x_i)$, with $B_\varepsilon(x_i) := \{y \in \mathbf{R}^d, |y - x_i| \leq \varepsilon\}$. This implies

$$\widetilde{g}_s(X_s) = \zeta^s \exp\left(-\zeta|\Lambda|\right) \sum_{n \geq s} \frac{\left(\zeta|\Lambda \cap {}^c B_\varepsilon(X_s)|\right)^{n-s}}{(n-s)!} \prod_{1 \leq i \leq s} \mathbb{1}_{x_i \in \Lambda}.$$

If $B_\varepsilon(x_i) \subset \Lambda$ for all $1 \leq i \leq s$, then $|\Lambda| - |\Lambda \cap {}^c B_\varepsilon(X_s)| = |B_\varepsilon(X_s)|$. We obtain

(10.1.5) $$\widetilde{g}_s(X_s) = \zeta^s \exp\left(-\zeta |B_\varepsilon(X_s)|\right), \qquad \text{if } B_\varepsilon(X_s) \subset \Lambda.$$

Besides, by (10.1.1),

$$g_s(X_s) = \widetilde{g}_s(X_s)$$
$$+ \sum_{n=s}^{\infty} \sum_{m=1}^{n-s} \frac{n! C_{n-s}^m}{(n-s)!} \int_{\Delta_m(X_s)} \left(\int_{\mathbf{R}^{(n-s-m)d}} \mathbb{1}_{X_n \in \mathcal{D}_n^{s+m}} g_s(X_n)\, dX_{(s+m+1,n)} \right) dX_{(s+1,s+m)}.$$

By Fubini, we get

$$\sum_{n=s}^{\infty} \sum_{m=1}^{n-s} \frac{n! C_{n-s}^m}{(n-s)!} \int_{\Delta_m(X_s)} \left(\int_{\mathbf{R}^{(n-s-m)d}} \mathbb{1}_{X_n \in \mathcal{D}_n^{s+m}} \varphi_n(X_n)\, dX_{(s+m+1,n)} \right) dX_{(s+1,s+m)}$$

$$= \sum_{n=s}^{\infty} \sum_{m=1}^{n-s} \frac{n!}{(k-s)!(n-k)!} \int_{\Delta_{k-s}(X_s)} \left(\int_{\mathbf{R}^{(n-k)d}} \mathbb{1}_{X_n \in \mathcal{D}_n^k} \varphi_n(X_n)\, dX_{(k+1,n)} \right) dX_{(s+1,k)}$$

$$= \sum_{k=s+1}^{\infty} \frac{1}{(k-s)!} \sum_{n=k}^{\infty} \frac{n!}{(n-k)!} \int_{\Delta_{k-s}(X_s)} \left(\int_{\mathbf{R}^{(n-k)d}} \mathbb{1}_{X_n \in \mathcal{D}_n^k} \varphi_n(X_n)\, dX_{(k+1,n)} \right) dX_{(s+1,k)}$$

$$= \sum_{k=s+1}^{\infty} \frac{1}{(k-s)!} \int_{\Delta_{k-s}(X_s)} \widetilde{g}_k(X_k)\, dX_{(s+1,k)}.$$

We have proved that

(10.1.6) $$g_s(X_s) = \widetilde{g}_s(X_s) + \sum_{k=s+1}^{\infty} \frac{1}{(k-s)!} \int_{\Delta_{k-s}(X_s)} g_k(X_k)\, dX_{(s+1,k)}.$$

We now show how identities (10.1.4)-(10.1.5)-(10.1.6) imply the bounds (10.1.2)-(10.1.3).

We first retain only the contribution of $k = s + m$ in the right-hand side of (10.1.6). Given $\varepsilon > 0$ and $X_s \in \mathbf{R}^{ds}$, we choose Λ large enough so that $B_\varepsilon(Y) \subset \Lambda$ for all $Y \in \Delta_m(X_s)$. This gives

$$\zeta^s \geq \frac{1}{m!} \int_{\Delta_m(X_s)} \zeta^{s+m} \exp\left(-\zeta |B_\varepsilon(X_{s+m})|\right) dX_{(s+1,s+m)},$$

and now $|B_\varepsilon(X_{s+m})| \leq \kappa_d \varepsilon^d (s+m)$ implies (10.1.2).

We finally fix an integer $K \geq 2$ and choose $s = 1$ in (10.1.6). Given $\varepsilon > 0$ and $x_1 \in \mathbf{R}^d$, we choose Λ large enough so that $B_\varepsilon(Y_K) \subset \Lambda$ for all $Y_K \in \Delta_K(x_1)$. This gives

$$\zeta - \zeta \exp\left(-\zeta |B_\varepsilon(x_1)|\right) \geq \sum_{k=2}^{K} \int_{\Delta_{k-1}(x_1)} \zeta^k \exp\left(-\zeta |B_\varepsilon(X_k)|\right) dX_{(2,k)},$$

and bounding the volumes of balls from above, we find

$$\zeta \left(1 - \exp(-\zeta \kappa_d \varepsilon^d)\right) \geq \sum_{k=1}^{K-1} \frac{\zeta^{k+1}}{k!} \exp\left(-\zeta \kappa_d (k+1) \varepsilon^d\right) \int_{\Delta_k(x_1)} dX_{(2,k+1)}.$$

It then suffices to let $K \to \infty$ to find (10.1.3). This ends the proof of Lemma 10.1.1. $\qquad \square$

10.2. Functional spaces

To show the convergence of the series defining mild solutions (9.5.2) to the BBGKY hierarchy, we need to introduce some norms on the space of sequences $(\widetilde{f}^{(s)})_{s\geq 1}$. Given $\varepsilon > 0$, $\beta > 0$, an integer $s \geq 1$, and a continuous function $g_s : \Omega_s \to \mathbf{R}$, we let

$$(10.2.1) \qquad |g_s|_{\varepsilon,s,\beta} := \sup_{Z_s \in \Omega_s} \left(|g_s(Z_s)| \exp\left(\beta E_\varepsilon(Z_s)\right) \right)$$

where for $\varepsilon > 0$, the function E_ε is the s-particle Hamiltonian

$$(10.2.2) \qquad E_\varepsilon(Z_s) := \sum_{1 \leq i \leq s} \frac{|v_i|^2}{2} + \sum_{1 \leq i < k \leq s} \Phi_\varepsilon(x_i - x_k), \quad \text{with} \quad \Phi_\varepsilon(x) := \Phi\left(\frac{x}{\varepsilon}\right).$$

Notice that this norm does coincide with its counterpart defined in Paragraph 5.1 in the limit described in Remark 1.0.1.

Definition 10.2.1. — *For $\varepsilon > 0$ and $\beta > 0$, we denote $X_{\varepsilon,s,\beta}$ the Banach space of continuous functions $\Omega_s \to \mathbf{R}$ with finite $|\cdot|_{\varepsilon,s,\beta}$ norm.*

By Assumption 1.2.1, for $\varepsilon > 0$ (and $\beta > 0$) there holds $\exp(\beta E_\varepsilon(Z_s)) \to \infty$ as Z_s approaches $\partial\Omega_s$. This implies for $g_s \in X_{\varepsilon,s,\beta}$ the existence of an extension by continuity: $\bar{g}_s \in C^0(\mathbf{R}^{2ds}; \mathbf{R})$ such that $\bar{g}_s \equiv 0$ on $\partial\Omega_s$, and $\bar{g}_s \equiv g$ on Ω_s.

For sequences of functions $G = (g_s)_{s\geq 1}$, with $g_s : \Omega_s \to \mathbf{R}$, we let for $\varepsilon > 0$, $\beta > 0$, $\mu \in \mathbf{R}$,

$$\|G\|_{\varepsilon,\beta,\mu} := \sup_{s\geq 1} \left(|g_s|_{\varepsilon,s,\beta} \exp(\mu s) \right).$$

Definition 10.2.2. — *For $\varepsilon \geq 0$, $\beta > 0$, and $\mu \in \mathbf{R}$, we denote $\mathbf{X}_{\varepsilon,\beta,\mu}$ the Banach space of sequences $G = (g_s)_{s\geq 1}$, with $g_s \in X_{\varepsilon,s,\beta}$ and $\|G\|_{\varepsilon,\beta,\mu} < \infty$.*

As in (5.1.4), he following inclusions hold:

$$(10.2.3) \qquad \text{if } \beta' \leq \beta \text{ and } \mu' \leq \mu, \text{ then} \quad X_{\varepsilon,s,\beta'} \subset X_{\varepsilon,s,\beta}, \quad \mathbf{X}_{\varepsilon,\beta',\mu'} \subset \mathbf{X}_{\varepsilon,\beta,\mu}.$$

Finally similarly to Definition 5.1.4 we define norms of time-dependent functions as follows.

Definition 10.2.3. — *Given $T > 0$, a positive function $\boldsymbol{\beta}$ and a real valued function $\boldsymbol{\mu}$ defined on $[0, T]$ we denote $\mathbf{X}_{\varepsilon,\boldsymbol{\beta},\boldsymbol{\mu}}$ the space of functions $G : t \in [0, T] \mapsto G(t) = (g_s(t))_{1\leq s} \in \mathbf{X}_{\varepsilon,\beta(t),\mu(t)}$, such that for all $Z_s \in \mathbf{R}^{2ds}$, the map $t \in [0, T] \mapsto g_s(t, Z_s)$ is measurable, and*

$$(10.2.4) \qquad \|G\|_{\varepsilon,\boldsymbol{\beta},\boldsymbol{\mu}} := \sup_{0\leq t\leq T} \|G(t)\|_{\varepsilon,\beta(t),\mu(t)} < \infty.$$

Notice that the following conservation of energy properties hold, as for (5.2.1):

$$(10.2.5) \qquad |\mathbf{H}_s(t)g_s|_{\varepsilon,s,\beta} = |g_s|_{\varepsilon,s,\beta} \quad \text{and} \quad \|\mathbf{H}(t)G_N\|_{\varepsilon,\beta,\mu} = \|G_N\|_{\varepsilon,\beta,\mu},$$

for all parameters $\beta > 0$, $\mu \in \mathbf{R}$, and for all $g_s \in X_{\varepsilon,s,\beta}$, $G_N = (g_s)_{1\leq s\leq N} \in \mathbf{X}_{\varepsilon,\beta,\mu}$, and all $t \geq 0$.

10.3. Continuity estimates

We now establish bounds, in the above defined functional spaces, for the collision operators defined in (9.4.4)-(9.4.6), and for the total collision operator $\mathbf{C}_N$ defined in (9.5.3).

Notice that in the case when $m = 1$ the estimates are the same as in Chapter 5: in particular thanks to (10.2.5) the following bound holds:

$$(10.3.1) \qquad e^{s(\mu_0 - \lambda t)} \Big| \int_0^t \mathbf{H}_s(t - \tau)\mathcal{C}_{s,s+1}g_{s+1}(\tau)\,d\tau \Big|_{\varepsilon,s,\beta_0 - \lambda t} \leq \bar{c}(\beta_0, \mu_0, \lambda, T) \|G_N\|_{\varepsilon,\boldsymbol{\beta},\boldsymbol{\mu}},$$

for all $G_N = (g_{s+1})_{1 \leq s \leq N} \in \mathbf{X}_{\varepsilon,\boldsymbol{\beta},\boldsymbol{\mu}}$, with $\bar{c}(\beta_0, \mu_0, \lambda, T)$ computed explicitly in (5.3.11).

The following statement is the analogue of Proposition 5.3.1 in the hard spheres case, but in the present situation higher order correlations must be taken into account.

Proposition 10.3.1. — *Given $\beta > 0$ and $\mu \in \mathbf{R}$, for $m \geq 1$ and $1 \leq s \leq N - m$, the collision operators $\mathcal{C}_{s,s+m}$ satisfy the bounds, for all $G_N = (g_s)_{1 \leq s \leq N} \in \mathbf{X}_{\varepsilon,\beta,\mu}$,*

$$(10.3.2) \quad \big|\mathcal{C}_{s,s+m}g_{s+m}(Z_s)\big| \leq \varepsilon^{m-1}C_d e^{m\kappa_d}(\beta/C_d)^{-\frac{md}{2}}\Big(s\beta^{-\frac{1}{2}} + \sum_{1 \leq i \leq s}|v_i|\Big)e^{-\beta E_\varepsilon(Z_s)}|g_{s+m}|_{\varepsilon,s+m,\beta},$$

for some $C_d > 0$ depending only on d.

If $\varepsilon < e^{\mu - \kappa_d}(\beta/2\pi)^{d/2}$, then for all $0 < \beta' < \beta$ and $\mu' < \mu$, the total collision operator $\mathbf{C}_N$ satisfies the bound

$$(10.3.3) \qquad \|\mathbf{C}_N G_N\|_{\varepsilon,\beta',\mu'} \leq C_d(1 + \beta^{-\frac{1}{2}})\Big(\frac{1}{\beta - \beta'} + \frac{1}{\mu - \mu'}\Big)\|G_N\|_{\varepsilon,\beta,\mu}.$$

Considering the case $m > 1$ in (10.3.2), for which the upper bound is $O(\varepsilon)$, we see that higher-order interactions are negligible in the Boltzmann-Grad limit.

Proof. — We shall only consider the case $m \geq 2$, as the case $m = 1$ is dealt with exactly as in the proof of Proposition 5.3.1. From the definition of $G_{\langle i,s+1\rangle}^{(m-1)}$ in (9.4.6), we obtain

$$\big|G_{\langle i,s+1\rangle}^{(m-1)}(g_{s+m})(Z_{s+1})\big| \leq |g_{s+m}|_{\varepsilon,s+m,\beta}\int_{\Delta_{m-1}(x_{s+1})\times\mathbf{R}^{d(m-1)}}\exp\big(-\beta E_\varepsilon(Z_{s+m})\big)dZ_{(s+2,s+m)},$$

where the norm $|\cdot|_{\varepsilon,s,\beta}$ is defined in (10.2.1), and the Hamiltonian E_ε is defined in (10.2.2). For the collision operator defined in (9.4.4), this implies the bound

$$(10.3.4) \qquad |\mathcal{C}_{s,s+m}g_{s+m}(Z_s)| \leq mC_{N-s}^m|g_{s+m}|_{\varepsilon,s+m,\beta} \times \sum_{1 \leq i \leq s} I_{i,m}(V_s) \times J_{i,m}(X_s),$$

where $I_{i,m}$ is the velocity integral

$$I_{i,m}(V_s) := \int_{\mathbf{R}^{dm}}\big(|v_{s+1}| + |v_i|\big)\exp\Big(-\frac{\beta}{2}\sum_{j=1}^{s+m}|v_j|^2\Big)dV_{(s+1,s+m)},$$

and $J_{i,m}$ is the spatial integral

$$J_{i,m}(X_s) := \int_{S_\varepsilon(x_i)\times\Delta_{m-1}(x_{s+1})}\exp\Big(-\beta\sum_{1 \leq j < k \leq s+m}\Phi_\varepsilon(x_j - x_k)\Big)d\sigma(x_{s+1})dX_{(s+2,s+m)}.$$

The velocity integral is a product of Gaussian integrals and can be exactly computed:

$$(10.3.5) \qquad I_{i,m}(V_s) \leq (\beta/C_d)^{-\frac{md}{2}}\Big(|v_i| + \beta^{-\frac{1}{2}}\Big)\exp\Big(-\frac{\beta}{2}\sum_{1 \leq j \leq s}|v_j|^2\Big).$$

For the spatial integral, there holds

$$J_{i,m}(X_s) \le \exp\Big(-\beta \sum_{1\le j<k\le s} \Phi_\varepsilon(x_j-x_k)\Big) |S_\varepsilon(x_i)| \times \sup_x \int_{\Delta_{m-1}(x)} dX_{(1,m-1)}$$

$$\le \exp\Big(-\beta \sum_{1\le j<k\le s} \Phi_\varepsilon(x_j-x_k)\Big) \times \kappa_d \varepsilon^{d-1} \times \Big((m-1)!\,\varepsilon^{(m-1)d}\exp(m\kappa_d)\Big),$$

where in the last bound we used identity (10.1.2) from Lemma 10.1.1 with $s=1$ and $\zeta=\varepsilon^{-d}$. This implies

$$|\mathcal{C}_{s,s+m}g_{s+m}(Z_s)| \le C_d \varepsilon^{m-1}\big((N-s)\varepsilon^{d-1}\big)^m e^{m\kappa_d}(\beta/C_d)^{-\frac{md}{2}}\Big(s\beta^{-\frac12}+\sum_{1\le i\le s}|v_i|\Big)$$

$$\times e^{-\beta E_\varepsilon(Z_s)}|g_{s+m}|_{\varepsilon,s+m,\beta}\,.$$

In the Boltzmann-Grad scaling $N\varepsilon^{d-1}\equiv 1$, this gives (10.3.2). Above and in the following, C_d denotes a positive constant which depends only on d, and which may change from line to line.

We turn to the proof of (10.3.3), which is similar to the proof of (5.3.2) up to the control of higher correlations. From the pointwise inequality (5.3.3) we deduce for the above velocity integral $I_{i,m}(V_s)$ the bound, for $0<\beta'<\beta$,

$$\sum_{1\le i\le s}\exp\Big((\beta'/2)\sum_{1\le j\le s}|v_j|^2\Big)I_{i,m}(V_s) \le C_d(\beta/C_d)^{-\frac{md}{2}}\Big(s\beta^{-\frac12}+s^{\frac12}(\beta-\beta')^{-\frac12}\Big).$$

From the above bound in $J_{i,m}(X_s)$, we deduce immediately, for $0<\beta'<\beta$,

$$\max_{1\le i\le s}\exp\Big(\beta'\sum_{1\le j<k\le s}\Phi_\varepsilon(x_j-x_k)\Big)J_{i,m}(X_s) \le \kappa_d(m-1)!\,e^{m\kappa_d}\varepsilon^{md-1}\,.$$

With (10.3.4), these bounds yield, in the Boltzmann-Grad scaling,

$$e^{\beta' E_\varepsilon(Z_s)+\mu' s}|\mathcal{C}_{s,s+m}g_{s+m}(Z_s)| \le \varepsilon^{m-1}C_d(\beta/C_d)^{-\frac{md}{2}}e^{m\kappa_d}e^{\mu' s}\big(s\beta^{-\frac12}+s^{\frac12}(\beta-\beta')^{-\frac12}\big)$$

$$\times |g_{s+m}|_{\varepsilon,s+m,\beta}\,.$$

Summing over m, we finally obtain, for $\mathbf{C}_N$ defined in (9.5.3),

$$\|\mathbf{C}_N G_N\|_{\varepsilon,\beta',\mu'} \le C_d\|G_N\|_{\varepsilon,\beta,\mu}\sup_{1\le s\le N}\Big(\big(s\beta^{-\frac12}+s^{\frac12}(\beta-\beta')^{-\frac12}\big)e^{-(\mu-\mu')s}\Big)$$

$$\times \sum_{1\le m\le N-s}e^{-m(\mu-\kappa_d)}\varepsilon^{m-1}\,(\beta/C_d)^{-\frac{md}{2}}\,.$$

If ε is small enough so that $\varepsilon e^{\kappa_d-\mu}(C_d/\beta)^{d/2}<1$, then the above series is convergent, and

$$\sum_{1\le m\le N-s}e^{-m(\mu-\kappa_d)}\varepsilon^{m-1}(C_d/\beta)^{md/2} \le \frac{e^{\kappa_d-\mu}(C_d/\beta)^{d/2}}{1-\varepsilon e^{\kappa_d-\mu}(C_d/\beta)^{d/2}}\,.$$

We conclude as in the proof of Proposition 5.3.1. Proposition 10.3.1 is proved. □

Remark 10.3.1. — *We do not use the extra decay provided by the contribution of the potential in the exponential of the Hamiltonian. This is quite obvious in the bound for $J_{i,m}(X_s)$ in the proof of Proposition 10.3.1, where we bound $e^{-\beta\sum_{1\le j<k\le s+m}\Phi_\varepsilon(x_j-x_k)}$ by $e^{-\beta\sum_{1\le j<k\le s}\Phi_\varepsilon(x_j-x_k)}$. Then, we might be tempted to replace E_ε by the free Hamiltonian E_0 in the definition of the functional spaces. The kinetic energy, however, is not a conserved quantity, so that in $X_{0,s,\beta}$ spaces the conservation of energy does not hold.*

This leads to the following lemma, which is the key to the proof of the uniform bound stated in Theorem 9 in the next paragraph. It is the analogue of Lemma 5.3.1.

Lemma 10.3.2. — *Let $\beta_0 > 0$ and $\mu_0 \in \mathbf{R}$ be given. There is $T > 0$ depending only on β_0 and μ_0 such that for an appropriate choice of λ in $(0, \beta_0/T)$, there holds for all $t \in [0, T]$*

$$(10.3.6) \qquad \left\| \int_0^t \mathbf{H}(t-\tau) \mathbf{C}_N G_N(\tau) \, d\tau \right\|_{\varepsilon, \beta_0 - \lambda t, \mu_0 - \lambda t} \leq \frac{1}{2} \||G_N\||_{\varepsilon, \boldsymbol{\beta}, \boldsymbol{\mu}} \, .$$

Proof. — We follow closely the proof of Lemma 5.3.1. The difference is that here we take into account higher-order collision operators $\mathcal{C}_{s,s+m}$, with $m \geq 2$. Using notation (5.3.7), Estimate (10.3.2) from Proposition 10.3.1 gives

$$e^{\beta_0^\lambda(t) E_\varepsilon(Z_s)} |\mathcal{C}_{s,s+m} g_{s+m}(t', Z_s)|$$
$$\leq \varepsilon^{m-1} C_d e^{m\kappa_d} (2\pi/\beta_0^\lambda(t'))^{md/2} |g_{s+m}(t')|_{\varepsilon, s+m, \beta_0^\lambda(t')} \left(s(\beta_0^\lambda(t'))^{-d/2} + \sum_{1 \leq i \leq s} |v_i| \right) e^{\lambda(t'-t) E_\varepsilon(Z_s)} \, .$$

Using also (5.3.8) with $s + 1$ replaced by $s + m$, we get

$$(10.3.7) \qquad \begin{aligned} & \left\| \int_0^t \mathbf{H}(t-t') \mathbf{C}_N G_N(t') \, dt' \right\|_{\varepsilon, \beta_0^\lambda(t), \mu_0^\lambda(t)} \\ & \leq \||G_N\||_{\varepsilon, \boldsymbol{\beta}, \boldsymbol{\mu}} \Big(\sum_{1 \leq m \leq N-s} C_m \Big) \sup_{Z_s \in \mathbf{R}^{2ds}} \int_0^t \overline{C}(t, t', Z_s) \, dt' \, , \end{aligned}$$

where $C_m := C_d \varepsilon^{m-1} e^{m(\kappa_d - \mu_0^\lambda(T))} (C_d/\beta_0^\lambda(T))^{md/2}$, and $\overline{C}$ is defined in (5.3.9) and satisfies (5.3.10) which we recall here:

$$(10.3.8) \qquad \sup_{Z_s \in \mathbf{R}^{2ds}} \int_0^t \overline{C}(\tau, t, Z_s) \, d\tau \leq \frac{C_d}{\lambda} \left(1 + \frac{1}{(\beta_0^\lambda(T))^{d/2}} \right) ,$$

Under the assumption that

$$(10.3.9) \qquad \varepsilon_0 e^{\kappa_d - \mu_0^\lambda(T)} (2\pi/\beta_0^\lambda(T))^{d/2} < 1/2 \, ,$$

we find

$$(10.3.10) \qquad \sum_{1 \leq m \leq N-s} C_m \leq 2 C_d e^{-\mu_0^\lambda(T)} (\beta_0^\lambda(T))^{-d/2} \, .$$

The upper bounds in (10.3.8) and (10.3.10) are independent of s, and their product is equal to $2\bar{c}(\beta_0, \mu_0, \lambda, T)$. It then suffices to choose λ so that $2\bar{c}(\beta_0, \mu_0, \lambda, T) \leq 1/2$ and taking the supremum in s in (10.3.7) then yields the result. □

10.4. Uniform bounds for the BBGKY and Boltzmann hierarchies

The results of the previous section enable us, exactly as in the hard spheres case page 35, to deduce directly the following bounds on the BBGKY hierarchy defined in (9.5.4) page 72.

Theorem 9 (**Uniform bounds for the BBGKY hierarchy**). — *Let $\beta_0 > 0$ and $\mu_0 \in \mathbf{R}$ be given. There is a time $T > 0$ as well as two nonincreasing functions $\boldsymbol{\beta} > 0$ and $\boldsymbol{\mu}$ defined on $[0, T]$, satisfying $\boldsymbol{\beta}(0) = \beta_0$ and $\boldsymbol{\mu}(0) = \mu_0$, such that in the Boltzmann-Grad scaling $N\varepsilon^{d-1} \equiv 1$, any family of initial*

marginals $\widetilde{F}_N(0) = \left(\widetilde{f}_N^{(s)}(0)\right)_{1 \leq s \leq N}$ *in* $\mathbf{X}_{\varepsilon,\beta_0,\mu_0}$ *gives rise to a unique solution* $\widetilde{F}_N(t) = (\widetilde{f}_N^{(s)}(t))_{1 \leq s \leq N}$ *in* $\mathbf{X}_{\varepsilon,\boldsymbol{\beta},\boldsymbol{\mu}}$ *to the BBGKY hierarchy* (9.5.4) *satisfying the following bound:*

$$\|\|\widetilde{F}_N\|\|_{\varepsilon,\boldsymbol{\beta},\boldsymbol{\mu}} \leq 2\|\widetilde{F}_N(0)\|_{\varepsilon,\beta_0,\mu_0}\,.$$

In the case of the Boltzmann hierarchy associated with the collision operator (8.3.6), the same existence result as in Theorem 7 holds, again with the same proof.

Theorem 10 (**Existence for the Boltzmann hierarchy**). — *Assume the potential satisfies Assumption* 1.2.1. *Let* $\beta_0 > 0$ *and* $\mu_0 \in \mathbf{R}$ *be given. There is a time* $T > 0$ *as well as two nonincreasing functions* $\boldsymbol{\beta} > 0$ *and* $\boldsymbol{\mu}$ *defined on* $[0, T]$, *satisfying* $\boldsymbol{\beta}(0) = \beta_0$ *and* $\boldsymbol{\mu}(0) = \mu_0$, *such that any family of initial marginals* $F(0) = \left(f^{(s)}(0)\right)_{s \geq 1}$ *in* $\mathbf{X}_{0,\beta_0,\mu_0}$ *gives rise to a unique solution* $F(t) = (f^{(s)}(t))_{s \geq 1}$ *in* $\mathbf{X}_{0,\boldsymbol{\beta},\boldsymbol{\mu}}$ *to the Boltzmann hierarchy* (5.0.2), *satisfying the following bound:*

$$\|\|F\|\|_{0,\boldsymbol{\beta},\boldsymbol{\mu}} \leq 2\|F(0)\|_{0,\beta_0,\mu_0}\,.$$

CHAPTER 11

CONVERGENCE RESULT AND STRATEGY OF PROOF

The main goal of this chapter is to reduce the proof of Theorem 5 stated page 17 to the term-by-term convergence of some functionals involving a finite (uniformly bounded) number of marginals with only first-order collisions, bounded energies and a finite number of collision times, exactly as was performed in Chapter 7 (see Section 11.3).

Before doing so we define, as in the hard spheres case, the notion of admissible initial data in Section 11.1. We give the precise version of Theorem 5 in Section 11.2.

11.1. Admissible initial data

The characterization of admissible initial data is very similar to the hard spheres case studied in Paragraph 6.1.1. The only new aspect concerns the fact that marginals have been truncated, and that feature will be dealt with in this section.

Definition 11.1.1 (Admissible Boltzmann data). — *Admissible Boltzmann data are defined as families $F_0 = (f_0^{(s)})_{s \geq 1}$, with each $f_0^{(s)}$ nonnegative, integrable and continuous over Ω_s, such that*

$$(11.1.1) \qquad \int_{\mathbf{R}^{2d}} f_0^{(s+1)}(Z_s, z_{s+1}) \, dz_{s+1} = f_0^{(s)}(Z_s) \,,$$

and which are limits of BBGKY initial data $\widetilde{F}_{0,N} = (\widetilde{f}_{0,N}^{(s)})_{1 \leq s \leq N} \in \mathbf{X}_{\varepsilon, \beta_0, \mu_0}$ in the following sense: it is assumed that

$$(11.1.2) \qquad \sup_{N \geq 1} \|\widetilde{F}_{0,N}\|_{\varepsilon, \beta_0, \mu_0} < \infty \,, \quad \text{for some } \beta_0 > 0 \,, \, \mu_0 \in \mathbf{R} \,, \text{ as } N \varepsilon^{d-1} \equiv 1 \,,$$

and that for each given $s \in [1, N]$, the truncated marginal of order s defined by

$$(11.1.3) \qquad \widetilde{f}_{0,N}^{(s)}(Z_s) = \int_{\mathbf{R}^{2d(N-s)}} \mathbb{1}_{\mathcal{D}_N^s}(X_N) \widetilde{f}_{0,N}^{(N)}(Z_N) dZ_{(s+1,N)} \,, \quad 1 \leq s < N \,,$$

converges in the Boltzmann-Grad limit:

$$(11.1.4) \qquad \widetilde{f}_{0,N}^{(s)} \longrightarrow f_0^{(s)} \,, \quad \text{as } N \to \infty \text{ with } N \varepsilon^{d-1} \equiv 1 \,, \text{ locally uniformly in } \Omega_s \,.$$

The following result is proved very similarly to Proposition 6.1.1.

Proposition 11.1.1. — *The set of admissible Boltzmann data, in the sense of Definition 11.1.1, is the set of families of marginals F_0 as in (11.1.1) satisfying a uniform bound $\|F_0\|_{0,\beta_0,\mu_0} < \infty$ for some $\beta_0 > 0$ and $\mu_0 \in \mathbf{R}$.*

We shall not give the proof of that result, as the only difference with Proposition 6.1.1 lies in the presence of a truncation in the marginals, whose effect disappears asymptotically as stated in the following lemma.

Lemma 11.1.2. — *Given $\widetilde{F}_{0,N} = (\widetilde{f}_{0,N}^{(s)})_{1 \le s \le N}$ satisfying (11.1.2) and (11.1.3) from Definition 11.1.1, with associated family $F_{0,N} = (f_{0,N}^{(s)})_{1 \le s \le N}$ of untruncated marginals:*

$$(11.1.5) \qquad f_{0,N}^{(s)}(Z_s) = \int_{\mathbf{R}^{2d(N-s)}} f_{0,N}^{(N)}(Z_N) \, dZ_{(s+1,N)} \,, \quad 1 \le s < N, \quad Z_s \in \Omega_s, \qquad \widetilde{f}_{0,N}^{(N)} = f_{0,N}^{(N)} \,,$$

there holds the convergence

$$f_{0,N}^{(s)} - \widetilde{f}_{0,N}^{(s)} \longrightarrow 0, \qquad \text{for fixed } s \ge 1, \text{ as } N \to \infty \text{ with } N\varepsilon^{d-1} \equiv 1, \text{ uniformly in } \Omega_s \,.$$

Proof. — We apply identity (10.1.1) from Lemma 10.1.1 to $f_{0,N}^{(N)}$, and obtain after integration in the velocity variables

$$(11.1.6) \qquad f_{0,N}^{(s)}(Z_s) - \widetilde{f}_{0,N}^{(s)}(Z_s) = \sum_{m=1}^{N-s} C_{N-s}^m \int_{\Delta_m(X_s) \times \mathbf{R}^{dm}} \widetilde{f}_{0,N}^{(s+m)}(Z_{s+m}) dZ_{(s+1,s+m)} \,.$$

Then, denoting $C_0 = \sup_{M \ge 1} \|F_{0,M}\|_{\varepsilon,\beta_0,\mu_0}$, a finite number by assumption, from

$$f_{0,N}^{(s+m)}(Z_{s+m}) \le \exp\big(-\mu_0(s+m) - \beta_0 E_\varepsilon(Z_{s+m})\big) C_0$$
$$\le \exp\Big(-\mu_0(s+m) - (\beta_0/2) \sum_{1 \le i \le s} |v_i|^2\Big) C_0 \,,$$

we deduce, first by integrating the velocity gaussians and then by using the cluster bound (10.1.2) in Lemma 10.1.1 with $\zeta = \varepsilon^{-d}$, the bound

$$\int_{\Delta_m(X_s) \times \mathbf{R}^{dm}} f_{0,N}^{(s+m)}(Z_{s+m}) dZ_{(s+1,s+m)} \le (C_d/\beta_0)^{md/2} e^{-\mu_0(s+m)} C_0 \int_{\Delta_m(X_s)} dX_{(s+1,s+m)}$$
$$\le m!(C_d/\beta_0)^{md/2} \varepsilon^{md} e^{(\kappa_d-\mu_0)(s+m)} C_0 \,.$$

If $2\varepsilon e^{\kappa_d-\mu_0}(C_d/\beta_0)^{d/2} < 1$, then

$$\sum_{m=1}^{N-s} C_{N-s}^m m!(C_d/\beta_0)^{md/2} \varepsilon^{md} e^{(\kappa_d-\mu_0)(s+m)} \le e^{(\kappa_d-\mu_0)s} \sum_{m \ge 1} \big(2\varepsilon e^{\kappa_d-\mu_0}(C_d/\beta_0)^{d/2}\big)^m \longrightarrow 0$$

as $\varepsilon \to 0$, implying $f_{0,N}^{(s)} - \widetilde{f}_{0,N}^{(s)} \longrightarrow 0$ for fixed s, uniformly in Ω_s. $\qquad \square$

Remark 11.1.3. — *We can reproduce the above proof in the case of a time-dependent family of bounded marginals, i.e., $F_N \in \mathbf{X}_{\varepsilon,\boldsymbol{\beta},\boldsymbol{\mu}}$, with $\sup_{N \ge 1} \|F_N\|_{\varepsilon,\boldsymbol{\beta},\boldsymbol{\mu}} < \infty$, with the notation of Definition 10.2.1. This gives uniform convergence to zero, in time $t \in [0,T]$ and in space $X_s \in \Omega_s$, of the difference between truncated and untruncated marginals: $\widetilde{f}_N^{(s)} - f_N^{(s)} \to 0$.*

We consider therefore families of initial data: Boltzmann initial data $F_0 = (f_0^{(s)})_{s \in \mathbf{N}}$ such that

$$\|F_0\|_{0,\beta_0,\mu_0} = \sup_{s \in \mathbf{N}} \sup_{Z_s} \big(\exp(\beta_0 E_0(Z_s) + \mu_0 s) f_0^{(s)}(Z_s)\big) < +\infty$$

and for each N, BBGKY initial data $\widetilde{F}_{N,0} = (\widetilde{f}_{N,0}^{(s)})_{1 \leq s \leq N}$ such that

$$\sup_N \|\widetilde{F}_{N,0}\|_{\varepsilon, \beta_0, \mu_0} = \sup_N \sup_{s \leq N} \sup_{Z_s} \left(\exp(\beta_0 E_\varepsilon(Z_s) + \mu_0 s) \widetilde{f}_{N,0}^{(s)}(Z_s) \right) < +\infty \,,$$

satisfying (11.1.3) and (11.1.4). These give rise to a unique, uniformly bounded solution $\widetilde{F}_N$ to the BBGKY hierarchy thanks to Theorem 9 page 81, and to a unique solution F to the Boltzmann hierarchy thanks to Theorem 10 page 82.

11.2. Convergence to the Boltzmann hierarchy

Our main result is the following.

Theorem 11 (**Convergence**). — *Assume the potential satisfies Assumption* 1.2.1 *as well as* (8.3.1). *Let $\beta_0 > 0$ and $\mu_0 \in \mathbf{R}$ be given. There is a time $T > 0$ such that the following holds. For any admissible Boltzmann datum F_0 in $\mathbf{X}_{0,\beta_0,\mu_0}$ associated with a family $(\widetilde{F}_{0,N})_{N \geq 1}$ of BBGKY data in $\mathbf{X}_{\varepsilon,\beta_0,\mu_0}$, the solution $\widetilde{F}_N$ to the BBGKY hierarchy satisfies, in the sense of Definition* 6.2.1,

$$\widetilde{F}_N \xrightarrow{\sim} F$$

uniformly on $[0, T]$, where F is the solution to the Boltzmann hierarchy with data F_0.

Corollary 11.2.1. — *Assume the potential satisfies Assumption* 1.2.1 *as well as* (8.3.1). *Let $\beta_0 > 0$ and $\mu_0 \in \mathbf{R}$ be given. There is a time $T > 0$ such that the following holds. For any admissible Boltzmann datum F_0 in $\mathbf{X}_{0,\beta_0,\mu_0}$ associated with a family $(\widetilde{F}_{0,N})_{N \geq 1}$ of BBGKY data in $\mathbf{X}_{\varepsilon,\beta_0,\mu_0}$, the associate family of untruncated marginals F_N satisfies*

$$F_N \xrightarrow{\sim} F \,,$$

uniformly on $[0, T]$, where F is the solution to the Boltzmann hierarchy with data F_0.

Proof. — By Proposition 11.1.1, the family F_0 is an admissible Boltzmann datum. Denoting $\widetilde{F}_{0,N}$ an associated BBGKY datum, let $T > 0$ be an existence time for the BBGKY hierarchy $\widetilde{F}_N$ with datum $\widetilde{F}_{0,N}$, given by Theorem 9. By Theorem 11 the convergence $I_{\varphi_s}(\widetilde{f}_N^{(s)} - f^{(s)}) \to 0$ holds uniformly in $[0, T]$ and locally uniformly in Ω_s.

Then, by Lemma 11.1.2 and Remark 11.1.3, there holds $f_N^{(s)} - \widetilde{f}_N^{(s)} \to 0$, for fixed s, uniformly in $[0, T] \times \Omega_s$. By Lemma 6.2.2, this implies $I_{\varphi_s}(f_N^{(s)} - \widetilde{f}_N^{(s)}) \to 0$, uniformly in $[0, T]$ and locally uniformly in Ω_s. We conclude that $f_N^{(s)} \xrightarrow{\sim} f^{(s)}$, uniformly in $[0, T]$. $\qquad\square$

In the next paragraph we shall prove that in the sum defining $\widetilde{f}_N^{(s)}(t)$ one can neglect all higher-order interactions and restrict our attention to the case when $m_i = 1$ for each $i \in [1, n]$ and each $n \in \mathbf{N}$. Then we can, exactly as in the hard spheres case discussed in Chapter 7, consider only a finite number of collisions, and reduce the study to bounded energies and well separated collision times.

11.3. Reductions of the BBGKY hierarchy, and pseudotrajectories

In this paragraph, we first prove that the estimates obtained in Chapter 10 enable us to reduce the study of the BBGKY hierarchy to the equation

$$(11.3.1) \quad \widetilde{g}_N^{(s)}(t, Z_s) = \mathbf{H}_s(t)\widetilde{f}_N^{(s)}(0, Z_s) + \int_0^t \mathbf{H}_s(t-\tau)\mathcal{C}_{s,s+1}\widetilde{g}_N^{(s+1)}(\tau, Z_s)\,d\tau, \qquad 1 \leq s \leq N-1.$$

Estimate (10.3.2) in Proposition 10.3.1 shows indeed that higher-order collisions are negligible in the Boltzmann-Grad limit. For the solution to the BBGKY hierarchy, this translates as follows.

Proposition 11.3.1. — *Let $\beta_0 > 0$ and μ_0 be given. Then with the same notation as Theorem 9, in the Boltzmann-Grad scaling $N\varepsilon^{d-1} \equiv 1$, any family of initial marginals $\widetilde{F}_N(0) = \left(\widetilde{f}_N^{(s)}(0)\right)_{1\leq s\leq N}$ in $\mathbf{X}_{\varepsilon,\beta_0,\mu_0}$ gives rise to a unique solution $\widetilde{G}_N \in \mathbf{X}_{\varepsilon,\boldsymbol{\beta},\boldsymbol{\mu}}$ of (11.3.1) and there holds the bound*

$$\|\widetilde{G}_N\|_{\varepsilon,\boldsymbol{\beta},\boldsymbol{\mu}} \leq 2\|\widetilde{F}_N(0)\|_{\varepsilon,\beta_0,\mu_0}.$$

Besides, the solution $\widetilde{G}_N$ to the modified hierarchy (11.3.1) is asymptotically close to the solution $\widetilde{F}_N$ to the BBGKY hierarchy (9.5.4):

$$(11.3.2) \qquad\qquad \|\widetilde{G}_N - \widetilde{F}_N\|_{\varepsilon,\boldsymbol{\beta},\boldsymbol{\mu}} \leq 2\varepsilon\|\widetilde{F}_N(0)\|_{\varepsilon,\beta_0,\mu_0}.$$

Proof. — We find the bound for $\widetilde{G}_N$, in the same way as for Theorem 9. We turn to the proof of (11.3.2). There holds

$$\|\widetilde{G}_N - \widetilde{F}_N\|_{\varepsilon,\boldsymbol{\beta},\boldsymbol{\mu}} \leq \left\|\left|\int_0^t \left(\mathbf{H}_s(t-t')\mathcal{C}_{s,s+1}(\widetilde{g}_N^{(s+1)} - \widetilde{f}_N^{(s+1)})(t')\right)_{1\leq s\leq N} dt'\right|\right\|_{\varepsilon,\boldsymbol{\beta},\boldsymbol{\mu}}$$
$$+ \left\|\left|\int_0^t \left(\mathbf{H}_s(t-t')\sum_{2\leq m\leq N-s}\mathcal{C}_{s,s+m}f_N^{(s+m)}(t')\right)_{1\leq s\leq N} dt'\right|\right\|_{\varepsilon,\boldsymbol{\beta},\boldsymbol{\mu}}.$$

With (10.3.1), this implies

$$\|\widetilde{G}_N - \widetilde{F}_N\|_{\varepsilon,\boldsymbol{\beta},\boldsymbol{\mu}} \leq c_0 \left\|\left|\int_0^t \left(\mathbf{H}_s(t-t')\sum_{2\leq m\leq N-s}\mathcal{C}_{s,s+m}f_N^{(s+m)}(t')\right)_{1\leq s\leq N} dt'\right|\right\|_{\varepsilon,\boldsymbol{\beta},\boldsymbol{\mu}},$$

with $c_0 := \left(1 - \bar{c}(\beta_0, \mu_0, \lambda, T)\right)^{-1}$, which is indeed strictly positive by assumption. We conclude as in the proof of Proposition 10.3.1 and Lemma 10.3.2. $\qquad\square$

One has the following formulation for $\widetilde{g}_N^{(s)}$ in terms of the initial datum:

$$\widetilde{g}_N^{(s)}(t) = \sum_{n=0}^\infty \int_0^t \int_0^{t_1} \ldots \int_0^{t_{n-1}} \mathbf{H}_s(t-t_1)\mathcal{C}_{s,s+1}\mathbf{H}_{s+1}(t_1-t_2)\ldots\mathbf{H}_{s+n}(t_n)\widetilde{f}_N^{(s+n)}(0)\,dt_n\ldots dt_1.$$

We define the functional

$$I_s(t)(X_s) := \sum_{k=0}^\infty \int \varphi_s(V_s)\int_{\mathcal{T}_k(t)} \mathbf{H}_s(t-t_1)\mathcal{C}_{s,s+1}\mathbf{H}_{s+1}(t_1-t_2)\mathcal{C}_{s+1,s+2}$$
$$\ldots\mathcal{C}_{s+k-1,s+k}\mathbf{H}_{s+k}(t_k-t_{k+1})\widetilde{f}_{N,0}^{(s+k)}\,dT_k\,dV_s$$

and following Chapter 7, the reduced elementary functional

$$(11.3.3) \quad I_{s,k}^{R,\delta}(t)(X_s) := \int \varphi_s(V_s)\int_{\mathcal{T}_{k,\delta}(t)} \mathbf{H}_s(t-t_1)\mathcal{C}_{s,s+1}\mathbf{H}_{s+1}(t_1-t_2)\mathcal{C}_{s+1,s+2}$$
$$\ldots\mathcal{C}_{s+k-1,s+k}\mathbf{H}_{s+k}(t_k-t_{k+1})\mathbb{1}_{E_\varepsilon(Z_{s+k})\leq R^2}\widetilde{f}_{N,0}^{(s+k)}\,dT_k\,dV_s.$$

We can reproduce the proofs of Propositions 7.1.1, 7.2.1 and 7.3.1 to obtain the following result, as in Corollary 7.4.1.

Proposition 11.3.2. — *With the notation of Theorem* 9, *given* $s \in \mathbf{N}^*$ *and* $t \in [0, T]$, *there are two positive constants* C *and* C' *such that for all* $n \in \mathbf{N}^*$,

$$\Big\| I_s(t) - \sum_{k=0}^n I_{s,k}^{R,\delta}(t) \Big\|_{L^\infty(\mathbf{R}^{ds})} \leq C(2^{-n} + e^{-C'R^2} + \delta) \|\varphi\|_{L^\infty(\mathbf{R}^{ds})} \|\widetilde{F}_{N,0}\|_{\varepsilon,\beta_0,\mu_0} \,.$$

As in the hard-spheres case, in the integrand of the collision operators $\mathcal{C}_{s,s+1}$ defined in (9.4.4), we can distinguish between pre- and post-collisional configurations, as we decompose

$$\mathcal{C}_{s,s+1} = \mathcal{C}_{s,s+1}^+ - \mathcal{C}_{s,s+1}^-$$

where

(11.3.4)
$$\mathcal{C}_{s,s+1}^\pm \widetilde{g}^{(s+1)} = \sum_{m=1}^s \mathcal{C}_{s,s+1}^{\pm,m} \widetilde{g}^{(s+1)}$$

the index m referring to the index of the interaction particle among the s "fixed" particles, with the notation

$$\big(\mathcal{C}_{s,s+1}^{\pm,m} \widetilde{g}^{(s+1)}\big)(Z_s) := (N-s)\varepsilon^{d-1} \int_{\mathbf{S}_1^{d-1} \times \mathbf{R}^d} (\nu \cdot (v_{s+1} - v_m))_\pm \widetilde{g}^{(s+1)}(Z_s, x_m + \varepsilon\nu, v_{s+1})$$

$$\times \prod_{\substack{1 \leq j \leq s \\ j \neq m}} \mathbb{1}_{|x_j - x_{s+1}| \geq \varepsilon} \, d\nu dv_{s+1} \,,$$

the index $+$ corresponding to post-collisional configurations and the index $-$ to pre-collisional configurations, according to terminology set out in Chapter 8.

The elementary BBGKY observables we are interested in can therefore be decomposed as

(11.3.5)
$$I_{s,k}^{R,\delta}(t, X_s) = \sum_{J,M} \Big(\prod_{i=1}^k j_i \Big) I_{s,k}^{R,\delta}(t, J, M)(X_s)$$

where the elementary functionals $I_{s,k}^{R,\delta}(t, J, M)$ are defined by

$$I_{s,k}^{R,\delta}(t, J, M)(X_s) := \int \varphi_s(V_s) \int_{\mathcal{T}_{k,\delta}(t)} \mathbf{H}_s(t - t_1) \mathcal{C}_{s,s+1}^{j_1,m_1} \mathbf{H}_{s+1}(t_1 - t_2) \mathcal{C}_{s+1,s+2}^{j_2,m_2}$$

$$\dots \mathbf{H}_{s+k}(t_k - t_{k+1}) \mathbb{1}_{E_\varepsilon(Z_{s+k}) \leq R^2} \widetilde{f}_{N,0}^{(s+k)} dT_k dV_s \,,$$

with

$$J := (j_1, \dots, j_k) \in \{+, -\}^k \quad \text{and} \quad M := (m_1, \dots, m_k) \quad \text{with} \quad m_i \in \{1, \dots, s+i-1\} \,.$$

As in the hard spheres case, we still cannot study directly the convergence of $I_{s,n}^{R,\delta}(t, J, M) - I_{s,n}^{0,R,\delta}(t, J, M)$ since the transport operators $\mathbf{H}_k$ do not coincide everywhere with the free transport operators $\mathbf{S}_k$, which means – in terms of pseudo-trajectories – that there are recollisions. Note that, because the interaction potential is compactly supported, recollisions happen only for characteristics such that there exist $i, j \in [1, k]$ with $i \neq j$, and $\tau > 0$ such that

$$|(x_i - \tau v_i) - (x_j - \tau v_j)| \leq \varepsilon \,.$$

We shall thus prove that these recollisions arise only for a few pathological pseudo-trajectories, which can be eliminated by additional truncations of the domains of integration. This is the goal of Part IV, which deals with the hard-spheres and the potential case simultaneously.

PART IV

TERM-BY-TERM CONVERGENCE

CHAPTER 12

ELIMINATION OF RECOLLISIONS

This last part is the heart of our contribution. We prove the term-by-term convergence of the series giving the observables, both in the case of hard spheres and in the case of smooth hamiltonian systems.

We have indeed seen in Corollary 7.4.1 (for the hard-spheres case) and Proposition 11.3.2 (for the potential case) that the convergence of observables reduces to the convergence to zero of the elementary functionals $I_{s,k}^{R,\delta} - I_{s,k}^{0,R,\delta}$, where $I_{s,k}^{R,\delta}$ is defined in (7.3.1) in the hard-spheres case and in (11.3.3) for the potential case, and $I_{s,k}^{0,R,\delta}$ is defined in (7.3.1). These functionals correspond to dynamics

– involving only a finite number $s + k$ of particles,
– with bounded energies (at most R^2),
– such that the k additional particles are adjoined through binary collisions,
– at times separated at least by δ.

What we shall establish is that recollisions can occur only for very pathological pseudo-trajectories, in the sense that the velocities and impact parameters of the additional particles in the collision trees have to be chosen in small measure sets.

We point out the fact that, even in the case of hard spheres, these bad sets are generally not of zero measure because they are built as non countable unions of zero measure sets. The arguments are actually very similar whatever the precise nature of the microscopic interaction.

The only differences we shall see between the case of hard spheres and the case of smooth potentials are the following:

– the parametrization of collisions by the deflection angle is trivial in the case of hard spheres since it coincides exactly with the impact parameter;
– there is no time shift between pre-collisional and post-collisional configurations in the case of hard spheres since the reflection is instantaneous.

These two simplifications will enable us to obtain explicit estimates on the convergence rate in the case of hard spheres. For more general interactions, this convergence rate can be expressed as an implicit function depending on the potential.

12.1. Stability of good configurations by adjunction of collisional particles

In this paragraph we momentarily forget the BBGKY and Boltzmann hierarchies, and focus on the study of pseudo-trajectories.

Definition 12.1.1 (Good configuration). — *For any constant $c > 0$, we denote by $\mathcal{G}_k(c)$ the set of "good configurations" of k particles, separated by at least c through backwards transport: that is the set of $(X_k, V_k) \in \mathbf{R}^{dk} \times B_R^k$ such that the image of (X_k, V_k) by the backward free transport satisfies the separation condition*

$$\forall \tau \geq 0, \quad \forall i \neq j, \quad |x_i - x_j - \tau(v_i - v_j)| \geq c,$$

in particular it is never collisional.

We recall that $B_R^k := \left\{ V_k \in \mathbf{R}^{dk} \,/\, |V_k| \leq R \right\}$ and in the following we write $B_R := B_R^1$.

Our aim is to show that "good configurations" are stable by adjunction of a collisional particle provided that the deflection angle and the velocity of the additional particle do not belong to a small pathological set. Furthermore the set to be excluded can be chosen in a uniform way with respect to the initial positions of the particles in a small neighborhood of any fixed "good configuration".

Notation 12.1.2. — *In all the sequel, given two positive parameters η_1 and η_2, we shall say that*

$$\eta_1 \ll \eta_2 \ \text{ if } \eta_1 \leq C\eta_2$$

for some large constant C which does not depend on any parameter.

In the following we shall fix three parameters $a, \varepsilon_0, \eta \ll 1$ such that

$$(12.1.1) \qquad\qquad\qquad a \ll \varepsilon_0 \ll \eta\delta \,.$$

We recall that the parameter δ scales like time while we shall see that η, like R, scales like a velocity. The parameters a and ε_0, just like ε, will have the scaling of a distance.

Proposition 12.1.1. — *Let $a, \varepsilon_0, \eta \ll 1$ satisfy (12.1.1). Given $\overline{Z}_k \in \mathcal{G}_k(\varepsilon_0)$, there is a subset $\mathcal{B}_k(\overline{Z}_k)$ of $\mathbf{S}_1^{d-1} \times B_R$ of small measure: for some fixed constant $C > 0$ and some constant $C(\Phi, \eta, R) > 0$,*

$$\left| \mathcal{B}_k(\overline{Z}_k) \right| \leq Ck \left(R\eta^{d-1} + R^d \left(\frac{a}{\varepsilon_0} \right)^{d-1} + R \left(\frac{\varepsilon_0}{\delta} \right)^{d-1} \right)$$

in the case of hard spheres

$(12.1.2)$

$$\left| \mathcal{B}_k(\overline{Z}_k) \right| \leq Ck \left(R\eta^{d-1} + C(\Phi, R, \eta) \, R^d \left(\frac{a}{\varepsilon_0} \right)^{d-1} + C(\Phi, R, \eta) \, R \left(\frac{\varepsilon_0}{\delta} \right)^{d-1} \right)$$

in the case of a smooth interaction potential Φ ,

and such that good configurations close to $\overline{Z}_k$ are stable by adjunction of a collisional particle close to $\bar{x}_k$ and not belonging to $\mathcal{B}_k(\overline{Z}_k)$, in the following sense.

Consider $(\nu, v) \in (\mathbf{S}_1^{d-1} \times B_R) \setminus \mathcal{B}_k(\overline{Z}_k)$ and let Z_k be a configuration of k particles such that $V_k = \overline{V}_k$ and $|X_k - \overline{X}_k| \leq a$.

- *If $\nu \cdot (v - \bar{v}_k) < 0$ then for all $\varepsilon > 0$ sufficiently small,*

$$(12.1.3) \qquad \forall \tau \geq 0, \quad \begin{cases} \forall i \neq j \in [1, k], & |(x_i - \tau\bar{v}_i) - (x_j - \tau\bar{v}_j)| \geq \varepsilon, \\ \forall j \in [1, k], & |(x_k + \varepsilon\nu - \tau v) - (x_j - \tau\bar{v}_j)| \geq \varepsilon. \end{cases}$$

Moreover after the time δ, the $k + 1$ particles are in a good configuration:

$$(12.1.4) \qquad (X_k - \delta \overline{V}_k, \overline{V}_k, x_k + \varepsilon \nu - \delta v, v) \in \mathcal{G}_{k+1}(\varepsilon_0/2)\,.$$

- *If $\nu \cdot (v - \bar{v}_k) > 0$ then define for $j \in [1, k-1]$*

$$(z_k^{\varepsilon*}, z^{\varepsilon*}) := \sigma^{-1}\big(z_k, (x_k + \varepsilon \nu, v)\big) \quad and \quad z_j^{\varepsilon*} := (x_j - \bar{v}_j, \bar{v}_j)$$

in the hard-spheres case, where σ is defined in (4.4.2), and

$$(z_k^{\varepsilon*}, z^{\varepsilon*}) := \sigma_\varepsilon^{-1}\big(z_k, (x_k + \varepsilon \nu, v)\big) \quad and \quad z_j^{\varepsilon*} := (x_j - t_\varepsilon \bar{v}_j, \bar{v}_j)$$

in the potential case, where σ_ε is the scattering operator as in Definition 8.2.1 and where $t_\varepsilon < \delta$ denotes the scattering time between z_k and $(x_k + \varepsilon \nu, v)$. Then for all $\varepsilon > 0$ sufficiently small,

$$(12.1.5) \qquad \forall \tau \geq 0\,, \quad \begin{cases} \forall i \neq j \in [1, k]\,, & |(x_i^{\varepsilon*} - \tau v_i^{\varepsilon*}) - (x_j^{\varepsilon*} - \tau v_j^{\varepsilon*})| \geq \varepsilon\,, \\ \forall j \in [1, k]\,, & |(x^{\varepsilon*} - \tau v^{\varepsilon*}) - (x_j^{\varepsilon*} - \tau v_j^{\varepsilon*})| \geq \varepsilon\,. \end{cases}$$

Moreover after the time δ, the $k + 1$ particles are in a good configuration:

$$(12.1.6) \qquad \Big(X_k^{\varepsilon*} - (\delta - t_\varepsilon)V_k^{\varepsilon*}, V_k^{\varepsilon*}, x^{\varepsilon*} - (\delta - t_\varepsilon)v^{\varepsilon*}, v^{\varepsilon*}\Big) \in \mathcal{G}_{k+1}(\varepsilon_0/2)\,,$$

with $t_\varepsilon := 0$ in the hard-spheres case.

The proof of the proposition may be found in Section 12.3. It relies on some elementary geometrical lemmas, stated and proved in the next section. The first one describes the bad trajectories associated with (free) transport. The other ones explain how they are modified by collisions, both in the case of hard spheres and in the case of smooth interactions.

Remark 12.1.3. — *For the sake of simplicity, we have assumed in the statement of Proposition 12.1.1 that the additional particle collides with the particle numbered k. Of course, a simple symmetry argument shows that an analogous statement holds if the new particle is added close to any of the particles in Z_k.*

The proof of Proposition 12.1.1 shows that if $Z_k = \overline{Z}_k$ then the factor $\varepsilon_0/2$ in (12.1.4) and (12.1.6) may be replaced by ε_0. The loss if $Z_k \neq \overline{Z}_k$ comes from the fact that the set to be excluded has to be chosen in a uniform way with respect to the initial positions of the particles in a small neighborhood of $\overline{X}_k$.

12.2. Geometrical lemmas

We first consider the case of two particles moving freely, and describe the set of velocities v_2 leading possibly to collisions (or recollisions).

Here and in the sequel, we denote by $K(w, y, \rho)$ the cylinder of origin $w \in \mathbf{R}^d$, of axis $y \in \mathbf{R}^d$ and radius $\rho > 0$ and by $B_\rho(y)$ the ball centered at y of radius ρ.

12.2.1. Bad trajectories associated to free transport. —

Lemma 12.2.1. — *Given $\bar{a} > 0$ satisfying $\varepsilon \ll \bar{a} \ll \varepsilon_0$, consider $\bar{x}_1, \bar{x}_2$ in $\mathbf{R}^d$ such that $|\bar{x}_1 - \bar{x}_2| \geq \varepsilon_0$, and $v_1 \in B_R$. Then for any $x_1 \in B_{\bar{a}}(\bar{x}_1)$, any $x_2 \in B_{\bar{a}}(\bar{x}_2)$ and any $v_2 \in B_R$, the following results hold.*

- *If $v_2 \notin K(v_1, \bar{x}_1 - \bar{x}_2, 6R\bar{a}/\varepsilon_0)$, then*

$$\forall \tau \geq 0, \quad |(x_1 - v_1\tau) - (x_2 - v_2\tau)| > \varepsilon \, ;$$

- *If $v_2 \notin K(v_1, \bar{x}_1 - \bar{x}_2, 6\varepsilon_0/\delta)$*

$$\forall \tau \geq \delta, \quad |(x_1 - v_1\tau) - (x_2 - v_2\tau)| > \varepsilon_0 \, .$$

Proof. — • Assume that there exists τ_* such that

$$|(x_1 - v_1\tau_*) - (x_2 - v_2\tau_*)| \leq \varepsilon \, .$$

Then, by the triangular inequality and provided that ε is sufficiently small,

$$|(\bar{x}_1 - \bar{x}_2) - \tau_*(v_1 - v_2)| \leq \varepsilon + 2\bar{a} \leq 3\bar{a} \, .$$

This means that $(v_1 - v_2)$ belongs to the cone of vertex 0 based on the ball centered at $\bar{x}_1 - \bar{x}_2$ and of radius $3\bar{a}$, which is a cone of solid angle $(3\bar{a}/|\bar{x}_1 - \bar{x}_2|)^{d-1}$ (since $\bar{a} \ll \varepsilon_0$).

The intersection of this cone and of the sphere of radius $2R$ is obviously embedded in the cylinder of axis $\bar{x}_1 - \bar{x}_2$ and radius $6R\bar{a}/\varepsilon_0$, which proves the first result.

- Similarly assume that there exists $\tau^* \geq \delta$ such that

$$|(x_1 - v_1\tau_*) - (x_2 - v_2\tau_*)| \leq \varepsilon_0 \, .$$

Then, by the triangular inequality again,

$$|(\bar{x}_1 - \bar{x}_2) - \tau_*(v_1 - v_2)| \leq \varepsilon_0 + 2\bar{a} \leq 3\varepsilon_0 \, .$$

In particular, for any unit vector n orthogonal to $\bar{x}_1 - \bar{x}_2$,

$$\tau^* |n \cdot (v_1 - v_2)| = |n \cdot ((\bar{x}_1 - \bar{x}_2) - \tau_*(v_1 - v_2))| \leq 3\varepsilon_0 \, .$$

This tells us exactly that $v_1 - v_2$ belongs to the cylinder of axis $\bar{x}_1 - \bar{x}_2$ and radius $3\varepsilon_0/\delta$.

The lemma is proved. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

12.2.2. Modification of bad trajectories by hard sphere reflection. —

We now consider the case when particles 1 and 2 undergo a hard sphere collision before being transported, and look at impact parameters ν and velocities v_2 leading possibly to collisions (or recollisions).

Lemma 12.2.2. — *Consider $\rho \ll R$, and $(y, w) \in \mathbf{R}^d \times B_R$. For any v_1 in B_R, define*

$$\mathcal{N}^*(w, y, \rho)(v_1) := \big\{ (\nu, v_2) \in \mathbf{S}_1^{d-1} \times B_R \, / \, (v_2 - v_1) \cdot \nu > 0 \, ,$$

$$v_1^* \in K(w, y, \rho) \text{ or } v_2^* \in K(w, y, \rho) \big\} \, ,$$

where

$$v_1^* := v_1 - \nu \cdot (v_1 - v_2)\nu \quad and \quad v_2^* := v_2 + \nu \cdot (v_1 - v_2)\nu \, .$$

Then

$$|\mathcal{N}^*(w, y, \rho)(v_1)| \leq C_d R \rho^{d-1} \, ,$$

where the constant C_d depends only on the dimension d.

Proof. — Denote by $r = |v_1 - v_2| = |v_1^* - v_2^*|$. The reflection condition shows that, as ν varies in $\mathbf{S}_1^{d-1}$, the velocities v_1^* and v_2^* range over a sphere of diameter r.

The solid angle of the intersection of such a sphere with the cylinder $K(w, y, \rho)$ is less than

$$C_d \min\left(1, \left(\frac{\rho}{r}\right)^{d-1}\right)$$

which implies that

$$\left|\left\{(\nu, v_2) \,/\, v_1^* \in K(w, y, \rho) \text{ or } v_2^* \in K(w, y, \rho)\right\}\right| \leq C_d \int r^{d-1} \min\left(1, \left(\frac{\rho}{r}\right)^{d-1}\right) dr$$

$$\leq C_d R \rho^{d-1}.$$

This proves Lemma 12.2.2. □

12.2.3. Modification of bad trajectories by the scattering associated to Φ. —

The last geometrical lemma requires the use of notation coming from scattering theory, introduced in Chapter 8: it states that if two particles z_1, z_2 in $\mathbf{R}^{2d}$ are in a post-collisional configuration and if v_1 or v_2 belong to a cylinder as in Lemma 12.2.1, then the pre-image z_2^* of z_2 through the scattering operator belongs to a small set of $\mathbf{R}^{2d}$.

Lemma 12.2.3. — *Consider two parameters $\rho \ll R$ and $\eta \ll 1$, and $(y, w) \in \mathbf{R}^d \times B_R$. For any v_1 in B_R, define*

$$\mathcal{N}^*(w, y, \rho)(v_1) := \left\{(\nu, v_2) \in \mathbf{S}_1^{d-1} \times B_R \,/\, (v_2 - v_1) \cdot \nu > \eta,\right.$$

$$\left. v_1^* \in K(w, y, \rho) \text{ or } v_2^* \in K(w, y, \rho)\right\},$$

where $(\nu^*, v_1^*, v_2^*) = \sigma_0^{-1}(\nu, v_1, v_2)$ with the notations of Chapter 8. Then

$$|\mathcal{N}^*(w, y, \rho)(v_1)| \leq C(\Phi, R, \eta) R \rho^{d-1}$$

where the constant depends on the potential Φ through the L^∞ norm of the cross-section b on the compact set $B_{2R} \times [\eta/2R, \pi/2]$ defined in Chapter 8.

Proof. — Denote by $r = |v_1 - v_2| = |v_1^* - v_2^*|$, and by ω the deflection angle.

From the proof of the previous lemma, we deduce that

$$\left|\left\{(\omega, v_2) \,/\, v_1^* \in K(w, y, \rho) \text{ or } v_2^* \in K(w, y, \rho)\right\}\right| \leq C_d \int r^{d-1} \min\left(1, \left(\frac{\rho}{r}\right)^{d-1}\right) dr$$

$$\leq C_d R \rho^{d-1}.$$

According to Chapter 8, the change of variables $(\nu, v_1 - v_2) \mapsto (\omega, v_1 - v_2)$ is a Lipschitz diffeomorphism away from $\nu \cdot (v_1 - v_2) = 0$. We therefore get the expected estimate. □

Remark 12.2.4. — *Note that the geometrical Lemmas 12.2.1 to 12.2.3 consist in eliminating sets in the velocity variables and deflection angles only, and do not concern the position variables.*

12.3. Proof of the geometric proposition

In this section we prove Proposition 12.1.1. We fix a good configuration $\overline{Z}_k \in \mathcal{G}_k(\varepsilon_0)$, and we consider a configuration $Z_k \in \mathbf{R}^{2dk}$, with the same velocities as $\overline{Z}_k$, and neighboring positions: $|X_k - \overline{X}_k| \leq a$. In particular we notice that for all $\tau \geq 0$ and all $i \neq j$,

$$(12.3.7) \qquad |x_i - x_j - \tau(\bar{v}_i - \bar{v}_j)| \geq |\bar{x}_i - \bar{x}_j - \tau(\bar{v}_i - \bar{v}_j)| - 2a \geq \varepsilon_0/2$$

since $a \ll \varepsilon_0$. This implies that $Z_k \in \mathcal{G}_k(\varepsilon_0/2)$. Next we consider an additional particle $(x_k + \varepsilon\nu, v_{k+1})$ and we shall separate the analysis into two parts, depending on whether the situation is pre-collisional (meaning $\nu \cdot (v_{k+1} - \bar{v}_k) < 0$) or post-collisional (meaning $\nu \cdot (v_{k+1} - \bar{v}_k) > 0$).

12.3.1. The pre-collisional case. — We assume that

$$\nu \cdot (v_{k+1} - \bar{v}_k) < 0 \,,$$

meaning that $(x_k + \varepsilon\nu, v_{k+1})$ and z_k form a pre-collisional pair. In particular we have for all times $\tau \geq 0$ and all $\varepsilon > 0$

$$\left| \left(x_k + \varepsilon\nu - v_{k+1}\tau \right) - \left(x_k - \bar{v}_k\tau \right) \right| \geq \varepsilon \,.$$

Furthermore up to excluding the ball $B_\eta(\bar{v}_k)$ in the set of admissible v_{k+1}, we may assume that

$$|v_{k+1} - \bar{v}_k| > \eta \,.$$

Under that assumption we have for all $\tau \geq \delta$ and all $\varepsilon > 0$ sufficiently small,

$$\left| \left(x_k + \varepsilon\nu - v_{k+1}\tau \right) - \left(x_k - \bar{v}_k\tau \right) \right| \geq \tau |v_{k+1} - \bar{v}_k| - \varepsilon$$
$$\geq \delta\eta - \varepsilon > \varepsilon_0/2 \,.$$

Furthermore we know that Z_k belongs to $\mathcal{G}_k(\varepsilon_0/2)$ thanks to (12.3.7).

Now let $j \in [1, k-1]$ be given. According to Lemma 12.2.1, we find that for any v_{k+1} belonging to the set $B_R \setminus K(\bar{v}_j, \bar{x}_j - \bar{x}_k, 6Ra/\varepsilon_0 + 6\varepsilon_0/\delta)$, we have

$$\forall \tau \geq 0 \,, \quad |(x_k + \varepsilon\nu - v_{k+1}\tau) - (x_j - \bar{v}_j\tau)| > \varepsilon \,,$$

and

$$\forall \tau \geq \delta \,, \quad |(x_k + \varepsilon\nu - v_{k+1}\tau) - (x_j - \bar{v}_j\tau)| > \varepsilon_0 \,.$$

Notice that

$$\left| B_R \cap K(\bar{v}_j, \bar{x}_j - \bar{x}_k, 6Ra/\varepsilon_0 + 6\varepsilon_0/\delta) \right| \leq C \left(R^d \left(\frac{a}{\varepsilon_0} \right)^{d-1} + R \left(\frac{\varepsilon_0}{\delta} \right)^{d-1} \right) \,.$$

Defining $\mathcal{M}^-(\overline{Z}_k) := \bigcup_{j \leq k-1} K(\bar{v}_j, \bar{x}_j - \bar{x}_k, 6Ra/\varepsilon_0 + 6\varepsilon_0/\delta)$ and

$$\mathcal{B}_k^-(\overline{Z}_k) := \mathbf{S}_1^{d-1} \times \left(B_\eta(\bar{v}_k) \cup \mathcal{M}^-(\overline{Z}_k) \right)$$

we find that

$$\left| \mathcal{B}_k^-(\overline{Z}_k) \right| \leq Ck \left(\eta^d + R^d \left(\frac{a}{\varepsilon_0} \right)^{d-1} + R \left(\frac{\varepsilon_0}{\delta} \right)^{d-1} \right)$$

and (12.1.3) and (12.1.4) hold as soon as $(\nu, v_{k+1}) \notin \mathcal{B}_k^-(\overline{Z}_k)$.

12.3.2. The post-collisional case with hard sphere reflection. —

We now assume that

$$\nu \cdot (v_{k+1} - \bar{v}_k) > 0 \,,$$

meaning that $(x_k + \varepsilon \nu, v_{k+1})$ and z_k form a post-collisional pair. In particular, at time $\tau = 0+$, the configuration is changed and we have the pre-collisional pair $(x_k + \varepsilon \nu, v_{k+1}^*)$ and (x_k, v_k^*) where v_k^* and v_{k+1}^* are defined by the usual reflection condition. Furthermore, we have for all times $\tau \geq 0$ and all $\varepsilon > 0$

$$\left| \left(x_k + \varepsilon \nu - v_{k+1}^* \tau \right) - \left(x_k - v_k^* \tau \right) \right| \geq \varepsilon \,.$$

We can then repeat the same arguments as inthe pre-collisional case replacing $\bar{v}_k, v_{k+1}$ by v_k^*, v_{k+1}^*.

Excluding the ball $B_\eta(\bar{v}_k)$ in the set of admissible v_{k+1}, we find that

$$\begin{aligned} \left| \left(x_k + \varepsilon \nu - v_{k+1}^* \tau \right) - \left(x_k - v_k^* \tau \right) \right| &\geq \tau |v_{k+1} - \bar{v}_k| - \varepsilon \\ &\geq \delta \eta - \varepsilon > \varepsilon_0/2 \,. \end{aligned}$$

According to Lemma 12.2.1, if v_k^*, v_{k+1}^* belong to the set $B_R \setminus K(\bar{v}_j, \bar{x}_j - \bar{x}_k, 6Ra/\varepsilon_0 + 6\varepsilon_0/\delta)$, we have

$$\forall \tau \geq 0, \quad |(x_k + \varepsilon \nu - v_{k+1}^* \tau) - (x_j - \bar{v}_j \tau)| > \varepsilon \,,$$
$$|(x_k - v_k^* \tau) - (x_j - \bar{v}_j \tau)| > \varepsilon \,,$$

and

$$\forall \tau \geq \delta, \quad |(x_k + \varepsilon \nu - v_{k+1}^* \tau) - (x_j - \bar{v}_j \tau)| > \varepsilon_0$$
$$|(x_k - v_k^* \tau) - (x_j - \bar{v}_j \tau)| > \varepsilon_0 \,.$$

Combining Lemmas 12.2.1and 12.2.2, we therefore obtain that (12.1.3) and (12.1.4) hold as soon as $(\nu, v_{k+1}) \notin \mathcal{B}_k^+(\overline{Z}_k)$ where

$$\mathcal{B}_k^+(\overline{Z}_k) := \mathbf{S}_1^{d-1} \times B_\eta(\bar{v}_k) \cup \bigcup_{j \leq k-1} \mathcal{N}^*(\bar{v}_j, \bar{x}_j - \bar{x}_k, 6Ra/\varepsilon_0 + 6\varepsilon_0/\delta)(\bar{v}_k) \,.$$

In particular,

$$\left| \mathcal{B}_k^+(\overline{Z}_k) \right| \leq Ck \left(\eta^d + R^d \left(\frac{a}{\varepsilon_0} \right)^{d-1} + R \left(\frac{\varepsilon_0}{\delta} \right)^{d-1} \right) \,.$$

12.3.3. The post-collisional case with smooth scattering. —

In the case of a smooth interaction potential, dealing with the post-collisional case is a little bit more intricate because of the time shift. Furthermore, using Lemma 12.2.3 instead of Lemma 12.2.2, we lose the explicit estimate for the bad set $\mathcal{B}_k^+(\overline{Z}_k)$.

Let us first define

$$(12.3.8) \qquad C(\overline{Z}_k) := \left\{ (\nu, v_{k+1}) \in \mathbf{S}_1^d \times B_R \,, \nu \cdot (v_{k+1} - \bar{v}_k) \leq \eta \right\} \,,$$

which satisfies

$$|C(\overline{Z}_k)| \leq CR\eta^{d-1} \,.$$

Choosing $(\nu, v_{k+1}) \in (\mathbf{S}_1^d \times B_R) \setminus C(\overline{Z}_k)$ ensures that the cross-section is well defined (see Definition 8.3.3), and that the scattering time t_ε is of order $C(\Phi, R, \eta)\varepsilon$ by Proposition 8.2.1.

Considering the formulas (8.2.2) expressing $(z_k^{\varepsilon*}, z_{k+1}^{\varepsilon*})$ in terms of $\big(z_k, (x_k + \varepsilon\nu, v_{k+1})\big)$, we know that

$$|x_k^{\varepsilon*} - x_k| \leq \frac{1}{2}|x_k^{\varepsilon*} - x_{k+1}^{\varepsilon*}| + \frac{1}{2}|(x_k^{\varepsilon*} + x_{k+1}^{\varepsilon*}) - (x_k + x_{k+1})| + \frac{1}{2}|(x_k - x_{k+1})|$$

$$\leq Rt_\varepsilon + \varepsilon \leq C(\Phi, R, \eta)\varepsilon,$$

(12.3.9)

$$|x_{k+1}^{\varepsilon*} - (x_k + \varepsilon\nu)| \leq \frac{1}{2}|x_k^{\varepsilon*} - x_{k+1}^{\varepsilon*}| + \frac{1}{2}|(x_k^{\varepsilon*} + x_{k+1}^{\varepsilon*}) - (x_k + x_{k+1})| + \frac{1}{2}|(x_k - x_{k+1})|$$

$$\leq Rt_\varepsilon + \varepsilon \leq C(\Phi, R, \eta)\varepsilon.$$

Note that due to (12.3.7), all particles x_j with $j \leq k-1$ are at a distance at least $\varepsilon_0/2 - \varepsilon \geq \varepsilon_0/3$ of the particles x_k and $x_k + \varepsilon\nu$. Since they have bounded velocities, they cannot enter the protection spheres of these post-collisional particles during the interaction time t_ε, provided that ε is small enough:

$$Rt_\varepsilon \ll \varepsilon_0/3.$$

Since the dynamics of the particles $j \leq k-1$ is not affected by the scattering, we get that $Z_{k-1}^{\varepsilon*}$ belongs to $\mathcal{G}_{k-1}(\varepsilon_0/2)$:

(12.3.10) $\forall \tau \geq 0, \ \forall (i,j) \in [1, k-1]^2$ with $i \neq j, \quad |x_i^{\varepsilon*} - x_j^{\varepsilon*} - \tau(v_i^{\varepsilon*} - v_j^{\varepsilon*})| \geq \varepsilon_0/2.$

The pair $(z_k^{\varepsilon*}, z_{k+1}^{\varepsilon*})$ is a pre-collisional pair by definition, so we know that for all $\tau \geq 0$,

$$|(x_k^{\varepsilon*} - \tau v_k^{\varepsilon*}) - (x_{k+1}^{\varepsilon*} - \tau v_{k+1}^{\varepsilon*})| \geq \varepsilon.$$

Excluding the ball $B_\eta(\bar{v}_k)$ in the set of admissible v_{k+1}, we find as above that

$$\forall \tau \geq \delta, \quad |x_k^{\varepsilon*} - x^{\varepsilon*} - \tau(v_k^{\varepsilon*} - v_{k+1}^{\varepsilon*})| \geq \eta\delta - \varepsilon \geq \varepsilon_0,$$

for ε sufficiently small, since $\varepsilon_0 \ll \eta\delta$.

Next for $j \leq k-1$ we have for ε sufficiently small, recalling that the uniform, rectilinear motion of the center of mass as described in (8.1.3),

$$|x_j^{\varepsilon*} - \bar{x}_j| \leq |x_j^{\varepsilon*} - x_j| + |x_j - \bar{x}_j| \leq Rt_\varepsilon + a \leq 2a$$

$$|x_k^{\varepsilon*} - \bar{x}_k| \leq |x_k^{\varepsilon*} - x_k| + |x_k - \bar{x}_k| \leq Rt_\varepsilon + \varepsilon + a \leq 2a$$

$$|x_{k+1}^{\varepsilon*} - \bar{x}_k| \leq |x_{k+1}^{\varepsilon*} - x_{k+1}| + |x_k + \varepsilon\nu - \bar{x}_k| \leq Rt_\varepsilon + 2\varepsilon + a \leq 2a.$$

By Lemma 12.2.1, provided $v_k^{\varepsilon*}$ and $v_{k+1}^{\varepsilon*}$ do not belong to

$$K(\bar{v}_j, \bar{x}_j - \bar{x}_k, 12Ra/\varepsilon_0 + 12\varepsilon_0/\delta) \cap B_R,$$

we get since $v_j^{\varepsilon*} = \bar{v}_j$,

$$\forall \tau \geq 0, \quad |x_k^{\varepsilon*} - x_j^{\varepsilon*} - \tau(v_k^{\varepsilon*} - v_j^{\varepsilon*})| \geq \varepsilon,$$

$$\text{and} \quad |x_{k+1}^{\varepsilon*} - x_j^{\varepsilon*} - \tau(v_{k+1}^{\varepsilon*} - v_j^{\varepsilon*})| \geq \varepsilon$$

as well as

$$\forall \tau \geq \delta/2, \quad |x_k^{\varepsilon*} - x_j^{\varepsilon*} - \tau(v_k^{\varepsilon*} - v_j^{\varepsilon*})| \geq \varepsilon_0/2,$$

$$\text{and} \quad |x_{k+1}^{\varepsilon*} - x_j^{\varepsilon*} - \tau(v_{k+1}^{\varepsilon*} - v_j^{\varepsilon*})| \geq \varepsilon_0/2.$$

Lemma 12.2.3 bounds from the above the size of the set $\mathcal{N}^*(\bar{v}_j, \bar{x}_j - \bar{x}_k, \rho)$ of all (ν, v_{k+1}) belonging to $(\mathbf{S}_1^d \times B_R) \setminus C(\overline{Z}_k)$ such that $v_k^{\varepsilon*}$ or $v_{k+1}^{\varepsilon*}$ belongs to $K(\bar{v}_j, \bar{x}_j - \bar{x}_k, \rho)$. We let $\rho = 12Ra/\varepsilon_0 + 12\varepsilon_0/\delta$, and define

$$\mathcal{B}_k^+(\overline{Z}_k) := C(\overline{Z}_k) \cup \big(\mathbf{S}_1^{d-1} \times B_\eta(\bar{v}_k)\big) \bigcup_{j \leq k-1} \mathcal{N}^*(\bar{v}_j, \bar{x}_j - \bar{x}_k, 12Ra/\varepsilon_0 + 12\varepsilon_0/\delta)(\bar{v}_k).$$

By Lemma 12.2.3,

$$\left|\mathcal{B}_k^+(\overline{Z}_k)\right| \leq CkR\eta^{d-1} + C(\Phi, R, \eta)R\big(R\frac{a}{\varepsilon_0} + \frac{\varepsilon_0}{\delta}\big)^{d-1}$$

and (12.1.5) and (12.1.6) hold as soon as $(\nu, v) \notin \mathcal{B}_k^+(\overline{Z}_k)$. Proposition 12.1.1 is proved. $\qquad\square$

Note that, in order to prove that pathological sets have vanishing measure as $\varepsilon \to 0$, we have to choose η small enough, and then a and ε_0 even smaller in order that (12.1.1) is satisfied and that (12.1.2) is small. Moreover, if we want to get a rate of convergence, we need to have more precise bounds on the cross-section b in terms of the truncation parameters R and η.

CHAPTER 13

TRUNCATED COLLISION INTEGRALS

Our goal in the present chapter is to slightly modify (in a uniform way) the functionals $I_{s,k}^{R,\delta}$ (defined in (7.3.1) in the hard-spheres case and in (11.3.3) for the potential case) and $I_{s,k}^{0,R,\delta}$, defined in (7.3.1), in order for the corresponding pseudo-trajectories to be decomposed as a succession of free transport and binary collisions, without any recollision. This will be possible thanks to Proposition 12.1.1. We then expect to be able to compare these approximate observables, which will be done in the next chapter.

13.1. Initialization

The first step consists in preparing the initial configuration Z_s so that it is a good configuration. We define

$$\Delta_s(\varepsilon_0) := \left\{ Z_s \in \mathbf{R}^{ds} \times B_R^s \,/\, \inf_{1 \leq \ell < j \leq s} |x_\ell - x_j| \geq \varepsilon_0 \right\},$$

and we shall assume from now on that Z_s belongs to $\Delta_s(\varepsilon_0)$. We also define for convenience

$$\Delta_s^X(\varepsilon_0) := \left\{ X_s \in \mathbf{R}^{ds} \,/\, \inf_{1 \leq \ell < j \leq s} |x_\ell - x_j| \geq \varepsilon_0 \right\}.$$

Proposition 13.1.1. — *For all $X_s \in \Delta_s^X(\varepsilon_0)$, there is a subset $\mathcal{M}_s(X_s)$ of $\mathbf{R}^{ds}$ such that*

$$\left| \mathcal{M}_s(X_s) \right| \leq CRs^2 \left(\left(R\frac{\varepsilon}{\varepsilon_0} \right)^{d-1} + \left(\frac{\varepsilon_0}{\delta} \right)^{d-1} \right),$$

and defining $\mathcal{P}_s := \left\{ Z_s \in \Delta_s(\varepsilon_0) \,/\, V_s \notin \mathcal{M}_s(X_s) \right\}$, then

$$\forall \tau \geq 0, \quad \mathbb{1}_{\mathcal{P}_s} \circ \mathbf{T}_s(\tau) \equiv \mathbb{1}_{\mathcal{P}_s} \circ \mathbf{S}_s(\tau)$$
$$\text{in the hard-spheres case,}$$

(13.1.1)
$$\forall \tau \geq 0, \quad \mathbb{1}_{\mathcal{P}_s} \circ \mathbf{H}_s(\tau) \equiv \mathbb{1}_{\mathcal{P}_s} \circ \mathbf{S}_s(\tau)$$
$$\text{in the potential case, and}$$

$$\forall \tau \geq \delta, \quad \mathbb{1}_{\mathcal{P}_s} \circ \mathbf{S}_s(\tau) \equiv \mathbb{1}_{\mathcal{P}_s} \circ \mathbf{S}_s(\tau) \circ \mathbb{1}_{\mathcal{G}_s(\varepsilon_0)}.$$

denoting abusively by $\mathbb{1}_A$ the operator of multiplication by the indicator of A.

Proof. — The proof is very similar to the arguments of the previous chapter. For any Z_s in $\Delta_s(\varepsilon_0)$, we apply Lemma 12.2.1 which shows that outside a small measure set $\mathcal{M}_s(X_s) \subset \mathbf{R}^{ds}$ of velocities $(v_1, \ldots, v_s)$, with

$$|\mathcal{M}_s(X_s)| \le CRs^2 \left(\left(R\frac{\varepsilon}{\varepsilon_0} \right)^{d-1} + \left(\frac{\varepsilon_0}{\delta} \right)^{d-1} \right),$$

the backward nonlinear flow is actually the free flow and the particles remain at a distance larger than ε to one another for all times:

$$\forall \tau > 0, \quad \forall \ell \ne \ell' \in \{1, \ldots, s\}, \quad |(x_\ell - v_\ell \tau) - (x_{\ell'} - v_{\ell'} \tau)| > \varepsilon,$$

and that

$$\forall \tau \ge \delta, \quad \forall \ell \ne \ell' \in \{1, \ldots, s\}, \quad |(x_\ell - v_\ell \tau) - (x_{\ell'} - v_{\ell'} \tau)| \ge \varepsilon_0.$$

By construction, $\mathcal{M}_s(X_s)$ depends continuously on X_s; the result follows by definition of $\mathcal{P}_s$. □

13.2. Approximation of the Boltzmann functional

We recall that we consider a family of initial data $F_0 = (f_0^{(s)})$ satisfying

$$\|F_0\|_{0,\beta_0,\mu_0} := \sup_{s \in \mathbf{N}} \sup_{Z_s} \left(\exp(\beta_0 E(Z_s) + \mu_0 s) f_0^{(s)}(Z_s) \right) < +\infty$$

and after the reductions of Chapters 7 and 11, the observable we are interested in is the following:

$$(13.2.2) \quad I_{s,k}^{0,R,\delta}(t, J, M)(X_s) := \int \varphi_s(V_s) \int_{\mathcal{T}_{k,\delta}(t)} \mathbf{S}_s(t - t_1) \mathcal{C}_{s,s+1}^{0,j_1,m_1} \mathbf{S}_{s+1}(t_1 - t_2) \mathcal{C}_{s+1,s+2}^{0,j_2,m_2}$$

$$\ldots \mathbf{S}_{s+k}(t_k - t_{k+1}) \mathbb{1}_{E_0(Z_{s+k}) \le R^2} f_0^{(s+k)} dT_k dV_s,$$

By Proposition 13.1.1, up to an error term of order $CRs^2 \left(\left(R\frac{\varepsilon}{\varepsilon_0} \right)^{d-1} + \left(\frac{\varepsilon_0}{\delta} \right)^{d-1} \right)$, we can assume that the initial configuration Z_s is a good configuration, meaning that

$$I_{s,k}^{0,R,\delta}(t, J, M)(X_s) = \int_{B_R \setminus \mathcal{M}_s(X_s)} \varphi_s(V_s) \int_{\mathcal{T}_{k,\delta}(t)} \mathbf{S}_s(t - t_1) \mathcal{C}_{s,s+1}^{0,j_1,m_1} \mathbf{S}_{s+1}(t_1 - t_2) \mathcal{C}_{s+1,s+2}^{0,j_2,m_2}$$

$$\ldots \mathcal{C}_{s+k-1,s+k}^{0,j_k,m_k} \mathbf{S}_{s+k}(t_k - t_{k+1}) \mathbb{1}_{|E_0(Z_{s+k})| \le R^2} f_0^{(s+k)} dT_k dV_s$$

$$+ O\left(c_{k,J,M} Rs^2 \left(\left(R\frac{\varepsilon}{\varepsilon_0} \right)^{d-1} + \left(\frac{\varepsilon_0}{\delta} \right)^{d-1} \right) \|F_0\|_{0,\beta_0,\mu_0} \right),$$

where $\displaystyle\sum_k \sum_{J,M} c_{k,J,M} = 1$ and

$$\left(\mathcal{C}_{s,s+1}^{0,-,m} f^{(s+1)} \right)(Z_s) = \int_{\mathbf{S}_1^{d-1} \times \mathbf{R}^d} ((v_{s+1} - v_m) \cdot \nu_{s+1})_- f^{(s+1)}(Z_s, x_m, v_{s+1}) \, d\nu_{s+1} dv_{s+1} \quad \text{and}$$

$$\left(\mathcal{C}_{s,s+1}^{0,+,m} f^{(s+1)} \right)(Z_s) = \int_{\mathbf{S}_1^{d-1} \times \mathbf{R}^d} ((v_{s+1} - v_m) \cdot \nu_{s+1})_+ f^{(s+1)}(z_1, \ldots, x_m, v_m^*, \ldots, z_s, x_m, v_{s+1}^*) \, d\nu_{s+1} dv_{s+1}.$$

Now let us introduce some notation which we shall be using constantly from now on: given $Z_s \in \Delta_s(\varepsilon_0)$, we call $Z_s^0(\tau)$ the position of the backward free flow initiated from Z_s, at time $t_1 \le \tau \le t$. Then given $j_1 \in \{+, -\}$, $m_1 \in [1, s]$, a deflection angle ν_{s+1} and a velocity v_{s+1} we call $Z_{s+1}^0(\tau)$ the position at time $t_2 \le \tau < t_1$ of the Boltzmann pseudo-trajectory initiated by the adjunction of the particle (ν_{s+1}, v_{s+1}) to the particle $z_{m_1}^0(t_1)$ (which is simply free-flow in the pre-collisional case $j_1 = -$, and free-flow after scattering of particles $z_{m_1}^0(t_1)$ and (ν_{s+1}, v_{s+1}) in the post-collisional case $j_1 = +$).

Similarly by induction given $Z_s \in \Delta_s(\varepsilon_0)$, T, J and M we denote for each $1 \leq k \leq n$ by $Z_{s+k}^0(\tau)$ the position at time $t_{k+1} \leq \tau < t_k$ of the pseudo-trajectory initiated by the adjunction of the particle (ν_{s+k}, v_{s+k}) to the particle $z_{m_k}^0(t_k)$ (which is simply free-flow in the pre-collisional case $j_k = -$, and free-flow after scattering of particles $z_{m_k}^0(t_k)$ and (ν_{s+k}, v_{s+k}) in the post-collisional case $j_k = +$).

Notice that $\tau \mapsto Z_{s+k}^0(\tau)$ is pointwise right-continuous on $[0, t_k]$.

With this notation, the elementary functional $I_{s,k}^{0,R,\delta}$ may be reformulated as

$$I_{s,k}^{0,R,\delta}(t, J, M)(X_s) = \int_{B_R \setminus \mathcal{M}_s(X_s)} dV_s \varphi_s(V_s) \int_{\mathcal{T}_{k,\delta}(t)} dT_k \int_{\mathbf{S}_1^{d-1} \times B_R} d\nu_{s+1} dv_{s+1} ((v_{s+1} - v_{m_1}^0(t_1) \cdot \nu_{s+1})_+$$

$$\dots \int_{\mathbf{S}_1^{d-1} \times B_R} d\nu_{s+k} dv_{s+k} ((v_{s+k} - v_{m_k}^0(t_k) \cdot \nu_{s+k})_+ \mathbb{1}_{E_0(Z_{s+k}^0(0)) \leq R^2} f_0^{(s+k)}(Z_{s+k}^0(0))$$

$$+ O\left(c_{k,J,M} R s^2 \left(\left(R \frac{\varepsilon}{\varepsilon_0}\right)^{d-1} + \left(\frac{\varepsilon_0}{\delta}\right)^{d-1}\right) \|F_0\|_{0,\beta_0,\mu_0}\right),$$

where $\sum_k \sum_{J,M} c_{k,J,M} = 1$. Let $a, \varepsilon_0, \eta \ll 1$ be such that

$$a \ll \varepsilon_0 \ll \eta\delta.$$

According to Proposition 12.1.1, for any good configuration $\overline{Z}_{s+k-1} \in \mathbf{R}^{2d(s+k-1)}$, we can define a set

$${}^c\mathcal{B}_{s+k-1}^{m_k}(\overline{Z}_{s+k-1}) := \left(\mathbf{S}_1^{d-1} \times B_R\right) \setminus \mathcal{B}_{s+k-1}^{m_k}(\overline{Z}_{s+k-1}),$$

such that good configurations $Z_{s+k-1} = (X_{s+k-1}, \overline{V}_{s+k-1})$ with $|X_{s+k-1} - \overline{X}_{s+k-1}| \leq Ca$ are stable by adjunction of a collisional particle $z_{s+k} = (x_{m_k} + \varepsilon \nu_{k+s}, v_{k+s})$ with $(\nu_{k+s}, v_{k+s}) \in {}^c\mathcal{B}_{s+k-1}^{m_k}(\overline{Z}_{s+k-1})$.

We further notice that thanks to Remark 12.1.3, if the adjoined pair (ν_{s+k}, v_{s+k}) belongs to the set ${}^c\mathcal{B}_{s+k-1}^{m_k}(Z_{s+k-1}^0(t_k))$ with $Z_{s+k-1}^0(t_k) \in \mathcal{G}_{s+k-1}(\varepsilon_0)$, then $Z_{s+k}^0(t_{k+1})$ belongs to $\mathcal{G}_{s+k}(\varepsilon_0)$.

As a consequence we may define recursively the approximate Boltzmann functional

$$(13.2.3) \quad \begin{aligned} J_{s,k}^{0,R,\delta}(t, J, M)(X_s) = &\int_{B_R \setminus \mathcal{M}_s(X_s)} dV_s \varphi_s(V_s) \int_{\mathcal{T}_{k,\delta}(t)} dT_k \\ &\int_{{}^c\mathcal{B}_s^{m_1}(Z_s^0(t_1))} d\nu_{s+1} dv_{s+1} (v_{s+1} - v_{m_1}^0(t_1) \cdot \nu_{s+1})_{j_1} \\ &\dots \int_{{}^c\mathcal{B}_{s+k-1}^{m_k}(Z_{s+k-1}^0(t_k))} d\nu_{s+k} dv_{s+k} (v_{s+k} - v_{m_k}^0(t_k) \cdot \nu_{s+k})_{j_k} \\ &\times \mathbb{1}_{E_0(Z_{s+k}^0(0)) \leq R^2} f_0^{(s+k)}(Z_{s+k}^0(0)). \end{aligned}$$

The following result is an immediate consequence of Proposition 12.1.1, together with the continuity estimates for the Boltzmann collision operator in Proposition 5.3.2.

Proposition 13.2.1. — *Let $a, \varepsilon_0, \eta \ll 1$ satisfying (12.1.1). Then, we have the following error estimates for the observables associated to the Boltzmann dynamics:*
– with the cross-section associated to hard-spheres,

$$\left|\sum_{k=0}^n \sum_{J,M} \mathbb{1}_{\Delta_s(\varepsilon_0)} \left(I_{s,k}^{0,R,\delta} - J_{s,k}^{0,R,\delta}\right)(t, J, M)\right| \leq Cn(s+n)$$

$$\times \left(R\eta^{d-1} + R^d \left(\frac{a}{\varepsilon_0}\right)^{d-1} + R \left(\frac{\varepsilon_0}{\delta}\right)^{d-1}\right) \|F_0\|_{0,\beta_0,\mu_0};$$

– *with the cross-section associated with a smooth short-range potential* Φ,

$$\left| \sum_{k=0}^{n} \sum_{J,M} \mathbb{1}_{\Delta_s(\varepsilon_0)} \left(I_{s,k}^{0,R,\delta} - J_{s,k}^{0,R,\delta} \right)(t,J,M) \right| \leq Cn(s+n)$$

$$\times \left(R\eta^{d-1} + C(\Phi,\eta,R)R^d \left(\frac{a}{\varepsilon_0} \right)^{d-1} + C(\Phi,\eta,R)R \left(\frac{\varepsilon_0}{\delta} \right)^{d-1} \right) \|F_0\|_{0,\beta_0,\mu_0}.$$

13.3. Approximation of the BBGKY functional

We recall that after the reductions of Chapters 7 and 11, the elementary functionals we are interested in are
– in the case of hard spheres:

$$I_{s,k}^{R,\delta}(t,J,M)(X_s) := \int \varphi_s(V_s) \int_{\mathcal{T}_{k,\delta}(t)} \mathbf{T}_s(t-t_1) \mathcal{C}_{s,s+1}^{j_1,m_1} \mathbf{T}_{s+1}(t_1-t_2) \mathcal{C}_{s+1,s+2}^{j_2,m_2}$$

$$\dots \mathcal{C}_{s+k-1,s+k}^{j_k,m_k} \mathbf{T}_{s+k}(t_k - t_{k+1}) \mathbb{1}_{E_\varepsilon(Z_{s+k}) \leq R^2} f_{N,0}^{(s+k)} dT_k dV_s,$$

where $F_{N,0} = (f_{N,0}^{(s)})_{1 \leq s \leq N}$ satisfies

$$\|F_{N,0}\|_{\varepsilon,\beta_0,\mu_0} := \sup_{s \in \mathbf{N}} \sup_{Z_s \in \mathcal{D}_s} \left(\exp(\beta_0 E_0(Z_s) + \mu_0 s) f_{N,0}^{(s)}(Z_s) \right) < +\infty;$$

– in the case of a smooth interaction potential Φ:

$$I_{s,k}^{R,\delta}(t,J,M)(X_s) := \int \varphi_s(V_s) \int_{\mathcal{T}_{k,\delta}(t)} \mathbf{H}_s(t-t_1) \mathcal{C}_{s,s+1}^{j_1,m_1} \mathbf{H}_{s+1}(t_1-t_2) \mathcal{C}_{s+1,s+2}^{j_2,m_2}$$

$$\dots \mathcal{C}_{s+k-1,s+k}^{j_k,m_k} \mathbf{H}_{s+k}(t_k - t_{k+1}) \mathbb{1}_{E_\varepsilon(Z_{s+k}) \leq R^2} \widetilde{f}_{N,0}^{(s+k)} dT_k dV_s,$$

where $\widetilde{F}_{N,0} = (\widetilde{f}_{N,0}^{(s)})_{1 \leq s \leq N}$ satisfies

$$\|\widetilde{F}_{N,0}\|_{\varepsilon,\beta_0,\mu_0} := \sup_{s \in \mathbf{N}} \sup_{Z_s} \left(\exp(\beta_0 E_\varepsilon(Z_s) + \mu_0 s) \widetilde{f}_{N,0}^{(s)}(Z_s) \right) < +\infty.$$

Since both formulas are quite similar, we shall deal with the case of smooth potentials and will indicate
– if need be – simplifications arising in the case of hard spheres.

Thanks to Proposition 13.1.1, we have

$$I_{s,k}^{R,\delta}(t,J,M)(X_s) = \int_{B_R \backslash \mathcal{M}_s(X_s)} \varphi_s(V_s) \int_{\mathcal{T}_{k,\delta}(t)} \mathbf{S}_s(t-t_1) \mathbb{1}_{\mathcal{G}_s(\varepsilon_0)} \mathcal{C}_{s,s+1}^{j_1,m_1} \mathbf{H}_{s+1}(t_1-t_2) \mathcal{C}_{s+1,s+2}^{j_2,m_2}$$

$$\dots \mathcal{C}_{s+k-1,s+k}^{j_k,m_k} \mathbf{H}_{s+k}(t_k - t_{k+1}) \mathbb{1}_{E_\varepsilon(Z_{s+k}(0)) \leq R^2} \widetilde{f}_{N,0}^{(s+k)} dT_k dV_s$$

$$+ O \left(c_{k,J,M} Rs^2 \left(\left(R\frac{\varepsilon}{\varepsilon_0} \right)^{d-1} + \left(\frac{\varepsilon_0}{\delta} \right)^{d-1} \right) \|\widetilde{F}_{N,0}\|_{\varepsilon,\beta_0,\mu_0} \right),$$

where recall that $c_{k,J,M}$ denotes a sequence of positive real numbers satisfying $\sum_k \sum_{J,M} c_{k,J,M} = 1$.

Then using the notation introduced in the previous paragraph for the Boltzmann pseudo-trajectory, let us define the approximate functionals

$$J_{s,k}^{R,\delta}(t,J,M)(X_s) := \int_{B_R \backslash \mathcal{M}_s(X_s)} \varphi_s(V_s) \int_{\mathcal{T}_{k,\delta}(t)} \mathbf{S}_s(t-t_1) \mathbb{1}_{\mathcal{G}_s(\varepsilon_0)} \widetilde{\mathcal{C}}_{s,s+1}^{j_1,m_1} \mathbf{H}_{s+1}(t_1-t_2)$$

$$\dots \widetilde{\mathcal{C}}_{s+k-1,s+k}^{j_k,m_k} \mathbf{H}_{s+k}(t_k - t_{k+1}) \mathbb{1}_{E_\varepsilon(Z_{s+k}(0)) \leq R^2} \widetilde{f}_0^{(s+k)} dT_k dV_s,$$

where the modified collision operators are obtained by elimination of the pathological set of impact
parameters and velocities

$$\big(\widetilde{\mathcal{C}}_{s+k-1,s+k}^{\pm,m_k}g^{(s+k)}\big)(Z_{s+k-1}) := (N-s-k+1)\varepsilon^{d-1}\int_{^c\mathcal{B}_{s+k-1}^{m_k}(Z_{s+k-1}^0(t_k))} (\nu_{s+k}\cdot(v_{s+k}-v_{m_k}(t_k)))_{\pm}$$

$$\times g^{(s+k)}(\cdot,x_{m_k}(t_k)+\varepsilon\nu_{s+k},v_{s+k}(t_k)) \prod_{\substack{1\le j\le s+k-1\\ j\ne m_k}} \mathbb{1}_{|(x_j-x_{m_k})(t_k)-\varepsilon\nu_{s+k}|\ge\varepsilon}\, d\nu_{s+k}dv_{s+k}\,.$$

By construction, we know that the remaining collision trees are nice, in the sense that collisions involve
only two particles and are well-separated in time. Using the pre/post-collisional change of variables,
we can rewrite the gain terms as follows

$$\mathbb{1}_{\mathcal{G}_{s+k-1}(\varepsilon_0/2)}\big(\widetilde{\mathcal{C}}_{s+k-1,s+k}^{+,m_k}\mathbf{H}_{s+k}(t_k-t_{k+1})g^{(s+k)}\big)(Z_{s+k-1})$$

$$:= (N-s-k+1)\varepsilon^{d-1}\mathbb{1}_{\mathcal{G}_{s+k-1}(\varepsilon_0/2)}\int_{^c\mathcal{B}_{s+k-1}^{m_k}(Z_{s+k-1}^0(t_k))} (\nu_{s+k}\cdot(v_{s+k}-v_{m_k}(t_k)))_{+}$$

$$\times\,\mathbf{H}_{s+k}(t_k-t_{k+1}-t_\varepsilon(Z_{s+k}))g^{(s+k)}(\cdot,x_{m_k}^*,v_{m_k}^*,\dots,x_{s+k}^*,v_{s+k}^*)$$

$$\times \prod_{\substack{1\le j\le s+k-1\\ j\ne m_k}} \mathbb{1}_{|(x_j-x_{m_k})(t_k)-\varepsilon\nu_{s+k}|\ge\varepsilon}\, d\nu_{s+k}dv_{s+k}.$$

denoting as previously by $(x_{m_k}^*,v_{m_k}^*,x_{s+k}^*,v_{s+k}^*)$ the pre-image by the scattering operator σ_ε of the
point $(x_{m_k},v_{m_k}(t_k),x_{m_k}(t_k)+\varepsilon\nu_{s+k},v_{s+k}(t_k))$.

Note that this last step is obvious in the case of hard spheres since there is no time shift : $t_\varepsilon\equiv 0$.

As in the Boltzmann case described above, the following result is an immediate consequence of Proposition 12.1.1 together with the continuity estimates for the BBGKY collision operator in Propositions 5.3.1 and 10.3.1.

Proposition 13.3.1. — *Let* $a,\varepsilon_0,\eta\ll 1$ *satisfying* (12.1.1). *Then, for* ε *sufficiently small, we have
the following error estimates for the observables associated to the BBGKY dynamics:*
– *in the case of hard-spheres*

$$\Big|\sum_{k=0}^n\sum_{J,M}\mathbb{1}_{\Delta_s(\varepsilon_0)}\big(I_{s,k}^{R,\delta}-J_{s,k}^{R,\delta}\big)(t,J,M)\Big|\le Cn(s+n)\Big(R\eta^{d-1}+R^d\Big(\frac{a}{\varepsilon_0}\Big)^{d-1}+R\Big(\frac{\varepsilon_0}{\delta}\Big)^{d-1}\Big)\|F_{N,0}\|_{\varepsilon,\beta_0,\mu_0}\,,$$

– *in the case of some smooth short-range potential* Φ

$$\Big|\sum_{k=0}^n\sum_{J,M}\mathbb{1}_{\Delta_s(\varepsilon_0)}\big(I_{s,k}^{0,R,\delta}-J_{s,k}^{0,R,\delta}\big)(t,J,M)\Big|\le Cn(s+n)$$

$$\times\Big(R\eta^{d-1}+C(\Phi,\eta,R)R^d\Big(\frac{a}{\varepsilon_0}\Big)^{d-1}+C(\Phi,\eta,R)R\Big(\frac{\varepsilon_0}{\delta}\Big)^{d-1}\Big)\|\widetilde{F}_{N,0}\|_{\varepsilon,\beta_0,\mu_0}\,.$$

The functional $J_{s,k}^{R,\delta}$ can be written in terms of pseudo-trajectories, as in (13.2.3). Let us therefore
introduce some notation which we shall be using constantly from now on: given $Z_s\in\Delta_s(\varepsilon_0)$, we
call $Z_s^0(\tau)$ the position of the backward free flow initiated from Z_s, at time $t_1\le\tau\le t$. Then
given $j_1\in\{+,-\}$, $m_1\in[1,s]$, an angle ν_{s+1} (or equivalently a position $x_{s+1}=x_{m_1}^0(t_1)+\varepsilon\nu_{s+1}$) and
a velocity v_{s+1} we call $Z_{s+1}^\varepsilon(\tau)$ the position at time $t_2\le\tau<t_1$ of the BBGKY pseudo-trajectory
initiated by the adjunction of the particle z_{s+1} to the particle $z_{m_1}^0(t_1)$.

Similarly by induction given $Z_s \in \Delta_s(\varepsilon_0)$, T, J and M we denote for each $1 \leq k \leq n$ by $Z_{s+k}^\varepsilon(\tau)$ the position at time $t_{k+1} \leq \tau < t_k$ of the BBGKY pseudo-trajectory initiated by the adjunction of the particle z_{s+k} to the particle $z_{m_k}(t_k)$. We have

$$J_{s,k}^{R,\delta}(t,J,M)(X_s) = \frac{(N-s)!}{(N-s-k)!} \varepsilon^{k(d-1)} \int_{B_R \setminus \mathcal{M}_s(X_s)} dV_s \varphi_s(V_s) \int_{\mathcal{T}_{k,\delta}(t)} dT_k$$

$$\int_{{}^c\mathcal{B}_s(Z_s^0(t_1))} d\nu_{s+1} dv_{s+1} \left(\nu_{s+1} \cdot (v_{s+1} - v_{m_1}(t_1))\right)_{j_1} \prod_{\substack{1 \leq j \leq s \\ j \neq m_1}} \mathbb{1}_{|(x_j - x_{m_1})(t_1) - \varepsilon\nu_{s+1}| \geq \varepsilon}$$

(13.3.4)

$$\ldots \int_{{}^c\mathcal{B}_{s+k-1}^{j_k}(Z_{s+k-1}^0(t_k))} d\nu_{s+k} dv_{s+k} \left(\nu_{s+k} \cdot (v_{s+k} - v_{m_k}(t_k))\right)_{j_k}$$

$$\times \prod_{\substack{1 \leq j \leq s+k-1 \\ j \neq m_k}} \mathbb{1}_{|(x_j - x_{m_k})(t_k) - \varepsilon\nu_{s+k}| \geq \varepsilon} \mathbb{1}_{E_\varepsilon(Z_{s+k}(0)) \leq R^2} \widetilde{f}_{N,0}^{(s+k)}(Z_{s+k}^\varepsilon(0)).$$

Thanks to Propositions 13.2.1 and 13.3.1 the proof of Theorems 8 and 11 reduces to the proof of the convergence to zero of $J_{s,k}^{R,\delta} - J_{s,k}^{0,R,\delta}$. This is the object of the next chapter.

CHAPTER 14

CONVERGENCE PROOF

In this chapter we conclude the proof of Theorems 8 and 11 by proving that $J_{s,k}^{R,\delta} - J_{s,k}^{0,R,\delta}$ goes to zero in the Boltzmann-Grad limit, with the notation of the previous chapter, namely (13.2.3) and (13.3.4). The main difficulty lies in the fact that in contrast to the Boltzmann situation, collisions in the BBGKY configuration are not pointwise in space (nor in time in the case of the smooth Hamiltonian system). At each collision time t_k a small error is therefore introduced, which needs to be controlled.

We recall that, as in the previous chapters, we consider dynamics
– involving only a finite number $s + k$ of particles,
– with bounded energies (at most $R^2 \gg 1$),
– such that the k additional particles are adjoined through binary collisions at times separated at least by $\delta \ll 1$.
The additional truncation parameters $a, \varepsilon_0, \eta \ll 1$ satisfy (12.1.1).

14.1. Proximity of Boltzmann and BBGKY trajectories

This paragraph is devoted to the proof, by induction, that the BBGKY and Boltzmann pseudo-trajectories remain close for all times, in particular that there is no recollision for the BBGKY dynamics.

We recall that the notation $Z_k^0(t)$ and $Z_k(t)$ were defined in Paragraphs 13.2 and 13.3 respectively.

Lemma 14.1.1. — *Fix $T \in \mathcal{T}_{n,\delta}(t)$, J, and M and given Z_s in $\Delta_s(\varepsilon_0)$, consider for all $i \in \{1, \ldots n\}$, an impact parameter ν_{s+i} and a velocity v_{s+i} such that $(\nu_{s+i}, v_{s+i}) \notin \mathcal{B}_{s+i-1}(Z_{s+i-1}^0(t_i))$. Then, for ε sufficiently small, for all $i \in [1,n]$, and all $k \leq s + i$,*
– for the hard sphere dynamics

$$(14.1.1) \qquad |x_k^\varepsilon(t_{i+1}) - x_k^0(t_{i+1})| \leq \varepsilon i \quad and \quad v_k(t_{i+1}) = v_k^0(t_{i+1}),$$

– for the hamiltonian dynamics associated to Φ

$$(14.1.2) \qquad |x_k^\varepsilon(t_{i+1}) - x_k^0(t_{i+1})| \leq C(\Phi, R, \eta)\varepsilon i \quad and \quad v_k(t_{i+1}) = v_k^0(t_{i+1}),$$

where the constant $C(\Phi, R, \eta)$ depends only on Φ, R, and η.

Proof. — We proceed by induction on i, the index of the time variables t_{i+1} for $0 \leq i \leq n$.

We first notice that by construction, $Z_s(t_1) - Z_s^0(t_1) = 0$, so (14.1.2) holds for $i = 0$. The initial configuration being a good configuration, we indeed know – by definition – that there is no possible recollision.

Now let $i \in [1, n]$ be fixed, and assume that for all $\ell \leq i$

$$(14.1.3) \qquad \forall k \leq s + \ell - 1, \qquad |x_k^\varepsilon(t_\ell) - x_k^0(t_\ell)| \leq C\varepsilon(\ell - 1) \quad \text{and} \quad v_k(t_\ell) = v_k^0(t_\ell),$$

with $C = 1$ for hard spheres.

Let us prove that (14.1.3) holds for $\ell = i + 1$. We shall consider two cases depending on whether the particle adjoined at time t_i is pre-collisional or post-collisional.

• As usual, the case of pre-collisional velocities $(v_{s+i}, v_{m_i}(t_i))$ at time t_i is the most simple to handle. We indeed have $\forall \tau \in [t_{i+1}, t_i]$

$$\forall k < s + i, \quad x_k^0(\tau) = x_k^0(t_i) + (\tau - t_i)v_k^0(t_i), \qquad v_k^0(\tau) = v_k^0(t_i),$$
$$x_{s+i}^0(\tau) = x_{m_i}^0(t_i) + (\tau - t_i)v_{s+i}, \qquad v_{s+i}^0(\tau) = v_{s+i}.$$

Now let us study the BBGKY trajectory. We recall that the particle is adjoined in such a way that (ν_{s+i}, v_{s+i}) belongs to ${}^c\mathcal{B}_{s+i-1}(Z_{s+i-1}^0(t_i))$. Provided that ε is sufficiently small, by the induction assumption (14.1.3), we have

$$\forall k \leq s + i - 1, \quad |x_k^\varepsilon(t_i) - x_k^0(t_i)| \leq C\varepsilon(i - 1) \leq a,$$

with $C = 1$ for hard spheres.

Since $Z_{s+i-1}^0(t_i)$ belongs to $\mathcal{G}_{s+i-1}(\varepsilon_0)$ (see Paragraph 13.2), we can apply Proposition 12.1.1 which implies that backwards in time, there is free flow for Z_{s+i}^ε. In particular,

$$\forall k < s + i, \quad x_k(\tau) = x_k(t_i) + (\tau - t_i)v_k(t_i), \qquad v_k(\tau) = v_k(t_i),$$
$$x_{s+i}(\tau) = x_{m_i}(t_i) + \varepsilon\nu_{s+i} + (\tau - t_i)v_{s+i}, \qquad v_{s+i}(\tau) = v_{s+i}.$$

We therefore obtain

$$(14.1.4) \qquad \forall k \leq s + i, \quad \forall \tau \in [t_{i+1}, t_i], \quad v_k(\tau) - v_k^0(\tau) = v_k(t_i) - v_k^0(t_i) = 0,$$

and

$$(14.1.5) \qquad \forall k \leq s + i, \quad \forall \tau \in [t_{i+1}, t_i], \quad |x_k(\tau) - x_k^0(\tau)| \leq C\varepsilon(i - 1) + \varepsilon,$$

with $C = 1$ in the case of hard spheres.

• The case of post-collisional velocities $(v_{s+i}, v_{m_i}(t_i))$ at time t_i for the hard sphere dynamics is very similar. We indeed have $\forall \tau \in [t_{i+1}, t_i[$

$$\forall k < s + i, \quad k \neq m_i, \quad x_k^0(\tau) = x_k^0(t_i) + (\tau - t_i)v_k^{0*}(t_i), \qquad v_k^0(\tau) = v_k^0(t_i),$$
$$x_{m_i}^0(\tau) = x_{m_i}^0(t_i) + (\tau - t_i)v_{m_i}^{0*}(t_i), \qquad v_{m_i}^0(\tau) = v_{m_i}^{0*}(t_i),$$
$$x_{s+i}^0(\tau) = x_{m_i}^0(t_i) + (\tau - t_i)v_{s+i}^*, \qquad v_{s+i}^0(\tau) = v_{s+i}^*.$$

Now let us study the BBGKY trajectory. We recall that the particle is adjoined in such a way that (ν_{s+i}, v_{s+i}) belongs to ${}^c\mathcal{B}_{s+i-1}^{m_i}(Z_{s+i-1}^0(t_i))$. Provided that ε is sufficiently small, by the induction assumption (14.1.3), we have

$$\forall k \leq s + i - 1, \quad |x_k^\varepsilon(t_i) - x_k^0(t_i)| \leq \varepsilon(i - 1).$$

Since $Z_{s+i-1}^0(t_i)$ belongs to $\mathcal{G}_{s+i-1}(\varepsilon_0)$ (see Paragraph 13.2), we can apply Proposition 12.1.1 which implies that backwards in time, there is free flow for Z_{s+i}^ε. In particular,

$$(14.1.6) \qquad \forall k \leq s+i\,, \quad \forall \tau \in [t_{i+1}, t_i[\,, \quad v_k(\tau) - v_k^0(\tau) = v_k(t_i^-) - v_k^0(t_i^-) = 0\,,$$

and

$$(14.1.7) \qquad \forall k \leq s+i\,, \quad \forall \tau \in [t_{i+1}, t_i[\,, \quad |x_k(\tau) - x_k^0(\tau)| \leq \varepsilon(i-1) + \varepsilon \leq i\varepsilon\,.$$

• The case of post-collisional velocities is a little more complicated since there is a (small) time interval during which interaction occurs.

Let us start by describing the Boltzmann flow. By definition of the post-collisional configuration, we know that the following identities hold:

$$\forall t_{i+1} \leq \tau < t_i\,, \begin{cases} (v_{m_i}^0, v_{s+i}^0)(\tau) = (v_{m_i}^{0*}(t_i), v_{s+i}^*) \text{ with } (\nu_{s+i}^*, v_{m_i}^{0*}(t_i), v_{s+i}^*) := \sigma_0^{-1}(\nu_{s+i}, v_{m_i}^0(t_i), v_{s+i}) \\ x_{m_i}^0(\tau) = x_{m_i}^0(t_i) + (\tau - t_i)v_{m_i}^{0*}(t_i)\,, \; x_{s+i}^0(\tau) = x_{s+i}^0(t_i) + (\tau - t_i)v_{s+i}^* \\ \forall j \notin \{m_i, s+1\}\,, \quad v_j^0(\tau) = v_j^0(t_i)\,, \; x_j^0(\tau) = x_j^0(t_i) + (\tau - t_i)v_j^0(t_i)\,, \end{cases}$$

where σ_0 denotes the scattering operator defined in Definition 8.2.1 in Chapter 8.

First, by Proposition 12.1.1, we know that for $j \notin \{m_i, s+i\}$ and $\forall \tau \in [t_{i+1}, t_i]$,

$$x_j(\tau) = x_j(t_i) + (\tau - t_i)v_j(t_i)\,, \qquad v_j(\tau) = v_j(t_i)\,,$$

so that by the induction assumption (14.1.3) we obtain

$$(14.1.8) \qquad \begin{aligned} \forall j \notin \{m_i, s+i\}\,, \forall \tau \in [t_{i+1}, t_i]\,, \quad |x_j(\tau) - x_j^0(\tau)| = |x_j(t_i) - x_j^0(t_i)| \leq C\varepsilon(i-1) \\ \text{and} \quad v_j(\tau) = v_j^0(\tau)\,. \end{aligned}$$

We now have to focus on the pair $(s+i, m_i)$. According to Chapter 8, the relative velocity evolves under the nonlinear dynamics on a time interval $[t_i - t_\varepsilon, t_i]$ with $t_\varepsilon \leq C(\Phi, R, \eta)\varepsilon$ (recalling that by construction, the relative velocity $|v_{s+i} - v_{m_i}(t_i)|$ is bounded from above by R and from below by η, and that the impact parameter is also bounded from below by η). Then, for all $\tau \in [t_{i+1}, t_i - t_\varepsilon]$,

$$(14.1.9) \qquad v_{s+i}(\tau) = v_{s+i}^* = v_{s+i}^0(\tau)\,, \quad v_{m_i}(\tau) = v_{m_i}^*(t_i) = v_{m_i}^{0*}(t_i) = v_{m_i}^0(\tau)\,.$$

In particular,

$$(14.1.10) \qquad v_{s+i}(t_{i+1}) = v_{s+i}^0(t_{i+1}) \quad \text{and} \quad v_{m_i}(t_{i+1}) = v_{m_i}^0(t_{i+1})\,.$$

The conservation of total momentum as in Paragraph 12.3.3 shows that

$$\left| \frac{1}{2}(x_{m_i}^\varepsilon(t_i - t_\varepsilon) + x_{s+i}^\varepsilon(t_i - t_\varepsilon)) - \frac{1}{2}(x_{m_i}^0(t_i - t_\varepsilon) + x_{s+i}^0(t_i - t_\varepsilon)) \right|$$

$$= \left| \frac{1}{2}(x_{m_i}^\varepsilon(t_i) + x_{s+i}^\varepsilon(t_i) - \frac{1}{2}(x_{m_i}^0(t_i) + x_{s+i}^0(t_i)) \right|$$

$$= \left| x_{s+i}^\varepsilon(t_i) - x_{s+i}^0(t_i) \right| + \frac{\varepsilon}{2} \leq C\varepsilon(i-1) + \frac{\varepsilon}{2}\,.$$

On the other hand, by definition of the scattering time t_ε,

$$|x_{m_i}^\varepsilon(t_i - t_\varepsilon) - x_{s+i}^\varepsilon(t_i - t_\varepsilon)| = \varepsilon\,,$$

$$|x_{m_i}^0(t_i - t_\varepsilon) - x_{s+i}^0(t_i - t_\varepsilon)| = t_\varepsilon |v_{m_i}^* - v_{s+i}^*| \leq C(\Phi, R, \eta)\,\varepsilon\,.$$

We obtain finally

$$(14.1.11) \qquad |x_{m_i}^\varepsilon(t_i - t_\varepsilon) - x_{m_i}^0(t_i - t_\varepsilon)| \leq C\varepsilon i \text{ and } |x_{s+i}^\varepsilon(t_i - t_\varepsilon) - x_{s+i}^0(t_i - t_\varepsilon)| \leq C\varepsilon i$$

provided that C is chosen sufficiently large (depending on Φ, R and η).

Now let us apply Proposition 12.1.1, which implies that for all $\tau \in [t_{i+1}, t_i - t_\varepsilon]$ the backward in time evolution of the two particles $x_{s+i}^\varepsilon(t_i - t_\varepsilon)$ and $x_{m_i}^\varepsilon(t_i - t_\varepsilon)$, is that of free flow: we have therefore, using (14.1.9),

$$x_{m_i}^\varepsilon(t_{i+1}) - x_{m_i}^0(t_{i+1}) = x_{m_i}^\varepsilon(t_i - t_\varepsilon) - x_{m_i}^0(t_i - t_\varepsilon),$$
$$x_{s+i}^\varepsilon(t_{i+1}) - x_{s+i}^0(t_{i+1}) = x_{s+i}^\varepsilon(t_i - t_\varepsilon) - x_{s+i}^0(t_i - t_\varepsilon).$$

From (14.1.11) we therefore deduce that the induction assumption is satisfied at time step t_{i+1}, and the proposition is proved. $\square$

Note that, by construction,

$$Z_{s+k}^0(0) \in \mathcal{G}_{s+k}(\varepsilon_0),$$

so that an obvious application of the triangular inequality leads to

$$Z_{s+k}^\varepsilon(0) \in \mathcal{G}_{s+k}(\varepsilon_0/2).$$

Note also that the indicator functions are identically equal to 1 for good configurations. We therefore have the following

Corollary 14.1.2. — *Under the assumptions of Lemma* 14.1.1, *the functional* $J_{s,n}^{R,\delta}(t, J, M)$ *defined in* (13.3.4) *may be written as follows:*

$$J_{s,k}^{R,\delta}(t, J, M)(X_s) = \frac{(N-s)!}{(N-s-k)!} \varepsilon^{k(d-1)} \int_{B_R \setminus \mathcal{M}_s(X_s)} dV_s \varphi_s(V_s) \int_{\mathcal{T}_{k,\delta}(t)} dT_k$$
$$\int_{{}^c\mathcal{B}_s^{m_1}(Z_s^0(t_1))} d\nu_{s+1} dv_{s+1} \left(\nu_{s+1} \cdot (v_{s+1} - v_{m_1}(t_1))\right)_{j_1}$$
$$\cdots \int_{{}^c\mathcal{B}_{s+k-1}^{m_k}(Z_{s+k-1}^0(t_n))} d\nu_{s+k} dv_{s+k} \left(\nu_{s+n} \cdot (v_{s+k} - v_{m_k}(t_k))\right)_{j_k}$$
$$\times \mathbb{1}_{E_\varepsilon(Z_{s+k}(0)) \leq R^2} \mathbb{1}_{Z_{s+k}(0) \in \mathcal{G}_{s+k}(\varepsilon_0/2)} \widetilde{f}_{N,0}^{(s+k)}(Z_{s+k}^\varepsilon(0)).$$

14.2. Proof of convergence for the hard sphere dynamics: proof of Theorem 8

In this section we prove Theorem 8, which concerns the case of hard spheres. The potential case will be treated in the following section.

From Corollary 7.4.1, we know that any observable associated to the BBGKY hierarchy can be approximated by a finite sum : more precisely, given s and $t \in [0, T]$, there are two positive constants C and C' such that

$$(14.2.12) \qquad \left\| I_s(t) - \sum_{k=0}^n I_{s,k}^{R,\delta}(t) \right\|_{L^\infty(\mathbf{R}^{ds})} \leq C(2^{-n} + e^{-C'R^2} + \delta) \|\varphi\|_{L^\infty(\mathbf{R}^{ds})} \|F_{N,0}\|_{\varepsilon,\beta_0,\mu_0}.$$

Similarly, for the Boltzmann hierarchy, we get

$$(14.2.13) \qquad \left\| I_s^0(t) - \sum_{k=0}^n I_{0,s,k}^{R,\delta}(t) \right\|_{L^\infty(\mathbf{R}^{ds})} \leq C(2^{-n} + e^{-C'R^2} + \delta) \|\varphi\|_{L^\infty(\mathbf{R}^{ds})} \|F_0\|_{0,\beta_0,\mu_0}.$$

Then, from Propositions 13.2.1 and 13.3.1, we obtain the error terms corresponding to the elimination of pathological velocities and impact parameters

$$
(14.2.14) \quad \left| \mathbb{1}_{\Delta_s(\varepsilon_0)} \sum_{k=0}^{n} \sum_{J,M} \left(I_{s,k}^{0,R,\delta} - J_{s,k}^{0,R,\delta} \right)(t,J,M) \right| \leq Cn(s+n)
$$
$$
\times \left(R\eta^{d-1} + R^d \left(\frac{a}{\varepsilon_0} \right)^{d-1} + R \left(\frac{\varepsilon_0}{\delta} \right)^{d-1} \right) \|F_0\|_{0,\beta_0,\mu_0} \|\varphi\|_{L^\infty(\mathbf{R}^{ds})},
$$

and

$$
(14.2.15) \quad \left| \mathbb{1}_{\Delta_s(\varepsilon_0)} \sum_{k=0}^{n} \sum_{J,M} \left(I_{s,k}^{R,\delta} - J_{s,k}^{R,\delta} \right)(t,J,M) \right| \leq Cn(s+n)
$$
$$
\times \left(R\eta^{d-1} + R^d \left(\frac{a}{\varepsilon_0} \right)^{d-1} + R \left(\frac{\varepsilon_0}{\delta} \right)^{d-1} \right) \|F_{N,0}\|_{\varepsilon,\beta_0,\mu_0} \|\varphi\|_{L^\infty(\mathbf{R}^{ds})}.
$$

The end of the proof of Theorem 8 consists in estimating the error terms in $J_{s,k}^{R,\delta} - J_{s,k}^{0,R,\delta}$ coming essentially from the micro-translations described in the previous paragraph and from the initial data.

14.2.1. Error coming from the initial data. —

Let us replace the initial data in $J_{s,k}^{R,\delta}$ by that of the Boltzmann hierarchy, defining:

$$
\widetilde{J}_{s,k}^{R,\delta}(t,J,M)(X_s) := \frac{(N-s)!}{(N-s-k)!} \varepsilon^{k(d-1)} \int_{B_R \setminus \mathcal{M}_s(X_s)} dV_s \varphi_s(V_s) \int_{\mathcal{T}_{k,\delta}(t)} dT_k
$$
$$
\int_{{}^c\mathcal{B}_s^{m_1}(Z_s^0(t_1))} d\nu_{s+1} dv_{s+1} \left(\nu_{s+1} \cdot (v_{s+1} - v_{m_1}(t_1)) \right)_{j_1}
$$
$$
\dots \int_{{}^c\mathcal{B}_{s+k-1}^{m_k}(Z_{s+k-1}^0(t_k))} d\nu_{s+k} dv_{s+k} \left(\nu_{s+k} \cdot (v_{s+k} - v_{m_k}(t_k)) \right)_{j_k}
$$
$$
\times \mathbb{1}_{E_0(Z_{s+k}(0)) \leq R^2} \mathbb{1}_{Z_{s+k}^\varepsilon(0) \in \mathcal{G}_{s+k}(\varepsilon_0/2)} f_0^{(s+k)}(Z_{s+k}(0)).
$$

Since, by definition of admissible Boltzmann data, we have for any fixed s

$$
f_{0,N}^{(s)} \longrightarrow f_0^{(s)} \quad \text{as } N \to \infty \text{ with } N\varepsilon^{d-1} \equiv 1, \text{ locally uniformly in } \Omega_s,
$$

we expect that

$$
J_{s,k}^{R,\delta}(t,J,M)(X_s) - \widetilde{J}_{s,k}^{R,\delta}(t,J,M)(X_s) \to 0
$$

as $N \to \infty$ with $N\varepsilon^{d-1} \equiv 1$, locally uniformly in Ω_s.

Lemma 14.2.1. — *Let F_0 be an admissible Boltzmann datum and $F_{0,N}$ an associated BBGKY datum. Then, in the Boltzmann-Grad scaling $N\varepsilon^{d-1} = 1$, for all fixed $s, k \in \mathbf{N}$ and $t < T$,*

$$
J_{s,k}^{R,\delta}(t,J,M)(X_s) - \widetilde{J}_{s,k}^{R,\delta}(t,J,M)(X_s) \to 0,
$$

locally uniformly in Ω_s.

For tensorized initial data

$$
f_{0,N}^{(N)}(Z_N) = \mathcal{Z}_N^{-1} \mathbb{1}_{Z_N \in \mathcal{D}_N} f_0^{\otimes N}(Z_N) \quad \text{with} \quad \left\| f_0 \exp(\beta_0 |v|^2) \right\|_{L^\infty} < +\infty,
$$

we further have the following error estimate :

$$\left| \mathbb{1}_{\Delta_s^X(\varepsilon_0)} \sum_{k=0}^n \sum_{J,M} (J_{s,k}^{R,\delta} - \widetilde{J}_{s,k}^{R,\delta})(t,J,M)(X_s) \right| \leq C\varepsilon(s+n)\|F_0\|_{0,\beta_0,\mu_0}\|\varphi\|_{L^\infty(\mathbf{R}^{ds})}.$$

Proof. — By definition of the good sets $\mathcal{G}_k(c)$, the positions in the argument of $f_{N,0}^{(s+k)} - f_0^{(s+k)}$ satisfy the separation condition $|x_i - x_j| \geq \varepsilon_0/2 > \varepsilon$ for $i \neq j$:

$$\mathbb{1}_{\mathcal{G}_{s+k}(\varepsilon_0/2)}(f_{N,0}^{(s+k)} - f_0^{(s+k)}) = \mathbb{1}_{\mathcal{G}_{s+k}(\varepsilon_0/2)}\mathbb{1}_{\Delta_{s+k}^X(\varepsilon_0/2)}(f_{N,0}^{(s+k)} - f_0^{(s+k)}).$$

So we can write

$$(J_{s,k}^{R,\delta}(t,J,M) - \widetilde{J}_{s,k}^{R,\delta}(t,J,M))(X_s) = \frac{(N-s)!}{(N-s-k)!}\varepsilon^{k(d-1)}\int_{B_R\setminus\mathcal{M}_s(X_s)} dV_s\varphi_s(V_s)\int_{\mathcal{T}_{k,\delta}(t)} dT_k$$

$$\int_{^c\mathcal{B}_s^{m_1}(Z_s^0(t_1))} d\nu_{s+1}dv_{s+1}(\nu_{s+1}\cdot(v_{s+1}-v_{m_1}(t_1)))_{j_1}$$

$$\cdots\int_{^c\mathcal{B}_{s+k-1}^{m_k}(Z_{s+k-1}^0(t_k))} d\nu_{s+k}dv_{s+k}\,(\nu_{s+k}\cdot(v_{s+k}-v_{m_k}(t_k)))_{j_k}$$

$$\times \mathbb{1}_{E_\varepsilon(Z_{s+k}^\varepsilon(0))\leq R^2}\mathbb{1}_{\Delta_{s+k}(\varepsilon_0/2)}(f_{N,0}^{(s+k)} - f_0^{(s+k)}),$$

and we find directly that

$$\left| \mathbb{1}_{\Delta_s^X(\varepsilon_0)}(J_{s,k}^{R,\delta}(t,J,M) - \widetilde{J}_{s,k}^{R,\delta}(t,J,M))(X_s) \right| \leq C\frac{R^{k(d+1)}t^k}{k!}\left\|\mathbb{1}_{\Delta_{s+k}(\varepsilon_0/2)}(f_{N,0}^{(s+k)} - f_0^{(s+k)})\right\|_{L^\infty}.$$

Note that, summing all the elementary contributions (i.e. summing over J, M and k), we get the convergence to 0, but with a very bad dependence with respect to R and n.

In the case of tensorized initial data, this estimate can be improved using some explicit control on the convergence of the initial data. Looking at the proof of Proposition 6.1.2, we indeed see that

$$\mathbb{1}_{Z_s\in\mathcal{D}_s}f_0^{\otimes s} - f_{0,N}^{(s)} = \left(1 - \mathcal{Z}_N^{-1}\mathcal{Z}_{N-s}\right)\mathbb{1}_{Z_s\in\mathcal{D}_s}f_0^{\otimes s} + \mathcal{Z}_N^{-1}\mathcal{Z}_{(s+1,N)}^\flat\mathbb{1}_{Z_s\in\mathcal{D}_s}f_0^{\otimes s}$$

with

$$\left|1 - \mathcal{Z}_N^{-1}\mathcal{Z}_{N-s}\right| \leq (1 - \varepsilon\kappa_d|f_0|_{L^\infty L^1})^{-s} - 1 \leq \varepsilon s\kappa_d|f_0|_{L^\infty L^1}\left(1 - \varepsilon\kappa_d|f_0|_{L^\infty L^1}\right)^{-(s+1)}$$

according to Lemma 6.1.2, and

$$\mathcal{Z}_N^{-1}\mathcal{Z}_{(s+1,N)}^\flat \leq \varepsilon s\kappa_d|f_0|_{L^\infty L^1}\left(1 - \varepsilon\kappa_d|f_0|_{L^\infty L^1}\right)^{-(s+1)}.$$

Using the continuity estimate in Proposition 5.3.1, we then deduce that

$$\left|\mathbb{1}_{\Delta_s^X(\varepsilon_0)}(J_{s,k}^{R,\delta}(t,J,M) - \widetilde{J}_{s,k}^{R,\delta}(t,J,M))(X_s)\right|$$

$$\leq \varepsilon(s+k)\kappa_d|f_0|_{L^\infty L^1}\|F_0\|_{0,\beta_0,\mu_0}\|\varphi\|_{L^\infty(\mathbf{R}^{ds})}c_{k,J,M}.$$

denoting by $(c_{k,J,M})$ a sequence of nonnegative real numners such that $\sum_k\sum_{J,M}c_{k,J,M} = 1$. This concludes the proof of Lemma 14.2.1. $\qquad\square$

14.2.2. Error coming from the prefactors in the collision operators. —

As $\varepsilon \to 0$ in the Boltzmann-Grad scaling, we have

$$\frac{(N-s)!}{(N-s-k)!}\varepsilon^{k(d-1)} \to 1\,.$$

Defining

$$\overline{J}_{s,k}^{R,\delta}(t, J, M)(X_s) = \int_{B_R \setminus \mathcal{M}_s(X_s)} dV_s \varphi_s(V_s) \int_{\mathcal{T}_{k,\delta}(t)} dT_k$$

(14.2.16)
$$\int_{{}^c\mathcal{B}_s^{m_1}(Z_s^0(t_1))} d\nu_{s+1} dv_{s+1} \left(\nu_{s+1} \cdot (v_{s+1} - v_{m_1}(t_1))\right)_{j_1}$$

$$\ldots \int_{{}^c\mathcal{B}_{s+k-1}^{m_k}(Z_{s+k-1}^0(t_k))} d\nu_{s+k} dv_{s+k} \left(\nu_{s+k} \cdot (v_{s+k} - v_{m_k}(t_k))\right)_{j_k}$$

$$\times \mathbb{1}_{E_0(Z_{s+k}(0)) \leq R^2} \mathbb{1}_{Z_{s+k}^\varepsilon(0) \in \mathcal{G}_{s+k}(\varepsilon_0/2)} f_0^{(s+k)}(Z_{s+k}(0))\,,$$

and using again the continuity estimate in Proposition 5.3.1, we have the following obvious convergence.

Lemma 14.2.2. — *In the Boltzmann-Grad scaling $N\varepsilon^{d-1} = 1$,*

$$\left| \mathbb{1}_{\Delta_s^X(\varepsilon_0)} \sum_{k=0}^n \sum_{J,M} (\widetilde{J}_{s,k}^{R,\delta} - \overline{J}_{s,k}^{R,\delta})(t, J, M)(X_s) \right| \leq C \frac{(s+n)^2}{N} \|\varphi\|_{L^\infty(\mathbf{R}^{ds})} \|F_0\|_{0,\beta_0,\mu_0}\,.$$

14.2.3. Error coming from the divergence of trajectories. —

We can now compare the definition (13.2.3) of $J_{s,k}^{0,R,\delta}(t, J, M)$:

$$J_{s,k}^{0,R,\delta}(t, J, M)(X_s) = \int_{B_R \setminus \mathcal{M}_s(X_s)} dV_s \varphi_s(V_s) \int_{\mathcal{T}_{k,\delta}(t)} dT_k \int_{{}^c\mathcal{B}_s^{m_1}(Z_s^0(t_1))} d\nu_{s+1} dv_{s+1} ((v_{s+1} - v_{m_1}^0(t_1) \cdot \nu_{s+1})_{j_1}$$

$$\ldots \int_{{}^c\mathcal{B}_{s+k-1}^{m_k}(Z_{s+k-1}^0(t_k))} d\nu_{s+k} dv_{s+k} ((v_{s+k} - v_{m_k}^0(t_n) \cdot \nu_{s+k})_{j_k}$$

$$\times \mathbb{1}_{E_0(Z_{s+k}^0(0)) \leq R^2} f_0^{(s+k)}(Z_{s+k}^0(0))\,.$$

and the formulation (14.2.16) for the approximate BBGKY hierarchy.

Lemma 14.1.1 implies that at time 0 we have

$$|X_{s+k}(0) - X_{s+k}^0(0)| \leq Ck\varepsilon\,, \quad \text{and} \quad V_{s+k}(0) = V_{s+k}^0(0)\,.$$

Since $f_0^{(s+k)}$ is continuous, we obtain the expected convergence as stated in the following lemma.

Lemma 14.2.3. — *In the Boltzmann-Grad scaling $N\varepsilon^{d-1} = 1$, for all fixed $s, k \in \mathbf{N}$ and $t < T$,*

$$\overline{J}_{s,k}^{R,\delta}(t, J, M)(X_s) - J_{s,k}^{0,R,\delta}(t, J, M)(X_s) \to 0\,.$$

For tensorized Lipschitz initial data, we further have the following error estimate :

$$\left| \mathbb{1}_{\Delta_s^X(\varepsilon_0)} \sum_{k=0}^n \sum_{J,M} (\overline{J}_{s,k}^{R,\delta} - J_{s,k}^{0,R,\delta})(t, J, M)(X_s) \right| \leq C\varepsilon n \|\nabla_x f_0\|_\infty \|F_0\|_{0,\beta_0,\mu_0} \|\varphi\|_{L^\infty(\mathbf{R}^{ds})}\,.$$

Notice that putting together Lemmas 14.2.1, 14.2.2 and 14.2.3, along with the estimates (14.2.12)-(14.2.13) and (14.2.14)-(14.2.15), end the proof of Theorem 8 up to the rate of convergence. This is the object of the next paragraph.

14.2.4. Optimization for tensorized Lipschitz initial data. — We can now conclude the proof of Theorem 8. Gathering the results of Lemmas 14.2.1, 14.2.2 and 14.2.3, together with the estimates (14.2.12)-(14.2.13) and (14.2.14)-(14.2.15), we get

$$
\begin{aligned}
\left\|I_s(t) - I_s^0(t)\right\|_{L^\infty(\mathbf{R}^{ds})} \leq & C(2^{-n} + e^{-C'R^2} + \delta)\|\varphi\|_{L^\infty(\mathbf{R}^{ds})} \sup_N \|F_{N,0}\|_{\varepsilon,\beta_0,\mu_0} \\
& + Cn(s+n)\Big(R\eta^{d-1} + R^d\Big(\frac{a}{\varepsilon_0}\Big)^{d-1} + R\Big(\frac{\varepsilon_0}{\delta}\Big)^{d-1}\Big)\|F_{N,0}\|_{\varepsilon,\beta_0,\mu_0}\|\varphi\|_{L^\infty(\mathbf{R}^{ds})} \\
& + C\varepsilon(s+n)\|F_0\|_{0,\beta_0,\mu_0}\|\varphi\|_{L^\infty(\mathbf{R}^{ds})} \\
& + C\frac{(s+n)^2}{N}\|\varphi\|_{L^\infty(\mathbf{R}^{ds})}\|F_0\|_{0,\beta_0,\mu_0} \\
& + Cn\varepsilon\|\nabla_x f_0\|_{L^\infty}\|\varphi\|_{L^\infty(\mathbf{R}^{ds})}\|F_0\|_{0,\beta_0,\mu_0}
\end{aligned}
$$

Therefore, choosing

$$
n \sim C_1|\log\varepsilon|, \quad R^2 \sim C_2|\log\varepsilon|
$$

for some sufficiently large constants C_1 and C_2, and

$$
\delta = \varepsilon^{(d-1)/(d+1)}, \quad \varepsilon_0 = \varepsilon^{d/(d+1)}
$$

we find that the total error is smaller than $C\varepsilon^\alpha$ for any $\alpha < (d-1)/(d+1)$.

This ends the proof of Theorem 8.

14.3. Convergence in the case of a smooth interaction potential: proof of Theorem 11

Let us now prove Theorem 11.

The same arguments as in the previous section provide the convergence for any smooth short-range potential satisfying (8.3.1). Let us only sketch the proof and point out how to deal with the following minor differences.

– The elimination of multiple collisions gives an additional error term : from Propositions 11.3.1 and 11.3.2, we indeed deduce the analogue of (14.2.12):

$$
(14.3.17) \qquad \left\|I_s(t) - I_{s,n}^{R,\delta}(t)\right\|_{L^\infty(\mathbf{R}^{ds})} \leq C(\varepsilon + 2^{-n} + e^{-C'R^2} + \delta)\|\varphi\|_{L^\infty(\mathbf{R}^{ds})}\|\widetilde{F}_{N,0}\|_{\varepsilon,\beta_0,\mu_0}.
$$

– The error term coming from the elimination of pathological velocities and impact parameters depends (in a non trivial way) on the local L^∞ norm of the cross-section: estimate (14.2.14) becomes

$$
\begin{aligned}
&\Big|\mathbb{1}_{\Delta_s(\varepsilon_0)} \sum_{k=0}^n \sum_{J,M} \big(I_{s,k}^{0,R,\delta} - J_{s,k}^{0,R,\delta}\big)(t,J,M)\Big| \\
&\leq Cn(s+n)\Big(R\eta^{d-1} + C(\Phi,R,\eta)R^d\Big(\frac{a}{\varepsilon_0}\Big)^{d-1} + C(\Phi,R,\eta)R\Big(\frac{\varepsilon_0}{\delta}\Big)^{d-1}\Big)\|F_0\|_{0,\beta_0,\mu_0}\|\varphi\|_{L^\infty(\mathbf{R}^{ds})}.
\end{aligned}
$$

– Additional error terms come from the difference between truncated marginals and true marginals (namely on the initial data) : by Lemma 11.1.2, there holds the convergence

$$f_{0,N}^{(s)} - \widetilde{f}_{0,N}^{(s)} \longrightarrow 0, \qquad \text{for fixed } s \geq 1, \text{ as } N \to \infty \text{ with } N\varepsilon^{d-1} \equiv 1, \text{ uniformly in } \Omega_s.$$

Together with Lemma 14.2.1, this implies that

$$J_{s,k}^{R,\delta}(t, J, M)(X_s) - \widetilde{J}_{s,k}^{R,\delta}(t, J, M)(X_s) \to 0.$$

– The micro-translations between the "good" Boltzmann and BBGKY pseudo-trajectories depend on the maximal duration of the interactions to be considered

$$|X_{s+k}(0) - X_{s+k}^0(0)| \leq C(\Phi, R, \eta)k\varepsilon, \quad \text{and} \quad V_{s+k}(0) = V_{s+k}^0(0),$$

so that the convergence

$$\bar{J}_{s,k}^{R,\delta}(t, J, M)(X_s) - J_{s,k}^{0,R,\delta}(t, J, M)(X_s) \to 0$$

may be very slow.

Combining all estimates shows that for any fixed $s \in \mathbf{N}$ and any $t < T$

$$I_s(t)(X_s) - I_s^0(t)(X_s) \to 0$$

locally uniformly in Ω_s, which concludes the proof of Theorem 11.

CHAPTER 15

CONCLUDING REMARKS

15.1. On the time of validity of Theorems 9 and 8

Let us first note that, for any fixed N, the BBGKY hierarchy has a global solution since it is formally equivalent to the Liouville equation in the phase space of dimension $2Nd$, which is nothing else than a linear transport equation. The fact that we obtain a uniform bound on a finite life span only, is therefore due to the analytical-type functional spaces $\mathbf{X}_{\varepsilon,\beta,\mu}$ we consider. Belonging to such a functional space requires indeed a strong control on the growth of marginals.

An important point is that the time of convergence is exactly the time for which these uniform a priori estimates hold. By definition of the functional spaces, we are indeed in a situation where the high order correlations can be neglected (see (14.2.12) and (14.3.17)), so that we only have to study the dynamics of a finite system of particles. The term-by-term convergence relies then on geometrical properties of the transport in the whole space, which do not introduce any restriction on the time of convergence.

A natural question is therefore to know whether or not it is possible to get better uniform a priori estimates and thus to improve the time of convergence. Let us first remark that such a priori estimates would hold for the Boltzmann hierarchy and thus for the nonlinear non homogeneous Boltzmann equation. As mentioned in Chapter 2, the main difficulty is to control the possible spatial concentrations of particles, which would contradict the rarefaction assumption and lead to an uncontrolled collision process.

15.2. More general potentials

A first natural extension to this work concerns the case of a compactly supported, repulsive potential, but no longer satisfying (8.3.1). As explained in Chapter 8, that assumption guarantees that the cross section is well defined everywhere, since the deflection angle is a one-to-one function of the impact parameter. If that is no longer satisfied, additional decompositions are necessary to split the integration domain in subdomains where the cross-section is well-defined : we then expect to be able to extend the convergence proof, up to some technical complications due to the resummation procedures (see [39] for an alternative method). Note that, if the deflecttion angle can be locally constant as a function of the impact parameter, the method does not apply, which is consistent with the fact that

we do not expect the Boltzmann equation to be a good approximation of the dynamics (see the by now classical counterexample by Uchiyama [**15**]).

From a physical point of view it would be more interesting to study the case of long-range potentials. Then the cross section actually becomes singular, so a different notion of limit must be considered, possibly in the spirit of Alexandre and Villani [**3**]. One intermediate step, as in [**16**], would be to extend this work to the case when the support of the potential goes to infinity with the number of particles. Then one could try truncating the long-range potential and showing that the tail of the potential has very little effect in the convergence.

Note that in the case when grazing collisions become predominant, then the Boltzmann equation should be replaced by the Landau equation, whose derivation is essentially open; a first result in that direction was obtained very recently by A. Bobylev, M. Pulvirenti and C. Saffirio in [**4**], where a time zero convergence result is established.

15.3. Other boundary conditions

As it stands, our analysis is restricted to the whole space (namely $X_N \in \mathbf{R}^{dN}$). It is indeed important that free flow corresponds to straight lines (see in particular Lemmas 12.2.1 and 12.2.3 as well more generally as the analysis of pathological trajectories in Chapter 12).

It would be very interesting to generalize this work to more general geometries. A first step in that direction is to study the case of periodic flows in X_N. The geometric lemmas must be adapted to that framework, and in particular it appears that a finite life span must a priori be given before the surgery of the collision integrals may be performed (see [**5**]).

The case of a general domain is again much more complicated, and results from the theory of billiards would probably need to be used.

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NOTATION INDEX

B_R, ball of radius R centered at zero in $\mathbf{R}^d$, page 26

B_R^s, ball of radius R centered at zero in $\mathbf{R}^{ds}$, page 26

$B_R(x)$, ball of radius R centered at x in $\mathbf{R}^d$, page 76

$\mathcal{B}_k(\overline{Z}_k)$ a small set of angles and velocities of a particle adjoined to $\overline{Z}_k$ (or a neighboring configuration), leading to pathological trajectories, page 92

$b(w,\omega)/|w|$, cross-section, page 64

$\mathbf{C}_N$, BBGKY hierarchy collision operator, page 29 for the hard-spheres case and page 64 for the potential case

$\mathbf{C}^0$, Boltzmann hierarchy collision operator, page 31 for the hard-spheres case and page 73 for the potential case

$\mathcal{C}_{s,s+1}$, BBGKY collision operator, page 29 for the hard-spheres case and page 71 for the potential case

$\mathcal{C}_{s,s+m}$, BBGKY collision operator involving m additional particles, page 71

$\mathcal{C}_{s,s+1}^0$, Boltzmann collision operator, page 31 for the hard-spheres case and page 64 for the potential case

$\mathcal{D}_N$, domain on which the hard-spheres dynamics take place, page 25

$\mathcal{D}_N^s$, artificial set on which the Hamiltonian dynamics take place, page 66

$\Delta_m(X_s)$, m-particle cluster based on X_s, page 70

Δ_s, well-separated initial configurations, page 101

Δ_s^X, well-separated initial positions, page 101

$d\sigma_N^{i,j}$, surface measure on $\Sigma_N^s(i,j)$, page 67

$d\sigma$, surface measure on $S_\varepsilon(x_i)$, page 71

$dZ_{(i,j)}$, $2d(j-i+1)$-dimensional Lebesgue measure, page 43

$E(X_s, X_n)$, ε-closure of X_s in X_N, page 70

$E_{<i_0,j_0>}(X_s, X_n)$, ε-closure of X_s in X_N with a weak link at (i_0, j_0), page 70

$E_\varepsilon(Z_s)$, s-particle Hamiltonian, page 78

$E_0(Z_s)$, s-particle free Hamiltonian, page 33

$f_N^{(s)}$, marginal of order s of the N-particle distribution function, page 27 for the hard-spheres case, page 65 for the potential case

$\widetilde{f}_N^{(s)}$, truncated marginal of order s of the N-particle distribution function, page 66

$f^{(s)}$, marginal of order s associated with the Boltzmann hierarchy, page 31

$\boldsymbol{\Phi}_\varepsilon$, rescaled potential, page 78

$\mathcal{G}_k$, set of good configurations of k particles, page 92