/tikz/,/tikz/graphs/
conversions/canvas coordinate/.code=1 , conversions/coordinate/.code=1

École Polytechnique
Université Paris-Saclay

Internship Report

by

Daniel PEREZ

under the supervision of

Cédric LORCÉ

Title:

A Geometrical Overview of Results in Theoretical Physics

2018

Contents

Chapter 1

Introduction

In this report, we will derive a variety of results in theoretical physics from a geometrical standpoint. In particular, we seek to show that many of the classical results in physics are intrinsically linked to the symmetries of gauge space or of spacetime itself.

In order to achieve the latter, we use the language of differential geometry and of representation theory, which will allow us to beautifully describe and generalize some notions in physics. The accompanying appendices are included for the sake of completeness and the reader is kindly invited to read them as an introduction if he is not well-versed in some of the mathematical concepts which we will use intensively throughout this report. We will focus on three main topics, which are all perfectly understood in geometrical terms.

First, we will discuss dressing fields and give a geometrical picture to understand how these objects naturally arise from the foliation of a reducible principal bundle. This language is particularly well-suited to properly understand spontaneous symmetry breaking, as illustrated by an explicit calculation of the Brout-Englert-Higgs mechanism in the electroweak sector of the Standard Model. Our construction will also bring many of the perspectives that have been taken by some people in the community [12, 13] under a single geometrical umbrella. Helpful appendices for this section will be Appendix B and parts of appendix C.2.

Second, we will formulate an extension of General Relativity – the Einstein-Cartan Theory of Gravity – which arises from purely geometrical considerations and the implementation of the Einstein-Hilbert action. Following the steps of [21, 22, 23], we include torsion in this theory, as it turns out that the inclusion of torsion has the benefit of coupling to spin, which in turn makes the singularities disappear from the theory in the presence of high densities of fermionic matter. However, we focus on the derivation and the geometrical concepts behind the theory, which will allow us to see the naturality of General Relativity and this possible extension of it from the geometrical point of view. A particularly helpful appendix for this chapter will be appendix C.

Third, we look at manifolds which allow a so-called spin structure and explore the geometrical nature of spin as well as some consequences of this nature in physical contexts. In particular, we provide a geometrical derivation of the Dirac equation an give an approach which generalizes it to all dimensions and to curved spacetimes, provided that the base manifold satisfies some topological constraints. Here, the useful appendices will be appendix C, which covers the geometrical framework we employ, and appendix E.1, which gives an in depth mathematical description of spin.

Finally, appendix A contains some useful definitions and facts included for completeness purposes and to set some vocabulary in place. Appendix D details how we can use the same geometrical approaches to enlargen the applicability of non-relativistic Quantum Mechanics to curved spaces. Last but not least, in appendix E.2, we provide a new graphical method to compute traces of γ-matrices and we provide an interpration for what these γ matrices actually are, as well as what the trace computations mean geometrically.

Chapter 2

Dressing Fields

The concept of "gauge invariance" is particularly important in physics, that is to say, whenever possible, physicists try to ensure that the objects being manipulated are "gauge invariant". For this reason, giving a precise definition of what we mean by "gauge invariance" is imperative in order to have a consistent treatment of gauge theories. Unfortunately, this is rarely done in the physics community, which employs this term extremely widely in a variety of different contexts. So widely, in fact, that one could argue that the term has become devoid of meaning, much like the word "local". Certain people in the community would argue that such pedantry is utterly useless, but, as shown in Appendix D, being precise in this respect can prove extremely helpful for certain applications.

This ambiguity on the definition of a gauge often is the source of different debates in physics which could be entirely avoided or whose resolution would become trivial if the community could agree on a single definition. This is what drove us to define what a gauge is in the context of Klein and later Cartan geometries based on a physical situation in section C.1.5. For the rest of this section, we will take the following definition (which is slightly altered with respect to the one in the discussion of section C.1.5) of a gauge. We refer the reader to appendix C.1.5 for physical and geometrical motivations for this definition.

Definition 2.0.1. Let P be a principal G-bundle over a manifold M and let $P \times_G V$ be an associated bundle with representation (V, ρ). In this context, we define a **choice of gauge** to be a choice of section $s : M \to P \times_G V$ from the base manifold into the associated bundle.

This definition is motivated by the fact that the objects which actually get gauge-transformed in physics correspond exactly to these sections over an associated bundle with representation V. With it, we understand that the term "gauge invariance" should really be "gauge *co*variance", as per the discussion given in C.1.5. This gauge covariance can be understood explicitly by the fact that results don't depend on the choice of a particular gauge that is made, as one is always free to translate along the fibres of the principal bundle in a consistent way to retrieve an analogous result in a different gauge. With the semantics out of the way, we are now ready to consistently treat a problem which has arisen time and time again and which has generated much debate within the community concerning the gauge invariance of spontaneous symmetry breaking.

In order to treat this problem, we will first give a geometrical picture of what "spontaneous symmetry breaking" looks like in a geometrical setting, after which we will have the tools to study the particular case of the Higgs mechanism. In particular, with our constructions, we will be able to naturally give rise to the maps presented in [12, 13], which render things explicitly gauge covariant according to the authors. Having retrieved these tools, it will be possible to use the results found in the previous references to perform the classical calculation of spontaneous symmetry breaking of the Lagrangian of the electroweak sector of the Standard Model.

To be more explicit, the tools from [12, 13] we talked about are the so-called *dressing fields*, which have been a concept studied widely by physicists for a number of years. Indeed, the very first person to consider these constructions was P.A.M. Dirac [15], but the concept has been rediscovered and discussed recurrently ever since [12, 13, 16, 17, 18]. The difference with our treatment is that we will have a consistent geometrical picture and interpretation of these dressing fields. Many attempts to formalize this concept have been explored [12, 13], but in the author's humble opinion, these attempts dwell too much on technicalities which can hinder if not completely impede understanding. For this reason, we hereby provide a new construction in the hope that it will help clarify how different methods employed in the symmetry reduction of principal bundles are connected with each other.

Let us now give a brief picture of how these dressing fields have been interpreted in the physics community. Intuitively and very loosely speaking, the dressing field supposes the existence of a "God-given basis" which exists in the gauge internal space. This "God-given basis" can then be used to measure "physical quantities" with respect to it. In other words, a quantity is called "physical" if it is measured with respect to the "God-given basis". Talking about "physical quantities" here is perhaps also a great misnomer, since objects in gauge space are never *actually* physically measurable and since the choice of what is physical will turn out to be completely arbitrary. This implies that the presence of this basis makes everything "gauge invariant", at least under passive transformations, since a coordinate change will not affect the geometrical quantity measured, which remains the difference between the God-given basis and the measurement. Yet, under an active transformation, the physical quantity changes, since this God-given basis stays fixed no matter the transformation. In this way, we have a clear distinction between what constitutes a "passive" transformation and an "active" one.

This reflects the reason of why it is important to talk about "gauge covariance" instead of "invariance", as it is important that the quantities all change in the right way such that they compensate each other in order not to affect the "physical" result.

2.1 Geometrical Setting

The language of differential geometry is naturally adapted to treat this kind of question, in particular the language of foliations, distributions and principal bundles is perfectly adapted to obtain the results we seek. In what will follow, we will briefly introduce these concepts. For notation, we will take P to be a G-principal bundle with base manifold M, unless otherwise specified. The statements below are presented for the sake of completeness, but we will not show the proofs, unless they are of conceptual value. If the reader is interested, a more in-depth treatment of this subject can be found in [1, 3, 11].

Theorem 2.1.1 (Constant rank map theorem)**.** *Let M and N be m and n dimensional manifolds respectively and let $f : M \to N$ be a smooth map with constant rank r. For each point $p \in M$, there exists connected charts (U, φ) and (V, ψ) around p and $f(p)$ respectively such that $\varphi(p) = 0$, $\psi(f(p)) = 0$, $f(U) \subset V$ and $\psi f \varphi^{-1}$ is a restriction of the canonical map:*

$$\begin{aligned} \mathbb{R}^r \times \mathbb{R}^{m-r} &\longrightarrow \mathbb{R}^r \times \mathbb{R}^{n-r} \\ (x, y) &\longmapsto (x, 0) \end{aligned} \tag{2.1.1}$$

Definition 2.1.1. Let $f : M \to N$ be a smooth map with constant rank. Then, f is called an **immersion** if the rank is m, and a **submersion** if the rank is n.

Definition 2.1.2. An **embedding** is a one-to-one immersion $f : M \to N$ such that the mapping $f : M \to f(M)$ is a homeomorphism (where the topology on $f(M)$ is the subspace topology inherited from N)

Definition 2.1.3. Let M be an m-dimensional smooth manifold as before. A **q-codimensional foliated atlas on** M is an atlas $\mathcal{A}$ such that if (U, φ) and $(V, \psi) \in \mathcal{A}$, then the coordinate changes $\Phi = \psi\varphi^{-1}$ have the form:

$$\begin{aligned} \Phi : \mathbb{R}^{m-q} \times \mathbb{R}^q &\longrightarrow \mathbb{R}^{m-q} \times \mathbb{R}^q && (2.1.2)\\ (x, y) &\longmapsto (\Phi_1(x, y), \Phi_2(y)) && (2.1.3) \end{aligned}$$

that is, such that the last q coordinates only depend on the last q variables.

Definition 2.1.4. Two foliated atlases are said to be **equivalent** if their union is a foliated atlas. A **foliation** on M is an equivalence class of foliated atlases.

2.1.1 Bundle Reductions

With the previous definitions, we are ready to attack our problem.

Definition 2.1.5. Let G be a Lie group and $H \subset G$ be a subgroup. Let P be a principal G-bundle over M. A **G-reduction** of P is a submanifold $P_0 \subset P$ such that $P_0 \to M$ is an H-bundle and the action of H on P_0 is the restriction of the action of H on P.

Lemma 2.1.2. Let $\mu : G \times Q \to Q$, $(g, x) \mapsto g * x$ be a smooth left action of a Lie group G on a connected smooth manifold Q. Then, every orbit $X \subset Q$ of this action is a submanifold. Moreover, if the action is proper, then X is a proper submanifold.

Proof. Fix an orbit $X \subset Q$ and choose $x_0 \in X$. Set $H = \{ g \in G \mid g * x_0 = x_0 \}$. It suffices to show the three following things:

1. H is a closed subgroup of G;

2. The following induced map is an injective immersion with image X

$$\begin{aligned} G/H &\longrightarrow Q && (2.1.4)\\ gH &\longmapsto g * x_0\,; && (2.1.5) \end{aligned}$$

3. If the original action is proper, then $G/H \to Q$ is a proper embedding.

Step 1: Since $\mu^{-1}(x_0)$ is closed and $H \times \{x_0\} = \mu^{-1} \cap (G \times \{x_0\})$, it follows that $H \times \{x_0\}$ is closed in $G \times Q$ and thus that H is closed in G. The fact that this map is injective and with image X is clear.

Step 2: Define the map $\Psi : G \to Q$ by $g \mapsto g * x_0$. Note that $\Psi^{-1}(x_0) = H$. Set $V := \ker \Psi_{*e} \subset \mathfrak{g}$, the Lie algebra of G. Let $\mathfrak{h}$ denote the Lie algebra of the subgroup H. Since Ψ is constant on H, it follows that $\Psi_{*e}(\mathfrak{h}) = 0$ and hence that $\mathfrak{h} \subset V$. We now proceed to show the other inclusion. Recall that L_g (the left translation on G) and $\ell_g : Q \to Q$, the left translation on Q defined by $x \mapsto g * x$, are diffeomorphisms. Then, consider the following commuting diagrams:

$$\begin{array}{ccc} G & \xrightarrow{\Psi} & Q \\ \downarrow{\scriptstyle L_g} & & \downarrow{\scriptstyle \ell_g} \\ G & \xrightarrow{\Psi} & Q \end{array} \qquad\qquad \begin{array}{ccc} T_eG & \xrightarrow{\Psi_{*e}} & T_{x_0}Q \\ \downarrow{\scriptstyle L_{g*}} & & \downarrow{\scriptstyle \ell_{g*x_0}} \\ T_gG & \xrightarrow{\Psi_{*g}} & T_{g*x_0}Q \end{array}$$

from which it is clear that $\ker \Psi_{*g} = L_{g*} \ker \Psi_{*e} = L_{g*}V$ for all $g \in G$. Thus, on G, Ψ has constant rank $r = \dim \mathfrak{g} - \dim V$. By theorem 2.1.1 the components of the level surfaces of Ψ foliate G with codimension r. The component of the identity in $\Psi^{-1}(x_0) = H$ is the identity component subgroup H_0 with Lie algebra $\mathfrak{h}$. So $V = \mathfrak{h}$. The injectivity of the map comes from considering the following commutative diagram:

$$\begin{tikzcd}
T_gG \arrow[d] \arrow[dr, "\Psi_{*g}"] & \\
T_{gh}(G/H) \arrow[r] & T_{g*x_0}M
\end{tikzcd}$$

since the two arrows pointing downwards have kernel $L_{g*e}V$, we have that the bottom map is necessarily injective, so that Ψ is an immersion.

Step 3: The proof of this can be found in [1]. ∎

We apply this lemma in the following proposition which gives us a simple and sufficient condition for the existence of a reduction.

Proposition 2.1.1. Let P be a smooth G-principal bundle and let Q be a manifold equipped with a right G-action. Let $f : P \to Q$ be a smooth equivariant map (*i.e.* $f(p * g) = f(p) * g$ for all $g \in G$). Fix $q_0 \in Q$ and set:

$$H_0 := \{h_0 \in G \mid q_0 * h_0 = q_0\} = \mathrm{Stab}(q_0) \tag{2.1.6}$$

Furthermore, suppose that $q_0 \in Q$ lies in the image under f of each fibre of P, then:

1. $P_0 = f^{-1}(q_0)$ is an H_0-reduction of P;

Moreover, fix another $q_1 \in Q$ and set $H_1 = \mathrm{Stab}(q_1)$ and, as before, suppose that q_1 lies in the image under f of each fibre of P so that by the previous point $P_1 = f^{-1}(q_1)$ is an H_1-reduction. Then:

2. There exists $g \in G$ such that $H_1 = g^{-1}H_0g$, and for any such element g, $P_1 = P_0 * g$.

Proof. Proof of 1: We start by showing that P_0 is a submanifold of P. For this, it is once again sufficient to show that f has constant rank. Since $\forall x \in M$, $q_0 \in f(\pi^{-1}(x))$, it follows that $\mathrm{Im}(f) = X = q_0 * G = \mathrm{Orb}(q_0) \subset Q$. By the previous lemma, since the action $Q \times H \to Q$ is proper, all the orbits, and in particular X, are submanifolds of Q. Since f takes values in X, it follows that $\mathrm{rank}_p(f) \leq \dim X$. We show equality by proving $\mathrm{rank}(f|_{\mathrm{fibre}}) = \dim X$. This follows from the fact that the restriction of the map f to any given fibre is nothing but the map $G \to X$ sending $g \mapsto q_0 * g$, which has rank $\dim X$ by the lemma. Thus, f has constant rank and P_0 is a submanifold of P.

Remark that $P_0 * H_0 = P_0$, *i.e.* that P_0 is stable under the action of H_0. We now show that this submanifold is indeed a principal bundle. Indeed, this follows rather easily by considering p and p' lying on the same fibre of P_0. Then there is a $g \in G$ such that $p' = p * g \in P_0$. Thus $q_0 = f(p * g) = f(p) * g = q_0 * g$, which implies that $g \in H_0$. It follows that the induced action of H_0 on P_0 is transitive on each fibre of P_0. Since $P_0 = f^{-1}(q_0)$ is closed in P and H_0 is closed in G, the induced action of H_0 in P_0 is proper and free. By theorem 4.2.4 of [1], P_0 is a principal H_0-bundle over M and is therefore an H_0-reduction of P.

Proof of 2: Let $P_1 = f^{-1}(q_1)$. Both q_0 and q_1 lie in the image under f of each fibre of P, this means that $\forall x \in M$, $q_0 \in f(\pi^{-1}(x))$ and $q_1 \in f(\pi^{-1}(x))$, so that we may pick two representatives p_0 and p_1 in each fibre of P such that $q_0 = f(p_0)$ and $q_1 = f(p_1)$. Since both p_0

and p_1 lie in the same fibre, it follows that $\exists\, g \in G$ so that $p_1 = p_0 * g$. It is clear by the right equivariance of f that this yields: $q_1 = f(p_1) = f(p_0 * g) = f(p_0) * g = q_0 * g$. Then, if $h_1 \in H_1$, we have $q_0 * gh_1 = q_0 * g$, which implies $q_0 * gh_1 g^{-1} = q_0$, but by definition, this implies $gh_1 g^{-1} \in H_0$ so that $H_1 = g^{-1} H_0 g$. Finally, it is clear that $P_1 = f^{-1}(q_0 * g) = f^{-1}(q_0) * g = P_0 * g$. ■

With this criterion under our belts, we provide a construction for the reduction of symmetry we can apply in a general sense to understand exactly where the dressing fields come from. Start by taking an associated bundle to P with representation (V, ρ) of G, and write this associated bundle as $P \times_G V$. This associated bundle is naturally equipped with a G-right action given by:

$$[p, f] * g = [p * g, \rho(g^{-1}) f] \tag{2.1.7}$$

Furthermore, if there exists a global section $\varphi : M \to P \times_G V$, by theorem B.2.1, we may regard this global section as being an equivariant function $\varphi : P \to V$ whose equivariance condition is given by $\varphi(p * g) = \rho(g^{-1})\varphi(p)$. It is clear that we may see $V = \mathbb{C}^2$ as a manifold naturally equipped with a right G-action given previously. Taking a suitable vector v_0, we may then simply consider its stabilizer H. Proposition 2.1.1 then yields that $P_0 = \varphi^{-1}(v_0)$ is a reduction of P and is a principal H-bundle.

Notice that the foliation provided by the reduction of the bundle induces natural Ehresmann connections on the principal bundle P. This is because an Ehresmann connection is fully determined by its distribution. If we decide then to set $\ker \varpi = TP \setminus TP_0$ as our horizontal distribution, or, equivalently, define TP_0 to be the vertical distribution, we have a natural Ehresmann connection, which can be defined in terms of the projectors onto these respective spaces. Notice also that if $i : P_0 \hookrightarrow P$ is the natural inclusion, we may pullback forms, functions, *etc.* from P onto P_0 via this mapping. Therefore, we can look at the pullback of the corresponding Ehresmann connection under this light as simply being $i^*\varpi$.

At this point, it might be good to illustrate this procedure of reduction with a neat example, which will help us understand a little bit better the nature of what will come in the following chapters.

2.1.2 Cartan Geometries and Ehresmann Connections

We will now briefly examine the link between Cartan geometries modelled on (G, H) and Ehresmann connections over a principal G-bundle. For the sake of simplicity, we consider here the case of Euclidean space, for which we won't need to worry about topological obstructions which might otherwise cause trouble. Thus, let P denote a principal bundle with group $ISO(n)$ and let $P \times_{ISO(n)} V$ be the associated bundle to P with a representation of $ISO(n)$, (V, ρ). Now, suppose the existence of a global section $s : M \to P \times_{ISO(n)} V$, which may in turn be regarded as a function $s : P \to V$. As per our previous discussion, we consider a vector $v_0 \in V$ and seek to calculate its stabilizer with respect to the right action of the group $ISO(n)$ on V. We quickly realize, however, that a vector with an interesting stabilizer in V is the origin itself, since the rest of the vectors are in general changed by a generic rotation. From a simple geometrical consideration, it's obvious that $\mathrm{Stab}(v_0) = O(n)$. By proposition 2.1.1, we have that the bundle gets reduced to an $O(n)$-bundle, P_0. In fact, notice that the origin is also stable by $SO(n)$, which means that we may further reduce the bundle to the group $SO(n)$. On the other hand, we may only do so if the base manifold is orientable as there are otherwise topological obstructions in trying to do so.

Following the lines of our previous reasoning concerning the Ehresmann connection induced by the foliation provided by the reduction, we have a bijective correspondence between Ehresmann connections on P whose kernel does not meet $i_* TP_0$ and Cartan connections themselves.

The proof of this as well as a more geometrical way of understanding Ehresmann connections in all their generality may be found in the appendices of Sharpe's book [1]. However, this illustration helps us understand what happens with the connections. Indeed, we see how Cartan connections arise from the reduction of a general principal bundle over a manifold. In some sense, it's as if part of the symmetry imposed on the system is redundant and, in the language of a true physicist, can be "gauged away". This also helps us see how Cartan geometries in some sense are indeed imposing local Euclidean (or Poincaré) symmetries on the manifold M.

2.2 Application to Electroweak Theory

Having already given the example of how the construction above applies to the particular case of Cartan geometries, we now apply the same construction in a more abstract setting, namely to the spontaneous symmetry breaking mechanism present in electroweak theory. In so doing, we will show that this mechanism can indeed be achieved in a gauge covariant manner.

2.2.1 Spontaneous Symmetry Breaking

In an analogous fashion to what we previously did, we now must find a vector $v_0 \in V$ such that $\forall\, x \in M$ we may find $v_0 \in \varphi(\pi^{-1}(x))$. This is in particular the case for $v_0 = \begin{pmatrix} 0 & 1 \end{pmatrix}^T$. Of course, we could have picked any vector in V, but many of these happen to have trivial stabilizers under the action of G. Motivated by the previous proposition, we seek a vector which has a non-trivial stabilizer in order to be able to reduce the bundle. We could have taken any vector in $\mathrm{Orb}(v_0)$ in order to achieve what we are about to do. Let us now calculate $\mathrm{Stab}(v_0)$.

$$\begin{aligned} \mathrm{Stab}(v_0) &= \left\{ h \in G = SU(2) \times U(1) \mid \rho(h^{-1}) \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\} && (2.2.8) \\ &= \left\{ e^{i\theta} \begin{pmatrix} \alpha & -\bar{\beta} \\ \beta & \bar{\alpha} \end{pmatrix} \in G \mid e^{i\theta} \begin{pmatrix} \alpha & -\bar{\beta} \\ \beta & \bar{\alpha} \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\} && (2.2.9) \\ &= \left\{ e^{i\theta} \begin{pmatrix} \alpha & -\bar{\beta} \\ \beta & \bar{\alpha} \end{pmatrix} \in G \mid e^{i\theta} \begin{pmatrix} -\bar{\beta} \\ \bar{\alpha} \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\} && (2.2.10) \\ &= \left\{ e^{i\frac{\theta}{2}} e^{i\theta\frac{\sigma_z}{2}} \in SU(2) \times U(1) \mid \theta \in \mathbb{R} \right\} && (2.2.11) \end{aligned}$$

From this, we see that the stabilization of the vacuum is provided by a $U(1)$ group provided by the above construction which is a combination of the $SU(2)$ and $U(1)$ parts of $SU(2) \times U(1)$. This is completely consistent with the classical result on the generator of the $U(1)$-electromagnetic symmetry we encounter in particle physics. In this case, though, we appreciate the geometrical origin of this symmetry.

For the particular leaf $\mathfrak{L}$ of the foliation implicitly chosen by our choice of v_0, we express explicitly what its elements look like:

$$\left(x \,,\, \frac{1}{\eta} \begin{pmatrix} \bar{\varphi}_2 & \varphi_1 \\ -\bar{\varphi}_1 & \varphi_2 \end{pmatrix} \right) \in \mathfrak{L}_p \tag{2.2.12}$$

where we have decomposed $\varphi = \begin{pmatrix} \varphi_1(x, e) & \varphi_2(x, e) \end{pmatrix}^T$ and we take $|\varphi_1|^2 + |\varphi_2|^2 = \eta^2$. We can see this by noticing that we must have $\varphi\left(\varphi^{-1}\left(\begin{pmatrix} 0 \\ 1 \end{pmatrix} \right) \right) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. Having said this, we can use this map in order to pullback everything onto the leaf $\mathfrak{L}$. In doing so, we obtain a very similar procedure to the one described in [12, 13]. These are the so-called dressing fields u discussed therein. To be fully clear and explicit, we have:

$$u(p) = \frac{1}{\eta}\begin{pmatrix} \bar{\varphi}_2(p) & \varphi_1(p) \\ -\bar{\varphi}_1(p) & \varphi_2(p) \end{pmatrix} \in SU(2) \tag{2.2.13}$$

However, contrarily to both papers, we have given here a fully geometrical picture of where these dressing fields stem from. Through this procedure, we have illustrated that the "method of dressing fields" may be embedded and properly understood in the context of reductions of principal bundles and that the pullbacks under these dressing fields are nothing other than pullbacks to particular leaves of the foliation induced by the reduction itself.

Since we have shown the geometric origin of this construction, we may now follow Attard's and François's papers in order to retrieve the familiar results of electroweak symmetry breaking. We see clearly that this is completely gauge independent, as we could have picked any v in the orbit of v_0, as prescribed by proposition 2.1.1. In this sense, the mechanism is gauge covariant.

2.2.2 Explicit Calculation Using Dressing Fields

In this section, we follow closely the developments carried out in [12, 13]. A lot more details can be found in the latter.

We start by considering a principal $SU(2) \times U(1)$-bundle and its endowed Ehresmann connection ω and curvature Ω. Notice here that since the Lie algebra of the gauge group is nothing more than $\mathfrak{su}(2) \oplus \mathfrak{u}(1) = \mathfrak{su}(2) \oplus \mathbb{R}$, it quickly follows that all the forms may be split into a $\mathfrak{su}(2)$ and a $\mathfrak{u}(1)$ component. This means in particular that we may choose local representatives (*i.e.* pullbacks under a local section $s : U \to P$) for ω which we may write $A = a + b$ and similarly for the curvature $F = f_a + f_b$.

We consider the associated bundle $E = P \times_{SU(2)\times U(1)} (\mathbb{C}^2, \rho)$, where $(\mathbb{C}^2, \rho)$ is nothing other than the fundamental representation of the gauge group. Furthermore, we consider the associated function φ to the section of this principal bundle $\Phi : U \subset M \to \mathbb{C}^2$ provided by virtue of theorem B.2.1. The covariant derivative is:

$$D\varphi = d\varphi + (g'a + gb)\varphi \tag{2.2.14}$$

with g' and g being the coupling constants of $U(1)$ and $SU(2)$ respectively. The electroweak Lagrangian may be simply expressed as:

$$\mathcal{L}(a, b, \varphi) = \frac{1}{2}\mathrm{Tr}(f_a \wedge *f_a) + \frac{1}{2}\mathrm{Tr}(f_b \wedge *f_b) + \langle D\varphi, *D\varphi\rangle - \underbrace{(\mu^2\langle\varphi,\varphi\rangle + \lambda\langle\varphi,\varphi\rangle^2)\mathrm{vol}}_{=:V(\varphi)} \tag{2.2.15}$$

We are now interested in using the dressing field u in order to dress the fields which take part in the Lagrangian (*i.e.* , the pulled back connection A, strength tensor F, *etc.*). Indeed, if let *i.e.* $A \mapsto \widehat{A}$ denote the newly dressed fields, we have:

$$\widehat{A} = u^{-1}Au + \frac{1}{g}u^{-1}du = u^{-1}(a + b)u + \frac{1}{g}u^{-1}du = a + \underbrace{u^{-1}bu + \frac{1}{g}u^{-1}du}_{=:B} \tag{2.2.16}$$

Since we have that $u^{-1}au = a$, simply from decomposition of A and the fact that u is $SU(2)$-valued. Next, we could examine the dressed $\widehat{F}$ and $\widehat{\varphi}$:

$$\widehat{F} = u^{-1}Fu = u^{-1}(f_a + f_b)u = f_a + u^{-1}f_bu =: f_a + G \tag{2.2.17}$$

$$\widehat{\varphi} = u^{-1}\varphi = \eta\begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{2.2.18}$$

Notice here that $G = dB + B \wedge B$. We then proceed to consider how these B fields transform under a residual $U(1)$-transformation. In what will follow, we denote $\alpha = e^{i\theta}$ and $\widehat{\alpha} = \begin{pmatrix} \alpha & 0 \\ 0 & 1 \end{pmatrix}$ to be the matrix corresponding to the residual $U(1)$-transformation. With this notation we obtain:

$$\begin{aligned} B^\alpha &= (u^\alpha)^{-1} \underbrace{b^\alpha}_{=b} (u^\alpha) + \frac{1}{g}(u^\alpha)^{-1} d(u^\alpha) \\ &= (u\widehat{\alpha})^{-1} b (u\widehat{\alpha}) + \frac{1}{g}(u\widehat{\alpha})^{-1} d(u\widehat{\alpha}) \\ &= \underbrace{\widehat{\alpha}^{-1} u^{-1} b u \widehat{\alpha} + \frac{1}{g}\widehat{\alpha}^{-1}(u^{-1}du)\widehat{\alpha}}_{=\widehat{\alpha}^{-1}B\widehat{\alpha}} + \frac{1}{g}\widehat{\alpha}^{-1} d\widehat{\alpha} \\ &= \widehat{\alpha}^{-1} B \widehat{\alpha} + \frac{1}{g}\widehat{\alpha}^{-1} d\widehat{\alpha} \end{aligned} \tag{2.2.19}$$

In order to retrieve the classical result of electroweak theory, we decompose this $B = B_a\sigma^a$, where σ^a are just the Pauli matrices. We thus obtain the following decomposition:

$$B = \begin{pmatrix} B_3 & B_1 - iB_2 \\ B_1 + iB_2 & -B_3 \end{pmatrix} := \begin{pmatrix} B_3 & \sqrt{2}\, W^+ \\ \sqrt{2}\, W^- & -B_3 \end{pmatrix} \tag{2.2.20}$$

Remark 2.2.1. Notice that if we really want B to be $\mathfrak{su}(2)$-valued, we have to have $B_a \in i\mathbb{R}$, which explains why $\overline{B}_a = -B_a$. Additionally, this condition further yields the relation $(W^+)^\dagger = W^-$.

With this choice of basis for the Lie algebra $\mathfrak{su}(2)$, we can then simply write the transformed fields after a $U(1)$ transformation:

$$B^\alpha = \begin{pmatrix} B_3 + \frac{1}{g}\alpha^{-1} d\alpha & \alpha^{-1}\sqrt{2}\, W^+ \\ \alpha\sqrt{2}\, W^- & -B_3 \end{pmatrix} \tag{2.2.21}$$

We here notice that the B_3 transforms in the same way that a residual $U(1)$-connection would. At first sight, it looks like the integration of a mass term will be difficult. On the other hand, the fields W^+ and W^- seem to transform vectorially, which means we can include a mass term.

2.2.3 Changing Variables and Results

To solve the mass problem of B_3, start by recalling that we still have the a part of the $U(1)$-connection $\widehat{A}$ lying around. In particular, we may at this stage further perform a change of variables. We define two 1-forms A and Z^0 (we stress here is different than the A we considered in the beginning, which denotes the connection) . We use the letter A because of standard physics notation to denote the electromagnetic field and Z^0 as follows:

$$\begin{pmatrix} A \\ Z^0 \end{pmatrix} := \begin{pmatrix} \cos\theta_W & \sin\theta_W \\ -\sin\theta_W & \cos\theta_W \end{pmatrix} \begin{pmatrix} a \\ B_3 \end{pmatrix} \tag{2.2.22}$$

where θ_W, the Weinberg angle, is determined by $\cos\theta_W = \frac{g}{\sqrt{g^2+g'^2}}$ and $\sin\theta_W = \frac{g'}{\sqrt{g^2+g'^2}}$. At this point, it is then obvious that both of these 1-forms are $SU(2)\times U(1)$ invariant given their construction. Finally, we give the transformation law of the field strength G:

$$G^\alpha = \widehat{\alpha}^{-1} G \widehat{\alpha} \tag{2.2.23}$$

which is once again good, because this should transform as a curvature. This now allows us to express the $\mathrm{Tr}(F \wedge *F)$ term in the Lagrangian:

$$\mathrm{Tr}(\widehat{F} \wedge *\widehat{F}) = \mathrm{Tr}(f_a \wedge *f_a) + \mathrm{Tr}(G \wedge *G) \tag{2.2.24}$$

At this point we could for example compute G given the matrix expressions we were given. It is straightforward to see that by taking the inverse matrix of equation 2.2.22, and after a long and tedious calculation, we retrieve that:

$$\begin{aligned}
\mathrm{Tr}(\widehat{F} \wedge *\widehat{F}) \quad & = dZ^0 \wedge *dZ^0 + dA \wedge *dA + dW^+ \wedge *dW^- \\
& + 2g \left[\sin\theta_W \left(dA \wedge *(W^+W^-) + dW^+ \wedge *(W^-A) + dW^- \wedge *(AW^+)\right)\right. \\
& \left. + \cos\theta_W \left(dZ^0 \wedge *(W^+W^-) + dW^+ \wedge *(W^-Z^0) + dW^- \wedge *(Z^0W^+)\right)\right] \\
& + 4g^2 \left[\sin^2\theta_W \,(AW^+) \wedge *(W^-A) + \cos^2\theta_W \,(Z^0W^+) \wedge *(W^-Z^0)\right. \\
& + \sin\theta_W \cos\theta_W \left\{(AW^+) \wedge *(W^-Z^0) + (Z^0W^+) \wedge *(W^-A)\right\} \\
& \left. + \frac{1}{4}(W^+W^-) \wedge *(W^+W^-)\right]
\end{aligned} \tag{2.2.25}$$

where there is an implicit wedge in between the 1-forms associated to each particle. This immediately gives us the coupling that is associated to each particle. As expected, we get no coupling between the A field and the Z^0 field and we retrieve the usual couplings that we encounter in electroweak theory.

Now, we look at the potential term of equation 2.2.15. We immediately appreciate that we may describe this potential in terms of the field η previously defined. Indeed the terms of the potential simply become:

$$V(\varphi) = \mu^2\langle\varphi, \varphi\rangle + \lambda\langle\varphi, \varphi\rangle^2 = V(\eta) = \mu^2\eta^2 + \lambda\eta^4 \tag{2.2.26}$$

Notice in particular that for this $\mathbb{R}^+$-valued field η, $V(\eta)$ has a *unique* minimum value, in this context, there is no choice or "spontaneous symmetry breaking" in the sense that the system does not determine an arbitrary minimum value–since in this case it is unique. We say that η is the **residual field** after extraction of the dressing field u from the scalar auxiliary field φ.

Finally, note that we still have to consider the kinetic term in the Lagrangian 2.2.15, $\langle D\varphi, *D\varphi\rangle$. To do this, we first consider simply what the dressed covariant derivative, $\widehat{D}$, is on the residual field η in this case:

$$\widehat{D}\eta = d\begin{pmatrix} 0 \\ \eta \end{pmatrix} + g\begin{pmatrix} B_3 & W^+ \\ W^- & -B_3 \end{pmatrix}\begin{pmatrix} 0 \\ \eta \end{pmatrix} + g'\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}\begin{pmatrix} 0 \\ \eta \end{pmatrix} = \begin{pmatrix} g\,W^+ \wedge \eta \\ d\eta - g\,B_3 \wedge \eta + g'\,a \wedge \eta \end{pmatrix} \tag{2.2.27}$$

Finally, we compute the norm of $\langle D\varphi, *D\varphi\rangle = \widehat{D}\eta^\dagger \wedge *\widehat{D}\eta$:

$$\begin{aligned}
\widehat{D}\eta^\dagger \wedge *\widehat{D}\eta \quad & = (-g\,W^- \wedge \eta\ ,\ d\eta + g\,B_3 \wedge \eta - g'\,a \wedge \eta) \wedge * \begin{pmatrix} g\,W^+ \wedge \eta \\ d\eta - g\,B_3 \wedge \eta + g'\,a \wedge \eta \end{pmatrix} \\
& = d\eta \wedge *d\eta \underbrace{- g^2\eta^2\,W^- \wedge *W^+ - (g^2 + g'^2)\eta^2\,Z^0 \wedge *Z^0}_{\text{Mass terms after proper expansion of } \eta}
\end{aligned} \tag{2.2.28}$$

As per the remark in the previous equation, we may now choose to expand η around its *unique* minimum value, which will in turn give rise to the usual mass terms which appear in the Lagrangian.

2.3 Remarks on the Higgs Field

Our geometrical picture provided us with much insight about what the scalar field's role is in spontaneous symmetry breaking. Indeed, it constitutes a choice of a *global* section over an associated bundle to the principal bundle with gauge group G. The existence of this Higgs field itself gives us information about the cohomology of space and indeed of its global properties as well, since the existence of such a global section can imply topological restrictions on the classes of manifolds on which such a structure can exist. What we gain with this geometrical picture is thus a hint of where to look further. Incidentally, although we didn't go over this, the avid reader will quickly realize that reductions of G-structures such as principal bundles quickly give rise to the study of different kinds of manifolds (orientable, Kähler, spin, hyperkähler, Calabi-Yau, *etc.*). It is often said that these manifolds come with an integrability condition. But now we understand fully why. Indeed the foliation problem can be posed as an integrability problem, thus the existence or non-existence of a reduction can be understood as the satisfaction or not of a particular integrability condition. The reader is welcome to consult Haelfliger's work [11] on the link between foliations and integrability for more details about what we have just discussed.

Chapter 3

Einstein-Cartan Theory of Gravity

3.1 Modelling Geometry

In this section, we will treat the Einstein-Cartan Theory of Gravity. Our main assumption will be that spacetime can be regarded as a manifold M which exhibits local (infinitesimal) Poincaré symmetry. In other words, we will be looking at a Cartan Geometry modelled on the Klein geometry $(ISO(n,1), SO(n,1))$ on the manifold M. We are thus here looking at a model of Minkowskian geometry.

In particular, the Klein pair associated to the Cartan geometry is simply given by $(\mathfrak{iso}(n,1), \mathfrak{so}(n,1))$ with group $H = SO(n,1)$. We notice that this geometry is reductive, since we may decompose $\mathfrak{g} = \mathfrak{iso}(n,1)$ as a direct sum of $\mathfrak{h}$ and an $\mathrm{Ad}(SO(n,1))$-module $\mathfrak{p}$ complementary to $\mathfrak{h}$. Thus, we have the decomposition $\mathfrak{iso}(n,1) \cong \mathbb{R}^{n,1} \oplus \mathfrak{so}(n,1)$, where we set $\mathfrak{p} = \mathbb{R}^{n,1}$ and $\mathfrak{h} = \mathfrak{so}(n,1)$ in the notations of definition C.2.8.

We start by picking a basis of $\mathfrak{p} = \mathbb{R}^{n,1}$. Thus, let $\{e_i\}_{i\in I} \subset \mathfrak{p}$ be the standard basis in $\mathfrak{p}$ and let $\{J_i{}^j\}_{i,j\in I} \subset \mathfrak{h}$ denote the unique elements satisfying

$$\mathrm{ad}(J_i{}^j)e_k = \delta^j_k e_i - \delta^i_k e_j \tag{3.1.1}$$

recalling that the action $\mathrm{ad}(A)$ on $\mathfrak{g}/\mathfrak{h} \cong \mathfrak{p}$ is defined as $\mathrm{ad}(A)v = Av$. This identification is unique since $\mathfrak{p}$ is an $\mathrm{Ad}(H)$-invariant complement of $\mathfrak{h}$ in $\mathfrak{g}$ and in particular it is an H-module. This means that:

$$\mathrm{ad}(J_i{}^j) = e_i \otimes e_j^* - e_j \otimes e_i^* =: e_i \otimes e^j - e_j \otimes e^i \tag{3.1.2}$$

under the isomorphism $\mathrm{End}(\mathfrak{g}/\mathfrak{h}) \cong \mathrm{End}(\mathfrak{p}) \cong \mathfrak{p}^* \otimes \mathfrak{p}$. Thus, $\{J_i{}^j\}_{i,j\in I}$ for $i < j$ is the standard basis of $\mathfrak{h}$, *i.e.* the generators of the Lorentz transformations. With this choice of basis made and since the Lie algebra $\mathfrak{g}$ is reductive, the Cartan connection 1-form ϖ and its associated curvature form $\tilde{\Omega}$ (and more generally any $\mathfrak{g}$-valued form) split as $\varpi = \varpi_{\mathfrak{p}} + \varpi_{\mathfrak{h}}$ and $\tilde{\Omega} = \tilde{\Omega}_{\mathfrak{p}} + \tilde{\Omega}_{\mathfrak{h}}$. In particular, we may write the following in accordance with our choice of basis:

$$\varpi = \begin{pmatrix} 0 & 0 \\ \varpi_{\mathfrak{p}} & \varpi_{\mathfrak{h}} \end{pmatrix} =: \begin{pmatrix} 0 & 0 \\ \theta & \omega \end{pmatrix} \quad \text{and} \quad \tilde{\Omega} = \begin{pmatrix} 0 & 0 \\ \tilde{\Omega}_{\mathfrak{p}} & \tilde{\Omega}_{\mathfrak{h}} \end{pmatrix} =: \begin{pmatrix} 0 & 0 \\ \Theta & \Omega \end{pmatrix} \tag{3.1.3}$$

The structural equation of the geometry gives us the relation $\tilde{\Omega} = d\varpi + \varpi \wedge \varpi$, which in terms of the matrices translates to:

$$\begin{pmatrix} 0 & 0 \\ \Theta & \Omega \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ d\theta + \omega \wedge \theta & d\omega + \omega \wedge \omega \end{pmatrix} \tag{3.1.4}$$

In accordance with our choice of basis, we may express all of the forms in terms of their components in the Lie algebra basis above as:

$$\theta = \theta^i e_i \quad \omega = \omega^i{}_j J_i^j \quad \Theta = \Theta^i e_i \quad \Omega = \Omega^i{}_j J_i^j \tag{3.1.5}$$

where we have used the Einstein summation convention.

Remark 3.1.1. The summation convention used in the case of the double indices written out above for the forms in $\mathfrak{h}$ in terms of the basis picked for the Lie algebra are to be understood as sums over ordered indices, where we always take $i < j$. Explicitly we have that:

$$\omega^i{}_j J_i^j = \sum_{0 \leq i < j \leq n} \omega^i{}_j J_i^j \tag{3.1.6}$$

This will only be the case for summations over these generators.

At this point, an important remark is in order.

Remark 3.1.2. The existence of such a Cartan connection relies heavily on the existence of a so-called *spin structure* on the base manifold M. We will talk more about spin structures and spin manifolds in chapter 4. For now, it is sufficient to understand that the existence of a spin structure on the base M imposes topological constraints on M. As we will see in what will follow, this is intimately related to the topological dependence of the existence of a Lorentzian metric on the manifold M as well.

3.2 Metric Equivalence Problem

If our main goal is to look at General Relativity and find an extension of it, it is unavoidable to consider metric structures on manifolds. On the other hand, at this point of our geometrical analysis of the situation, it might not be obvious to the reader to see the correspondence between metric manifolds and our geometrical picture so far. In fact, it will turn out that there is a complete equivalence between metric geometries on the base manifold M and torsion free Cartan geometries modelled on Minkowskian geometry. This thus motivates the point of view adopted in General Relativity of setting the torsion to zero. On the other hand, the study of this correspondence will give us good hints on how we can extend the General Relavity formalism in the most natural way possible in order to include this torsion. As we will see in the next chapter, it follows rather intuitively from the fact that since there is a $SO(n, 1)$ symmetry around, this introduced torsion should couple to spin. For now, we will take this result for granted having the advantage of hindsight [21, 22, 23].

In what will follow, we will explore the equivalence problem of Lorentzian metric manifolds and manifolds equipped with a Minkowskian Cartan geometry. Keeping the topological comments made in remark 3.1.2, our first important result is the following:

Proposition 3.2.1. A Minkowskian geometry on M determines a Lorentzian metric on M up to a constant scale factor.

Proof. The adjoint action of H on $\mathfrak{g}$ induces an action on $\mathfrak{g}/\mathfrak{h} \cong \mathfrak{p} \cong \mathbb{R}^{n,1}$, given by $\mathrm{ad}(A)v = Av$. This action preserves the standard quadratic form on $\mathfrak{g}/\mathfrak{h}$, the Lorentzian quadratic form η. In fact, η is the only quadratic form on $\mathfrak{g}/\mathfrak{h}$ preserved by H, up to scale. Furthermore, it is possible to use the isomorphisms provided by theorem C.2.3, $\varphi_p : T_xM \to \mathfrak{g}/\mathfrak{h}$ (where $p \in \pi^{-1}(x)$) to transport this quadratic form η to a quadratic form on T_xM, that we will imaginatively call η_p, given explicitly by:

$$\eta_p(w) := \eta(\varphi_p(w)) \quad \text{for } w \in T_xM \tag{3.2.7}$$

Since $\varphi_{ph} = \mathrm{Ad}(h^{-1})\varphi_p$, it follows that this is well-defined since:

$$\eta_{ph}(w) = \eta(\varphi_{ph}(w)) = \eta(\mathrm{Ad}(h^{-1})\varphi_p(w)) = \eta(\varphi_p(w)) = \eta_p(w) \tag{3.2.8}$$

implies that the value of $\eta_p(w)$ does not depend on our choice of point in the fibre $\pi^{-1}(x)$. We only need to prove that η_p determines a g which is smooth over the manifold M. This follows easily by considering the following diagrams:

$$\begin{tikzcd} T_pP \arrow[r, "\omega_p"] \arrow[d, "\pi_{*p}"] & \mathfrak{g} \arrow[d] \\ T_xM \arrow[r, "\varphi_p"] \arrow[dr, "\eta_p"] & \mathfrak{g}/\mathfrak{h} \arrow[d, "\eta"] \\ & \mathbb{R} \end{tikzcd} \qquad \begin{tikzcd} TP \arrow[r, "\omega"] \arrow[d, "\pi_*"] & \mathfrak{g} \arrow[d] \\ TM \arrow[dr, "g"] & \mathfrak{g}/\mathfrak{h} \arrow[d, "\eta"] \\ & \mathbb{R} \end{tikzcd}$$

The diagrams commute by virtue of the definition of η_p and φ_p. The upper composite in the second diagram going from $TP \to \mathbb{R}$ is clearly smooth. Since π_* is a submersion, the smoothness of g follows. ■

The opposite correspondence holds in the case where we consider a torsion-free Minkowskian geometry on M. To see this, we need the following lemma:

Lemma 3.2.1 (Cartan's Lemma)**.** Let V be an n-dimensional vector space, and let $\{\zeta_i\}_{i\in I} \subset V^*$ be a basis of V^*. Furthermore, let $\mu_i \in \Lambda^2 V^*$ be arbitrary. Then, there exists a unique collection of elements $\{\zeta_{ij}\}_{i,j\in I} \in V^*$ satisfying:

1. $\zeta_{ij} + \zeta_{ji} = 0 \quad \forall i, j \in I$
2. $\mu^i + \zeta^i{}_j \wedge \zeta^j = 0$

Furthermore, suppose that $\zeta_i \in A^1(U)$ where $U \subset \mathbb{R}^n$ is open and that $\mu_i \in A^2(U)$, then the forms guaranteed by the lemma are smooth, *i.e.* $\zeta_{ij} \in A^1(U)$.

Proof. We will prove existence. Smoothness will be a rather obvious consequence of existence, and uniqueness is left to the reader. We start by writing $\mu_i = A_{ijk}\,\zeta_j \wedge \zeta_k$, where we may assume that $A_{ijk} + A_{ikj} = 0$, so that the A_{ijk}'s are uniquely determined. Set the elements ζ_{ij} to be:

$$\zeta_{ij} := -(A_{jik} + A_{ikj} - A_{kji})\,\zeta_k \tag{3.2.9}$$

With this definition, it is easy to see by simple plugging into the equations of the lemma that the statement holds. The smoothness follows from the expression of the ζ_{ij}'s given in equation 3.2.9. ■

Remark 3.2.1. It will turn out that this structure of the ζ_{ij}'s is very similar to the index structure within the definition of the Christoffel symbols in terms of the metric. In some sense, this is not a coincidence, as we will remark in the next theorem.

Theorem 3.2.2. *Let (M, g) be a smooth manifold equipped with a metric g. There is exactly one torsion-free Cartan geometry modelled on Minkowskian geometry whose associated metric is, up to scale, g.*

Proof. Let $\{e(x)_i\}_{i\in I}$ be any choice of an *orthonormal* frame field on an open neighbourhood $U \subset M$ with respect to the metric g on the manifold. The determination of this orthonormal frame field immediately provides us with an orthonormal coframe field $\{\theta(x)_i\}_{i\in I}$ via the duality of T_x^*U and T_xU. Cartan's Lemma (3.2.1) guarantees the existence and smoothness of forms ${\Gamma^i}_j$ such that:

$$d\theta^i + {\Gamma^i}_j \wedge \theta^j = 0 \tag{3.2.10}$$

These Γ_{ij} are furthermore uniquely determined. We then set:

$$\omega = \begin{pmatrix} 0 & 0 \\ \theta^i & {\Gamma^i}_j \end{pmatrix} \in A^1(U, \mathfrak{g}) \tag{3.2.11}$$

It is clear that in this way, ω is a $\mathfrak{g}$-valued 1-form on U. Furthermore, since $\{e(x)_i\}_{i\in I}$ is a frame, the composite:

$$T_xU \xrightarrow{\theta_x} \mathfrak{g} \xrightarrow{\text{proj}} \mathfrak{g}/\mathfrak{h}$$

is an isomorphism, which means that ω is a Cartan gauge (*cf.* definition C.2.3). It is clear that since the manifold is covered by some cover $\{U_\alpha\}_{\alpha\in\mathcal{I}}$ and that on every such U_α we have a Cartan gauge as prescribed above, the argument holds for the entire manifold.

We must now check two things:

1. That the metric induced on U from the standard inner Lorentzian metric present on $\mathfrak{p}$ corresponds to the one set by g, at least up to scale.

2. That a different choice of orthonormal frame $\{f(x)_i\}_{i\in I}$ is gauge equivalent to the one we have just prescribed and in this way, we will have determined that there is a unique torsion-free Cartan geometry yielding the desired metric on M, g.

We start by checking the first point. Indeed, since the frame is orthonormal, we have simply that $g(e_i, e_j) = \lambda\eta_{ij}$, where η_{ij} is the standard quadratic form on $\mathfrak{p}$ and $\lambda \in \mathbb{R}$. It is thus obvious from this that the induced metric on M via the standard quadratic form on $\mathfrak{p}$ is completely equivalent (up to scale) to g.

Second, we must check that the orthonormal frame is gauge equivalent to any other. We give ourselves another orthonormal frame, $\{f(x)_i\}_{i\in I}$ (with corresponding coframe $\{\psi(x)^i\}_{i\in I}$) with the same orientation as $\{e(x)_i\}_{i\in I}$ (we restrict the orientation of our frames because of the topological considerations to the A's prescribed by equation C.1.7), the Cartan gauge ω transforms as:

$$\begin{aligned} \omega' &= \mathrm{Ad}(h^{-1})\omega + h^*\omega_H = \begin{pmatrix} 1 & 0 \\ 0 & h^{-1} \end{pmatrix}\begin{pmatrix} 0 & 0 \\ \theta^i & \star \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & h \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & \star \end{pmatrix} \\ &= \begin{pmatrix} 0 & 0 \\ h^{-1}\theta^i & \star \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ \psi^i & \star \end{pmatrix} \end{aligned} \tag{3.2.12}$$

However, since this new gauge must also be torsion free, the forms denoted by this $\star$ are immediately prescribed by the Cartan Lemma. This means that the two frames are gauge equivalent, as desired. ■

Remark 3.2.2. We can see here the motivation behind remark 3.2.1. The ${\Gamma^i}_j$'s correspond in some way to the Christoffel symbols. Furthermore, the link with the metric dependence is now clear thanks to the theorem above. We can clearly see how, given a metric and assuming the Cartan geometry is torsion free, we easily retrieve the Levi-Civita result we know.

In some sense, the last theorem justifies the assumption of no torsion in classical General Relativity. Indeed, we see here that the determination of the metric on the manifold M as well as some topological global conditions specify the geometry of the manifold completely. It is thus little to no surprise that we retrieve only something which depends on the metric in the Einstein-Hilbert Lagrangian. However, we note that if we want to include torsion, then there is no single Cartan geometry which is specified in this configuration. The remaining degree of freedom, torsion, must thus be fixed by the theory. This means in particular that it must appear in some way in the Lagrangian. Recall, however, that the torsion Θ stems from the soldering form θ, it will thus be sufficient to include this Θ in the Lagrangian in some way to completely fix the geometry, at least up to scale.

3.2.1 The Soldering Form and Vielbeine

In the following discussion, it is perhaps simpler at this point to take the more traditional approach of physicists and finally choose some local coordinates to perform calculations. Over an open neighbourhood $U \subset M$, we have an orthonormal frame field as defined in the proof of theorem 3.2.2, $\{e(x)_i\}_{i\in I}$. If $\{x^\mu\}_{\mu\in I}$ are local coordinates, then we have a linear transformation at every point mapping these local coordinates onto the orthonormal frame field chosen. By abuse of notation we will note this linear map with an e. In terms of indices, we thus have:

$$e_i = e_i^\mu \partial_\mu \quad \text{and} \quad \theta^i = e^i_\mu dx^\mu \tag{3.2.13}$$

From the proof of theorem 3.2.2, it becomes clear from the orthonormality condition on e_i that given a metric, g, we have:

$$g(e_i, e_j) = g(e_i^\mu \partial_\mu, e_j^\nu \partial_\nu) = g_{\mu\nu} e_i^\mu e_j^\nu = \eta_{ij} \tag{3.2.14}$$

which is a well-known relation in the context of tetrads. However, we now understand where exactly this stems from. The link between the tetrads and the soldering form (or any frame field) is now clear. However, the reader should understand that tetrads are nothing other than an object which occurs from a choice of coordinates and a frame on the base manifold. The natural geometrical corresponding object is really the soldering form (or more generally, the Cartan connection).

Remark 3.2.3. Some authors seem to like equation 3.2.14 so much that they wish to promote it to a definition of the "tetrad" field and this is valid as shown in the proof of proposition 3.2.1. However, it is important to stress one more time that, while in the torsion-free case the soldering form is enough to determine the entire geometry, this is no longer the case in the presence of torsion. We also need to add constraints to the vertical part of the Cartan connection $\omega_{\mathfrak{h}}$.

We revisit the objects introduced in section 3.1 under this choice of coordinates. Indeed, since $\{dx^\mu\}_{\mu\in I}$ is a basis for T_x^*U, in a Cartan gauge, we may always write:

$$\omega^i{}_j = \omega^i{}_{j\mu}\, dx^\mu\,, \qquad \Theta^i = T^i{}_{\mu\nu}\, dx^\mu \wedge dx^\nu \quad \text{and} \quad \Omega^i{}_j = \Omega^i{}_{j\mu\nu}\, dx^\mu \wedge dx^\nu \tag{3.2.15}$$

3.3 Fixing the Lagrangian of Einstein-Cartan Gravity

As discussed above, we want to find a Lagrangian which fixes the geometry. This, in particular, means that we must choose a Lagrangian containing terms depending on Ω and on θ. To find a reasonable form of the Lagrangian, we take inspiration from the Einstein-Hilbert action and realize that the form of the action we are seeking must reduce to Einstein-Hilbert in the torsionless case. Recall that the Einstein-Hilbert action takes the form:

$$S = \frac{1}{2\kappa}\int_M \mathcal{R} \tag{3.3.16}$$

where $\mathcal{R}$ is the Ricci scalar curvature. While it is possible to have an in-depth discussion of the description of the scalar curvature in the setting of representation theory, for the sake of brevity, we will do computations in local coordinates following the reasonings discussed in section 3.2.1. We may write the Ricci scalar in terms of the connection form as simply being:

$$\mathcal{R} = e_i^\alpha e_j^\beta \Omega^{ij}{}_{\alpha\beta} \tag{3.3.17}$$

This expression actually generalizes to the torsion case, except that, in that case, the Ricci curvature depends on the connection 1-form ω, which might carry a part that includes torsion (*i.e.* the forms deviate from the canonical forms guaranteed by Cartan's Lemma). It is practical at this point to split the two independent degrees of freedom that enter the action in order to be able to compute the variations in an easier manner. For this, we recognize our previous discussion of the determination of torsionless geometries by the metric. We conclude that the remaning freedom that we have left must thus be found in the connection coefficients ω. Thus, by virtue of theorem 3.2.2, we may find the forms Γ, which guarantee vanishing torsion given the forms θ in a certain Cartan gauge. In this way, we may write that the more general connection coefficients ω can be written as:

$$\omega = \Gamma + K \tag{3.3.18}$$

In the literature, K is known as the contorsion tensor. Writing the structural equations for this ω yields:

$$\Omega = \underbrace{d\Gamma + \Gamma\wedge\Gamma}_{R} + dK + K\wedge K + [\Gamma, K] \tag{3.3.19}$$

$$\Theta = \underbrace{d\theta + \Gamma\wedge\theta}_{=0} + K\wedge\theta \tag{3.3.20}$$

where R is the Levi-Civita Riemann curvature tensor. Notice that because of the last equation regarding torsion, we may use Cartan's Lemma once more in order to find a unique expression for the K forms in terms of the torsion tensor. If we write $\Theta^i = T^i{}_{jk}\,\theta^j\wedge\theta^k$, we have that:

$$K_{ij} = (T_{jik} + T_{ikj} - T_{kji})\,\theta^k \tag{3.3.21}$$

This point of view will become handy when we will vary the action with respect to the torsion later on. It is also helpful to give a full description of this in terms of coordinates:

$$K_{ij\mu} = (T_{ji\mu} + T_{i\mu j} - T_{\mu ji}) \tag{3.3.22}$$

Furthermore, after some lengthy but easy calculations that the reader is welcome to check, we can detail the Bianchi identities in the presence of torsion. These yield:

$$R\wedge\theta = 0 \tag{3.3.23}$$

$$dR = [R, \Gamma] \tag{3.3.24}$$

Which means that we can write the action in terms of coordinates as simply being:

$$S = \frac{1}{2\kappa}\int_M e_i^\alpha e_j^\beta (R^{ij}{}_{\alpha\beta} + (dK)^{ij}{}_{\alpha\beta} + K^{ik}{}_\alpha K^j{}_{k\beta} - K^{ik}{}_\beta K^j{}_{k\alpha}) \tag{3.3.25}$$

where the last term with the graded commutator $[\Gamma, K]$ present in equation 3.3.19 vanishes when we express it in local coordinates.

3.4 Variation in Vacuum

Immediately, we notice that variation of this equation with respect to the vielbein e^{α}_{i} yields a result similar in form to the Einstein Field Equations. This is because of the form of the Ricci scalar in terms of the curvature form Ω as prescribed by equation 3.3.17. The result after variation with respect to the vielbein is the equation:

$$e^{\beta}_{j}\Omega^{ij}{}_{\alpha\beta} - \frac{1}{2}e^{\beta}_{m}e^{\gamma}_{n}e^{i}_{\alpha}\Omega^{mn}{}_{\beta\gamma} = 0 \tag{3.4.26}$$

Now we notice that this curvature tensor Ω does not present all the nice symmetries that the Riemann tensor usually does. Hence, because this Ω contains non-trivial factors arising from the choice of vertical connection, this Ω is not to be confused with the ordinary Levi-Civita Riemann tensor, R. At this point, it is possible to decouple this equation and to explicitly write it in terms of the Riemann tensor along with a correction provided by the 1-form connection K, by using equation 3.3.19. In this way, it becomes clear that when torsion vanishes, we retrieve the normal Einstein Field Equations.

In an analogous way, we may vary the action S with respect to $K^{ij}{}_{\alpha}$. The contribution of the last term is simply:

$$e^{[a}_{m}\, e^{b]}_{n}\, \delta^{m}_{[i}\, \delta^{k}_{j]}\, K^{n}{}_{kb} = 0 \tag{3.4.27}$$

where the bracket denotes the antisymmetrization of indices. It is possible to express this equation in terms of the torsion tensor with which the reader might be more familiar. After some tedious calculations using the definition of the K's in terms of the torsion coefficients T_{ijk} and renaming of the indices, we arrive at our final result, which is simply:

$$2(T^{c}{}_{ab} + g^{c}{}_{a}\, T^{d}{}_{bd} - g^{c}{}_{b}\, T^{d}{}_{ad}) = 0 \tag{3.4.28}$$

where g is the metric determined by the vielbeine e.

Remark 3.4.1. The condition on the torsion coefficients as well as the symmetries of the torsion tensor imply simply that if we are not in the presence of fermionic matter, the torsion vanishes. In particular, notice also that the variation with respect to torsion yields an *algebraic* constraint. This combined with the field equations implies that torsion does not propagate outside of matter as a wave, as does curvature.

With this remark, we can establish that in the vacuum, the torsion vanishes, which implies in turn that $K = 0$, which means that the curvature equation reads:

$$e^{\gamma}_{j}R^{ij}{}_{\alpha\gamma} - \frac{1}{2}e^{\gamma}_{m}e^{\delta}_{n}e^{i}_{\alpha}R^{mn}{}_{\gamma\delta} = 0 \tag{3.4.29}$$

which after multiplication by $e_{i\beta}$ simply yields the normal Einstein Field Equations:

$$R_{\alpha\beta} - \frac{1}{2}g_{\alpha\beta}R = 0. \tag{3.4.30}$$

Remark 3.4.2. That a Schwarzschild solution exists for this modified version of General Relativity is trivial, since we know of the existence of this solution for normal GR. Since this solution is found in vacuum, the equations reduce to the ordinary Einstein Field Equations in this scenario.

3.5 Variation in Matter

In reality, if we consider the gravitation Lagrangian and we add a matter term $\mathcal{L}_M$, then, under variation with respect to the K's (or alternatively the torsion) we get that:

$$T^c{}_{ab} + g^c{}_a T^d{}_{bd} - g^c{}_b T^d{}_{ad} = \kappa\sigma^c{}_{ab} \tag{3.5.31}$$

where the tensor σ can be shown to be related to the spin tensor. Notice that if the torsion is set to zero (meaning we are in the vacuum), the curvature equation simply reduces down to the normal Einstein Field Equations, as expected. Of course, this is somewhat by construction.

Similarly, we also find that the energy-momentum tensor inherited from $\mathcal{L}_M$ couples to the curvature equation:

$$e^\beta_j \Omega^{ij}{}_{\alpha\beta} - \frac{1}{2} e^\beta_m e^\gamma_n e^i_\alpha \Omega^{mn}{}_{\beta\gamma} = \kappa P^i{}_\alpha \tag{3.5.32}$$

On the other hand, note that this tensor $P^i{}_\alpha$ is **not** symmetric and does not correspond to the canonical energy-momentum tensor prescribed by the Belifante-Rosenfeld procedure. Indeed this includes the effects of fermionic matter and it can be shown that the antisymmetric part of this tensor can be related to the spin tensor. If we think about it, this makes complete sense, since we have seen how the Ω-form splits into the Riemann tensor and the correction which takes into account the torsion.

The reader might wonder at this point what is gained from this point of view, aside from the obvious consequence of the inclusion of spin in the formalism. An additional bonus of this inclusion is that we may get non-trivial spin-spin interactions in the presence of high densities of fermionic matter. In particular, this implies that singularities are avoided in this modified version of General Relativity. At the same time, we preserve the experimental accuracy of GR, as the contributions of spin-spin interactions in normal experiments at reasonable densities (say, that of a star) remain negligible, so that the theory still agrees quite well with experiment.

3.6 Outlook

As we have seen, Einstein-Cartan theory is perhaps one of the most natural extensions one could think of for General Relativity. With the help of Cartan geometries, we were able to easily generalize the geometrical concepts that arise in General Relativity, as well as see why we may assume that torsion vanishes in the context of determining the full geometry of the manifold given a metric. Even with this generalization, it became clear that, at least in vacuum, torsion always disappears. In so doing, we were able to fix the irregularities that arise in General Relativity, as well as providing a clear depiction of where all of these ideas stem from.

As discussed previously, the above results basically all stem from an infinitesimal Poincaré invariance on the manifold at every point. In the sense of the discussion presented in remark C.2.1, one can geometrically picture this notion as seeing a Klein geometry rolling without twisting or slipping on the manifold M. In our particular case, this Klein geometry is nothing other than Minkowski space, which in some way justifies the point of view of seeing Einstein-Cartan theory as attaching Minkowski space everywhere. On the other hand, while this picture is helpful to visualize what is going on, we are not done exploiting this infinitesimal Poincaré symmetry, which still holds many consequences, as we will see in the next chapter.

Chapter 4

Spin Manifolds

Spin is a notion that has confused mathematicians and physicists since Cartan first introduced it in 1913 [26]. While the algebraic formalism is well-understood, it is a geometrical concept which has remained elusive. In the words of Michael Atiyah: "No one fully understands spinors. Their algebra is formally understood but their general significance is mysterious. In some sense they describe the "square root" of geometry and, just as understanding '$\sqrt{-1}$' took centuries, the same might be true of spinors" [27].

In this chapter, we continue our exploitation of infinitesimal Poincaré symmetry in order to find some interesting consequences, which come about in a purely geometrical way, but which have non-trivial physical implications. For starters, we give an abstract construction of the spin representations of $\mathfrak{so}(V,Q)$, the Lie algebra of the orthogonal group over a vector space V equipped with a quadratic form Q, and we explore Clifford algebras and their behaviour. In so doing, we will be able to give some non-trivial results about the computation of different objects which come about recurrently in Quantum Field Theory (QFT), in particular concerning the product of γ-matrices as well as their traces. We then proceed to give a purely geometrical derivation of the Dirac equation in flat space in the context of Cartan connections based on a Cartan geometry modelled on $(\mathfrak{iso}(V,Q), \mathfrak{so}(V,Q))$. This will show that the Dirac equation is actually the only first-order rotationally invariant differential equation applicable to particles of spin $\frac{1}{2}$. From this point, we will proceed to generalize our findings to curved space and give the expression of a natural generalization of the Dirac equation for curved space-times in arbitrary dimension and signature (provided a matrix representation of the Clifford algebra can be found and also provided the manifold obeys the proper topological constraints related to the signature).

4.1 The Dirac Equation(s) from Symmetry Principles

In this section, we will derive the Dirac equations as natural geometrical consequences of imposing infinitesimal Poincaré symmetry (in the sense of Cartan geometries), or at the very least, of imposing local Lorentz symmetry (or rather $Spin(3,1)$ symmetry, to be completely accurate). This assumption alone, together with the discussion in appendix C.2.6, provides us with a natural framework to extract all the natural first order differential operators which we can associate to a rotationally invariant system.

We shall thus consider a manifold M on which we have a Cartan geometry modelled on $(\mathfrak{g} = \mathfrak{iso}(V,Q), \mathfrak{h} = \mathfrak{so}(V,Q))$. In particular, the Minkowski and Euclidean case are the main geometries we will be concerned with here. The consideration of a Minkowski geometry, however, comes with certain topological constraints on M, as previously remarked in chapter 3.

4.1.1 Geometrical Setting

We start by defining exactly what kind of manifold we will be dealing with in the rest of this discussion; we specify this for the sake of completeness, but we will not dwell on the definition and meaning of the topological classes that will be involved in defining such objects. It will be sufficient for us to know that topological constraints exist and that we know of sufficient and necessary topological constraints to guarantee the existence of a spin structure on a manifold.

Definition 4.1.1. A **spin manifold** M is a manifold on which a **spin structure** exists. That is, there exists *Spin*-principal bundle over M. In particular, a spin manifold is a manifold on which the second Stiefel-Whitney class of M vanishes [29].

Now that we know exactly what kind of object we are dealing with, we may neglect any kind of topological obstruction when we consider spinorial representations of $O(V, Q)$ in an associated bundle.

Having said this, let M be an oriented spin manifold and let P be the oriented orthonormal frame bundle over M, *i.e.* $P = (M, O(V, Q))$ equipped with projection $\pi : P \to M$. At least locally, we may always think of P as being:

$$P = \{(x, e_1, e_2, \cdots, e_n), \quad x \in M \text{ and } e_1, \cdots, e_n \in T_x M\} \tag{4.1.1}$$

where $\{e_1, e_2, \cdots, e_n\}$ denotes an orthonormal basis. It is trivial to see that $\pi(x, e_1, \cdots, e_n) = x$. On such a principal bundle, we may induce a right action of $SO(V, Q)$ on P given by:

$$(x, e_1, \cdots, e_n) * R(s) \longmapsto (x, (e_1, \cdots, e_n) \cdot R(s)) \tag{4.1.2}$$

where $\cdot$ now denotes simple matrix multiplication. Recall from section 3.1 that we may express the Cartan connection form on P related to the Cartan geometry we considered as:

$$\varpi = \begin{pmatrix} 0 & 0 \\ \theta & \omega \end{pmatrix} \in A^1(P, \mathfrak{iso}(V, Q)) \tag{4.1.3}$$

We immediately notice that $\forall \nu \in T_p P$, we have $\pi_*(\nu) = \theta^i(\nu) e_i$. Just as before, we may decompose ω into the generators of $\mathfrak{so}(V, Q)$ which we will note ${J^i}_j$ as in the previous chapter. Notice that it is always possible to find vector fields on P such that these vector fields are duals of the ${\omega^i}_j$'s and the θ^i's. We call these dual vector fields $X_{{J^i}_j}$ and X_{θ^i}. In this setting, the following lemma is easy to see, since we can take the X_{θ^i}'s to be horizontal lifts of the e_i's and that the $X_{{J^i}_j}$'s are completely vertical.

Lemma 4.1.1. Using the same notation as above,

$$\pi_* X_{\theta^i} = e_i \quad \text{and} \quad \pi_* X_{{J^i}_j} = 0 \tag{4.1.4}$$

Furthermore, let X be an arbitrary vector field on M, then its horizontal lift is simply given by $\langle X | e_i \rangle X_{\theta^i}$, where we consider the metric to be the one fully determined by the Cartan geometry, as we saw in the previous chapter.

Lemma 4.1.2.

1. Fix $p \in P$ and define $c : [0, 1] \to P$ by considering $c(s) = p * R(s)$, where $R(s) = e^{s{J^j}_k}$, then $c'(0) = X_{{J^j}_k}|_p$.
2. Let $F : P \to V_k$ transform according to $F(p * h) = \rho(h^{-1}) F(p)$, where V_k denotes the representation space of highest weight k and $h \in SO(V, Q)$. Then,

$$(X_{{J^j}_k} F)(p) = -\rho_k({J^j}_k) F(p) \tag{4.1.5}$$

3. Let X be a vector field on M and consider the functions $\langle X|e_i\rangle : P \to \mathbb{R}$, then :

$$X_{J^j{}_k} \begin{pmatrix} \langle X|e_0\rangle \\ \vdots \\ \langle X|e_n\rangle \end{pmatrix} = -\rho_1(J^j{}_k) \begin{pmatrix} \langle X|e_0\rangle \\ \vdots \\ \langle X|e_n\rangle \end{pmatrix} \tag{4.1.6}$$

where ρ_1 denotes the vector representation.

From the above, it follows that we may decompose the universal covariant derivative $\tilde{D}F : P \to \mathfrak{g}^*$ we explored in appendix C.2.6 as follows:

$$\tilde{D}F = X_{\theta^i}(F) \otimes (e_i)^* + X_{J^i{}_j}(F) \otimes (J^i{}_j)^* \tag{4.1.7}$$

where there is an implicit summation over the generators and basis vectors of the Lie algebra and where $*$ denotes the dual of the element, since we are considering elements in $\mathfrak{g}^*$. This gives us some fundamental facts we will need later on in order to retrieve the Dirac operator.

4.1.2 Natural Differential Operators

Before we derive the Dirac equations, we will take a detour through the natural differential operators which stem from the universal covariant derivative we defined previously and examine the case of operators in dimension 3. This will be for illustrative purposes, to show how this all fits together and what kind of operators we can expect from such a decomposition. While the full treatment of what is to follow requires some representation theory to decompose tensor products of irreducible representations, we will not go into the details of the representation theory at hand, but it has been treated extensively by Cartan [10] and other authors. The results obtained in this section will serve as a blueprint for the necessary steps to take for the derivation of the Dirac equations in flat space and will hint at some of the problems we might run into when trying to generalize the Dirac equation to curved spaces.

Euclidean 3D space can be regarded as a Cartan geometry modelled on $(\mathfrak{iso}(3), \mathfrak{so}(3))$. The covariant derivative over the Cartan geometry we have considered can essentially be understood as the part of the universal covariant derivative which is horizontal with respect to the base manifold M. In other words, if $F : P \to V_k$ is a smooth function from a principal bundle into a representation V_k of highest weight k, the covariant derivative can be seen as $DF : P \to V_k \otimes \mathfrak{p}^*$. Furthermore, $\mathfrak{p}^* \cong \mathfrak{p} \cong \mathbb{R}^3$ can be seen as nothing other than the representation V_1, *i.e.* the vector representation of the orthogonal group. With an analogous choice of basis as above, we may decompose the covariant derivative in terms of the vector fields we specified above, namely:

$$DF = X_{\theta^i}(F) \otimes (e_i)^* \sim X_{\theta^1}(F) \otimes e_1 + X_{\theta^2}(F) \otimes e_2 + X_{\theta^3}(F) \otimes e_3 \tag{4.1.8}$$

Notation 4.1.1. In what will follow, we will note X_i the vector field that we previously wrote X_{θ^i} in order not to clutter the notation too much. We may do this without ambiguity since we are here only considering the horizontal part of the universal covariant derivative.

The interesting things happen when we apply the above concepts to a particular irreducible representation. For example, let F be a vector-valued smooth function (which can be seen as a section of an associated bundle with typical fibre $V_1 \cong \mathbb{R}^3$). In this case, because of the representation theory of the orthogonal group in 3D, we know that the universal covariant derivative takes values in:

$$V_1 \otimes \mathfrak{p}^* \cong V_1 \otimes V_1 = V_2 \oplus V_1 \oplus V_0 \tag{4.1.9}$$

This means that the universal covariant derivative naturally decomposes into three irreducible representations and has components in V_2, V_1 and V_0 respectively. For the sake of notational brevity, we introduce the following definition:

Definition 4.1.2. We denote $\Gamma(k)$ the space of smooth sections over an associated bundle with representation V_k with highest weight k.

Remark 4.1.1. This is completely analogous to talking about smooth functions from a principal bundle P into the representation V_k by virtue of theorem B.2.1.

In this case, the decomposition takes the form:

$$D : \Gamma(1) \to \Gamma(2) \oplus \Gamma(1) \oplus \Gamma(0) \tag{4.1.10}$$

yielding the following maps rig : $\Gamma(1) \to \Gamma(2)$, $\nabla\times : \Gamma(1) \to \Gamma(1)$, $\nabla\cdot : \Gamma(1) \to \Gamma(1)$, given by:

$$\mathrm{rig} \sim \begin{pmatrix} X_1 & 0 & -X_3 \\ 0 & X_2 & -X_3 \\ 0 & X_3 & X_2 \\ X_3 & 0 & X_1 \\ X_2 & X_1 & 0 \end{pmatrix} \qquad \nabla\times \sim \begin{pmatrix} 0 & -X_3 & X_2 \\ X_3 & 0 & -X_1 \\ -X_2 & X_1 & 0 \end{pmatrix} \qquad \text{and} \qquad \nabla\cdot \sim \begin{pmatrix} X_1 & X_2 & X_3 \end{pmatrix} \tag{4.1.11}$$

The suggestive naming of these representations hints at their geometric meaning in the case where M is the Euclidean 3D plane. However, in order to be able to interpret these findings and conclude that, it is useful to consider the following lemma, which will allow us to perform the computation of these objects in a particular gauge (we look at the case of 3D, but the proof easily generalizes to arbitrary dimension):

Lemma 4.1.3. Fix a local section $\sigma : M \to P$ such that $\sigma(x) = (x, e_1, e_2, e_3)$ such that (e_1, e_2, e_3) is an orthonormal frame on M. Then,

1. At points of P in the image of σ, we have $X_i = \sigma_*(e_i) - \omega^j{}_k(\sigma_*(e_i))X_{J^j{}_k}$. Now let

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ \theta^1 & 0 & -\omega^1{}_2 & \omega^1{}_3 \\ \theta^2 & \omega^1{}_2 & 0 & -\omega^2{}_3 \\ \theta^3 & -\omega^1{}_3 & \omega^2{}_3 & 0 \end{pmatrix} \tag{4.1.12}$$

denote the infinitesimal gauge corresponding to σ.

2. Under the conditions above:

$$X_i\langle X|e_j\rangle = e_i\langle X|e_j\rangle - \omega^k{}_\ell(e_i)X_{J^k{}_\ell}\langle X|e_j\rangle \tag{4.1.13}$$

where we may use lemma 4.1.2 to further simplify the last expression.

Proof. We start by proving the first statement. By lemma 4.1.1, $\pi_*(X_i) = e_i$ so that $\pi_*(X_i - \sigma_* e_i) = 0$. It follows that $X_i = \sigma_*(e_1) + \lambda^k{}_\ell X_{J^k{}_\ell}$ for some functions $\lambda^k{}_\ell$. Applying $\omega^k{}_\ell$ to this equation, we have $0 = \omega^k{}_\ell(\sigma_*(e_i)) + \lambda^k{}_\ell$, which proves the first result.

The second result follows from considering:

$$X_i\langle X|e_j\rangle = (e_i - \omega^k{}_\ell(e_i)X_{J^k{}_\ell})\langle X|e_j\rangle \tag{4.1.14}$$

∎

With this we are finally able to give meaning in a given gauge to the operators we got from representation theory. To make our results more readable, we let

$$\omega^1{}_2 = \alpha, \quad \omega^1{}_3 = \beta, \quad \text{and} \quad \omega^2{}_3 = \gamma \tag{4.1.15}$$

Then:

Theorem 4.1.4. *Given the splitting of the universal covariant derivative as shown above, and taking X to be the vector field on M which corresponds to the function $F : P \to V_1$ given by:*

$$F(x, e_1, e_2, e_3) = \begin{pmatrix} \langle X|e_1\rangle \\ \langle X|e_2\rangle \\ \langle X|e_3\rangle \end{pmatrix} \tag{4.1.16}$$

The following assertions hold:

1. *If we interpret the operator $\nabla\cdot : \Gamma(1) \to \Gamma(0)$ as a map from vector fields on M to functions on M, with respect to the gauge above, it assumes the form:*

$$\begin{aligned} \nabla \cdot F \quad &= (e_1 - \alpha(e_2) + \beta(e_3)) \langle X|e_1\rangle \\ &\quad +(e_2 + \alpha(e_1) - \gamma(e_3)) \langle X|e_2\rangle \\ &\quad +(e_3 - \beta(e_1) + \gamma(e_2)) \langle X|e_3\rangle \end{aligned} \tag{4.1.17}$$

2. *The operator $\nabla\times : \Gamma(1) \to \Gamma(1)$ as a map from vector fields on M onto vector fields on M with respect to the same gauge takes the form:*

$$\nabla \times F = \begin{pmatrix} \beta(e_2) - \alpha(e_3) & -e_3 - \gamma(e_2) & e_2 - \gamma(e_3) \\ e_3 + \beta(e_1) & -\alpha(e_3) - \gamma(e_1) & -e_1 + \beta(e_3) \\ e_2 - \alpha(e_1) & -e_1 - \alpha(e_2) & \beta(e_2) + \gamma(e_1) \end{pmatrix} \begin{pmatrix} \langle X|e_1\rangle \\ \langle X|e_2\rangle \\ \langle X|e_3\rangle \end{pmatrix} \tag{4.1.18}$$

3. *In the particular case where $M = \mathbb{R}^3$, α, β and γ are all identically zero. Moreover, if we take the canonical basis for $\mathbb{R}^3$, we simply have the divergence and curl operators for the operators of the spin 0 and spin 1 representations, respectively. Finally, a similar analysis can be carried for the spin 2 representation, this will yield the operators of rigid motion in 3D Euclidean space.*

This means in particular that if we equate the above to zero, we get rotationally invariant equations. In the case of spin zero this simply yields:

$$\nabla \cdot F = 0 \tag{4.1.19}$$

which is indeed rotationally invariant. Doing the same for the spin 2 representation, we get the equations of rigid motion in 3D Euclidean space:

$$\operatorname{rig} F = 0 \tag{4.1.20}$$

Finally, for vector (spin 1) representation, the more general rotationally invariant equation we can consider is given by:

$$\nabla \times F = mF \tag{4.1.21}$$

This is due to the fact that the representations of both sides transform in the same way. Under a reversal, we get that the sign in front of the m coefficient gets flipped. This remark will be of importance when we derive the Dirac equation.

4.1.3 The Dirac Equation

The discovery of spinors has had not only the well-known developments in physics, but has also had important consequences in our understanding of geometry and mathematics as well. It is important to stress that the Dirac equation's existence does not depend on a particular physical theory at all, but is rather a consequence of the symmetries of space. In the physical case, the only assumption needed for its derivation is infinitesimal Poincaré symmetry and nothing else. We will find out in this section that because of the geometrical source of the Dirac equations, their structure can be expected to always be the same.

We will now show that, at least in flat space, we can see the Dirac equation as stemming from the decompositions of the universal covariant derivative. The first important remark to make in order to get there is that there is no reason to limit ourselves to the representations of $\mathfrak{so}_3$ which are integer valued over a spin manifold. We may instead consider the half-spin representations as well. For starters, let us take the simplest half-spin representation, $V_{\frac{1}{2}}$, which corresponds to the fundamental representation of $\mathfrak{sl}_2$ as we showed earlier. As before, motivated by geometrical reasons, we consider the tensor product decomposition of $V_{\frac{1}{2}} \otimes V_1 = V_{\frac{3}{2}} \oplus V_{\frac{1}{2}}$. This implies that the covariant derivative for the spinors can in some way be seen as $D : \Gamma(\frac{1}{2}) \to \Gamma(\frac{3}{2}) \oplus \Gamma(\frac{1}{2})$. In our case, we will only worry about the mapping $\Gamma(\frac{1}{2}) \to \Gamma(\frac{1}{2})$ as this is the part which yields the Dirac equation. By a similar procedure as in the case of the decomposition of the vector representations, the representation theory decomposition yields:

$$\partial\!\!\!/ = \begin{pmatrix} X_3 & X_1 - iX_2 \\ X_1 + iX_2 & -X_3 \end{pmatrix} \tag{4.1.22}$$

which we may then apply on the spinors. However, notice that contrarily to the previous case where we could immediately relate the function $F : P \to V_1$ to a vector field X which we could then project the different components, the case for spinors is somewhat trickier, since they are, in a sense, the square root of a vector. We have been able to appreciate this twice already. It turns out that even from the point of view of representation theory, such a conception of the square root of a vector is true since, for example, if we place ourselves in dimension $2n + 1$, the tensor product of the spinor representation with itself gives back $S \otimes S = \bigwedge_{k=0}^{n} V$, thus we may reconstruct the vectors of the original space if we start from certain spinors. In the case of 3D, we have a special case in which $S \otimes S = V \oplus \mathbb{C}$. Thus every spinor can be considered as the square root of a vector

In the case of Euclidean space, the above is no mystery and $\partial\!\!\!/ = \sigma^i \partial_i$, which may be readily applied on the spinors themselves, since the action of the vectors σ when we see it as an element of the Clifford algebra has already been readily specified previously on the space of spinors. In that case, since elements of $V_{\frac{1}{2}}$ transform in the same way under rotation as the spinors themselves, it follows that the equation:

$$(\partial\!\!\!/ - m)\psi = 0 \tag{4.1.23}$$

is invariant under rotation. Under reversal, we simply get the same equation, with the sign in front of m changed. At this point a few remarks are in order:

Remark 4.1.2. The fact that the Dirac equation shows up is of no surprise, this is because this is the only first order operator which acts on spinors which is invariant under rotations. We clearly see here that the Dirac equation is nothing other than a geometrical consequence of the geometry of flat space. The reader should notice at this point that it is also obvious that $\partial\!\!\!/\partial\!\!\!/ = \nabla^2$, since ∂_i is clearly a vector and as we have seen previously, in the Clifford algebra, we simply have that any vector squares to $Q(v)$.

Remark 4.1.3. The generalization for curved spaces here is somewhat tricky, due to the presence of the correcting factors coming from the connection. Since these factors identically vanish in the case of flat space, we have no problem here. Ideally, we would like to use what we previously saw in the case of the V_1 representation, but the problem is that, this time, we are dealing with spinors, which are not straightforward quantities to describe in terms of the vectors as we have discussed previously. We may, however, take inspiration from what is happening in flat space in order to generalize this in a straight forward way to the case of curved space. We will do this in section 4.2

The case of 4D Minkowski Space

In this case, we are looking at the group $\mathfrak{so}(3,1)$, of which the representation theory is well known. A caveat is, however, that since the Lorentz group is non-compact, its the representation theory changes. On the other hand, using Weyl's unitary trick, we may find a one-to-one correspondence between the representation theory of $\mathfrak{sl}_2(\mathbb{C}) \oplus \mathfrak{sl}_2(\mathbb{C})$ and the one of $\mathfrak{so}(3,1)$. Overall, we won't concern ourselves too much with the details of the representation theory of the Lorentz group, however we will give some handwavy arguments and insights as to why this is in a way not very surprising. This identification can be understood from a geometrical and from a physical point of view as well. Both perspectives are worth looking at, as they provide different kinds of insight as to what is actually going on.

Let us take a look at the geometry first and examine $\mathfrak{so}(4,\mathbb{C})$ for the sake of simplicity. There are multiple angles from which we can understand this isomorphism from a geometrical point of view, but we will take a similar approach to the one we took in section E.1.2 when we explored the case of the matrix representations Clifford algebras in the case of 3D. As before, we can consider the action of the orthogonal group over $\mathbb{P}(V)$. In this way, we can realize $PSO_4(\mathbb{C}) \cong SO_4(\mathbb{C})/\mathbb{Z}_2$ as the connected component of the identity of the group of motions of $\mathbb{P}^3(\mathbb{C})$ carrying the quadric hypersurface $-t^2+x^2+y^2+z^2=0$ onto itself. However, a quadric hypersurface over $\mathbb{P}^3(\mathbb{C})$ has two rulings, which means that we may identify the quadric hypersurface with $\mathbb{P}^1(\mathbb{C}) \times \mathbb{P}^1(\mathbb{C})$. This means that $PSO_4(\mathbb{C})$ acts on $\mathbb{P}^1(\mathbb{C}) \times \mathbb{P}^1(\mathbb{C})$, but the connected component at the identity of the automorphism group of $\mathbb{P}^1(\mathbb{C}) \times \mathbb{P}^1(\mathbb{C})$ is nothing but $PGL_2(\mathbb{C}) \times PGL_2(\mathbb{C})$, so that we get the inclusion:

$$PSO_4(\mathbb{C}) \to PGL_2(\mathbb{C}) \times PGL_2(\mathbb{C}) \tag{4.1.24}$$

To get the reverse inclusion, we consider $V = U \otimes W$ where U and W are the pullbacks of the fundamental representations onto the first and second factor of $\mathfrak{sl}_2 \times \mathfrak{sl}_2$. Then, clearly, the action of $PGL_2 \times PGL_2$ on $\mathbb{P}(U \otimes W)$ will preserve decomposable tensors, *i.e.* points of the form $[u \otimes w]$. However, the locus of all such points must be a quadric hypersurface because of Segre embedding, thus giving us the reverse inclusion.

Next, we take a look at the physical point of view, which is a nice interpretation of the use of Lorentz symmetry in physics and is inherently linked to the geometrical picture we just talked about. We start by noting that $SO(3,1) \cong PGL_2(\mathbb{C})$, so that we may (handwavingly) see this as simply being the projection onto one of the factors of $SO_4(\mathbb{C})$. Then, using the embedding of $\mathbb{P}^1 \times \mathbb{P}^1$ above into the quadric hypersurface, we render explicit the isomorphism of the rulings and the quadric via a similar procedure to the one we employed in the case of 3D in section E.1.2. In this case, we only need one of the rulings because we are concerned with $SO(3,1)$, thus if $[\xi : \zeta] \in \mathbb{P}^1(\mathbb{C})$, then:

$$[\xi : \zeta] \longmapsto [\xi^2 + \zeta^2 + 1 : 2\xi : -2\zeta : \xi^2 + \zeta^2 - 1] \tag{4.1.25}$$

The physical interpretation of this map is rather neat. Consider very distant, fixed stars so that we may identify these stars with points on the celestial sphere, to which we may assign coordinates on the Riemann sphere. We may write these coordinates $[\xi : \zeta]$, since the Riemann sphere is actually $\mathbb{P}^1(\mathbb{C})$. The line specified in V by the homogeneous coordinates above corresponds to to the line of sight of a particular event in Minkowski spacetime to the distant fixed star having coordinates $[\xi : \zeta]$. A Lorentz transformation, seen as an element of PGL_2, acting on the locus of all such points in $\mathbb{P}^1(\mathbb{C})$ is then equivalent to stating that this is how this observer sees the appearence of the night sky change as he or she travels at relativistic speeds. This picture reconciles the geometric reasoning above, as well as providing some physical intuition as to why the light cone plays such an important role in the story. While a bit unrelated, one can also understand the connection between the light cone and spinors in the physical case as being linked to the causal limits of the theory as well as to the fact that chiral spinors travel along the light cone.

Having said the above, the construction of the actual matrix representation of the Clifford algebra can be done in an analogous way to the 3D case and we give here only some guidelines and leave the details out for the sake of conciseness. We start by noticing that in this case, due to the even dimensionality of the Minkowski case, the decomposition of V into maximally isotropic subspaces will look like $V = W \oplus W'$ with $\dim W = 2$. The space of spinors is then $\bigwedge W$. Again, due to the fact we are in even dimension, there will be two irreducible representations of the *special* orthogonal group, which correspond to $S^+ = \Lambda^{\text{even}} W$ and $S^- = \Lambda^{\text{odd}} W$, which are the two irreducible chiral representations. As far as the full orthogonal group is concerned though, no such splitting occurs. This motivates our consideration of $\bigwedge W$ as a whole instead of splitting it explicitly into the irreducible components of $SO(3,1)$. We can then simply use the accidental isomorphism we just provided to identify what the chiral representations should look like. Finally, after distiguishing both representations appropriately, we fall back on the γ-matrices we know and love.

To get the Dirac equation in Minkowski flat space, it is then sufficient to consider the splitting of the tensor product of the vector representation and the half-spin representation. If we do so, we have to take care of the fact that we have indexing of these representations with 2 numbers now, since we have $\mathfrak{so}(3,1) \cong \mathfrak{sl}_2 \oplus \mathfrak{sl}_2$. Examining the the $(\frac{1}{2}, 0) \oplus (0, \frac{1}{2})$ component of this tensor product decomposition, we get the usual Dirac equation in exactly the same way we did in the case of 3D:

$$(\partial\!\!\!/ - m)\psi = 0 \tag{4.1.26}$$

where here we now have $\partial\!\!\!/ = \gamma^\mu \partial_\mu$ as expected.

Remark 4.1.4. The reader might be wondering where the factor i in front of $\partial\!\!\!/$ present in the usual Dirac equation went. We recall that this factor i came from trying to factorize the Klein-Gordon operator, which reads $\Box + m^2$. Thus the factor i is necessary in order to obtain this factorization. Without the factor i we have:

$$(\partial\!\!\!/ - m)(\partial\!\!\!/ + m) = \Box - m^2 \tag{4.1.27}$$

so we see that for us to obtain a consistent treatment, we need an extra factor i either in front of $\partial\!\!\!/$ or simply from the square of m^2. In our case, this factor has been implicitly absorbed by this m factor in the equation. We recall that this m for us was nothing other than a constant figuring in the most general first order differential operator operating on spinors which we found with the use of representation theory. We considered it then as an arbitrary parameter. Of course, in the case of physics, we want to recover Klein-Gordon back, in which m plays the role of the mass, so we must add this factor i by hand. In what will follow, for the sake of respecting convention, we will always add this factor i in front of the /'ed operator.

With this last remark, we have thus obtained the form of the Dirac equation in both Euclidean 3D and Minkowski spacetime. In both cases, they read:

$$(i\partial\!\!\!/ - m)\psi = 0 \tag{4.1.28}$$

4.2 Dirac Operators and Curved Space

In order to apply the same reasoning to curved space, we must do a little bit more work, as the generalization does not come in a straight forward manner due to the strange square root-like nature of spin. We start by making a series of remarks which bullet point important aspects and problems which we will have to solve. These remarks will then be the main building pillars to our generalization of the above to curved space.

Remark 4.2.1. The first remark to make is that the gamma matrices γ^μ present over in flat space are inherently linked to a choice of an orthonormal basis, since these γ's are honest to goodness vectors from the point of view of the Clifford algebra. This means that γ^μ is a direction which agrees with the direction of the orthogonal basis taken for the ∂_μ.

Remark 4.2.2. We may see these γ-matrices as acting on the spinor space, and indeed this was also the case when we constructed the matrix representation, the vectors in the Clifford algebra act on $\bigwedge W$ via the action we had denoted with an $*$ in section E.1.

Remark 4.2.3. The problem we ran into previously came about when we considered the connection as acting on the spinor components. In order to fix it, we must take a closer look at the structure of the connection. The $\mathfrak{h}$-valued part of the Cartan connection is nothing other than an $\mathfrak{h}$-valued 1-form on the principal bundle. This has as a consequence that we must pick a Lie algebra basis in order to express this 1-form. So far, these sound like trivialities, but the solution to the problem comes precisely at the stage of picking a basis for the Lie algebra. Recall that in our previous construction, we had made the point to choose generators $J^i{}_j$ along which we decided to express our connection coefficients. For spinors, the story must be different. We recall, however, that because of the construction of the Clifford algebra, we may always give the expression of the generators of $\mathfrak{so}(V, Q)$ in terms of $\sigma^{\mu\nu} = \gamma^\mu \wedge \gamma^\nu = \frac{1}{2}[\gamma^\mu, \gamma^\nu]$. With this, we now know we should write the connection form as simply being given by:

$$\omega = \omega^{ij}\, \gamma_i \wedge \gamma_j \tag{4.2.29}$$

Thus, it is clear how the connection coefficients must act on the spinor components, since there is no ambiguity as to how things act on the spinors anymore due to our previous discussion concerning the action of vectors on the spinorial representation. We once again appreciate the square root-like nature of the spinors, since in order to find the corresponding action on spinor space, we had to take the "wedge square root" of the generators of the rotation group.

Remark 4.2.4. In order to give consistent treatment in curved space we *must* absolutely have a representation of dimension 4, because otherwise there is a problem as far as the connection is concerned. Indeed, the $\mathfrak{h}$-valued part of the Cartan connection must keep the correct dimension. However, if we have bigger half-spin representations, the gamma compatibility condition absolutely requires a bigger matrix, which just doesn't fit into this framework. This imposes limitations on the spins that fundamental particles can carry and forbids anything beyond the $(\frac{3}{2}, 0)$ and $(0, \frac{3}{2})$ representations of the Lorentz group. In particular, it rules out the existence of the graviton as a fundamental particle.

With all of the remarks above, we are now ready to generalize our treatment of vector representation splitting of the universal covariant derivative to the case of spinors. Since we will mainly be concerned with the half-spin representation $(\frac{1}{2}, 0) \oplus (0, \frac{1}{2})$ for the case of a geometry modelled on Minkowski spacetime, we focus a curved space equivalent to the operator $\not{\partial}$ we found earlier, which we will denote $\not{\nabla}$. Notice that in that case, we were dealing with spanning vector fields over P which we previously denoted X. These were taken to be the duals of the Cartan connection forms. We can thus see that picking these vector fields is nothing other than finding a lift of a connection present on M to the principal bundle on P. With this remark, we are ready to take a look at general Dirac operators.

In light of the remarks above, let us start by picking a section from the base manifold onto P which we can take to be: $\sigma : M \to P$ given by $\sigma(x) = (x, \{e_i\}_{i\in I})$, where the $\{e_i\}_{i\in I}$'s are an orthonormal basis. Pushing everything down with this choice of section, the Dirac operator in the case of curved space can then be associated with:

$$\not{\nabla} = e_i \nabla_{e_i} \tag{4.2.30}$$

where ∇ here denotes the connection induced on the manifold M by the Cartan connection ϖ and where there is an implicit summation over the i's. Notice also that we understand this multiplication to really be between elements in the Clifford algebra. Actually, this is because we may see these vector fields (at least locally) as being embedded in the Clifford algebra itself. Over an open neighbourhood, we may always find a chart relating the manifold to $\mathbb{R}^n$, which we can take as our vector space, on which we may then impose the Clifford algebra in the usual way. In this manner, we notice that actually this is truly Clifford multiplication taking place over a general manifold M. Another picture we can have is to use the Clifford matrix representation. We can then write this as:

$$\not{\nabla} = \gamma^i \nabla_{e_i} \tag{4.2.31}$$

On the other hand, we must always remember that this equation only holds for an orthogonal basis. If we wish to think in terms of local coordinates, we once again must play the vielbein game and let $e_i = e_i^\mu \partial_\mu$, then we may rewrite the above as simply being:

$$\not{\nabla} = \gamma^i e_i^\mu \nabla_\mu \tag{4.2.32}$$

It is possible to interpret these results in terms of frame fields. This is somewhat akin to the famous vielbeins we had in the previous chapter. We can see the gamma matrices or the Pauli matrices as simply being the link between two bases living in two different spaces. In this way, any vector a may be transformed into $\not{a}$, which basically embeds it into a Clifford algebra. We may thus consider $/ : V \hookrightarrow \mathcal{C}\ell(V, Q)$ as being the natural inclusion. This map, however, requires the choice of an orthogonal basis (although it is independent of this choice and thus is a natural mapping in the categorical sense). This provides a handwavy geometrical motivation as to why spin manifolds must be orientable, as this choice of orthogonal basis must not change orientations from point to point over the manifold M.

Choosing a section to pull everything back to the base manifold, we are now ready to write the general curved Dirac equation, which reads, perhaps unsurprisingly:

$$(i\not{\nabla} - m)\psi \tag{4.2.33}$$

Expanding everything in terms of our previous definitions, and recalling that ψ is a section over an associated bundle, we have that:

$$i\gamma^i e_i^\mu \nabla_\mu \psi = m\psi \quad \iff \quad i\gamma^i e_i^\mu \left[\partial_\mu - \frac{1}{2}\omega^{\alpha\beta}{}_\mu \, \gamma_\alpha \wedge \gamma_\beta\right] \psi = m\psi \tag{4.2.34}$$

which is the desired equation in curved space.

Chapter 5

Conclusion

Throughout this report, we have treated many different aspects of physics under a geometrical light. Indeed, starting from a description of spontaneous symmetry breaking in the electroweak sector, through a formulation of an extension of General Relativity and ending in a description of spin and a generalization of the Dirac equation, we have shown that geometry provides powerful tools to treat problems in physics. This might not hint at a similarity between these different topics per se, but on the other hand, what we can say is that the geometrical approach is so general as to being able to encompass them all.

The reason for this is not so mysterious either. In the end, it relies on the fact that nature seems to be inherently symmetric and the manifestations of these symmetries are perfectly described in the language of differential geometry. We stress that despite the fact that we were hereby restricted to cover only three topics, many other results can be understood from a geometrical standpoint.

For example, the Gribov ambiguity and Singer's approach to proving the impossibility of the existence of a global section corresponding to the Lorentz gauge fixing condition was geometrical in nature. In fact, many other such difficult problems can be understood under this light, which might lead to eventual possible solutions of them.

In the author's opinion, geometry is perhaps one of the most powerful – and yet underrated – tools that physicists posses to understand nature, as shown by illustration in this report. In any case, the consequences of a good understanding of the geometry might be worth the detour to study it further and, in particular, to seek new ways of implementing the geometrical point of view in various other branches of physics.

Appendix A

Definitions and Facts

A.1 Group Theory

Definition A.1.1. Let X be a set and G be a group. A **(right) action** of G on X is a function:

$$\begin{aligned} * : X \times G &\longrightarrow \quad X && \text{(A.1.1)} \\ (x, g) &\longmapsto \quad x * g && \text{(A.1.2)} \end{aligned}$$

Such that $*$ satisfies the following (if $e \in G$ is the identity element on G):

1. $\forall x \in X, \quad x * e = x$ (compatibility with the identity)
2. $\forall x \in X,\ \forall g, h \in G, \quad x * (gh) = (x * g) * h$ (compatibility with the group operation)

If such a function exists, we say that G acts on X and we will note: $G \curvearrowright X$.

Definition A.1.2. We characterize actions in the following ways. We say that a group action $*$ is:

- **Transitive** or that G acts transitively on X if $X \neq \emptyset$ and if $\forall (x, y) \in X^2,\ \exists\, g \in G$ such that $x * g = y$.
- **Free** or that G acts freely on X if $x * g = x \implies g = e$.
- **Effective (or faithful)** if for each $e \neq g \in G,\ \exists\, x \in X$ such that $x * g \neq x$ (a free action on a non-empty set is always faithful)

Remark A.1.1. We could've easily defined a left action in an analogous fashion. There exists a sort of duality on right and left actions, which is given by the simple fact that if $G \curvearrowright X$ on the left, then one can induce a right action by simply composing $*$ with the inverse operation of the group. This is simply a direct consequence of the formula $(gh)^{-1} = h^{-1}g^{-1}$.

Definition A.1.3. The **orbit** of a point $x \in X$ under the right action of G on a set X is defined to be:

$$\mathrm{Orb}(x) := \{y \in X \mid y = x * g,\ \forall g \in G\} \tag{A.1.3}$$

The **stabilizer** of $x \in X$ under a right G-action on a set X is defined as being the set:

$$\mathrm{Stab}(x) := \{g \in G \mid x = x * g\} \tag{A.1.4}$$

A.2 Lie Groups and Algebras

Notation A.2.1. We will note the right and left actions of an element of a group $g \in G$ on a set X by: R_g and L_g respectively.

Definition A.2.1. A subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ is called a **Cartan subalgebra** if $\mathfrak{h}$ is nilpotent and $\mathfrak{h} = I(\mathfrak{h})$, where:

$$I(\mathfrak{h}) := \{x \in \mathfrak{g} \,|\, [y, x] \in \mathfrak{h} \ \ \forall y \in \mathfrak{h}\} \tag{A.2.5}$$

As we have seen, if $\mathfrak{g}$ is finite dimensional, it is always possible to find a Cartan subalgebra of $\mathfrak{g}$. If we further impose a condition of being simple, then it is possible to decompose the Lie Algebra in terms of a Cartan Subalgebra $\mathfrak{h}$ and one-dimensional vector spaces $\mathfrak{g}_\alpha$ indexed by the positive roots of the Lie Algebra. In this way, we obtain that the Lie Algebra $\mathfrak{g}$ can simply be written as:

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Phi} \mathfrak{g}_\alpha \tag{A.2.6}$$

It is noteworthy that Φ is not a linearly independent set, however it turns out that the set spans $\mathfrak{h}^*$. Additionally, we can pick a basis of $\mathfrak{h}^*$ in a particular way using Φ. Indeed, we can form set $\Pi \subset \Phi$ which forms a basis with the additional condition that for every $\alpha \in \Phi$, we can express α as a linear combination of all positive or all negative integers of vectors in the set Π. The choice of Π is not unique, however, once it is set, we have that Φ can be partitioned into:

$$\Phi = \Phi^+ \sqcup \Phi^- \tag{A.2.7}$$

Where $\Phi^\pm$ refer to the roots made out of positive (resp. negative) integer linear combinations of vectors in the set Π.

Definition A.2.2. The **Killing form** K is the bilinear symmetric non-degenerate quadratic form $K : \mathfrak{g} \times \mathfrak{g} \to \mathbb{C}$ given by:

$$(x, y) \mapsto \langle x|y\rangle_K = \langle x|K|y\rangle \text{ where } K := \mathrm{Tr}(\mathrm{ad}_x \mathrm{ad}_y) \tag{A.2.8}$$

Notice here that by abuse of notation we denote K as the matrix corresponding to this bilinear form which can be constructed in $\mathfrak{g}^* \otimes \mathfrak{g}$.

Remark A.2.1. We notice here that the duality between $\langle\,|$ and $|\,\rangle$ is given with respect to the standard basis of the vector spaces in which they live. This means that if we want to map $|x\rangle \in \mathfrak{h} \to \mathfrak{h}^*$ via the inner product defined by the Killing form, the correct formula is simply given by: $|x\rangle \mapsto \langle x|K$.

The Killing form is important for many reasons, but it is particularly interesting because it allows us to define the concepts of orthogonality and angles between vectors. We may restrict the Killing form to $\mathfrak{h}$ and it turns out that the form remains non-degenerate on $\mathfrak{h}$. Thus, there exists an isomorphism given by K from $\mathfrak{h} \to \mathfrak{h}^*$, explicitly this gives us:

$$\begin{aligned} &\mathfrak{h}^* \times \mathfrak{h}^* \to \mathbb{C} \\ &(\langle x|, \langle y|) \mapsto \langle x|y\rangle_K = \langle x|K|y\rangle \end{aligned}$$

In what will follow however, we will stick to the notation $\langle x|y\rangle_K$ for the inner product on $\mathfrak{h}^*$ defined by the Killing form. It turns out that the geometry defined by the Killing form will play an important role in the classification of simple Lie Algebras.

Remark A.2.2. It shouldn't come to us as a surprise that this geometry is what will characterize the algebra since the Killing form contains all the information about the commutation relations characterizing the Lie Algebra at hand.

We may further introduce another definition that will be practical later on.

Definition A.2.3. Let $\langle\alpha_j|, \langle\alpha_i| \in \Pi$. Consider the elements $|h_i\rangle \in \mathfrak{h}$ corresponding to the element such that:

$$\frac{2\langle\alpha_i|}{\langle\alpha_i|\alpha_i\rangle_K} \mapsto |h_i\rangle \text{ which is just } |h_i\rangle = \frac{2K|\alpha_i\rangle}{\langle\alpha_i|\alpha_i\rangle_K} \tag{A.2.9}$$

via the Killing duality we exposed earlier, then we have that for any root $\alpha_j \in \mathfrak{h}^*$, we have that:

$$\langle\alpha_j|h_i\rangle = \frac{2\langle\alpha_i|\alpha_j\rangle_K}{\langle\alpha_i|\alpha_i\rangle_K} \in \mathbb{Z} \tag{A.2.10}$$

We call the set of $|h_i\rangle$'s the **fundamental coroots** it follows from this duality that the set $\{h_i\}_{i\in\Pi}$ forms a basis of $\mathfrak{h}$.

Definition A.2.4. $\forall\alpha \in \Phi$, let $s_\alpha : \mathfrak{h}^* \to \mathfrak{h}^*$ be the reflection with respect to the hyperplane orthogonal to α defined by:

$$\lambda \mapsto s_\alpha(\lambda) = \lambda - 2\frac{\langle\alpha|\lambda\rangle_K}{\langle\alpha|\alpha\rangle_K}\alpha \tag{A.2.11}$$

This set of reflections constitutes a group acting on $\mathfrak{h}^*$. We call this group the **Weyl Group** and will be denoted by $\mathfrak{W}$.

The Weyl group has interesting properties, but notably it has the property to act freely and transitively on the set of roots (more accurately on the Weyl chambers, a notion we will encounter later on). This set of properties and the definition of s_α provide motivation for the following definition:

Definition A.2.5. The **Cartan Matrix** A is the matrix whose components are given by:

$$A_{ij} = \frac{2\langle\alpha_i|\alpha_j\rangle_K}{\langle\alpha_i|\alpha_i\rangle_K} \in \mathbb{Z} \tag{A.2.12}$$

Furthermore, if $i \neq j$ then $A_{ij} \leq 0$ and $A_{ii} = 2$ trivially.

Remark A.2.3. Notice that the Cartan Matrix is not always symmetric, this is only true when all the roots have the same length with respect to the norm induced by the Killing inner product.

A.2.1 Dynkin Diagrams

Now that we have these definitions in our toolkit, more can be said about the structure of simple Lie algebras by looking at the geometry of its roots. To start, we may define the angle between two roots α_i and α_j using the Killing inner product as:

$$\cos\theta_{ij} = \frac{\langle\alpha_i|\alpha_j\rangle_K}{\sqrt{\langle\alpha_i|\alpha_i\rangle_K\langle\alpha_j|\alpha_j\rangle_K}} \tag{A.2.13}$$

This angle may be written in terms of the Cartan Matrix. After some trivial algebraic manipulations we arrive at:

$$4\cos^2\theta_{ij} = A_{ij}A_{ji} \tag{A.2.14}$$

In turn, this tells us about the nature of the elements in the Cartan Matrix. Indeed, since $\cos^2(x)$ is bounded, the elements of $n_{ij} := A_{ij}A_{ji}$ are bounded and are integers between 0 and 4. The only possibilities are thus $n_{ij} \in \{0, 1, 2, 3\}$, since 4 would mean that the two roots are colinear.

To illustrate the procedure further, it is helpful to look at an example at this point.

Example A.2.1. *Let us consider $\mathfrak{g} = \mathfrak{sl}_3(\mathbb{C})$. The Cartan subalgebra $\mathfrak{h}$ is simply given by the diagonal matrices of trace zero (in fact, this holds for $\mathfrak{sl}_n(\mathbb{C})$). We may then choose the ordered basis of $\mathfrak{h}$ to be the following two matrices:*

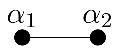
$$E_{1,2} := \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \text{ and } E_{2,3} := \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \tag{A.2.15}$$

Furthermore, we notice that a basis of $\mathfrak{h}^*$ can be obtained by considering the dual elements of $E_{i,i+1}$:

$$E^*_{i,i+1} : \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix} \mapsto \lambda_i - \lambda_{i+1} \tag{A.2.16}$$

With this duality, it is easy to see that we have two roots in the set Π defined earlier. We call them α_i. Furthermore it is easy to compute the Killing form and restrict it to $\mathfrak{h}$, in so doing we obtain:

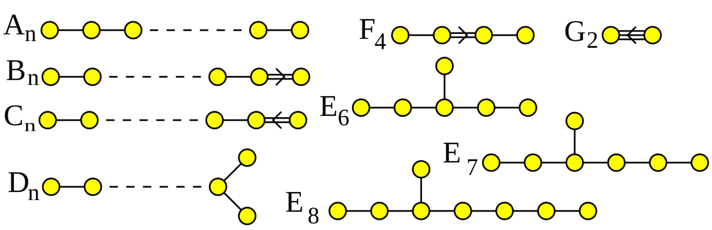
$$K = \begin{bmatrix} 12 & -6 \\ -6 & 12 \end{bmatrix} \implies A = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \tag{A.2.17}$$

We can thus see that $n_{12} = 1$. We may represent this graphically as: 

which illustrates quite simply that there are two fundamental roots α_1, α_2 in $\mathfrak{sl}_3(\mathbb{C})$. The number of edges connecting roots i and j indicate the value entry n_{ij} in the matrix n introduced above. Since there is one edge linking the two fundamental roots in our example, it means that the n matrix has form: $\begin{bmatrix} 4 & 1 \\ 1 & 4 \end{bmatrix}$

Definition A.2.6. We call diagrams constructed as explicited above **Dynkin diagrams**. Notice that the diagrammatic representation in the above example is only dependent on $\mathfrak{g}$ and nothing else.

Figure A.1: All possible Dynkin diagrams, each corresponding to a simple Lie Algebra



It turns out that we may characterize all simple Lie Algebras using this kind of diagram. Indeed we have:

Theorem A.2.1. *Let Δ be the Dynking diagram of a Lie Algebra $\mathfrak{g}$, then:*

1. *Δ is a connected graph $\iff$ $\mathfrak{g}$ is a non-trivial simple Lie Algebra;*

2. *Two nodes are connected by at most 3 bonds;*

3. *Let $Q(x_1, \cdots, x_l)$ be the quadratic form defined by:*

$$Q(x_1, \cdots, x_l) = 2\sum_{i=1}^{l} x_i^2 - \sum_{i\neq j} \sqrt{n_{ij}} x_i x_j \tag{A.2.18}$$

   We notice that the quadratic form defined only depends on the diagram and nothing else, furthermore it is positive definite.

We complement this theorem with a beautiful theorem of the classification of Lie Algebras (which even we will not use, but the result is beautiful given its simplicity)

Theorem A.2.2. *If Δ is a graph satisfying the properties given in Theorem A.2.1, then we have that Δ must be one of the diagrams in Figure A.1*

Actually, it is possible just from the diagram to go backwards and retrieve the whole structure of the Lie Algebra, which is why Dynkin diagrams are a powerful tool in Lie and Representation theory.

A.3 Finite dimensional irreducible modules

In what will follow, we will interest ourselves in the representation theory of Lie Algebras. It turns out the same formalism we used to classify simple Lie algebras is also useful to classify the associated irreducible representations. In particular, the following theorem is of utmost importance:

Theorem A.3.1. *Let $\langle\lambda| \in \mathfrak{h}^*$ and let $|h_i\rangle$ be a fundamental coroot of the simple Lie Algebra $\mathfrak{g}$. Then there is an irreducible representation associated to $\langle\lambda|$ and the dimension of it is finite if and only if:*

- *$\langle\lambda|h_i\rangle \in \mathbb{Z}$, i.e. if the weight is* ***integral****;*

- *$\langle\lambda|h_i\rangle \geq 0 \quad \forall i \in \{0, \cdots, \#\Pi\}$, then we say the weight is* ***dominant integral****.*

This means that we can completely characterize an irreducible representation (of finite dimension) by examining the weights $\langle\lambda|$ associated to it. In particular the dominant integral weights $\lambda \in \mathfrak{h}^*$ are interesting as we will see in the following theorem attributed to Élie Cartan:

Theorem A.3.2. *Every finite dimensional irreducible $\mathfrak{g}$-module (representation) has the form $L(\lambda)$ for some $\lambda \in \mathfrak{h}^*$ dominant integral.*

Thus by considering all integral weights we have all the representations (of finite dimension) and there is nothing else. To further characterize these dominant integral weights, it is useful to pull-back the fundamental coroots back to $\mathfrak{h}^*$.

Definition A.3.1. Using the canonical inner product, that is: let $\langle\omega_i|h_j\rangle = \delta_{ij}$. We call the ω_i the **fundamental weights** and they also (trivially) form a basis of $\mathfrak{h}^*$.

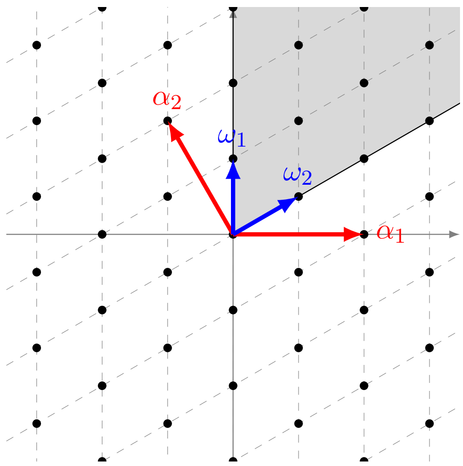


Figure A.2: Fundamental roots and weights of $\mathfrak{sl}_3(\mathbb{C})$, the lattice Λ_R (resp. Λ_W) is the one spanned by the roots α (resp. by the weights ω, which correspond to irreducible representations of the algebra). Notice we have highlighted here the first Weyl chamber.

At this point, it is useful to consider what the basis transformation between the ω_i's and the fundamental roots α_i looks like. It turns out that they are connected through the Cartan Matrix and the reader is encouraged to verify this. We have that:

$$\langle \alpha_i | = \sum_j A_{ij} \langle \omega_j | \tag{A.3.19}$$

Definition A.3.2. A **lattice** Λ, of a basis of vectors $\{e_i\}_{i \in I}$ is simply the integer linear combinations of the e_i's, *i.e.* :

$$\Lambda = \sum_{i \in I} \mathbb{Z} e_i \tag{A.3.20}$$

Furthermore it is practical to define:

Definition A.3.3. The open **Weyl chambers** for a given root system $(\mathfrak{h}, \alpha_i)$ are the connected components of $\mathfrak{h} \setminus \cup_\alpha V_\alpha$ where the V_α are the hyperplanes through the origin perpendicular to α. In particular, the **first Weyl chamber** is:

$$\mathcal{W}_1 = \{\lambda \in \mathfrak{h} \,|\, \langle \alpha_i | \lambda \rangle_K > 0 \quad \forall\, i\} \tag{A.3.21}$$

A.4 Differential Geometry

Definition A.4.1. A **fibre bundle** is a structure (E, B, π, F), where E, B and F are topological spaces which are called the **total space**, the **base space** and the **fibre** respectively and $\pi : E \to B$ is a continuous surjection satisfying a condition of **local triviality** condition, *i.e.* such that $\forall x \in E, \exists U \subset B$ an open neighbourhood such that there is a homeomorphism ϕ, such that the following diagram commutes (π_1 denotes the canonical projection onto the first coordinate):

$$\begin{tikzcd} \pi^{-1}(U) \arrow[r, "\exists \phi"] \arrow[dr, "\pi"'] & U \times F \arrow[d, "\pi_1"] \\ & U \end{tikzcd}$$

In particular, if the structure is such that F is a vector space, we call this particular kind of fibre bundle a **vector bundle**.

Definition A.4.2. Given a general fibre bundle (E, B, π, F), we define a **section** of this bundle to be a continuous function $s : B \to E$ such that $s \circ \pi$ is the identity map on B.

Notation A.4.1. We will note $\Gamma(E)$ to be the set of all (smooth) sections of E.

Definition A.4.3. Let V be a vector space. We will call an ordered basis $(v_1, \cdots, v_n) \in V^n$ a **frame** in V.

Definition A.4.4. A V-valued **Koszul connection** on a smooth vector bundle E with base M is a $\mathbb{R}$-linear function:

$$\nabla : \Gamma(E) \longrightarrow \Gamma(E \otimes V^*) \tag{A.4.22}$$

Such that the Leibniz rule holds $\forall f \in C^\infty(M)$ and $\forall \sigma \in \Gamma(E)$, *i.e.* :

$$\nabla(\sigma f) = (\nabla\sigma) f + \sigma \otimes df \tag{A.4.23}$$

In particular, note that a Koszul connection naturally induces a **covariant derivative** on the manifold by considering the following map for $X \in TM$:

$$\nabla_X : \Gamma(E) \longrightarrow \Gamma(E) \tag{A.4.24}$$

which is obtained by considering: $\nabla_X \sigma = \nabla\sigma(X)$. This covarint derivative then satisfies the following (for $X_1, X_2, Y_1, Y_2 \in TM$, $f \in C^\infty(M)$ and $a \in \mathbb{R}$):

1. $\nabla_{fX_1+X_2} Y = f\,\nabla_{X_1} Y + \nabla_{X_2}$ ($C^\infty(M)$-linear in the first component)
2. $\nabla_X(aY_1 + Y_2) = a\nabla_X Y_1 + \nabla_X Y_2$ ($\mathbb{R}$-linear in the second component)
3. $\nabla_X(fY) = f\,\nabla_X Y + X(f)Y$ (Leibniz)
4. $\nabla_X(A \otimes B) = \nabla_X A \otimes B + A \otimes \nabla_X B$ ($\otimes$-Leibniz compatible)
5. For any contraction C, we have $\nabla_X \circ C = C \circ \nabla_X$ (Contraction compatible)

In particular, the compatibility with the tensor product and the contractions can actually be deduced from the first postulates. It will thus be sufficient in the future to check if two Koszul connections coincide on a basis of vectors of TM. We see immediately that the Levi-Civita connection is a particular case of a Koszul connection.

In the following, let $\phi : M \to N$ be a diffeomorphism between two smooth manifolds M and N. We are looking for some way to induce the manifold structure of N onto M via the morphism ϕ.

Definition A.4.5. The diffeomorphism ϕ induces a diffeomorphism

$$\phi^* : C^\infty(N) \longrightarrow C^\infty(M) \tag{A.4.25}$$

$$f \longmapsto f \circ \phi \tag{A.4.26}$$

We call this morphism ϕ^* the **pullback** associated with ϕ. Furthermore, this new morphism induces another one on the 1-forms on the manifold M (which we will still note ϕ^*) in such a way that the following diagram commutes:

$$\begin{tikzcd} C^\infty(N) \arrow[r, "\phi^*"] \arrow[d, "d"] & C^\infty(M) \arrow[d, "d"] \\ T^*N \arrow[r, "\phi^*"] & T^*M \end{tikzcd}$$

Similarly, we can carry the vector field and smooth structure that is present on M to N via the morphism ϕ.

Definition A.4.6. The morphism ϕ induces yet another morphism between the tangent vector bundles $TM \to TN$ called the **pushforward** by ϕ such that the following diagram commutes

$$\begin{array}{ccc} TM & \xrightarrow{\phi_*} & TN \\ \big\downarrow{\scriptstyle \pi_M} & & \big\downarrow{\scriptstyle \pi_N} \\ M & \xrightarrow{\phi} & N \end{array}$$

Explicitly, if the curve: $\gamma : (-\varepsilon, \varepsilon) \to M$ is an integral curve of vector field $X \in TM$ we have the formula:

$$\phi_*(X) := \left.\frac{d}{dt}\right|_{t=0} (\phi \circ \gamma) \tag{A.4.27}$$

Or, alternatively, if X is a vector field and $f \in C^\infty(N)$, we have that:

$$(\phi_* X)(f) = X(\phi^* f) = X(f \circ \phi) \tag{A.4.28}$$

Remark A.4.1. We may see the pushforward as a bundle map from $TM \to \phi^* TN$, which in turn can be viewed as a section of the bundle $\mathrm{Hom}(TM, \phi^* TN)$ over M. It is also noteworthy to state that pointwise we have that the induced morphism $\phi_* : T_x M \to T_{\phi(x)} N$.

A.5 Functional Analysis

Hilbert spaces are the generalization of Euclidean spaces to infinite dimensional spaces. In particular, they are Banach spaces whose norm stems from an inner product structure on the vector space. Lots of the results that hold in the finite dimensional case still hold for the infinite dimensional case, however, one must be careful about certain manipulations when dealing with a general Hilbert space structure.

Definition A.5.1. A **Hilbert space** $(\mathcal{H}, \langle \,|\, \rangle)$ is a real or complex inner product space that is also a complete metric space with respect to the distance function induced by the inner product.

In particular, this inner product induces also the notion of orthogonality of vectors in the space where it is defined. We now list some important results from the theory of Hilbert spaces.

Theorem A.5.1 (Projection onto a closed convex space)**.** *Let Γ be a convex closed subset of $\mathcal{H}$. Then for all $x \in \mathcal{H}$, there exists a unique point of Γ, $p_\Gamma(x)$ called **the projection of** x **onto** Γ such that:*

$$\|x - p_\Gamma(x)\| = \inf_{g \in \Gamma} \|x - g\| \,. \tag{A.5.29}$$

Proposition A.5.1. Let Γ be a closed convex subset of $\mathcal{H}$ and $x \in \mathcal{H}$ and let $p_\Gamma(x)$ be its projection. Additionally, let $\gamma \in \Gamma$. Then the following are true:

$$\Re\langle x - p_\Gamma(x) \,|\, p_\Gamma(x) - \gamma \rangle \geq 0 \quad \text{and} \quad \left\| p_\Gamma(x) - p_\Gamma(x') \right\| \leq \left\| x - x' \right\| \; \forall x' \in \mathcal{H} \tag{A.5.30}$$

Corollary 1. *Let F be a closed subspace of $\mathcal{H}$. Then: $\mathcal{H} = F \oplus F^\perp$.*

Definition A.5.2. Let $\mathcal{H}$ be a Hilbert separable space of infinite dimension. We call **a Hilbert basis or orthonormal basis of** $\mathcal{H}$ every sequence $(e_n)_{n\in\mathbb{N}}$ which is complete (total) and such that:

$$\langle e_i \mid e_j \rangle = \delta_{i,j} \tag{A.5.31}$$

Theorem A.5.2. *In a separable Hilbert space, such bases exist.*

Theorem A.5.3 (Bessel, Parseval)**.** *Let $\mathcal{H}$ be a separable Hilbert space and let $(e_n)_{n\in\mathbb{N}}$ be an orthonormal basis. The map defined by:*

$$\mathcal{H} \quad \longrightarrow \ell^2(\mathbb{N}) \tag{A.5.32}$$

$$x \quad \longmapsto (\langle x|e_n\rangle)_{n\in\mathbb{N}} \tag{A.5.33}$$

is a linear isometrical bijection. In particular, the following identities are true:

$$x = \sum_{n\in\mathbb{N}} \langle x \mid e_n\rangle e_n \quad \text{and} \quad \|x\|^2 = \sum_{n\in\mathbb{N}} |\langle x \mid e_n\rangle|^2. \tag{A.5.34}$$

Theorem A.5.4 (Riesz Representation Theorem)**.** *Let $\mathcal{H}^*$ be the continuous dual of the Hilbert space $\mathcal{H}$ (i.e. the space of all continuous linear functionals from $\mathcal{H}$ to $\mathbb{R}$ (or $\mathbb{C}$)). Then, the mapping $\Xi : \mathcal{H} \to \mathcal{H}^*$ defined by $\Xi(x) = \xi_x$, where $\xi_x(y) = \langle x|y\rangle$ is an isometric (antilinear) isomorphism.*

Appendix B

Principal Bundles

In physics, we are often concerned with the particular reference frame of any particular observer. Mathematically, if this observer lives on some manifold M, this reference frame can be seen as a basis choice for the tangent space T_pM at the point $p \in M$ at which the observer lies at a given instant.

For example, imagine an airplane travelling through $\mathbb{R}^3$. The plane is free to turn, move around and have different orientations at any given moment. Furthermore, the pilot is free to choose any particular set of reference axes at any moment, which we will assume are constrained to change in a smooth manner. In particular, these axes need not be orthogonal at any given moment.

This kind of physical situation immediately motivates the study of these frames and the way they behave as the observer moves around the manifold. We start by noticing that TM actually forms a vector bundle over M (*cf.* definition A.4.1). Usually, in physics, it is customary to consider a coordinate system over M and define an ordered basis of T_pM as the vectors corresponding to an infinitesimal change in each of the coordinates. Such a choice of local basis for TM over some neighborhood $U \subset M$ where the coordinates are defined is called a **natural basis**. However, we need not always choose a natural basis to study manifolds. As a matter of fact, in all generality, we shouldn't (consider again the case of the pilot continuously changing his set of axes). By noticing that we need not have a particular choice of coordinates on the manifold in order to have a particular frame, it allows us to obtain a much more powerful tool to study differential geometry:

Definition B.0.1. Let F_p denote the set of all frames at any given point $p \in M$. We define the **frame bundle** over M (which we will denote $F(M)$) as the disjoint union over all points in the manifold of these F_p's. In symbols:

$$F(M) := \coprod_{p \in M} F_p \tag{B.0.1}$$

In particular, we notice that at least locally, we can always determine an arbitrary basis $(X_1, \cdots, X_n)$ for TU. In that case, we note that we have that these X_i actually induce a function:

$$\begin{aligned} \phi : U &\longrightarrow F(U) \\ p &\longmapsto (X_1(p), \cdots, X_n(p)) \end{aligned}$$

Whose values are frames in various tangent spaces of U. In particular, this allows us to define:

Definition B.0.2. We define such a function ϕ to be a **moving frame** over U. In particular, notice that every moving frame determines a set of vector fields $(X_1, \cdots, X_n)$ and vice-versa.

Remark B.0.1. Notice that a given frame $(X_1, \cdots, X_n)$ might not correspond to the natural basis induced by a coordinate system over some open set $U \subset M$. This implies in particular that $[X_i, X_j]$ need not be zero for a general moving frame.

We note that $F(M)$ is equipped with a natural right action of $GL(n, \mathbb{R})$ induced by the action of the elements of this group on any particular frame of T_pM by simple matrix multiplication on the right. This action is free and transitive by a standard result from linear algebra, which states that there is a unique, invertible, linear transformation sending one basis to another. The idea of a group acting on a manifold will turn out to be the key to generalizing geometric concepts further. Going back to our pilot, this action is precisely his (smooth) freedom of choice of axes at any given instant. This whole discussion motivates the following definition:

Definition B.0.3. A **principal G-bundle** is a fibre bundle $\pi : P \to X$ together with a continuous right action $* : P \times G \to P$ such that if $x \in X$, $y \in \pi^{-1}(x) \implies y * g \in \pi^{-1}(x)$, *i.e.* the group action preserves the fibres of P. Furthermore, we demand that this action is free and transitive.

Note that the frame bundle $F(M)$ actually forms a principal bundle over M under the action of $GL(n, \mathbb{R})$.

Remark B.0.2. Notice here that the condition of the action being free and transitive actually implies that the fibres themselves are homeomorphic to the group G itself.

Remark B.0.3. In the above, we have defined the bundles in the general case, for what will follow, we will be concerned with bundles over manifolds, which means we must replace the continuity condition by one of smoothness of the maps above.

As usual in mathematics, it is useful to state explicitly what we mean by a (iso)morphism of principal bundles.

Definition B.0.4. Let (P, π, M, G) and (Q, π', N, H) be a principal G and H-bundle respectively. A **principal bundle (iso)morphism** is the given of smooth maps (bijective diffeomorphisms) (u, f, ρ) such that the diagram commutes:

$$\begin{tikzcd}
P \arrow[r, "u"] & Q \\
P \times G \arrow[u, "*_G"] \arrow[r, "u \times \rho"] & Q \times H \arrow[u, "*_H"] \\
P \arrow[u, "i_1"] \arrow[r, "u"] \arrow[d, "\pi"] & Q \arrow[u, "i_1"] \arrow[d, "\pi'"] \\
M \arrow[r, "f"] & N
\end{tikzcd}$$

where the morphism i_1 denotes the inclusion into the first coordinate and ρ is a group (iso)morphism $\rho : G \to H$.

Remark B.0.4. In the case where $H = G$ or if the bases are the same, the diagram simplifies considerably by taking the morphisms to simply being the identity.

B.1 Tensors

In physics, so-called *tensorial* quantities are often of interest. Heuristically, physicists understand tensors as being quantities which "transform under a certain way". In order to make this concept more precise, we introduce the following definition:

Definition B.1.1. Let (P, π, M, G) be a principal bundle over M. Furthermore, let (V, ρ) be a representation of G. We denote by $A^q(P,(V,\rho))$ (or $A^q(P,\rho)$ for short) the following set:

$$A^q(P,\rho) := \{\eta : \Lambda^q TP \to V \mid R_g^* \eta = \rho(g^{-1})\eta, \quad \forall g \in G\} \tag{B.1.2}$$

With the definition of this set $A^q(P,\rho)$, we are ready to give a more precise meaning to tensors. In particular, we want to characterize this "transformation under a certain way" using the precision provided by representation theory. It follows quite naturally, then, that we should define the following:

Definition B.1.2. Let (P, M, π, G) be a principal G-bundle and let (V, ρ) be a representation of G. A **tensorial form of type** (V, ρ) is an element of the set $A^q(P,\rho)$ (*cf.* definition B.1.1). In particular, a **tensor of type** (V, ρ) is nothing other than an element of $A^0(P,\rho)$.

In the end, we regard tensors as objects which are G-equivariant on a principal G-bundle in this particular sense.

B.2 Associated Bundles

We next introduce some practical definitions and facts about so-called associated bundles. These notions will become useful later when we explore how all these concepts are applicable in physics.

Definition B.2.1. Let (P, π, M, G) be a principal G-bundle and let F be a smooth manifold equipped with a *left* G-action. We define:

1. $P_F = P \times_G F := (P \times F)/\sim_G$, where $(p, f) \sim_G (p', f') \iff \exists\, g \in G$ such that $(p', f') = (p * g, g^{-1} * f)$. We note an element of P_F as $[p, f]$ where here the brackets are to be understood as the equivalence class of a point (p, f) in $P \times F$;

2. The well-defined map $\pi_F : P_F \to M$ which takes $[p, f] \mapsto \pi(p)$.

The **associated bundle** to (P, π, M, G) with fibre F, is the bundle (P_F, π_F, M).

It is important to give the notion of (iso)morphisms for any of the objects we introduce. For this, we have the following definition:

Definition B.2.2. Let (P_F, π_F, M) and (Q_F, π'_F, N) be the associated bundles to the principal G-bundles (P, π, M) and (Q, π', N). An **(iso)morphism of associated bundles** is a (bijective) bundle map $(\tilde{u}, v)$ such that for some u, (u, v) is a principal bundle morphism, *i.e.*

such that the following diagram commutes:

$$\begin{tikzcd}
P_F \arrow[r, "\tilde{u}"] \arrow[d, "\pi_F"] & Q_F \arrow[d, "\pi'_F"] \\
M \arrow[r, "v"] & N \\
P \arrow[u, "\pi"] \arrow[r, "u"] \arrow[d, "*_G"] & Q \arrow[u, "\pi'"] \arrow[d, "*_G"] \\
P \arrow[r, "u"] & Q
\end{tikzcd}$$

Remark B.2.1. Notice that it is always possible to construct such an associated bundle by simply considering F to be different smooth representations (which may in particular be linear) of G. These are naturally equipped with a left action of G.

Theorem B.2.1. *Let (P, π, M, G) be a principal G-bundle and (P_F, π_F, M) be its associated bundle. Let furthermore (U, x) be a chart of M. Then, there exists a bijection:*

$$\{\textit{Local sections } s : U \to P_F\} \underset{\sigma}{\overset{\psi}{\rightleftarrows}} \left\{G\textit{-equivariant } \phi : \pi^{-1}(U) \to F\right\}$$

Where the G-left equivariant condition is given by (if $*$ denotes the corresponding left and right actions):

$$\forall g \in G, \forall p \in \pi^{-1}(U) \; : \; \phi(p * g) = g^{-1} * \phi(p) \tag{B.2.3}$$

Remark B.2.2. This bijection allows us in particular to express the local sections of P_F in terms of functions ϕ defined from the base onto the fiber univocally.

Proof. We proceed by first expliciting the map σ. Let ϕ be a G-equivariant function from $\pi^{-1}(U) \to F$. Then:

$$\sigma : \phi \longmapsto (s_\phi : x \mapsto [p, \phi(p)]) \tag{B.2.4}$$

where $p \in \pi^{-1}(x)$. We start by checking that this map is well-defined. Indeed, we have that if $p, q \in \pi^{-1}(x)$, then there is a unique $g \in G$ such that $q = p * g$. By the underlying quotient present in P_F and the equivariance condition on ϕ, we have:

$$[p, \phi(p)] = [p * g, g^{-1} * \phi(p)] = [p * g, \phi(p * g)] = [q, \phi(q)] \tag{B.2.5}$$

And so, the map is well-defined. We now check that the condition of being a section is satisfied, but we have:

$$\pi_F([p, \phi(p)]) = \pi(p) = x \tag{B.2.6}$$

Thus the condition is satisfied, so s_ϕ is indeed a local section. Next we examine the reciprocal of ψ. First however, we introduce an intermediate map:

$$i_p : F \longrightarrow \pi_F^{-1}(\pi(p)) \tag{B.2.7}$$

$$f \longmapsto [p, f] \tag{B.2.8}$$

This map is clearly a bijection (surjection is clear and injectivity follows from the freedom of the action $*$). Furthermore, we have the property that:

$$i_p(f) = [p, f] = [p * g, g^{-1} * f] = i_{p*g}(g^{-1} * f) \tag{B.2.9}$$

With these two facts in mind, let $s : U \to P_F$ be a section, we take the map ψ to be:

$$\psi : s \longmapsto \left[\, \phi_s : p \mapsto i_p^{-1}(s(\pi(p)))\, \right] \tag{B.2.10}$$

We further need to check that ϕ_s is G-equivariant and that these two are inverses of each other. The equivariance can be shown by the property of the map i_p we uncovered previously, it is left to the reader as an exercice. Finally, we show that $\psi \circ \sigma = id$, the other order is left also to the reader. We aim to show that if $p \in \pi^{-1}(U)$ then $\psi(\sigma(\phi))(p) = \psi(\sigma(\phi(p))) = \phi(p)$:

$$\psi(\sigma(\phi))(p) = \psi(s_\phi)(p) = i_p^{-1}(s_\phi(\pi(p))) \tag{B.2.11}$$

Now recall that $s_\phi(\pi(p)) = [q, \phi(q)]$, where q is any element in $\pi^{-1}(\pi(p))$. In particular $p \in \pi^{-1}(\pi(p))$ so we have:

$$\psi(\sigma(\phi))(p) = i_p^{-1}([p, \phi(p)]) = \phi(p) \tag{B.2.12}$$

■

B.3 Connections in Principal Bundles

Now that we have a proper sense of the space in which we are working, we want to generalize concepts that we commonly encounter in differential geometry to these principal bundles. In particular, one of the first natural objects that we look at while considering manifolds are connections, covariant derivatives and similar objects. Before definining these concepts, it is interesting to take a look of the structure that the principal bundle construction provides us with in order to better understand the kind of objects we are dealing with. We start by introducing some notation and some definitions.

B.3.1 Lie Algebra Valued Forms

We start by first taking a small detour to define some notation that will become handy later on for different purposes.

Notation B.3.1. For two Lie algebra-valued k and q-forms η and ζ on a principal bundle P, we define a $(k+q)$-form $[\eta, \zeta]$ defined by:

$$[\eta,\zeta](v_1,\cdots,v_{p+q}) = \sum_{\sigma\in S_{k+q}} \mathrm{sgn}(\sigma)\left[\, \eta(v_{\sigma(1)},\cdots,v_{\sigma(k)})\,,?\,\zeta(v_{\sigma(k+1)},\cdots,v_{\sigma(k+q)})\,\right] \tag{B.3.13}$$

where S_{k+q} is the permutation group of $k+q$ elements and the $v_i \in T_pP$.

The notation as a commutator can be justified by the fact that if $\mathfrak{g}$ is a matrix algebra, then the operation $[\eta, \zeta]$ is nothing other than a graded commutator, *i.e.* :

$$[\eta,\zeta] = \eta\wedge\zeta - (-1)^{\deg\eta\,\deg\zeta}\zeta\wedge\eta \tag{B.3.14}$$

It follows from this **graded commutativity** that we may express:

$$[\zeta,\eta] = -(-1)^{\deg\eta\,\deg\zeta}[\eta,\zeta] \tag{B.3.15}$$

Sometimes, for convenience, or depending on the context, we may write things like (for forms of odd degree):

$$\omega\wedge\omega = \frac{1}{2}[\omega,\omega] \tag{B.3.16}$$

It is to be understood in such occasions that we implicitly use the matrix algebra structure of $\mathfrak{g}$ in such cases. For more general purposes, the notation with $[\,,]$ is preferred.

Lemma B.3.1. Let $\alpha, \beta \in A^p(P, \mathfrak{g})$ be homogeneous $\mathfrak{g}$-valued p-forms. Then:

$$[\alpha, [\beta, \beta]] = \begin{cases} 2[[\alpha, \beta], \beta], & \text{if } \deg \beta \text{ is odd} \\ 0, & \text{if } \deg \beta \text{ is even.} \end{cases} \tag{B.3.17}$$

Proof. Use the graded Jacobi identity, *i.e.* :

$$(-1)^{ba}[[\beta, \beta], \alpha] + (-1)^{b^2}[[\beta, \alpha], \beta] + (-1)^{ab}[[\alpha, \beta], \beta] = 0 \tag{B.3.18}$$

where $a = \deg \alpha$ and $b = \deg \beta$. The result above is a consequence of graded commutativity. ∎

Corollary 2. $[[\alpha, \alpha], \alpha] = 0$ *for all homogeneous* $\alpha \in A(P, \mathfrak{g})$.

Proof. We may without loss of generality assume that deg α is odd. So that:

$$[\alpha, [\alpha, \alpha]] \quad = 2[[\alpha, \alpha], \alpha] \text{ by the lemma} \tag{B.3.19}$$

$$= -2[\alpha, [\alpha, \alpha]] \text{ by graded commutativity} \tag{B.3.20}$$

∎

B.3.2 Vertical and Horizontal Spaces

Definition B.3.1. Consider a general principal bundle P whose base is a smooth manifold M and let $\pi : P \to M$ denote the canonical projection. Consider then a point $x \in M$ such that for $p \in P$ we have $\pi(p) = x$. Then we have that if the map $i : \pi^{-1}(x) \to P$ is the natural inclusion, then the image of the tangent space $i_*(T_p\pi^{-1}(x))$ is a subspace $T_pV \subset T_pP$ called the **vertical subspace at** p. We call all vectors $v \in T_pV$ **vertical tangent vectors** at p. It is clear then from the definition that $Y \in T_pV \iff \pi_*(Y) = 0$.

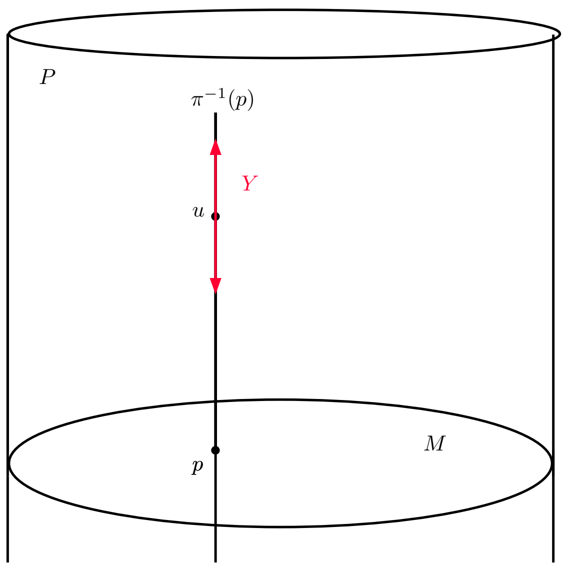


Figure B.1: An illustration of the definition of vertical tangent vectors

Remark B.3.1. More explicitly, recall the definition of the pushforward (*cf.* A.4.6) as the derivative of an integral curve of a vector field. Then it is clear that it is true. Indeed, since we are solely focusing on a single fibre, the flow generated will be in some way "vertical" to the base space M. It is then obvious that by projecting back the integral curve gets reduced to a single point, hence why the vector field induced by π_* gives zero.

Since we are focusing here on principal bundles, tautologically, we have a Lie group G acting on the smooth principal bundle P. In particular, we may consider the smooth mapping $\sigma_p : G \to P$ that takes $a \mapsto p \cdot a$. We may thus look at $\mathfrak{g}$, the Lie algebra of the Lie group, which also acts on the principal bundle on the right. Indeed, the action is naturally induced by the exponential of any element $A \in \mathfrak{g}$. Explicitly, we have that the map $t \mapsto e^{tA}$ induces a curve on G (recall G is in particular a manifold, since it is a Lie group). However, due to the presence of the action of G on P, it induces also induces a curve on the principal bundle by considering the curve $\gamma_p(t) = p \cdot e^{tA}$. As such, by taking $\gamma_p'(0)$, we have a vector that lives in T_pP. We denote this map $\sigma(A)(p) = \gamma_p'(0)$. We can thus see that the action of the Lie group induces an morphism between $\mathfrak{g}$ and the vector fields via $\sigma : \mathfrak{g} \to TP$. Notice that we may describe this morphism in an equivalent manner with:

$$\sigma_{p*}(A) = \sigma(A)(p) \tag{B.3.21}$$

Definition B.3.2. The **fundamental vector field**, $\sigma(A)$, is the vector field corresponding to A, $\forall A \in \mathfrak{g}$. In particular, $\forall p \in P$ we have that the map $A \mapsto \sigma(A)(p)$ is an isomorphism because G acts without fixed points.

Furthermore, the $\sigma(A)$'s satisfy the following proposition:

Proposition B.3.1. Let G act on the right on P, then we have the following:

1. $\sigma : \mathfrak{g} \to TP$ is linear;
2. $\sigma([A, B]) = [\sigma(A), \sigma(B)]$ for $A, B \in \mathfrak{g}$ (*i.e.* σ is a Lie algebra morphism);
3. If G acts transitively and freely and $A \neq 0$ then $\sigma(A)$ is not the zero vector field;
4. If G acts freely and $A \neq 0$, then $\sigma(A)$ is nowhere zero.

Remark B.3.2. Notice that since the action of G takes fibres to themselves, the set of all $\sigma(A)(p)$ is precisely the set of vertical vectors at T_pP, since we may write them as $\sigma_{p*}(A)$. We can then see that the integral curves are then restricted to the fibre of $\pi(p)$.

Definition B.3.3. The **adjoint mapping** (and we will note $\mathrm{Ad}(a)$ or Ad_a) from $\mathfrak{g} \to \mathfrak{g}$ is the mapping given by:

$$\mathrm{Ad}(a) = (L_a R_a^{-1})_* = (R_a^{-1} L_a)_* \tag{B.3.22}$$

Proposition B.3.2. If R_a denotes the function from $P \to P$ such that $p \mapsto p \cdot a$, then $\forall A \in \mathfrak{g}$ and $\forall a \in G$, the vector field $(R_a)_*\sigma(X)$ is the fundamental vector field given by:

$$(R_a)_*\sigma(A) = \sigma(\mathrm{Ad}(a^{-1})A) \tag{B.3.23}$$

The vector fields $\sigma(A)$ are difficult or even impossible to picture (mostly because picturing a principal bundle itself can be impossible due to dimensionality issues), however, in a way, they encode the G-structure that is present on the principal bundle. This is why they are particularly useful in the constructions that will follow and also why they play such an important role in the study of principal bundles.

We would now like to generalize the notion of a connection to a principal bundle. For this, we have the following definition given by Ehresmann in 1950:

Definition B.3.4. An **(Ehresmann) connection** over a principal bundle $\pi : P \to M$ over M with a group G is a C^∞ $\mathfrak{g}$-valued 1-form ω on P such that:

1. $\omega(\sigma(A)) = A \qquad \forall A \in \mathfrak{g}$;
2. $R_a^*\omega = \mathrm{Ad}(a^{-1})\omega \qquad \forall a \in G$

Definition B.3.5. The **curvature of the Ehresmann connection** ω is the $\mathfrak{g}$-valued 2-form Ω on P given by: $\Omega = d\omega + \frac{1}{2}[\omega, \omega]$

Proposition B.3.3. The curvature form behaves under transformation as: $\forall a \in G$ we have that: $R_a^*\Omega = \mathrm{Ad}(a^{-1})\Omega$.

Notice here that if ω is an Ehresmann connection, then the map $\omega_p : T_pP \to \mathfrak{g}$ is onto. In particular, $\ker(\omega_p)$ forms a subspace of T_pP.

Definition B.3.6. The subspace $\ker(\omega_p) := T_pH \subset T_pP$ is called the **horizontal subspace** at p (determined by the connection ω). Vectors in this subset are called **horizontal**.

Remark B.3.3. All Ehresmann connections thus give naturally rise to these distributions of horizontal subspaces TH. We will later on see that giving the connection form ω or giving the distribution of the horizontal spaces is basically equivalent, some authors even define the Ehresmann connection in terms of these C^∞-distributions they generate. We can kind of see these horizontal subspaces as being a "lifted" copy of TM into TP.

Proposition B.3.4. If H is the distribution given by the Ehresmann connection the following are true:

1. $T_pP = T_pV \oplus T_pH$;
2. $T_{p \cdot a}H = (R_a)_* T_pH$;
3. H is a C^∞-distribution.

Remark B.3.4. Using the decomposition of T_pP it is possible to express $Y \in TP$ as:

$$Y = h(Y) + v(Y) \tag{B.3.24}$$

where h and v are projectors onto the horizontal and vertical subspaces respectively. In particular, we may write the projector h and v as:

$$v(Y) = \sigma(\omega(Y))\,, \quad h(Y) = Y - \sigma(\omega(Y)) \tag{B.3.25}$$

since we saw in remark B.3.2 that the set $\sigma(\mathfrak{g})$ is the set of all vertical vectors. From this, it is clear that h and hence v are C^∞.

Remark B.3.5. We also note that from the decomposition $T_pP = T_pV \oplus T_pH$ and from the fact that $T_pV = \ker(\pi_*)$, the morphism $\pi_* : T_pH \to T_{\pi(p)}M$ is an isomorphism. Consequently, it follows that for every vector field $X \in TM$ there exists a unique vector field $\tilde{X} \in TP$ such that $\tilde{X}$ is everywhere horizontal and $\pi_*(\tilde{X}_p) = X_{\pi(p)}$, $\forall p \in P$. Heuristically, this means that, in some way, it's as if the horizontal spaces were the lifts of the tangent spaces $T_{\pi(p)}M$ up to some $p \in P$.

Definition B.3.7. We call this unique vector field $\tilde{X}$ the **lift** of X at point $p \in P$.

There are then two simple facts about lifts which we summarize in the following propositions:

Proposition B.3.5. $\tilde{X}$ is a C^∞-vector field on P stemming from a unique $X \in TM$ if and only if $\forall a \in G$, we have that $R_{a*}(\tilde{X}) = \tilde{X}$.

Lemma B.3.2. Consider a connection ω on a principal bundle P over M with group G. For any $A \in \mathfrak{g}$ and horizontal vector field Y on P, the vector $[\sigma(A), Y] \in TH$.

Proof. We look at the definition of the Lie derivative:

$$\lim_{t\to 0} \frac{1}{t}\left(Y - R_{e^{tA}*}Y\right), \tag{B.3.26}$$

notice that here Y is horizontal, but so is $R_{e^{tA}*}Y$, since the flow along the vertical vector fields preserves horizontality. Thus, the entire quantity is in TH for all t, which proves the lemma. ∎

Proposition B.3.6. If $\tilde{X}$ and $\tilde{Y}$ are the lifts of $X, Y \in TM$, then the following are true:

1. $\tilde{X} + \tilde{Y}$ is the lift of $X + Y$;
2. $\forall f \in C^\infty(M)$, then $\widetilde{(fX)} = (f \circ \pi)\tilde{X}$;
3. $h([\tilde{X}, \tilde{Y}]) = [\tilde{X}, \tilde{Y}]$.

With the machinery of the Ehresmann connection behind us, we now aim to generalize all the concepts previously seen applicable to connections, that is we seek to build a notion of parallel transport, covariant derivatives, the structural equations, the curvature and torsion tensors and finally the Bianchi identities.

B.4 Geometrical Concepts on Principal Bundles

B.4.1 Parallel Transport

Our aim is now to define what parallel transport is on principal bundles. To do this, we consider parallel transport along a piecewise C^1-curve $\gamma : [0,1] \to P$.

Definition B.4.1. We say that such a curve γ is **horizontal** if all the $\gamma'(t) \in T_{\gamma(t)}P$ are horizontal. We may further define a **lift** of a piecewise C^1-curve $\gamma : [0,1] \to M$, $\tilde{\gamma} : [0,1] \to P$ such that $\tilde{\gamma}$ **covers** γ, *i.e.* such that $\pi \circ \tilde{\gamma} = \gamma$.

Remark B.4.1. Given an initial condition, *i.e.* $p_0 \in P$ such that $\pi(p_0) = c(0)$, then there exists a unique lift of γ such that $\tilde{\gamma}(0) = p_0$.

We can now define parallel transport of the fibres of P along $\gamma : [0,1] \to M$.

Definition B.4.2. We have that $\forall p \in \pi^{-1}(\gamma(0))$, let the function $\tau_t(p) \in \pi^{-1}(\gamma(t))$ be simply the following function:

$$\tau_t : \pi^{-1}(\gamma(0)) \longrightarrow \pi^{-1}(\gamma(t)) \tag{B.4.27}$$

$$p \longmapsto \tilde{\gamma}_p(t) \tag{B.4.28}$$

Where $\tilde{\gamma}_p(t)$ denotes the lift with initial condition $p \in P$. Such a function τ_t is called the **parallel transport** of the fibres of P.

Remark B.4.2. If $\tilde{\gamma}$ is a lift of the curve γ, it is clear that $R_a\tilde{\gamma}$ is also a lift of γ. Consequently, this fact, along with the uniqueness of the lifts implies that $\tau_t \circ R_a = R_a \circ \tau_t$. Finally, τ_t is a diffeomorphism, whose inverse is simply parallel transport along the reverse portion of γ from $t \to 0$.

B.4.2 Covariant Derivative and Curvature

We first start by remarking that we may define a different kind of exterior derivative which is linked to the Ehresmann connection ω. With a proper definition of what parallel transport means on a principal bundle, it is possible to define the notion of a covariant derivative. Indeed:

Proposition B.4.1. If φ is a tensor r-form of P of type (V, ρ) (*i.e.* $\varphi \in A^r(P, \rho)$) and h is the horizontal projection induced by an Ehresmann connection ω, then:

1. The form φh defined by: $\varphi h(X_1, \cdots, X_n) = \varphi(hX_1, \cdots, hX_n)$ for $X_i \in T_pP$ is a tensor form of type (V, ρ);
2. $d\varphi$ is a tensorial $(r+1)$-form of type (V, ρ);
3. The $(r+1)$-form $d^\nabla \varphi$ defined as $d^\nabla \varphi = (d\varphi)h$ is a tensorial form of type (V, ρ).

Definition B.4.3. We call the $(r+1)$-form of proposition B.4.1 the **exterior covariant derivative** of φ. We call the operator $d^\nabla : \Omega^r(P, \rho) \to \Omega^{r+1}(P, \rho)$ an **exterior covariant differentiation**.

With this new differentiation, it is possible to rewrite the definition of the curvature of an Ehresmann connection in a neat way:

Proposition B.4.2. We may write $\Omega = d^\nabla \omega$.

Remark B.4.3. Note that for this exterior covariant derivative we do **not** necessarily have $(d^\nabla)^2 = 0$.

Definition B.4.4. Let (P, π, M, G) be a principal bundle over M and let ω be an Ehresmann connection on P. Moreover, let E be the associated bundle to a representation (V, ρ) of G (*i.e.* $P \times_G V$) and let Y be a tangent vector field on M and $\tilde{Y}$ be its horizontal lift to P. Then, the **covariant derivative or exterior connection** $D_Y : \Gamma(E) \to \Gamma(E)$ (*cf.* notation A.4.1) associated to this data is defined by the following diagram:

$$\begin{tikzcd} A^0(P,\rho) \arrow[r, "\iota_{\tilde{Y}} d^\nabla"] & A^0(P,\rho) \\ \Gamma(E) \arrow[u, "\psi"] \arrow[r, "D_Y"] & \Gamma(E) \arrow[u, "\psi"] \end{tikzcd}$$

Here, the isomorphism ψ is the one that we explicited in theorem B.2.1 and where the mapping ι is the canonical isomorphism, for V, W vector spaces such that if $w \in W$:

$$\iota : V \otimes W^* \longrightarrow \operatorname{Hom}(W, E) \tag{B.4.29}$$

$$v \otimes \zeta \longmapsto [w \mapsto \zeta(w)v] \tag{B.4.30}$$

and the map ι_w denotes the evaluation at w morphism given by:

$$\iota_w : V \otimes W^* \longrightarrow \operatorname{Hom}(W, E) \tag{B.4.31}$$

$$v \otimes \zeta \longmapsto \zeta(w)v \tag{B.4.32}$$

Although this definition is a functional one, we want to be a bit more explicit if we want to perform some calculations. To this end we have the following proposition:

Proposition B.4.3. When the representation (V, ρ) is effective (*cf.* definition A.1.2), then:

1. Given a fixed basis of V, $\{\bar{e}_i\}_{i\in I}$, there is a canonical interpretation of P as the bundle of frames given by:
$$p \in P \mapsto (e_1(p), \cdots, e_n(p)), \quad \text{where } e_i(p) = [p, \bar{e}_i] \in P_V = P \times_G V; \tag{B.4.33}$$
2. The covariant derivative $D_Y X$ of a section $X \in \Gamma(E)$ may be calculated as follows. Express X in terms of the chosen basis $e_i(p)$, *i.e.* $X = a^i(p)e_i(p)$. Let $\tilde{Y}_p \in T_pP$ be the horizontal lift of Y_x (such that $\pi(p) = x$), then we have that:
$$(D_Y X)_x = \tilde{Y}_p(a^i(p))e_i(p) \tag{B.4.34}$$
where we apply Einstein's summation convention in the equations.

It is instructive to have a careful look at the proof of this proposition, the reader is thus strongly suggested to attempt it and go through the calculation in parallel in order to really understand how to manipulate the notation and the concepts at hand.

Proof. The first point of the proposition is simply due to the identification of the G-principal bundle with its associated bundle. The more interesting point of the proof is in the second part. We now prove the second point.

Let us thus choose a basis on some open neighbourhood of P, such that we have that for each point $p \in P$, we have:
$$p = (e_1(p), \cdots, e_n(p)) \text{ where } e_i(q) = [p, \bar{e}_i] \in P \times_G V \tag{B.4.35}$$
where the $\bar{e}_i$ form a basis of V. Here, we use the same notation as in the proof of theorem B.2.1. With this choice of basis, we may then express the section $X \in \Gamma(E)$ as simply being given by $X = a^i(p)e_i(p)$, where the a_i are C^∞-functions. We then need to evaluate $\psi(X)$. However, as seen in theorem B.2.1, this is nothing other than:
$$\psi(X)(p) := f(p) = a^i(p)\bar{e}_i \tag{B.4.36}$$
The reader is welcome to check that this $f(p)$ is indeed in $A^0(P, V)$ (*i.e.* that $f(p * g) = \rho(g^{-1})f(p)$), but essentially, this is true due to the equivariance condition on the associated bundle.

We now embark in the actual computation of the covariant derivative. For this, we give ourselves a horizontal lift of the vector field $Y \in TM$, that we will denote $\tilde{Y} \in TH$. We recall that for any $x \in M$, this lift to point $p \in \pi^{-1}(x)$ is unique by virtue of remark B.3.5.

Then, according to the definition of the covariant derivative given in B.4.4, we apply the operator $\iota_{\tilde{Y}} d^\nabla$ to this function f and evaluate it at point $p \in P$. This yields the following:
$$(\iota_{\tilde{Y}} d^\nabla f)(p) = (\iota_{\tilde{Y}}(df \circ h))(p) = (df(h\tilde{Y}))(p) \tag{B.4.37}$$
But notice that the lift is a horizontal lift, in particular, this means that $h\tilde{Y} = \tilde{Y}$. Additionally, replacing with the definition of f in equation B.4.36 and recalling the natural identification $T^*P \otimes V \cong \mathrm{Hom}(TP, V)$, we may finally write:
$$(\iota_{\tilde{Y}} d^\nabla f)(p) = df_p(\tilde{Y}_p) = da^i_p(\tilde{Y}_p)\bar{e}_i \tag{B.4.38}$$
Finally, we take this expression back to $\Gamma(E)$ with ψ^{-1}, thus finally obtaining the expression of the covariant derivative $(D_Y X)_x$ at a point $x \in M$ (since the expression does not depend

of the lift p chosen due to equivariance of the expressions and because the representation is effective):

$$(D_Y X)_x = da_p^i(\tilde{Y}_p)[p, \bar{e}_i] = \tilde{Y}_p(a^i)[p, \bar{e}_i] \tag{B.4.39}$$

■

From the proof of this proposition, it is easy to see how one can generalize this to general tensorial forms, furthermore due to the linearity properties of all of the maps above and their behaviour, the proposition yields the following result:

Proposition B.4.4. The operator D is such that:

$$D : \Gamma(E) \to \Gamma(E \otimes T^*M) \text{ defined by } \iota_Y(Df) = (Df)(Y) := D_Y f \tag{B.4.40}$$

where D_Y is the operator defined in definition B.4.4 is a Koszul connection (*cf.* definition A.4.4).

Remark B.4.4. Notice that this amounts to saying that all the linearity conditions, the Leibniz condition as well as the compatibility with the tensor product are immediately obeyed by the covariant derivative we have defined since it is a Koszul connection. One can see this immediately from the proof of proposition B.4.3.

B.4.3 Torsion and its Discontents

At this point, we would like to define the torsion. However, it is in general not possible to define torsion for connections in all principal bundles P. The presence of torsion on a manifold requires extra structure. To be exact, it requires to equip the principal G-bundle with a representation (V, ρ) of same dimension as M and to look at the associated bundle P_V.

It is, however, important to note that one doesn't really notice this in an introductory class to differential geometry simply because the frame bundle $F(M)$ exhibits this extra structure in a canonical way, that is, we may choose the representation of $F(M)$ at any given point $x \in M$ to simply be T_xM, corresponding to the fundamental representation of $GL(n, \mathbb{R})$. In this particular case, looking at the associated bundle amounts to nothing other than considering TM.

Additionally, over any local trivialisation of the principal bundle $U \times G \hookrightarrow P$ we have that we may express any point $p = (x, g)$ for some $x \in U$ and $g \in G$. We may thus also define over this local trivialisation the projection onto the second coordinate, $\pi_2 : (x, g) \mapsto g$. With the above, it is possible to define the following:

Definition B.4.5. Let $p \in F(M)$ and $x \in U$ such that $\pi(p) = x$, we have the existence of a certain $\mathbb{R}^n \cong T_xM$-valued 1-form θ defined by:

$$\theta_p(Y_p) = \rho(\pi_2(p)^{-1})(\pi_* Y_p) \quad \forall p \in F(M) \tag{B.4.41}$$

We call this 1-form the **canonical form** or the **dual form** of the principal bundle $F(M)$.

Remark B.4.5. This definition may seem a little bit untangible at first, so let us unwind it to understand it a bit better. Any choice of section $s = (X_1, \cdots, X_n)$ is nothing other than a moving frame over the local trivialization $U \subset M$. Let furthermore $x \in M$ and $Y_x \in T_xM$. The pullback of the dual form under the section is:

$$s^*\theta_x(Y_x) := \theta_{s(p)}(s_* Y_x) = \rho(\pi_2(s(x))^{-1})(\pi_* s_* Y_x) = \rho(\pi_2(s(x))^{-1})((\pi \circ s)_* Y_x) \tag{B.4.42}$$

However, since s is a section we have the condition $\pi \circ s = \mathrm{id}$, which yields in turn:

$$s^*\theta_x(Y_x) = \rho(g)^{-1}(Y_x) \tag{B.4.43}$$

where this $\rho(g)$ is simply the matrix corresponding to the frame $(X_1(p), \cdots, X_n(p))$, since ρ is the fundamental representation. In other words, this yields nothing other than the coordinates of the vector Y_p in the $\{X_i(p)\}_i$ basis.

Remark B.4.6. It is sometimes more convenient to look at the particular case under a different light. Indeed, some authors consider (somewhat by abuse of notation) the point $p \in P$ itself to be a function $x \mapsto g$, where $g \in GL(n, \mathbb{R})$. In other words, we regard the point p as an element of $GL(n, \mathbb{R})$. Furthermore, because in the case of the frame bundle, it is clear we are dealing with the fundamental representation, we may also write, again by abuse of notation:

$$\theta_p(Y_p) := p^{-1}(\pi_* Y_p) \quad \forall p \in F(M) \tag{B.4.44}$$

It is important to be said that with these identifications, there is not really an ambiguity in the notation, so far as the reader is aware of what is going on. In what is to follow, we will sometimes find it convenient to use this notation for the sake of not complicating further the notation and for the sake of clarity.

Notation B.4.1. In what is to follow, up until definition B.4.7, we will focus on $F(M)$ to see how things behave in this particular case which provides a natural definition of torsion. This is to provide some motivation for the definitions, as well as providing the reader with a more readable approach to the concept before we proceed to generalize. We will thus from here on make the identification of $GL(n, \mathbb{R})$ with its fundamental representation whenever necessary, *i.e.* $g \in GL(n, \mathbb{R}) \sim \rho(g)$. We will also use the identification mentioned in remark B.4.6. Finally, $\cdot$ will denote usual matrix multiplication.

The connection ω on the frame bundle $F(M)$ allows us to define a particular set of vector fields in $F(M)$.

Definition B.4.6. For $\xi \in \mathbb{R}^n$, the **basic vector field corresponding to** ξ, $B(\xi)$, is defined by letting $B(\xi)_p$ be the unique horizontal vector at u such that $\pi_*(B(\xi)_p) = p(\xi)$. In particular, let $\{e_i\}_{i\in I}$ be the standard basis of $\mathbb{R}^n$ and $\{v_i\}_{i\in I}$ is a basis of T_pH, then $B(e_i)_p$ is the unique horizontal vector at p which covers p_i.

We will now state a couple of important results:

Proposition B.4.5. For all $\xi \in \mathbb{R}^n$, we have:

1. $\theta(B(\xi)) = \xi$;
2. $R_{g*}B(\xi) = B(g^{-1} \cdot \xi), \quad \forall g \in GL(n, \mathbb{R})$.

Moreover, if $\xi \neq 0$, then $B(\xi)$ is nowhere zero and if $(\xi_1, \cdots, \xi_n)$ is a basis of $\mathbb{R}^n$, then $(B(\xi_1), \cdots, B(\xi_n))_p$ is a basis for T_pH.

Lemma B.4.1. Consider the basic vector fields determined by a connection on $F(M)$. For all $N \in \mathfrak{gl}(n, R)$ and $\xi \in \mathbb{R}$, we have:

$$[\sigma(N), B(\xi)] = B(N \cdot \xi) \tag{B.4.45}$$

It is possible to introduce a generalization of this canonical form and thus to extend the realm of principal bundles over which we may speak about torsion. We are thus looking for a similar concept, which we want this time to be vector space valued. This vector space is thus required to have the same dimension as M and must have a left action of G on it if it is to exhibit a similar behaviour as the canonical form. We may thus define:

Definition B.4.7. Let (P, π, M) be a principal G-bundle and let (V, ρ) be a representation of dim M of the group G. A **soldering form** on P is a 1-form $\theta \in \Omega^1 P \otimes V$ such that:

1. $\forall X \in \Gamma(TP)$ we have $\theta(v(X)) = 0$;
2. $\forall g \in G, \quad L_g[(R_g)^*\theta] = \theta$;
3. TM and P_V are isomorphic as associated bundles.

A soldering form thus provides an identification of V with each T_xM. It is here useful to consider as an illustration of this concept the case of the canonical form on $F(M)$.

Example B.4.1. *Here, we take $V = \mathbb{R}^{\dim M}$, which is naturally equipped with a left action of $GL(n, \mathbb{R})$. Let $\{p_i\}$ be the basis of vectors of the frame $p \in F(M)$. In particular we may identify the point p with the map $p : \mathbb{R}^{\dim M} \to T_{\pi(p)}M \cong T_pH$, which sends $\xi \in \mathbb{R}^{\dim M} \mapsto \pi_*(B(\xi)_p)$. More explicitly, we may see this as the map taking $(x^1, \cdots, x^{\dim M}) \mapsto x^i p_i$. Notice that for the frame p, it is possible to define a coframe f, by simply taking the dual vectors of the basis $\{p_i\}$. We may then consider a map p^{-1} to simply be the map:*

$$p^{-1} : T_{\pi(p)}M \longrightarrow \mathbb{R}^{\dim M} \tag{B.4.46}$$
$$Y \longmapsto (f^1(Y), \cdots, f^{\dim M}(Y)) \tag{B.4.47}$$

Finally, we define the soldering form $\theta : \Gamma(TF(M)) \to \mathbb{R}^{\dim M}$ to simply be:

$$\theta : X \mapsto p^{-1}(\pi_* X) \tag{B.4.48}$$

Notice then that by virtue of propositions B.4.6 and B.4.5, this definition of θ obeys all the conditions imposed on a soldering form.

Definition B.4.8. We may define the **torsion** form to be the V-valued 2-form obtained by taking the exterior covariant differential on the soldering form. Explicitly:

$$\Theta = d^\nabla \theta \in \Omega^2(P) \otimes V \tag{B.4.49}$$

Proposition B.4.6. The torsion form and the soldering form transform according to the following transformation laws:

$$\forall g \in G \quad R_g^*\theta = \rho(g^{-1})\theta \quad \text{and} \quad R_g^*\Theta = \rho(g^{-1})\Theta \tag{B.4.50}$$

In other words, we have that $\theta \in A^1(P, \rho)$ and $\Theta \in A^2(P, \rho)$.

B.4.4 Structural Equations and Bianchi's Identities

Theorem B.4.2 (Structural equations)**.** *Let ω be an Ehresmann connection over the principal bundle P over M with group G. Then, if P is the bundle of frames, with the dual form θ and torsion form Θ, we have that the **first structural equation** is:*

$$d\theta = -\omega \,\bar{\wedge}\, \theta + \Theta \tag{B.4.51}$$

Where $\bar{\wedge}$ indicates that we let ω act on θ. Now let P be any principal bundle, and Ω be the curvature form of ω. Then we have the ***second structural equation****:*

$$d\omega = -\frac{1}{2}[\omega, \omega] + \Omega \tag{B.4.52}$$

Remark B.4.7. We would've ideally had an identical equation for Θ as we did for Ω. However, since θ and Θ are both V-valued and ω is $\mathfrak{g}$-valued, it is nonsensical to write $[\omega, \theta]$, since $[\,,]$ is defined for $\mathfrak{g}$-valued forms only. The operation $\bar{\wedge}$ is thus a compromise in between, which lets the connection act on θ.

More precisely, if G has a matrix structure, we pick as a basis for V the corresponding expressions of $\mathfrak{g}$ in the representation ρ. Recall that we may do this, because the condition on the soldering form is that its representation space V have same dimension as T_pM. We may thus pick a basis of V, $\{e_i\}_{i\in I}$ and we note $\{e^j\}_{j\in I}$ the dual basis for V^*. In that case, if G is a matrix group and since since V is a G-representation, there is a natural linear action of $\mathfrak{g}$ on V via ρ. This means that we can identify $\mathfrak{g} \subset \mathrm{Hom}(V, V) \cong V^* \otimes V$. We can take a basis for the latter to be $\{e^i \otimes e_j\}_{(i,j)\in I^2}$. Let then ${\omega^i}_j \in T^*P \otimes \mathfrak{g}$ be the components corresponding to the Ehresmann connection, *i.e.* such that:

$$\omega = {(\rho_*\omega)^i}_j \, e^j \otimes e_i \tag{B.4.53}$$

Furthermore, in this basis, it is clear that the expressions of θ and Θ should be:

$$\theta = \theta^i e_i \quad \text{and} \quad \Theta = \Theta^i e_i \tag{B.4.54}$$

This makes it clear what the symbol $\bar{\wedge}$ implies. However, sometimes by abuse of notation we do not write explicitly the presence of ρ, since usually it is clear which representation we are using. This will be illustrated in the following example. At the end of the day, the reader may regard (at least most of the time and for the standard cases) this operation as simply being matrix multiplication under the $\wedge$-product.

Remark B.4.8. These equations are nothing other than integrability conditions imposed on the manifold. A more careful analysis requires the introduction of *foliations* and the fully detailed constructions as well as the obtention of the structural equations from integrability considerations can be found in Sharpe's book [1], which the curious reader is encouraged to consult.

Example B.4.2. *In the case of $F(M)$ we may rewrite both results with respect to the standard basis $\{e_i\}_n$ of $\mathbb{R}^n$, indeed then we have that:*

$$\theta = \theta^i e_i \qquad \Theta = \Theta^i e_i \tag{B.4.55}$$

And similarly if $\{{E^i}_j\}_{(i,j)\in I^2} = \{e^i \otimes e_j\}_{(i,j)\in I^2}$ is the standard basis of $\mathfrak{gl}(n, \mathbb{R})$ in the fundamental representation, then we have that:

$$\omega = {\omega^i}_j {E^j}_i \qquad \Omega = {\Omega^i}_j {E^j}_i \tag{B.4.56}$$

One of the important results in Riemannian geometries are the so-called Bianchi identities. These turn out to also generalize beautifully in the case of principal bundles. We give the generalized version of the Bianchi identities:

Theorem B.4.3 (Bianchi identities)**.** *For a connection ω over the frame bundle we have:*

$$d^\nabla \Theta = \Omega \,\bar{\wedge}\, \theta \qquad \textit{and} \qquad d^\nabla \Omega = 0 \tag{B.4.57}$$

Proof. This is a good exercise for the reader to check if he is confortable with the notation. We give a couple of indications on how to obtain the result. Consider $d^2 = 0$ on the first and second structural equations, from which two identities will follow. Then, consider the exterior covariant derivative on Θ and Ω respectively applied on three arbitrary vector fields in TP and use the identities found previously. In the case of $d^\nabla \Omega$, recall that $\ker \omega = TH$. ■

B.5 Reductions

An important concept in dealing with principal bundles is the one about reductions. In some way, so far we have only considered a principal G-bundle. However, what if we only cared about a particular subgroup of H. Can we build an analogous bundle using the G-structure already present on the principal G-bundle? To answer this question, we have the following definitions and facts:

Definition B.5.1. Let G be a Lie group and $H \subset G$ be a subgroup. We consider (P, G, π, M) a principal G-bundle over M. An H**-reduction** of this bundle is a submanifold $P_0 \subset P$ such that $(P_0, H, \pi\ f|_H, M)$ is an H-bundle and the action of H on P_0 simply be the restriction of the G-action present on P to H.

Remark B.5.1. Actually, one may generalize this concept not only for subgroups but for any mapping $\varphi : H \to G$ which need not be the inclusion. This allows us to treat spin structures, which we will tackle later on.

The following lemma allows us to construct a submanifold given a left G-action on a *general* manifold M. Indeed:

Lemma B.5.1. Let $\mu : G \times M \to M$ be a smooth left action $(g, x) \mapsto g * x$ of a Lie group G on a connected smooth manifold M. Then every orbit $X \subset M$ of this action is a submanifold.

Similarly, the following proposition is also practical in the construction of reductions.

Proposition B.5.1. Suppose that (P, H, π, M) is a smooth principal H-bundle. Let Q be a manifold equipped with a smooth, proper right H-action. Let $f : P \to Q$ be a smooth equivariant map (*i.e.* $f(p * h) = f(p) * h$ for all $h \in H$). Fix $q_0 \in Q$ and set $H_0 = \mathrm{Stab}(q_0)$ (*cf.* definition A.1.3). Suppose furthermore that $q_0 \in f(F)$ for all fibres F of P. Then:

- $P_0 = f^{-1}(q_0)$ is a H_0-reduction of P;

Fix another point $q_1 \in Q$, we may then define an analogous group $H_1 = \mathrm{Stab}(q_1)$ and suppose that $q_1 \in f(F)$ for all fibres F of P. Then we have a P_1, an H_1-reduction of P and furthermore:

- $\exists\, h \in H$ such that $H_1 = \mathrm{Ad}(h^{-1})H_0$ and for any such h, we have $P_1 = P_0 h$.

Note that in particular, we may use different representations in order to build H-reductions of manifolds, since the associated bundles of representations are equipped with natural left actions, lemma B.5.1 allows us to use this to construct the corresponding reduced manifold. In practice, the concept of reduction helps us understand many phenomena we study in differential geometry, however these reductions may not always be possible. To understand why this is, we need a concept from topology:

Definition B.5.2. Let X be a topological space and A be a subspace of X. Then, a continuous (smooth) map $F : X \times [0, 1] \to A$ is a **deformation retraction** of a space X onto a subspace A if $\forall x \in X$ and $\forall a \in A$ we have:

$$F(x, 0) = x, \quad F(x, 1) \in A, \text{ and } F(a, 1) = a \tag{B.5.58}$$

In other words, a deformation retraction is a homotopy between X and A. It may be seen as a special case of a homotopy equivalence.

Remark B.5.2. Here we can see why homotopy groups play a crucial role in the determining topological obstructions to the existence of a reduction of principal bundles.

We illustrate this with some examples:

Example B.5.1. *Consider the frame bundle $F(M)$ over a manifold M. In this case, we have that $G = GL(n, \mathbb{R})$, the following reductions correspond to concepts in differential geometry:*

- *$GL^+(n, \mathbb{R}) < GL(n, \mathbb{R})$ is an **orientation** on M. Note that this is not always possible. In fact, we may only do this when the bundle is orientable;*
- *$SL^{\pm}(n, \mathbb{R}) < GL(n, \mathbb{R})$ is a **pseudo-volume form**. This is always possible to construct.*
- *$SL(n, \mathbb{R}) < GL(n, \mathbb{R})$ is a **volume form**. Indeed, since $SL(n, \mathbb{R})$ is a deformation retraction of $GL^+(n, \mathbb{R})$, this is also only possible if and only if the bundle is orientable;*
- *$O(n, \mathbb{R}) < GL(n, \mathbb{R})$ is a **Riemannian metric**. Indeed, notice here that this is actually always possible, since $O(n, \mathbb{R})$ is the maximally compact subgroup of $GL(n, \mathbb{R})$. Thus every smooth manifold M may be equipped with a Riemannian metric in this fashion;*
- *$O(1, n-1) < GL(n, \mathbb{R})$ is a **Lorentzian metric**. There is a topological obstruction to this. This reduction is possible if and only if the second Stiefel-Whitney class vanishes.*

Appendix C

Klein and Cartan Geometries

C.1 Klein Geometries

We will now use the machinery we developped in order to study the particular case of principal bundles on Lie groups themselves. Indeed, note that we may induce a principal bundle structure on a Lie group G by simply considering a closed subgroup H and taking G/H to be the base manifold and equipped with the natural right action provided by G. While this may seem very abstract and purely a mathematical game, the consequences for spaces to which physicists are normally accustomed to work in come out rather naturally from this formalism without having to do much effort at all. In fact, the study of this precisely was one of the main focus points of the celebrated *Erlangen Programm* developped by Klein in the 1870's [6]. This will not only provide a solid footing for the next chapter, in which we will address a generalization of what is to follow, but will also get the reader accustomed to the notation, while introducing interesting examples of what has been mentioned above.

The motivation behind the original study of Klein geometries was to undercover the underlying symmetries of a given space. This set of symmetries gives naturally rise to a Lie group which one might study. From there, the logical progression is to study the homogeneous spaces of a given Lie group G. That is, a manifold P identical to the group G but without preferred origin. Since this manifold P has the same structure as this G, the group G acts on it transitively and freely. This means that, in some way, Klein geometries are actually the most natural example one could think of for principal bundles. Indeed, the G action in this particular case is tautological by considering group multiplication.

Without further ado, we introduce the following definitions:

Definition C.1.1. A **Klein geometry** is a pair (G, H), where G is a Lie group and $H < G$ is a closed subgroup such that G/H is connected. G is called the **principal group of the geometry** (akin to the terminology of principal bundle). The **kernel** of a Klein geometry G/H is the largest subgroup $K < H$ such that K is normal in G. The geometry is said to be:

- **Effective** if $K = \{1\}$ and **locally effective** if the group K is discrete;
- **Geometrically oriented** if G is connected;
- **Primitive** if the identity component $H_e \subset H$ is maximal among the proper closed connected subgroups of G;
- **Reductive** if there is an Ad_H-module decomposition of $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{p}$, where $\mathfrak{g}$ and $\mathfrak{h}$ are the Lie algebras of G and H respectively.

Finally, the space $M = G/H$ is called **the space of the Klein geometry** or sometimes, by abuse of language, merely the **Klein geometry**.

Remark C.1.1. Strictly speaking, we consider that while the principal bundle P that is associated to a Klein geometry is isomorphic to G, an important distinction is that we consider that there is, in some way, no preferred point of P, we don't have a fixed choice of origin on M corresponding to the coset eH. On the other hand, by abuse of notation, we will often just write G/H for the base of the bundle. It is, however, important that the reader be aware of this subtle distinction. This is also compatible with the original motivation of the study of Klein geometries, which was the study of homogeneous spaces on which a Lie group G is free to act on.

Note that definition C.1.1 gives us right away a structure which is a principal H-bundle. Notice to this bundle, we can attribute a connection in the sense of Ehresmann. However, due to the natural extra G-structure we have on the geometry, a little bit more can be said. To establish the link with Ehresmann and principal bundles, this extra information provided by the G-structure can simply be understood as being given a solder form in addition to the usual principal connection present on the H-bundle, we will later come back to this correspondence when we look at the link between Cartan geometries and Ehresmann connections.

Remark C.1.2. While the study of general Klein geometries is interesting on its own right, we will focus mostly on reductive Klein geometries, which comprise a lot of the spaces that are of interest in physics.

C.1.1 Connection on a Klein Geometry

On Euclidean space, the parallel transport of vectors allows us to define when two vectors at two different points of $\mathbb{E}^n$ are equal. This notion is so fundamental that some authors define equality of vectors over an affine space by saying that two vectors are equal if there exists a translation carrying one to the other. In particular, this notion also allows us to trivialize the tangent bundle over the whole of $\mathbb{E}^n$. Notice, however, that this depends strongly on the existence of a group (namely the group of translations in this case) acting smoothly and transitively on $\mathbb{R}^n$ to be able to translate the vectors from one point to another.

This notion can be transposed in the case of Klein geometries and indeed to *any* Lie group G. Notice, however, that in the case where the group is not abelian, there are in general two ways of doing this, either by inducing a right or a left action. In what will follow, we study the case of the left action, knowing that we can treat the right action case in an analogous manner. The left action induced by multiplication on the left can thus be noted L_g and it induces an action on the tangent spaces (which correspond in this case to copies of the Lie algebra $\mathfrak{g}$) by considering, for example, L_{g*} for any $g \in G$.

We may thus study the case where we want to apply this construction to trivialize the tangent bundle of G. We have then that for $g \in G$, we may identify $\mathfrak{g} = T_gG$ and define a 1-form which is invariant by action on the left. Indeed:

Definition C.1.2. The left-invariant 1-form $\omega_G : T_gG \to \mathfrak{g}$ defined by:

$$\omega_G(v) := (L_{g^{-1}})_*(v), \quad \forall v \in T_gG \tag{C.1.1}$$

is called the left-invariant **Maurer-Cartan form** on G

Remark C.1.3. We can thus see that the construction of this 1-form is, in a way, almost tautological. In fact, the Maurer-Cartan form is actually the father of all left-invariant $\mathbb{R}$-valued forms on G. These may be obtained in full generality by taking the exterior powers

$\Lambda^k\omega_G : \Lambda^k T_g G \to \Lambda^k\mathfrak{g}$ and composing with some linear map $\Lambda^k\mathfrak{g} \to \mathbb{R}$. A particularly neat example of this is simply taking $\Lambda^n\omega_G$ where n is the dimension of the Lie algebra. This yields a volume form, which actually corresponds exactly to the Haar measure on G.

In particular, notice that this notion actually generalizes in a full manner what we meant by parallel transport in the case of $\mathbb{E}^n$. Indeed, we thus may use the Maurer-Cartan form as a way of characterizing a connection over G if we are considering a Cartan geometry. In the case of Klein geometries, it remains to be checked that the Maurer-Cartan form as defined above is actually compatible with the Ehresmann connection induced on the principal bundle $P \cong G$ with base $M \cong G/H$. For this, we have the following proposition:

Proposition C.1.1. Let (G, H) be a Klein geometry, the **Maurer-Cartan form** satisfies:

1. ω_G is a linear isomorphism on each fibre;
2. $R_h^*\omega_G = \mathrm{Ad}(h^{-1})\omega_G$ for all $h \in H$;
3. $\omega_G(\sigma(A)) = A$ for all $A \in \mathfrak{h}$.

Remark C.1.4. This proposition allows us to take this characterization of the Maurer-Cartan form as a definition of it in the case of Klein geometries. We want to adopt this as a definition, since later on when we generalize the concept to the case of Cartan geometries, the conditions of compatibility above will give us a good hint on how to define the Cartan connection.

Remark C.1.5. Note also that point 2 of proposition C.1.1 actually holds for any $g \in G$, however, we restrict the property to H since we are interested in the particular case of a Klein geometry. The reason for this restriction will become clear when we study Cartan geometries

C.1.2 Structural Equation

With respect to the Maurer-Cartan form, all Klein geometries are flat (*i.e.* $\Omega = 0$). This is exactly what the following theorem tells us.

Theorem C.1.1 (Structural Equation for Lie Groups)**.** *The structural equation for the Klein geometry (G, H) is given by:*

$$d\omega_G + \frac{1}{2}[\omega_G, \omega_G] = 0. \tag{C.1.2}$$

Remark C.1.6. Although it is easy to think about this formula as a mere application of the exterior derivative to the Maurer-Cartan form, it is important to understand that its meaning runs far deeper than that. Indeed, this equation is a fundamental defining structure of the group itself. For example, in the case where $d\omega_G = 0$, this implies the Lie group is abelian. Furthermore, it provides us with a local characterization of the Lie group itself. Indeed, the structural equation should really be understood as an integrability condition that we can then use in order to define the exponential map of the Lie algebra to provide a local description of the group itself.

To be more precise about the last point of the last remark, we have the following theorem:

Theorem C.1.2. *Let N be a connected smooth manifold and let $\mathfrak{g}$ be a Lie algebra. Let ω be a $\mathfrak{g}$-valued 1-form on N satisfying the following conditions:*

1. $d\omega + \frac{1}{2}[\omega, \omega] = 0$*;*
2. $\omega : TN \to \mathfrak{g}$ *is an isomorphism on each fibre;*

3. ω is complete.

Then:

1. *The universal cover N, $\pi : G \to N$, has, for an arbitrary choice $e \in G$, the structure of a Lie group with identity element e and algebra $\mathfrak{g}$ whose Maurer-Cartan form is $\pi^*\omega$;*
2. *The period group $\Gamma = \Phi_\omega(\pi_1(N, b)) \subset G$ acts by left multiplication on G as the group of covering transformations for the cover $\pi : G \to M$.*

Remark C.1.7. About this theorem, it is important to remark that the last point is there just for the sake of completeness, as we did not introduce what the period group was. The takeaway from this point is that we may identify the Lie group with the three conditions above alone *up to some covering*. It is also noteworthy to state the fact that the Maurer-Cartan form on a Lie group G is complete.

Remark C.1.8. Dropping condition 3 in theorem C.1.2 leaves us with a manifold M which is locally a Lie group. Similarly, dropping condition 1 in theorem C.1.2 yields a "deformation" of a Lie group. For example consider $\omega_t := \omega + t\eta$ such that $t \in [0, 1]$ and ω satisfies conditions 1, 2 and 3. For all t, we choose the form η such that ω_t still satisfies point 2. On the other hand, the form may still be able to satisfy 3, but it will in general not be able to satisfy 1 any longer, in this case, we no longer get a Lie group even locally. This remark will be of relevance when we consider the generalization of these concepts to Cartan geometries.

C.1.3 Homogeneous Spaces and Metric Klein Geometries

Theorem C.1.2 has in particular shown a sort of correspondence between the Lie algebra $\mathfrak{g}$ and the group G. In particular, if we go back to considering Klein geometries, it is interesting to note that due to the structural equation form and the properties of the Maurer-Cartan form, this motivates the definition of what the Klein geometry looks like locally in the light of theorem C.1.2. In the case of the Klein geometry (G, H) we can in particular associate the pair $(\mathfrak{g}, \mathfrak{h})$, the discussion aboves thus motivates us to define the following:

Definition C.1.3. An **infinitesimal Klein geometry** or a **Klein pair** $(\mathfrak{g}, \mathfrak{h})$ is a pair of Lie algebras such that $\mathfrak{h}$ is a subalgebra of $\mathfrak{g}$. The **kernel** $\mathfrak{k}$ of $(\mathfrak{g}, \mathfrak{h})$ is the largest ideal of $\mathfrak{g}$ contained in $\mathfrak{h}$. If $\mathfrak{k} = 0$, we say the pair is **effective**. Furthermore, if there is an $\mathfrak{h}$-module decomposition such that $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{p}$, we say $(\mathfrak{g}, \mathfrak{h})$ is **reductive**.

With this definition, we are tempted to find a way to characterize the tangent bundle of the Klein geometry given its Klein pair. Let us thus consider (U, ψ) a bundle chart for the principal H-bundle over G/H. This means in particular that we have a diffeomorphism $\psi : U \times H \to \pi^{-1}(U)$. It is clear, then, that the following diagram commutes:

$$\begin{tikzcd}
T(\pi^{-1}(U)) \arrow[dr, "\pi_*"'] & T(U \times H) \arrow[l, hook', "\psi_*"'] \arrow[r, "\pi_{U*} \times \pi_{H*}"] \arrow[d, "\pi_{U*}"] & TU \times TH \arrow[dl, "\mathrm{proj}_1"] \\
& TU &
\end{tikzcd}$$

This diagram, implies the following one. Let $g \in G$ and $\pi(g) = x$. Then the following diagram also commutes by virtue of exactness of its rows:

$$\begin{tikzcd}
T_g(gH) \arrow[r, hook] \arrow[d, "\omega_H"] & T_gG \arrow[r, two heads] \arrow[d, "\omega_G"] & T_x(G/H) \arrow[d, "\varphi_g"] \\
\mathfrak{h} \arrow[r, hook] & \mathfrak{g} \arrow[r, two heads] & \mathfrak{g}/\mathfrak{h}
\end{tikzcd}$$

The φ_g is simply the unique morphism making the diagram commute. We can thus identify the tangent space $T_x(G/H)$ with $\mathfrak{g}/\mathfrak{h}$ pointwise. However, notice that the morphism φ_g depends of the choice of the g over x. On the other hand, since $\pi R_h = \pi$ and the fact that ω_H is nothing other than the Ehresmann connection for the H-bundle (so $R^*\omega_H = \mathrm{Ad}(h^{-1})\omega_H$), we have that $\varphi_{gh} = \mathrm{Ad}(h^{-1})\varphi_g$. Recognizing this, we have the following proposition:

Proposition C.1.2. *$T(G/H) \cong G \times_H \mathfrak{g}/\mathfrak{h}$ as vector bundles over G/H.*

With this identification, we are finally ready to look at what all of this is good for. Indeed, while the way started introducing Klein geometries as a natural generalization of $\mathbb{R}^n$ gave us a taste for this, the power of Klein geometries is perhaps best appreciated through the help of examples of applications.

By considering the isometry groups of every homogeneous space (or maximally Killing space) it is possible to recover the structure of the Klein geometry in an obvious manner. The following diagram illustrates some of the most relevant examples to physics:

$$\begin{tikzcd} & SO(3,2) \arrow[dl, "\Sigma"'] \arrow[dr, "\mathrm{AdS}_4"] & \\ SO(3,1) \arrow[dr, "\mathcal{H}^3"] & & SO(3,1) \arrow[dl, "\mathcal{H}^3"] \\ & SO(3) & \end{tikzcd}$$

$$\begin{tikzcd} & ISO(n,1) \arrow[dl, "\Sigma"'] \arrow[dr, "\mathcal{M}^{n,1}"] & \\ ISO(n) \arrow[dr, "\mathbb{E}^n"] & & SO(n,1) \arrow[dl, "\mathcal{H}^n"] \\ & SO(n) \arrow[d, "\mathbb{S}^{n-1}"] & \\ & SO(n-1) & \end{tikzcd}$$

$$\begin{tikzcd} & SO(4,1) \arrow[dl, "\Sigma"'] \arrow[dr, "\mathrm{dS}_4"] & \\ SO(4) \arrow[dr, "\mathbb{S}^3"] & & SO(3,1) \arrow[dl, "\mathcal{H}^4"] \\ & SO(3) & \end{tikzcd}$$

We have here displayed three examples of groups in physics, notably anti-de Sitter, Minkowski and de Sitter space in the above. Here, the label above each arrow $G \xrightarrow{G/H} H$ is to be understood as being the base space corresponding to the Klein geometry G/H. Notice that this diagram is not necessarily transitive. The arrows are there to indicate the reductions in the groups that take place, they should be interpreted as meaning "H is a subgroup of G". Furthermore, here the Σ stands for the space of space-like hyperplanes.

Notice that for each of the diagrams, if we look at the branches on the left, from top to bottom, we have that we identified the space of space-like hyperplanes and then we identified the position space of the system. Similarly, for the branches on the right from top to bottom, we identify the "event space" and then the "velocity space". However, because all of these concepts can be expressed in terms of groups and of their quotients, their geometry is fully described by the theory of Klein geometries, which renders the above, in some way, trivial.

At this stage, it is noteworthy that because almost all of the Lie groups considered by physicists are semisimple (an important exception to this is the Poincaré group), in particular their geometries are also reductive. This has important consequences as noted above. Indeed, we may consider the particular case of de Sitter space, whose isometry group is simply $SO(4,1)$. In this case, since the Lie algebra $\mathfrak{so}(n,1)$ is semisimple, obtain a metric by simply considering the Killing form on it. This metric is given explicitly by:

$$\langle \psi | \phi \rangle = -\frac{1}{2}\mathrm{tr}(\psi\phi) \tag{C.1.3}$$

Which is proportional to the Killing form. This metric restricts in a proper manner to the induces space dS_n and thus we get the metric on the homogeneous space for free, but we have even more than that as we will see in the following section. The power of Klein geometries lies in the fact that it treats all homogeneous spaces on the same grounding. In so doing, it is easy to see how looking at different spaces simply amounts to understanding the Lie groups and algebras of the different isometry groups at hand. This method allows for powerful ways of obtaining results. In particular, for example, we know already what the Haar measure for all of these spaces just because we know what the Maurer-Cartan form is.

Perhaps more importantly, we know how we can bring everything back to the homogeneous space in which we are interested in given a choice of a section over the bundle. This will be later exposed when we talk about gauges in section C.1.5.

C.1.4 Tetrads and Connections

After the development of general relativity, Einstein looked for a way to include the phenomenon of spin into his theory. However, he was forced to do so in a rather ugly manner. This formalism is known as the "vierbein" or "tetrad" formalism (for higher dimensions, the term "vielbein" is used) and is applied relatively often in general relativity nowadays. These so-called tetrads appear for free in the context of Klein geometries (and in fact in full generality, they appear in Cartan geometries as well as we will se in the next section). We will illustrate how this happens precisely in the case of Klein geometries.

Recall that Klein geometries are equipped with a connection, the so-called Maurer-Cartan form ω_G. This connection induces, at least locally, an isomorphism $TG \xrightarrow{\sim} \mathfrak{g}$. However, since we are dealing with a principal H-bundle, there is an Ehresmann connection on it as well, that we will call ω. Furthermore, the base structure is simply given by (or at least is non-canonically isomorphic to) G/H. For now, we will assume (as we will see later without loss of much generality) that G is a simply connected group. We may summarize all this information by looking at the following commutative diagram:

$$\begin{tikzcd} & \mathfrak{h} & \\ TG \arrow[ur, "\omega"] \arrow[r, "\omega_G"] \arrow[dr, "e"'] & \mathfrak{g} \arrow[u, "\mathrm{proj}"'] \arrow[r, "\exp"] \arrow[d, "\pi_*"] & G \arrow[d, "\pi"] \\ & \mathfrak{g}/\mathfrak{h} \arrow[r, "\exp"] & G/H \end{tikzcd}$$

Notice then that actually, since ω_G is an isomorphism, we have that $\omega = \mathrm{proj}^*\omega_G$, thus recovering the Ehresmann connection as simply being the pullback of the Maurer-Cartan form of the Klein geometry, but we have additional structure. Indeed, this (suggestively named) map $e : TG \to \mathfrak{g}/\mathfrak{h} \cong T(G/H)$ is the so-called tetrad we encounter in relativity. It is nothing other than the pullback of the Maurer-Cartan form under the map π. It is perhaps also important to point out explicitly (and this turns out to be the case even in the case of Cartan geometries) that this is equivalent to choosing an appropriate soldering form on the principal H-bundle G.

Notice that this corresponds exactly to the physics view of somehow attaching a Minkowski space to every point, which seemed a somewhat *ad hoc* thing to do *a priori*. On the other hand, now we are really able to appreciate why this had to be the case by having this extra mathematical structure to guide us. The famous tetrad stemmed from nothing other than the fact that we are now considering the Klein pair $(\mathfrak{iso}(3,1), \mathfrak{so}(3,1))$. The space attached thus corresponds exactly to a Minkowski space, but we can see that this came naturally simply out of symmetry groups considerations and nothing else. The fact that it is impossible to describe spin without this extra structure implies that spin, in a way, is a consequence of the existence of these symmetry groups on the space we live in. More accurately (but getting ahead of ourselves), we can say that (curved) space is really nothing other but a "lumpy" version of the homogenous space described by the $(ISO(3,1), SO(3,1))$.

C.1.5 Gauges and Klein Geometries

Figure C.1: A gauge, in the proper sense of the word

In physics, we often deal with so-called "gauges", people rarely bother to define precisely what is meant by this terminology, which was coined by Hermann Weyl in 1918 [5]. To get an intuitive idea of what it is, consider an actual gauge as depicted in figure C.1 (Weyl's original term was referring to the gauge of a railway track and referred to a scale factor, however, due to a happy accident of language we may consider an actual gauge). It is constituted of two main parts:

1. A part which is fixed, but arbitrary, in the case of the gauge this simply constitutes the markings on the gauge. A choice of such markings is called a choice of gauge;

2. A moving part, which is characterized by the moving needle, which characterizes what we are actually interested in measuring.

To understand why this picture is relevant at all, let us go back to considering the pilot in his airplane. As outside observers, we might want to describe the motion of this pilot. Indeed in this case, it is obvious that the moving part is the pilot. However, it is necessary to compare the movement of the pilot with respect to some arbitary choice of movements for the observer himself. This thus amounts to choosing a particular section. This gives us a hint of what we should define and exactly how.

We make the ideas above more precise. Let us start by assuming that we have an open set U over the manifold $M = G/H$ such that there exists a global section s over U. Once one of these sections exists, it is trivial to see that many others do as well, simply by acting with elements of H on the global section s. However, if we are dealing with a principal bundle, there will be a smooth section relating the choice of gauges by virtue of the free and transitive action of H on the sections.

Definition C.1.4. We will call a choice of a map $s : U \to H$ a **choice of gauge**. This can be regarded as a choice of motions that vary in a smooth manner on a principal bundle P.

Remark C.1.9. Note that this definition is general to all principal bundles P, it is just terminology. In particular, we will employ it in the next section as well.

Remark C.1.10. It is important to stress that the choice of a gauge bears **no** geometrical meaning whatsoever.

Gauge Picture of Trivializations

So far, we have worked over a neighbourhood U to define the sections that define the choice of gauges. However, on such a chart (U, ψ), we have the trivialization $\psi : U \times H \to \pi^{-1}(U)$. In fact, due to the fact that this diffeomorphism ψ is right equivariant, specifying the particular trivialization is completely equivalent to choosing a section over U. Indeed, the trivialization ψ corresponds to the section s defined by $s(x) = \psi(x, e)$, $\forall x \in U$ and $e \in G$ is the identity element. Due to right equivariance, it holds then that $\psi(x, h) = s(x)h$.

We might thus be curious as to how exactly this behaves under change of coordinates. Let s_1 and s_2 be two sections over U, then there is a smooth map $k : U \to H$ linking both gauges by : $s_2(x) = s_1(x)k(x)$ for $x \in U$. Thus, if ψ_1 and ψ_2 are the trivializations corresponding to the sections s_1 and s_2, we have that:

$$\psi_2^{-1}\psi_1(x, h) = (u, k(x)^{-1}h) \tag{C.1.4}$$

These functions k are important enough to give them a name. For this we have the following definition:

Definition C.1.5. The functions $k : U \to H$ which map one choice of section, s_1, to another one, s_2, are called the **transition functions** or **gauge transformations**.

Remark C.1.11. Going back to our previous problem on tracking the airplane, we notice that we could approach the problem by describing the motion of one of its points, given by the curve $q(t) \in M$ for $t \in [0, 1]$. We may use the gauge s to track the motion of the plane. Notice here that if the internal configurations of the plane (in other words, its orientation) along its path are given by $g(t)$, then both $s(q(t))$ and $g(t)$ lie in the fibre over $q(t)$. They both must thus differ by an element $k(t)$. The latter simply describes how the plane turns around point $q(t)$. It is very important to once again emphasize that there is no intrinsic meaning to $k(t)$ or the choice of gauge s alone. However, combined, they describe the motion of the plane.

Remark C.1.12. From what we have seen, we notice that it is enough to give the functions k in order to be able to reconstruct the principal bundle G.

Maurer-Cartan Form Trivialization

One might be interested in seeing how the Maurer-Cartan form behaves under a local trivialization. The following proposition gives us this explicitly:

Proposition C.1.3. Over a local trivialization (U, ψ), the following diagram commutes, where:

$$\varphi(v, y) = \mathrm{Ad}(h)^{-1}\omega_G(v) + \omega_H(y) \tag{C.1.5}$$

$$\begin{tikzcd}
T_{(x,h)}(U \times H) \arrow[d, "\pi_{U*} \times \pi_{H*}"'] \arrow[dr, "\psi^*\omega_G"] & \\
T_xU \times T_hH \arrow[r, "\varphi"'] & \mathfrak{g}
\end{tikzcd}$$

Infinitesimal Gauges

We might be interested to see what happens to ω_G when we choose a particular gauge s. We can actually use the section $s : U \to G$ to pull back the Maurer-Cartan form. We define the 1-form $\eta := s^*\omega_G$. Then the structural equation pulls back and yields:

$$d\eta + \frac{1}{2}[\eta, \eta] = 0 \tag{C.1.6}$$

We note that not only does s determine η, but the opposite is almost true. Indeed, it is true by a generalized version of the fundamental theorem of calculus that η determines s up to multiplication of a fixed element of G. For this reason, we may refer to both s and η as gauges on M.

Definition C.1.6. To make the proper distinction between a gauge s corresponding to section over a local trivialization and its associated form η, we will call this η an **infinitesimal gauge**.

The advantage of these infinitesimal gauges is that they do not depend on the base point. The geometrical interpretation of this η is that to every tangent vector $v \in T_xM$, we assign an infinitesimal motion "$\mathrm{id} + \varepsilon\eta(v) \in G$" of M whose effect on x itself is to move it to $x + \varepsilon v$.

By varying the gauge s (or equivalently, varying the trivialization ψ), we change the infinitesimal gauge as well. In particular it is interesting to see how this infinitesimal gauge transforms. Let thus s_1, s_2 be two sections over $U \subset M$ and consider the associated gauge transformation from one to the other given by $k : U \to H$. Then by virtue of proposition C.1.3 the following is true:

$$\eta_2 = \mathrm{Ad}(k^{-1})\eta_1 + k^*\omega_H \tag{C.1.7}$$

Definition C.1.7. We call such a variation of the infinitesimal gauge θ an **infinitesimal change of gauge** and the two infinitesimal gauges are said to be **infinitesimally gauge equivalent**. Sometimes, we drop the "infinitesimal" in front of the terms hereby defined.

Throughout this chapter, we have been hinting at how we could generalize the above concepts in the case of the (now infamous) Cartan geometries. We will explore this beautiful generalization of Klein's ideas which was done by Cartan in the 1920's in an effort to reformulate the theory of gravitation under more mathematically sound concepts in the next section.

C.2 Cartan Geometries

Cartan generalized the ideas that Klein proposed during the celebrated Erlangen Programm of the 1870's to further formalize differential geometry [6]. In thus doing he introduced the idea of Cartan geometries, which generalize Klein geometries in the same sense that Riemannian geometry generalizes Euclidean geometry. Later on, a student of Cartan, Charles Ehresmann introduced the idea of the Ehresmann connection [7] we saw in the previous chapter in order to set Cartan's ideas under a more rigorous footing. However, there is some dissatisfaction among mathematicians as far this definition of a connection is concerned.

There are two main reasons for this. First, in some way, the Ehresmann connection is so general that it includes perhaps much more than what is actually interesting [8]. This has perhaps already been remarked by the reader during the previous section on Klein geometries, in which we naturally saw aspects of physics naturally arise in the particular case of homogeneous spaces. Second, although Ehresmann's formulation is certainly a lot more general, it sweeps under the rug the similarity that connections have to the Maurer-Cartan form on a Lie group.

This similarity is at the base of the intuition that helped Cartan generalize Klein geometries to a bigger class of geometries. This extra generality of the Ehresmann connection can thus have as a consequence to be an unnecessary obstacle to one's understanding of the subject the first time one learns it.

On the other hand, Ehresmann connections appear nonetheless as important components of the Cartan connection we will introduce in this chapter as well as being the rigorous foundation of covariant differentiation, so both concepts are definitely intertwined and decoupling them can be difficult, if not impossible. In some way, as Sharpe [1] so accurately put it, "the relation between Ehresmann connections and Cartan connections is somewhat analogous to the one between rings of algebraic integers and arbitrary commutative rings in that each degree of generality illuminates each other."

C.2.1 General Definitions

As we have previously discussed in the last section, Cartan geometries generalize the idea of Klein geometries. It follows that the definition is akin to looking at geometries that are "infinitesimally Klein", but globally different. We make this notion precise in the following definition:

Definition C.2.1. A **Cartan geometry**, $\xi = (P, \omega)$, on M modelled on $(\mathfrak{g}, \mathfrak{h})$ with group H is the given of:

1. A smooth manifold M
2. A principal right H-bundle P over M
3. A $\mathfrak{g}$-valued 1-form ω on P satisfying the following conditions:
   (a) For each $p \in P$, the linear map $\omega_p : T_pP \to \mathfrak{g}$ is an isomorphism;
   (b) $(R_h)^*\omega = \mathrm{Ad}(h^{-1})\omega, \quad \forall h \in H$;
   (c) $\omega(\sigma(B)) = B, \quad \forall B \in \mathfrak{h}$.

We may sometimes refer to this construction by abuse of notation as a Cartan geometry M. Additionally, motivated by our previous discussions on Klein geometries we define the following:

1. The **curvature form** is the $\mathfrak{g}$-valued 2-form given by $\Omega = d\omega + \frac{1}{2}[\omega, \omega]$.
2. Furthermore, if $\mathrm{proj} : \mathfrak{g} \to \mathfrak{g}/\mathfrak{h}$ is the canonical projection, then $\mathrm{proj}(\Omega)$ is called the **torsion form**.
3. If Ω takes values on $\mathfrak{h}$ only, we say that the geometry is **torsion free**.
4. The geometry is said to be **complete** if all the vector fields X such that $\omega(X)$ is a constant are such that all the integral curves of the vector field are complete (*i.e.* are defined over an infinite interval $(-\infty, \infty)$).
5. The geometry is **effective, primitive or reductive** respectively if the model Klein geometry (G, H) is effective, primite or reductive.

Remark C.2.1. The above conditions on the connection ω have a straightforward interpretation. We may perceive a Cartan connection as simply "rolling" a Klein geometry on a smooth manifold M. This view is compatible to the point of view given in remark C.1.8. A good illustration of this is given in figure C.2.

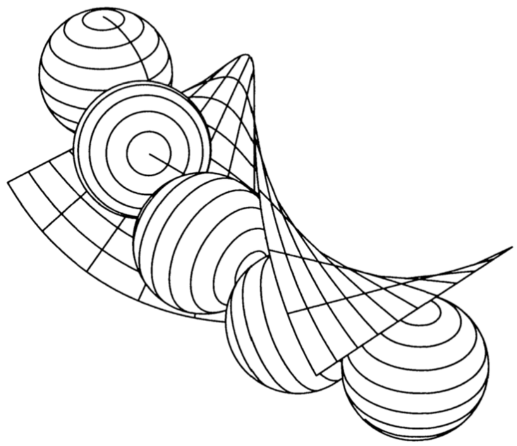


Figure C.2: A sphere rolling along the central line of a helicoid without slipping or twisting. This is a good picture for one's understanding of a Cartan geometry. In this case, the Klein geometry we are rolling along the manifold is simply $(SO(3), SO(2))$, which amounts to roll a sphere on the manifold.

1. That the connection is an isomorphism at each tangent space basically imposes a condition of no twisting and no slipping of the Klein geometry. This is simply because the configuration state of the so-called internal space (which corresponds to the Klein geometry) is uniquely determined by parallel transport, because of the presence of this isomorphism. Thus, no "twisting" or "slipping" of the geometry can occur;

2. $(R_h)^*\omega = \mathrm{Ad}(h^{-1})\omega, \quad \forall h \in H$. This condition is one of equivariance. Intuitively, it may be understood as simply the fact that the configurations of the internal space have no significance or importance;

3. $\omega(\sigma(B)) = B, \quad \forall B \in \mathfrak{h}$. This condition is actually equivalent to saying that the connection restricts to the Maurer-Cartan form on the vertical spaces. This is in some way simply stating that the Klein geometry structure does not vary as we roll it around the manifold M.

Remark C.2.2. In defining the Cartan geometry, we were very careful to define the curvature to go along with it. Indeed, this was Cartan's *coup* to generalizing the ideas of Klein. One should really understand this curvature form to characterize how much "lumpiness" the geometry exhibits.

Next as is customary, we want a way of comparing constructions. We thus introduce the notion of morphism for Cartan geometries.

Definition C.2.2. Let (P_1, ω_1) and (P_2, ω_2) be two Cartan geometries on M_1 and M_2 respectively modelled on the pair $(\mathfrak{g}, \mathfrak{h})$ with group H. Let $f : M_1 \to M_2$ be an immersion covered by the H-bundle map $\tilde{f} : P_1 \to P_2$ with the property that $\tilde{f}^*\omega_2 = \omega_1$. Then f is called a **local isomorphism of geometries**, or a **local geometric isomorphism**. Furthermore, if f is a diffeomorphism, then it is called an **isomorphism of geometries**, or a **geometric isomorphism**.

We notice that this definition is very similar to the context in which we built the theory of principal bundles. In fact, the remarkable difference is that now the connection form is

$\mathfrak{g}$-valued instead of $\mathfrak{h}$-valued. We have the following results that ensure that most of the results we previously found still hold in the case of Cartan geometries. In fact, as we will later see, this is little to no surprise as one can retrieve a Cartan geometry from the right principal bundle equiped with the right Ehresmann connection. Because of the similarity between Klein geometries and Cartan geometries, we quickly give some facts about Cartan geometries in the following lemma.

Lemma C.2.1. Let (P, ω) be a Cartan geometry on M modelled on $(\mathfrak{g}, \mathfrak{h})$ with group H. The following results are true:

1. Assume $\psi : P \to H$ is a smooth map. Define $f : P \to P$ by taking $f(p) = R_{\psi(p)}p$. Then, $f^*\Omega = \mathrm{Ad}(\psi(p))\Omega$;

2. The curvature form Ω may be regarded as a 2-form on π^*TM;

3. Let V be a vector subspace of the Lie algebra $\mathfrak{h}$. $\omega^{-1}(V)$ is an integrable distribution of P if and only if V is a subalgebra of $\mathfrak{h}$;

4. If ξ is torsion free and $V \subset \mathfrak{g}$ such that $\mathfrak{h} \in V$, then $\omega^{-1}(V)$ is an integrable distribution of P if and only if V is a subalgebra of $\mathfrak{g}$.

C.2.2 Gauges in Cartan Geometries

The concept of gauge seen in the case of Klein geometries can be naturally extended to Cartan geometries, the idea of this section is to see exactly how these notions generalize. In particular, we may consider what tensors, as defined in B.1.2 look like in a particular gauge. In the end, the notion of choosing a gauge is essential if we wish to express things in terms of indices and coordinates, which is often the case in physics. For this, we have the following definitions and facts:

Definition C.2.3. Let ξ be a Cartan geometry. A **Cartan gauge** with this model on a smooth manifold M is a pair (U, η) where $U \in M$ is an open set and η is a $\mathfrak{g}$-valued 1-form satisfying a regularity condition that:

$$\bar{\eta} : T_xU \xrightarrow{\eta} \mathfrak{g} \xrightarrow{\text{proj}} \mathfrak{g}/\mathfrak{h}$$

is a linear isomorphism for each $x \in U$ (one usually assumes that U is a coordinate neighbourhood of M but this is strictly speaking not necessary).

Definition C.2.4. Let (U, η) be a gauge of a Cartan geometry (P, ω) corresponding to a section $s : U \to P$ (*i.e.* $s^*\omega = \eta$). If $f : P \to V$ is a tensor of type (V, ρ), then $\phi = s^*f = f \circ s : U \to V$ is the **expression of the tensor in the gauge** (U, η).

The following lemma justifies our definition of a tensor and reconciles it with the usual notion of a tensor in physics, which is an object which (very) loosely speaking "transforms in a certain way".

Lemma C.2.2. Let $\phi : U \to V$ be the expression of a tensor f of type (V, ρ) in a Cartan gauge (U, η_1). We fix another gauge η_2, where $k : U \to H$ is the corresponding change of gauge between η_1 and η_2 (as prescribed in equation C.1.7). Then $\rho(k^{-1})\phi : U \to V$ is the expression of the same tensor in the new gauge η_2.

C.2.3 Tangent Bundle of a Cartan Geometry

The result of proposition C.1.2 we uncovered in the case of Klein geometries still holds in the case of Cartan geometries. This means in particular that just as in the Klein case, the tangent bundle can be expressed as a vector bundle associated to the principal H-bundle via the representation $\mathrm{Ad}_{\mathfrak{g}/\mathfrak{h}} : H \to \mathrm{End}(\mathfrak{g}/\mathfrak{h})$.

Theorem C.2.3. *Let (P, ω) be a Cartan geometry, then there is a canonical isomorphism $TM \cong P \times_H \mathfrak{g}/\mathfrak{h}$. Moreover, for each $p \in P$ such that $\pi(p) = x$, there is a canonical isomorphism $\varphi_p : T_xM \to \mathfrak{g}/\mathfrak{h}$ such that $\varphi_{ph} = Ad(h^{-1})\varphi_p$.*

This theorem has an important consequence we will use later on when we look at what connections and in particular what the covariant derivative looks like in the case of Cartan geometries.

Corollary 3. *Let (P, ω) be a Cartan geometry. The vector fields X on M are in bijective correspondence with the functions $f : P \to \mathfrak{g}/\mathfrak{h}$ transforming according to the adjoint representation (i.e. $f(ph) = Ad_{\mathfrak{g}/\mathfrak{h}}(h^{-1})f(p)$). The correspondence is given explicitly by:*

$$X \mapsto [f_X : p \mapsto \varphi_p(X_{\pi(p)})] \tag{C.2.8}$$

Along the same motivation to explore the representation theory of H later on, we introduce some terminology used in the literature. To give some motivation for the following definition, it is enough to consider any of the examples of applications. For instance, we may be interested in looking at wavefunctions (which may, for example, be part of a gauge theory). However, these objects take an element from the base manifold $x \in M$ and give us a vector valued quantity. We thus want a way of formally describe these spaces which can naturally be associated to the principal bundle in order to later make use of the machinery of representation theory. Having said this, we define the following:

Definition C.2.5. Let (P, ω) be a Cartan geometry. A vector bundle E with base P is called a **geometric vector bundle** if it is given the form $E = P \times_H V$ for some representation $\rho : H \to GL(V)$.

C.2.4 Bianchi identity

Taking into consideration lemma B.3.1 as well as its derived corollary 2, the Bianchi identity on a Cartan geometry is simply a formal consequence of those to facts, indeed we have:

Theorem C.2.4 (Bianchi identity)**.** $d\Omega = [\Omega, \omega]$.

Proof. The reader is welcome to check this is nothing other than a consequence of $d^2 = 0$, $\Omega = d\omega + \frac{1}{2}[\omega, \omega]$ and using the lemma and corollary above. ∎

Remark C.2.3. Actually, by continuing to take exterior derivatives of the definition of curvature, one obtains a chain of identities by successive differentiation. The next identity after Bianchi is simply given by $[\Omega, \Omega] + [[\Omega, \omega], \omega] = \frac{1}{2}[\Omega, [\omega, \omega]]$. The reader is welcome to check this as an exercise.

The reader might notice that in the case of Cartan geometries, we do not get two Bianchi identities as we did in the case of the Ehresmann connection, but simply one. In applications, we retrieve the expressions for the two separate Bianchi identities simply by virtue of representation theory.

C.2.5 Curvature Function

While the form Ω provides us with a two form, there is a function that also gives us a notion of curvature over the principal bundle P. This notion will become useful once we examine Riemannian geometry as an application of Cartan geometries. Having motivated its introduction, we have the following definition:

Definition C.2.6. Let Ω be the curvature form on the principal bundle P. The curvature function $K : P \to \mathrm{Hom}(\Lambda^2(\mathfrak{g}/\mathfrak{h}), \mathfrak{g})$ at a point $p? \in P$ is defined by:

$$K(p)(T_1, T_2) = \Omega_p(\omega^{-1}(T_1), \omega^{-1}(T_2)) \tag{C.2.9}$$

Lemma C.2.5. The curvature function is well-defined and satisfies the invariance property:

$$K(ph)(T_1, T_2) = \mathrm{Ad}(h^{-1})(K(p)(\mathrm{Ad}(h)T_1, \mathrm{Ad}(h)T_2)) \tag{C.2.10}$$

The following proposition gives us the geometrical interpretation of this curvature function K

Proposition C.2.1. The curvature function measures the difference between the Lie algebra bracket and the bracket of corresponding vector fields on P. More precisely we have that:

$$K(p)(T, R) = [T, R] - \omega_p([\omega^{-1}(T), \omega^{-1}(R)]) \tag{C.2.11}$$

Notice in particular that this torsion form gives us a nice characterization of torsion free Cartan geometries. Indeed, the following is true:

Proposition C.2.2. The Cartan geometry is torsion free if and only if the curvature function K takes values in the subrepresentation $\mathrm{Hom}(\Lambda^2(\mathfrak{g}/\mathfrak{h}), \mathfrak{h}) \subset \mathrm{Hom}(\Lambda^2(\mathfrak{g}/\mathfrak{h}), \mathfrak{g})$.

C.2.6 Universal Covariant Derivative

The notion of covariant derivative has been useful to us in the past. It is thus little to no surprise that we want to introduce a similar construction in the case of Cartan geometries. The reader might note that there is some similarity between the concept of a covariant derivative in the sense of Ehresmann as we saw in the previous chapter and the definition of the covariant derivative in the case of Cartan geometries. However, these are not strictly speaking the same notion, since in the case of a Cartan geometry, the principal bundle P has in a way "knowledge" of a bigger algebra $\mathfrak{g}$. It will thus be no surprise that we use the Cartan connection in the definition of the universal covariant derivative we will introduce in what will follow.

In order to make this introduction as smooth as possible, we shall introduce some natural isomorphisms which will help the reader understand exactly what is meant behind the notation employed.

Lemma C.2.6. Let (V, ρ) be a representation of the Lie algebra $\mathfrak{g}$ and let (P, ω) be a Cartan geometry modelled on $(\mathfrak{g}, \mathfrak{h})$. There is a natural isomorphism: $\varphi : A^k(P, V) \cong A^0(P, V \otimes \Lambda^k \mathfrak{g}^*)$ given by:

$$\varphi(\beta)(\xi_1, \cdots, \xi_k) = \beta(\omega^{-1}(\xi_1), \cdots, \omega^{-1}(\xi_k)) \tag{C.2.12}$$

Where $\beta \in A^k(P, V)$ and $\xi_j \in \mathfrak{g}$.

We note that this isomorphism is indeed equivariant due to the equivariance of ω. Furthermore, we know that there is the presence of the exterior derivative $d : A^k(P, V) \to A^{k+1}(P, V)$. If we take $k = 0$ in the natural isomorphism of lemma C.2.6 and compose with the exterior derivative we get a map:

Definition C.2.7. The **universal covariant derivative** is the map given by:

$$\tilde{D} := \varphi \circ d : A^0(P,V) \to A^0(P, V \otimes \mathfrak{g}^*) \tag{C.2.13}$$

Explicitly, for a function $f? \in A^0(P,V)$ and an $T \in \mathfrak{g}$, we have that:

$$\tilde{D}_T f = df(\omega^{-1}(T)) =: \omega^{-1}(T)f \tag{C.2.14}$$

Notice that $\tilde{D} = \iota_{T*}\tilde{D}$, where the natural isomorphism ι is the same as the one in definition B.4.4.

In fact, the canonical isomorphism given in lemma C.2.6 gives us a hint that there is yet another canonical isomorphism under the sheets. Indeed we have:

Proposition C.2.3. Let (V,ρ) be a representation of a Lie algebra $\mathfrak{g}$ corresponding to the maximal algebra of the set on which the Cartan geometry (P,ω) is modelled. There is a canonical isomorphism $\phi : A^k(P,\rho) \cong A^0(P, \rho \otimes \Lambda^k \mathrm{Ad}^*)$ (by abuse of notation we note the respective vector spaces by their associated representation homomorphisms).

Proof. Recall the natural isomorphism shown in lemma C.2.6. Further recall that there is a natural isomorphism $V^* \otimes W \cong \mathrm{Hom}(V,W)$ for any two vector spaces V and W (provided that either V or W are finite dimensional). It is thus sufficient to show how the morphism in lemma C.2.6 behaves under right action of an element $h \in H$. Consider thus $p \in P$, $\alpha \in A^k(P,\rho)$ and $\xi_i \in \mathfrak{g}$, then:

$$\begin{aligned} R_h^*\varphi(\alpha)(p)(\xi_1,\cdots,\xi_k) &= \varphi(\alpha)(ph)(\xi_1,\cdots,\xi_k) = \alpha_{ph}(\omega_{ph}^{-1}(\xi_1),\cdots,\omega_{ph}^{-1}(\xi_k)) &\text{(C.2.15)}\\ &= \underbrace{\rho(h^{-1})\alpha_p}_{\because \alpha\in A^k(P,\rho)}\underbrace{(\mathrm{Ad}(h)\omega_p^{-1}(\xi_1),\cdots,\mathrm{Ad}(h)\omega^{-1}(\xi_k))}_{\therefore \in \Lambda^k \mathrm{Ad}} &\text{(C.2.16)}\end{aligned}$$

It is thus clear that $\varphi(\alpha) \in A^0(P, \rho \otimes \Lambda^k \mathrm{Ad}^*)$ and the isomorphism condition holds due to lemma C.2.6. ∎

Remark C.2.4. For the case $k = 2$, this isomorphism yields exactly $\phi(\Omega) = K$, where K is the curvature function we had defined in section C.2.5.

Lemma C.2.7. For $B \in \mathfrak{h}$ and $f \in A^0(P,\rho)$, we have $\iota_{B*}(\tilde{D}f) = -\rho_*(B)f$, where ρ_* is the derivative at the identity of the representation $\rho : H \to GL(V)$.

Lemma C.2.8. $\tilde{D} : A^0(P,\rho) \to A^0(P, \rho \otimes \mathrm{Ad}^*)$.

Remark C.2.5. This last lemma is actually of non-trivial importance for the motivation that we hereby explain. Indeed, although the representation (V,ρ) might be irreducible, this need not be the case of the representation $(V \otimes \mathfrak{g}^*, \rho \otimes \mathrm{Ad}^*)$. Thus, it may be that the latter decomposes as a direct sum of irreducible representations yielding the decomposition $V \otimes \mathfrak{g}^* = W_1 \oplus \cdots \oplus W_r$. In that case, the breaking up of this representation will induce the break up of the operator $\tilde{D}$, the universal covariant derivative, as: $\tilde{D} = \tilde{D}_1 + \cdots + \tilde{D}_r$, where each of the $\tilde{D}_i$'s are projections of $\tilde{D}$ onto each subspace W_i.

This natural decomposition due to representation theory yields in turn all the important derivation operators we are usually interested in. What this all means is that thanks to the abstract machinery we have developed until now and a little bit of representation theory, we can naturally obtain the natural differential operators to treat our problem with.

Remark C.2.6. Because of the natural identification between $A^0(P,\rho) \cong A^0(M, P \times_H (V,\rho))$ given in theorem B.2.1, the universal covariant derivative may also be interpreted as a linear first-order differential operator $\tilde{\tilde{D}} = \tilde{D} : A^0(M, P \times_H (V,\rho)) \to A^0(M, (V \otimes \mathfrak{g}^*, \rho \otimes \mathrm{Ad}^*))$ (*cf.* lemma C.2.8) where we write the first equality by abuse of notation, since strictly speaking these are not the same operators. Note that this operator does not give the derivative of a section of $P \times_H (V,\rho)$ with respect to the direction of some tangent vector on M, but rather with respect to the direction of some vector on P. In other words, it is not enough to give the to give two nearby points of M to describe how a vector changes; we must instead give two *frames* of M.

Furthermore, because of remark C.2.5, if the representation $V \otimes \mathfrak{g}^*$ decomposes as a sum of irreducible components, then the bundle $F := P \times_H (V \otimes \mathfrak{g}^*, \rho \otimes \mathrm{Ad}^*)$ decomposes as a sum $F = F_1 \oplus \cdots \oplus F_r$, where the $F_i = P \times_H W_i$ and the induced $\tilde{D}_i$'s can be regarded as first-order linear differential operators on each F_i such that $\tilde{D}_i : A^k(M, W_i) \to A^k(M, F_i)$.

C.2.7 Covariant Derivative in the Reductive Case

The existence of the covariant derivative we all know and love (the one that generalizes the directional derivative in Euclidean space) actually requires that the Cartan geometry at hand be *reductive*. In other words, reductive geometries have, in this way, a much richer structure than non-reductive geometries. We recall what it means for a Cartan geometry to be *reductive*:

Definition C.2.8. A Cartan geometry modelled on $(\mathfrak{g}, \mathfrak{h})$ with group H is **reductive** if there is an H-module decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{p}$.

Remark C.2.7. Notice that the term *reductive* in the sense in which we are now employing clashes somewhat to the term employed in the classical theory of Lie algebras, which calls a Lie algebra reductive if it is the sum of a semi-simple and abelian ideal. This is not exactly what is meant in this definition, however, the two are closely related [9].

Because of this natural decomposition, we may express any form as one having a component in $\mathfrak{h}$ and a component in $\mathfrak{p}$ and by virtue of the logic exposed in remarks C.2.5 and C.2.6, we have that the covariant derivative will split into two different pieces as well, namely $\tilde{D}_{\mathfrak{h}}$ and $\tilde{D}_{\mathfrak{p}}$. Notice, however, that thanks to lemma C.2.7, we know how $\tilde{D}_{\mathfrak{h}} = -\rho$. We can see that the behaviour in the vertical vectors is not very interesting and is, in some way, trivial. On the other hand, we have that the behaviour on the horizontal vectors carries all the vital information, but this is nothing other than $\tilde{D}_{\mathfrak{p}}$. In some way, we have come full circle back to a definition similar to the one of the covariant derivative akin to the one that we gave in the case of the Ehresmann principal bundle, or at the very least, we will use the same ideas to construct the usual covariant derivative we know and love.

Remark C.2.8. Although the idea is the same, the case of Ehresmann is much more general, since the existence of the covariant derivative does not require any extra conditions, whereas in the case of Cartan geometries, we can only define the construction in the particular case of the geometry being reductive.

Just as before we define things on the bundle and then using theorem B.2.1, we define what it means to derive a section of the associated bundle.

Definition C.2.9. In a reductive geometry, the operator $\tilde{D}_{\mathfrak{p}}$ is called **(the bundle version of) the covariant derivative**. A function $f : P \to V$ is called **covariant constant** or **parallel** if $\tilde{D}_{\mathfrak{p}} f = 0$. If (V,ρ) is a representation of H and $P \times_H V$ is its associated bundle, for $X \in \Gamma(TM)$ we may define **(base version of) the covariant derivative** $D_X : \Gamma(E) \to \Gamma(E)$

by taking the composition of the bundle version of the covariant derivative with the ψ morphism of theorem B.2.1, akin to what we did in the case of Ehresmann connections (*cf.* definition B.4.4).

Remark C.2.9. This covariant derivative is also a Koszul connection (*cf.* definition A.4.4), since it is nothing but a particular case of what we saw in the Ehresmann exterior connection.

Now that we have a definition for the covariant derivative, we wish to be able to calculate it. In particular, this means understanding how the covariant derivative behaves under a particular choice of gauge. For this, we have the following proposition:

Proposition C.2.4. Let (U, θ) be a gauge for a reductive Cartan geometry, $X \in TU$ and ϕ be the expression of a tensor of type (V, ρ) in the gauge (U, θ). Then:

$$D_X \phi = d\phi(X) - \rho_*(\theta_{\mathfrak{h}}(X))\psi \tag{C.2.17}$$

where $\theta_{\mathfrak{h}}$ is the projection onto $\mathfrak{h}$ of the gauge and ψ is the isomorphism of theorem B.2.1.

Appendix D

Bundle View of Wave-functions

Another application of what we have seen so far is a correct mathematical description of non-relativistic wave-functions. However, if we want to treat this problem, it is necessary to have a perhaps pedantic approach to the topic.

D.1 Naive Approach and its Problems

Normally, physicists tend to define a wave-function as simply being a function $\psi : \mathbb{R}^n \to \mathbb{C}$ obeying some regularity condition, which most of the time is taken to be that the function be square integrable. This then leads to conclude that the wave functions $\psi \in L^2(\mathbb{R}^n, \mathbb{C})$ where we equip L^2 with its canonical inner product, namely:

$$\langle \psi | \phi \rangle := \int_{\mathbb{R}^n} \mathrm{d}\mu \; \overline{\psi} \phi \tag{D.1.1}$$

where $d\mu$ is the Lebesgue measure on $\mathbb{R}^n$. This choice is backed by physical arguments, which state that we should only consider normalized wave packets.

The particular choice of L^2 may seem somewhat arbitrary but can be motivated, among others by the fact that L^2 has a complete structure, which is nice if you are studying quantum mechanics in the usual sense, where the notion of a basis for the wave-function space comes over and over in different problems. Physicist thus like to think of these wave-functions as living in an abstract Hilbert space, which is most of the time –misleadingly and incorrectly– identified with L^2. To see why this identification is incorrect, we need to consider some of the structure that the space of these wave-functions actually require in order for normal quantum mechanical concepts to be consistent and then notice that L^2 doesn't really comply with all of these requirements.

Let us start by looking at the identification in a little bit more detail. Quantum Mechanics deals with operators, which act on these wave-functions, or more specifically which act on the Hilbert space (which we will for now suppose is L^2). It is then customary to define some particular self-adjoint operators with respect to the canonical inner product defined above. These operators are:

$$P_i : L^2(\mathbb{R}^n, \mathbb{C}) \to L^2(\mathbb{R}^n, \mathbb{C}) \quad \text{and} \quad Q^i : L^2(\mathbb{R}^n, \mathbb{C}) \to L^2(\mathbb{R}^n, \mathbb{C}) \tag{D.1.2}$$

The self-adjoint condition is simply that $(Q^i)^\dagger = Q^i$ and $P_i^\dagger = P_i$. Furthermore, we impose on these operators the following so-called **canonical commutation relations**:

$$[Q^i, P_j] = i\delta^i_j\,, \quad [Q^i, Q^j] = [P_i, P_j] = 0. \tag{D.1.3}$$

The standard approach is then to identify these operators as being defined by (in Cartesian coordinates): $(Q^i\psi)(\mathbf{x}) = x^i\psi(\mathbf{x})$ and $(P_i\psi)(\mathbf{x}) = (-i\partial_i\psi)(\mathbf{x})$. This is motivated by the analysis of the representation theory of the Heisenberg Lie Algebra constituted by the operators Q^i and P_j.

D.1.1 Target Problems

Already at this stage, there are some problems that come in. Indeed, by defining these operators on our space, we have already contradicted the fact that we are dealing with functions in $L^2(\mathbb{R}^n, \mathbb{C})$. This is because we would like our test functions ψ to be closed under operator transformations, but on the other hand, clearly $x^i\psi$ is not necessarily in $L^2(\mathbb{R}^n, \mathbb{C})$ since its norm may no longer be a finite quantity and $\partial_i\psi$ is clearly also not necessarily in $L^2(\mathbb{R}^n, \mathbb{C})$, since ψ need not be differentiable almost everywhere in order to be square integrable.

While these are most of the time considered to be mathematical pedantries by physicists, it is important to consider these facts with care if one is to elaborate an appropriate point of view of things. These functional analysis problems of domains and targets may be solved by considering the *Gelfand triple*. To introduce this notion, we need to take a brief detour to functional analysis. For a more in depth discussion about some of the facts hereby stipulated, the reader is welcome to consult section A.5 for some basic facts on functional analysis.

Definition D.1.1. A **rigged Hilbert space** is a pair $(\mathcal{H}, \Phi)$, with $\mathcal{H}$ a Hilbert space and Φ a dense subspace of $\mathcal{H}$ such that Φ is given a topological vector space structure for which the inclusion map $i : \Phi \to \mathcal{H}$ is continuous.

Recall from the Riesz representation theorem (*cf.* theorem A.5.4) that we may identify the Hilbert space with its continuous dual, *i.e.* $\mathcal{H} \cong \mathcal{H}^*$. We may thus define a map:

$$i^* : \mathcal{H} \xrightarrow{\Xi} \mathcal{H}^* \longrightarrow \Phi^*$$

This duality pairing is compatible with the inner product on $\mathcal{H}$, meaning that the inner product on Φ is nothing other than the restriction of the inner product in $\mathcal{H}$. Notice that we have $\mathcal{H} \subset \Phi^*$.

Remark D.1.1. Despite the fact that $\Phi \cong \Phi^*$, if Φ is a Hilbert subspace, then the isomorphism is **not** the same as the map given by:

$$i^*i : \Phi \hookrightarrow \mathcal{H} \xrightarrow{\sim} \mathcal{H}^* \hookrightarrow \Phi^*$$

Definition D.1.2. The triple $(\Phi, \mathcal{H}, \Phi^*)$ is called a **Gelfand triple**.

Thus if we take for example Φ to be the space of Schwartz test functions (for which the corresponding $\mathcal{H} = L^2$), then we may properly define the commutation relations as well as the self-adjointedness of the operators P_j and Q^i on these test functions without having target problems. We can simply then lift all the operators defined on Φ to Φ^*, thus solving our target problems. In this case, we say that the space Φ^* is called the space of **tempered distributions**.

Although these target issues are a problem in within themselves, the issue here concerns, in some way, more functional analysis than it does geometry. Having pointed out how it is possible to deal with these target problems, we will not focus on them in the rest of our discussion in order not to obscure the discussion too much. However, we remark that it is possible to regard both issues simultaneously.

D.1.2 Coordinate Dependence

In order to see where things go further wrong, we consider a typical calculation performed in any introductory Quantum Mechanics course. We may, for example, check that the definition of these operators indeed satisfies the commutation relations and that they are indeed self-adjoint with respect to the canonical inner product in L^2.

We look at the particular case of the self-adjointedness of the operator P_j.

$$\begin{aligned} \langle\phi|P_j\psi\rangle &= \int_{\mathbb{R}^n} \mathrm{d}^n\mathbf{x}\ \overline{\phi(\mathbf{x})}(-i\partial_j\psi(\mathbf{x})) = \underbrace{\int_{\mathbb{R}^n} \mathrm{d}^n\mathbf{x}\ (i\partial_j\overline{\phi(\mathbf{x})})\psi(\mathbf{x})}_{\text{Integration by parts}} \\ &= \int_{\mathbb{R}^n} \mathrm{d}^n\mathbf{x}\ \overline{(-i\partial_j\phi(\mathbf{x}))}\psi(\mathbf{x}) = \langle P_j\phi|\psi\rangle. \end{aligned} \tag{D.1.4}$$

$$\tag{D.1.5}$$

We thus retrieve the usual result obtained classically. However, this is very explicitly coordinate dependent. To show this, we perform the same calculation in polar coordinates (let us take the particular case $n = 2$, for the sake of argument. Then we have that:

$$\begin{aligned} \langle\phi|P_j\psi\rangle &= \int_{\mathbb{R}^2} \mathrm{d}r\,\mathrm{d}\theta\ r\overline{\phi(r,\theta)}(-i\partial_j\psi(r,\theta)) \\ &= \underbrace{\int_{\mathbb{R}^2} \mathrm{d}r\,\mathrm{d}\theta\ \overline{[-i\partial_j(r\phi)]}\psi}_{\text{Integration by parts}} \end{aligned} \tag{D.1.6}$$

We can clearly see at this point where the problem is. Indeed, in the case where $j = r$, the operator P_r is no longer self-adjoint ! We complete the calculation in order to get a result that we will later use when we provide a fix for the problem:

$$\langle\phi|P_j\psi\rangle = \begin{cases} \langle P_j\phi|\psi\rangle & \text{for } j = \theta; \\ \langle P_j\phi|\psi\rangle + i\int_{\mathbb{R}^2} \mathrm{d}r\,\mathrm{d}\theta\ \overline{\phi}\psi & \text{for } j = r. \end{cases} \tag{D.1.7}$$

Many times, people then thus claim that the canonical commutation relations only hold for Cartesian coordinates. This cop-out explanation, however, is not only wrong but also completely unphysical. After all, why should Cartesian coordinates play a particular role in physics at all?

Now that we have seen a manifestation of the problem, we understand that the key has to lie within differential geometry to fix it, since we wish to formulate the theory in a coordinate independent way. In the next section, we will see how to fix this problem and how principal bundles naturally enter the game.

D.2 Bundle Approach and Solutions

Since we reached a contradiction, we have to ask ourselves where everything went wrong. Actually, it turns out that what is wrong with the physicist's picture is simply the fact that we have regarded the wave-function as a *function* from the base manifold (in this case $\mathbb{R}^n$) onto the complex numbers. Let us instead consider $\mathbb{R}^n$ as the base manifold of a vector fibre bundle whose typical fibre will simply be $\mathbb{C}$. A wave-function under this light amounts to nothing other than a section over this fibre bundle. We then recover the picture of the wave function being a function, but only in a local sense (more accurately, under a local trivialization of the vector bundle), since the bundle need not necessarily be trivial. In what will follow, we shall thus consider a total space E with base $M = \mathbb{R}^n$ and typical fibre $\mathbb{C}$.

We can then see this vector bundle as simply being the associated bundle of a principal bundle. However, at this point, it is not exactly clear which is the specific principal bundle which we should attribute to this construction. On the other hand, we know what we want: invariance under change of coordinates. This strongly hints at the fact that we should be looking at the principal bundle prescribed by the frame bundle. Indeed, this is simply because coordinate transformations simply amounts to changing frames over the manifold. Change coordinates on the base manifold are thus induced by the changes of basis in the frames. Thus, we consider E as being the associated bundle to the frame bundle $F(M)$.

Why does this help us? Well, we know that if we have the frame bundle, we can establish an Ehresmann connection on this bundle. Then, we can define a covariant derivative on *sections* of any associated bundle, as we saw in chapter B. Indeed, recall that by theorem B.2.1, we can regard such sections as equivariant functions from the frame bundle $F(M)$ onto the fibre $\mathbb{C}$. So now we are looking at a wave-function ψ not as a section of the associated bundle, but rather as a $\mathbb{C}$-valued function, Ψ, not from M, but from the full $F(M)$. It turns out this subtlety will be crucial to solving the problem. We can then look at this covariant derivative on the sections as simply being the exterior covariant derivative defined on $F(M)$ (*cf.* definition B.4.4).

$$d^{\nabla}\Psi = d\Psi + \varpi \mathbin{\bar{\wedge}} \Psi = d\Psi + \varpi\Psi \tag{D.2.8}$$

where the last equality holds due to the fact that Ψ is nothing other than a function so the wedge operation is trivial and where ϖ is the Ehresmann connection on the principal bundle $F(M)$. Furthermore, recall that in the product $\omega\Psi$, we actually have the ω act on Ψ from the left, as we saw in remark B.4.7.

However, notice that now, when we push this all back down with a choice of section over P, this will naturally include some correction factors to the original simple derivatives we had considered in the case where we looked at simple functions alone. Furthermore, we can now see that the inclusion of the extra term is imperative in order to obtain a geometric quantity.

It is helpful at this point to state how these abstract geometric constructions get pulled back down to the base. Indeed, let us consider a choice of section s over a local chart $U \subset M$. The function $\Psi : F(M) \to \mathbb{C}$ can then be pulled back to U via this section. Let us then write $\psi = s^*\Psi$ for the pullback of Ψ via this section. The Ehresmann connection can also be pulled back down to U via the section s. We can thus write the expression of this pullback as $\omega = s^*\varpi$. Finally, the exterior covariant derivative of Ψ can also be pulled back down via the section simply by considering $s^*(d^{\nabla}\Psi)$ and we will write it as $\nabla\psi = s^*(d^{\nabla}\Psi)$. In particular it is helpful to consider what the expression of this ∇ looks like:

$$s^*(d^{\nabla}\Psi) = s^*(d\Psi + \varpi \mathbin{\bar{\wedge}} \Psi) = d(s^*\Psi) + s^*(\varpi) \mathbin{\bar{\wedge}} s^*(\Psi) = d\psi + \omega \mathbin{\bar{\wedge}} \psi \tag{D.2.9}$$

For some local coordinates $\{x^\alpha\}_{\alpha\in I}$, this is nothing other than: $\nabla_\alpha = \partial_\alpha + \omega_\alpha$, which we will call from here on a covariant derivative over the open set U. With these concepts properly defined, we are now ready to tackle the redefinition of the momentum operator.

D.2.1 Redefining the Momentum Operator

Because of the discussion above, we see that we want in some way define the momentum operator in terms of this exterior covariant derivative we had defined back in definition B.4.4. On the other hand, this requires the knowledge of the principal connection, for which we have *a priori* no knowledge. To build the actual expression of it, it will be necessary to inspire us from the nitty-gritty details of the calculations we performed on the self-adjointedness of the operator P_α in order to find a compatible definition for the local expression of the connection pulled back under a certain section.

With the above in mind, we can redefine the momentum operator in an arbitrary coordinate system $\{x^\alpha\}_{\alpha\in I}$ defined over $\mathbb{R}^n$, whose Jacobian is given by J to be:

$$P_\alpha = -i\nabla_\alpha = -i(\partial_\alpha + \omega_\alpha) \tag{D.2.10}$$

To give us an idea of what the expression of this ω_α should be, we can repeat the calculation for the self-adjointedness of P_α in this setting. We thus obtain that:

$$\begin{aligned}
\langle \phi \,|\, P_\alpha \psi\rangle &= \int_{\mathbb{R}^n} \mathrm{d}\mu\; \overline{\phi} P_\alpha \psi = \int_{\mathbb{R}^n} \mathrm{d}\mu\; \overline{\phi}(-i\nabla_\alpha\psi) \\
&= \int_{\mathbb{R}^n} \mathrm{d}^n\mathbf{x}\;\; (\det J)\; \overline{\phi}[-i(\partial_\alpha + \omega_\alpha)]\psi \\
&= \int_{\mathbb{R}^n} \mathrm{d}^n\mathbf{x}\;\; (\det J)\; \overline{\phi}(-i\partial_\alpha\psi) + \int_{\mathbb{R}^n} \mathrm{d}^n\mathbf{x}\;\; (\det J)\; \overline{\phi}\omega_\alpha\psi \\
&= \langle P_\alpha\phi \,|\, \psi\rangle - i\int_{\mathbb{R}^n} \mathrm{d}^n\mathbf{x}\;\; \det J\; \overline{\omega_\alpha\phi}\; \psi - i\int_{\mathbb{R}^n} \mathrm{d}^n\mathbf{x}\;\; (\det J)\, \omega_\alpha\, \overline{\phi}\psi \\
&\quad + i\int_{\mathbb{R}^n} \mathrm{d}^n\mathbf{x}\;\; \partial_\alpha(\det J)\; \overline{\phi}\psi \\
&= \langle P_\alpha\phi \,|\, \psi\rangle - i\int_{\mathbb{R}^n} \mathrm{d}^n\mathbf{x}\;\; \det J\; \underbrace{\left[\omega_\alpha + \overline{\omega}_\alpha - \frac{\partial_\alpha(\det J)}{\det J}\right]}_{\text{We want this to vanish}} \overline{\phi}\psi
\end{aligned} \tag{D.2.11}$$

And so, by setting the last bracket to zero, we obtain a condition on our ω_α, we have that:

$$\omega_\alpha + \overline{\omega}_\alpha = 2\Re(\omega_\alpha) = \partial_\alpha(\ln\det J) \tag{D.2.12}$$

Thus as we have seen, with an appropriate choice of these ω's, we can get rid of the problem of the coordinate dependence of the self-adjointedness.

Remark D.2.1. Notice that the specification of the real part of a holomorphic function determines the holomorphic function up to a purely imaginary constant that we may take to be zero and so this condition on the real part of the ω_α actually determine them up to a purely imaginary constant.

D.2.2 Generalization to Curved Spaces

We have considered the case of $\mathbb{R}^n$ and so far have had some success in generalizing it. However, now that we had the idea of looking at wave functions as simply being sections of an associated bundle to the frame bundle, we may generalize this construction to indeed any Riemannian manifold M. For this, we have to give ourselves a (Riemannian) metric on M, but on the other hand this is always possible to do as we saw in section B.5 when we talked about reductions and topological obstructions to such constructions.

Giving us this, we then consider the associated bundle $E = F(M)\times_{GL_n(\mathbb{R})}\mathbb{C}$ whose sections we may consider to be the our wave functions, just as before. The metric comes in when defining the inner product we need to define. Indeed, now we will have that the integration in $\mathbb{R}^n$ that previously took place with respect to the Lebesgue measure (which recall is actually the Haar measure for $\mathbb{R}^n$, which itself simply stems from the group structure of $\mathbb{R}^n$ by looking at the volume form induced by the Maurer-Cartan form on $\mathbb{R}^n$, *cf.* section C.1.3), we integrate with respect to the volume form naturally induced by the metric g on M. If we wish to make an analogy with the case of $\mathbb{R}^n$ and for some covering charts $\{(U_i, x_i^\alpha)\}_{i\in I}$:

$$\int_{\mathbb{R}^n} \mathrm{d}\mu \longmapsto \sum_{i\in I} \int_U \sqrt{\det g}\, dx_i^1 \wedge \cdots \wedge dx_i^n =: \int_M \mathrm{d}^n\mathbf{x} \sqrt{\det g} \tag{D.2.13}$$

At this point, it is worth remarking that what we have done in the calculation D.2.11 has not really used the fact that we are on $\mathbb{R}^n$, it is possible to use this calculation as a stencil for an analogous calculation with the above change in measure and situation. In the end, the result is simply that the expression of the connection term in the covariant derivative in this case is simply given by:

$$2\Re(\omega_\alpha) = \partial_\alpha(\ln \sqrt{\det g}) \tag{D.2.14}$$

D.2.3 Conservation of Commutation Relations

At this point one might worry that this definition will ruin the commutation relations we imposed on the operators. In principle, we have to check all possible combinations for the commutation relations. We will hereby illustrate only two cases:

Proposition D.2.1. The commutation relations $[Q^\alpha, P_\beta] = i\delta^\alpha_{beta}$ and $[P\alpha, P_\beta] = 0$ still hold. The rest of the commutation relations are left to the reader to check as an exercise (although it is straight forward to see that there will be no problems).

Proof. Let us thus consider, then, a smooth section $\psi \in \Gamma(E)$. Recall also that the action of ∇_α on any $f \in C^\infty(M)$ is simply given by the normal derivative on the function, since all differentiation notions coincide on the scalar functions. Then, we have the following:

$$\begin{aligned}
[Q^\alpha, P_\beta]\psi \quad &= q^\alpha(-i\nabla_\beta\psi) + i\nabla_\beta(q^\alpha\psi) \\
&= \cancel{q^\alpha(-i\nabla_\beta\psi)} + i(\nabla_\beta q^\alpha)\psi + \cancel{i(\nabla_\beta\psi)q^\alpha} \\
&= i(\nabla_\beta q^\alpha)\psi = i \underbrace{(\partial_\beta q^\alpha)}_{\because q^\alpha \in C^\infty(M)} \psi = i\delta^\alpha_\beta\psi
\end{aligned} \tag{D.2.15}$$

The key of this calculation yielding the right result was really to use the momentum operator as a covariant derivative. As we have seen in section B.4.2, the action of the covariant derivative is fully prescribed by the representation theory of different objects. In particular it is this dependence which allows us to state that the covariant derivative acts differently on scalar functions than on sections of the bundle. For scalar functions, since they transform with the trivial representation, the effect of the covariant derivative is simply reduced to the action of a normal derivative on the object as was shown in proposition B.4.3. This is, in fact, what played the key role in the calculation above as pointed out in the underbrace.

Another interesting commutation relation that we could consider is the momentum one. That is: $[P_\alpha, P_\beta] = 0$. This still holds in the most general case. Indeed, we then simply have the following:

$$[P_\alpha, P_\beta]\psi = (-i)^2[\nabla_\alpha, \nabla_\beta]\psi = -(\nabla_\alpha\nabla_\beta - \nabla_\beta\nabla_\alpha)\psi \tag{D.2.16}$$

We examine the case of $\nabla_\alpha\nabla_\beta\psi$ and then consider that because of the symmetry in the indices, we need only swap $\alpha \to \beta$ in order to obtain the result for $\nabla_\beta \nabla_\alpha$. Thus:

$$\begin{aligned}
\nabla_\alpha \nabla_\beta \quad &= \nabla_\alpha(\partial_\beta \psi) + \nabla_\alpha(\omega_\beta \psi) \\
&= (\partial_\alpha + \omega_\alpha)(\partial_\beta \psi) + \underbrace{(\nabla_\alpha \omega_\beta)}_{\partial_\alpha \omega_\beta} \psi + \omega_\beta \nabla_\alpha \psi \\
&= \partial_\alpha \partial_\beta \psi + \omega_\alpha \partial_\beta \psi + (\partial_\alpha \omega_\beta)\psi + \omega_\beta \partial_\alpha \psi + \omega_\beta \omega_\alpha \psi
\end{aligned} \tag{D.2.17}$$

We may thus summarize this by looking at the expression of both $\nabla_\alpha \nabla_\beta \psi$ and $\nabla_\beta \nabla_\alpha \psi$ in parallel:

$$\begin{aligned}
\nabla_\alpha \nabla_\beta \psi \quad &= \cancel{\partial_\alpha \partial_\beta \psi} + \cancel{\omega_\alpha \partial_\beta \psi} + (\partial_\alpha \omega_\beta)\psi + \cancel{\omega_\beta \partial_\alpha \psi} + \cancel{\omega_\beta \omega_\alpha \psi} \\
\nabla_\beta \nabla_\alpha \psi \quad &= \cancel{\partial_\beta \partial_\alpha \psi} + \cancel{\omega_\beta \partial_\alpha \psi} + (\partial_\beta \omega_\alpha)\psi + \cancel{\omega_\alpha \partial_\beta \psi} + \cancel{\omega_\alpha \omega_\beta \psi}
\end{aligned} \tag{D.2.18}$$

We thus have that the only term surviving is:

$$[\nabla_\alpha, \nabla_\beta]\, \psi = (\partial_\alpha \omega_\beta - \partial_\beta \omega_\alpha)\psi \tag{D.2.19}$$

which the naïve reader might be tempted at this point to relate to curvature or something similar because of the symbolic similarity of this expression to the one of the electromagnetic tensor. However, these terms also vanish. Indeed, consider the real part of ω_α, $\Re(\omega_\alpha)$. If we can show the relation is zero for the real part, then necessarily, it must be also zero for the imaginary part as well. Thus, we focus on the definition of $\Re(\omega_\alpha)$ given by equation D.2.14 and it becomes clear that:

$$\Re\left[\partial_\alpha \omega_\beta - \partial_\beta \omega_\alpha\right] = \frac{1}{4}\left[\partial_\alpha \partial_\beta (\ln \det g) - \partial_\beta \partial_\alpha (\ln \det g)\right] = 0 \tag{D.2.20}$$

Thus finally showing that $[P_\alpha, P_\beta] = 0$ as desired. ■

And so, we see that only by considering this construction of associated bundles and principal bundles over manifolds can we really get the preservation of the commutation relation *and* the coordinate invariance to marry each other. This mathematical subtlety might be considered by some as irrelevant or simply pedantic. However, it is a powerful tool that allows us to do quantum mechanical, non-relativisitic, calculations on any manifold and not just in flat space. It is actually relevant to note than in order to make our picture of quantum mechanics work with curved space, we had to go so far back as to completely giving up the notion that the wave function is a function, instead, it should be regarded instead as simply being a section over an associated bundle as we have hereby shown.

Remark D.2.2. A proper axiomatic treatment of quantum mechanics would *a priori* be necessary to properly treat everything consistently. In this case, we would have to specify exactly what we mean *mathematically* by a "state" of a system, a "measurement" and so forth and so on. This can be done, and indeed we quickly realize that if we want a consistent treatment of the topic, we need to, for example, completely give up the notion of a state of the system being described as an element of the Hilbert space. Although this is (incorrectly) widely alluded to in common literature, it is a mistake. In fact, a state of a system should be regarded as a *positive trace-class linear operator* $\rho : \mathcal{H} \to \mathcal{H}$ such that $\operatorname{Tr} \rho = 1$. This is one of the many subtleties that a correct formulation would entail.

On the other hand, a proper axiomatic treatment of quantum mechanics as a whole – while possible – is not in the general interest of this discussion and it is out of its scope. With this said, the above chapter was intended to give the reader some intuition as to how geometrical concepts help generalize some aspects of physics in a natural way, avoiding mistakes that might have been otherwise committed by a simple-minded generalization.

Appendix E

More About Spin

E.1 The Construction of Spin

This section will thus be consecrated to attempt to give some intuition and explain the geometrical concept to the best of the author's ability, as well as to give a formal construction of spinors we will later use in the rest of this discussion. We start by considering a Lie Group such that the following short exact sequence holds:

$$1 \longrightarrow \mathbb{Z}_2 \longrightarrow Spin(n) \xrightarrow{\eta} SO(n) \longrightarrow 1$$

In the above, we could also consider orthogonal groups of different signature. While this seems a little bit *ad hoc* at first, it is important to realize that there is a geometrical motivation for the existence of this Lie group $Spin(n)$. In particular, notice that by the virtue of the short exact sequence, we have that $\ker \eta = \mathbb{Z}_2$, which means that this map η is "2-to-1", *i.e.* it takes two elements to the same element in $SO(n)$.

For the sake of illustration, consider $n = 3$. It is clear that we can specify any 3D rotation as simply being a 2D rotation leaving an axis invariant. However, to specify the axis of rotation, we could either give a vector in the direction of the axis, or its negative. In both cases, we recover the same element of $SO(3)$. This is exactly what this short exact sequence is trying to capture, the spin group is the group which makes a clear distinction between these two elements, in the mathematical jargon, we often refer to this as a *double cover*.

We recall that the orthogonal group is defined as being the set of matrices fixing a quadratic form Q over a certain vector space V. In what will follow, we will focus on the development of the spinorial representations of $SO(n, \mathbb{C})$, knowing that most of the results hold in the case of $\mathbb{R}$, with some caveats coming about due to questions of signature. The main geometrical intuition behind the concepts can be understood and is perhaps better explained by considering an algebraically closed field such as $\mathbb{C}$.

E.1.1 A Detour Through Clifford Algebras

We will not try to immediately construct a double cover for $SO(V, Q)$. Instead, given a vector space V equipped with a quadratic form Q, we will construct a *Clifford algebra*, which will later turn out to contain a subgroup in its multiplicative group, which will turn out to the the double cover of $SO(V, Q)$ and in some cases of $O(V, Q)$.

The first step is to construct the Clifford algebra associated to the quadratic form Q, which can easily be constructed starting from the tensor algebra $T(V)$ quotiented by a two-sided ideal:

$$\mathcal{C}\ell(V, Q) := T(V)/\langle v \otimes v - Q(v)1 \rangle \tag{E.1.1}$$

In other words, the Clifford algebra $\mathcal{C}\ell(V)$ is nothing other than the freest algebra subject to the relation:

$$v^2 = Q(v)1 \quad \forall v \in V \tag{E.1.2}$$

where now the product is the Clifford multiplication and 1 denotes the multiplicative identity of $\mathbb{C}$ (or $\mathbb{R}$, depending on wheter the vector space V is real or complex). We will note this Clifford product $u \cdot v$. We notice that since the characteristic of the field $\mathbb{C}$ is not 2, the quadratic form induces a symmetric bilinear form given by polarization, *i.e.* :

$$\langle u|v\rangle := \frac{1}{2}(Q(u+v) - Q(u) - Q(v)) \tag{E.1.3}$$

In particular we see that this relation implies:

$$\begin{aligned} (u+v)^2 \qquad &= u^2 + v^2 + uv + vu = Q(u) + Q(v) + \{u, v\} = Q(u+v) \\ &\iff \{u, v\} = 2\langle u|v\rangle \end{aligned} \tag{E.1.4}$$

where $\{u, v\}$ denotes the anticommutator.

Lemma E.1.1. If $\{e_i\}_{i\in I}$ form a basis for V, then the products $e_{i_1} \cdot e_{i_2} \cdots e_{i_k}$ for $i_1 < \cdots < i_k$ and and $e_\emptyset = 1$ form a basis for $\mathcal{C}\ell(V)$. In particular, the dimension of the algebra is $2^{\dim V}$.

We will give a proof of this in section E.2.1. Since the ideal $\langle v \otimes v - Q(v)1\rangle$ is generated by elements of even degree, $\mathcal{C}\ell(V, Q)$ is $\mathbb{Z}_2$-graded. That is, we may split the Clifford algebra into two parts, namely:

$$\mathcal{C}\ell(V, Q) = \mathcal{C}\ell^{\text{even}}(V, Q) \oplus \mathcal{C}\ell^{\text{odd}}(V, Q) := \mathcal{C}\ell^{+}(V, Q) \oplus \mathcal{C}\ell^{-}(V, Q) \tag{E.1.5}$$

By virtue of the previous lemma, the dimension of $\mathcal{C}\ell^{\pm}$ is $2^{\dim V - 1}$. Furthermore, $\mathcal{C}\ell(V, Q)$ being an associative algebra, it determines a Lie algebra via Clifford multiplication. We can indeed define the Lie bracket as simply being:

$$[a, b] = a \cdot b - b \cdot a \tag{E.1.6}$$

For the rest of this report, we will assume that Q is non-degenerate (which will be the case for the physical cases we will be interested in). The spin representations of $\mathfrak{so}(V, Q)$ can be found in two steps from this point forward:

1. Embedding $\mathfrak{so}(V, Q)$ inside the Lie algebra of $\mathcal{C}\ell^{+}$;

2. Identifying the Clifford algebras with one or two copies of matrix algebras.

Embedding $\mathfrak{so}(V, Q)$ Inside the Lie Algebra of $\mathcal{C}\ell^{+}$

We start by recalling the definition of the Lie algebra $\mathfrak{so}(V, Q)$:

$$\mathfrak{so}(V, Q) := \{X \in \mathrm{End}(V) : \langle Xv|w\rangle + \langle v|Xw\rangle = 0 \quad \forall v, w \in V\} \tag{E.1.7}$$

It turns out that there exists an isomorphism between $\varphi : \Lambda^2 V \to \mathfrak{so}(V, Q)$. This isomorphism will help us relate $\mathfrak{so}(V, Q)$ to the Clifford algebra we defined before. It is given by:

$$a \wedge b \mapsto \varphi_{a\wedge b} \quad \text{where} \quad \varphi_{a\wedge b}(v) = 2\left(\langle b|v\rangle a - \langle a|v\rangle b\right) \tag{E.1.8}$$

It is possible to check [9] that the following then holds:

$$[\varphi_{a\wedge b}, \varphi_{c\wedge d}](v) = 2\langle b|c\rangle \varphi_{a\wedge d}(v) - 2\langle b|d\rangle \varphi_{a\wedge c}(v) - 2\langle a|d\rangle \varphi_{c\wedge b}(v) + 2\langle a|c\rangle \varphi_{d\wedge b}(v) \tag{E.1.9}$$

which implies that via the isomorphism, the Lie bracket on $\Lambda^2 V$ should take values:

$$[a\wedge b, c\wedge d] = 2\langle b|c\rangle a\wedge d - 2\langle b|d\rangle a\wedge c - 2\langle a|d\rangle c\wedge b + 2\langle a|c\rangle d\wedge b \tag{E.1.10}$$

On the Clifford algebra side, we have:

$$\begin{aligned} [a\cdot b, c\cdot d] \quad &= a\cdot b\cdot c\cdot d - c\cdot d\cdot a\cdot b \\ &= 2\langle b|c\rangle a\cdot d - 2\langle b|d\rangle a\cdot c - 2\langle a|d\rangle c\cdot b + 2\langle a|c\rangle d\cdot b \end{aligned} \tag{E.1.11}$$

We may then construct a map $\chi : \Lambda^2 V \to \mathcal{C}\ell(V,Q)$ defined by:

$$\chi(a\wedge b) = \frac{1}{2}(a\cdot b - b\cdot a) = a\cdot b - \langle a|b\rangle \tag{E.1.12}$$

which is bilinear and is alternating since $\chi(a\wedge a) = 0$, so that it defines a linear map on $\Lambda^2 V$.

With the morphism $\varphi^{-1} : \mathfrak{so}(V,Q) \to \Lambda^2 V$ and the map $\chi : \Lambda^2 V \to \mathcal{C}\ell(V,Q)$, we obtain finally a morphism of Lie algebras, which constitutes the embedding we wanted, namely: $\chi\circ\varphi^{-1}$. Thus, we have successfully shown that the Lie algebra $\mathfrak{so}(V,Q)$ is indeed embedded in the Clifford algebra as per our construction.

Remark E.1.1. Actually, this isomorphism χ can be extended to an isomorphism of vector spaces $\chi : \bigwedge V \to \mathcal{C}\ell(V,Q)$ and can also be shown for any field of characteristic $\mathrm{car}(\mathbb{K}) \neq 0$. In that case, we consider an orthonormal basis and send

$$e_1 \wedge \cdots \wedge e_k \mapsto e_1 \cdots e_k \tag{E.1.13}$$

It is then possible to show that this map is natural in the sense of category theory if $\mathrm{car}(\mathbb{K}) \neq 2$, since it doesn't depend on the orthonormal basis chosen. On the other hand, it may also be generalized to any basis in the case where $\mathrm{car}(\mathbb{K}) = 0$, by considering antisymmetrization. This will be seen in more detail in section E.2.

Remark E.1.2. An important fact is that, with the morphisms above, it is possible to check that the standard action of $\mathfrak{so}(V,Q)$ on the vector space V (that we will denote with $X \star v$ for $X \in \mathfrak{so}(V,Q)$ and $v \in V$) is compatible with the Clifford commutator, *i.e.* that:

$$X \star v = [X, v] \in V \subset \mathcal{C}\ell(V,Q) \tag{E.1.14}$$

what this means is that the Clifford commutator is in some way equivalent to infinitesimal rotations.

Pin and *Spin* Groups

The Clifford algebra is generated by elements of the vector space V. We claim that on $\mathcal{C}\ell(Q)$ there is a **conjugation morphism**, $*$, which is exactly analogous to the conjugation present for the complex numbers. This is because $\mathbb{C}$ itself can be regarded as a Clifford algebra constructed over a 1-dimensional real vector space with $-Q$ as a quadratic form, where Q is the standard quadratic form on $\mathbb{R}$ and the conjugation present in the complex field stems from the Clifford conjugation, which we will now define. Since the Clifford algebra is generated by elements of V, we give ourselves $v_1, \cdots, v_r \in V$ and define:

$$* : (v_1 \cdot v_2 \cdots v_r) \longmapsto (v_1 \cdot v_2 \cdots v_r)^* := (-1)^r (v_r \cdot v_{r-1} \cdots v_1) \tag{E.1.15}$$

This morphism may be decomposed as $* = \alpha \circ \tau$, where we define α and τ to be the **main involution** and the **reversing maps** respectively. Explicitly:

$$\alpha(v_1 \cdots v_r) := (-1)^r v_1 \cdots v_r \quad \text{and} \quad \tau(v_1 \cdots v_r) := v_r \cdots v_1 \tag{E.1.16}$$

With these homomorphisms, we are ready to define the *Pin* group, for which there turns out to be a 2-to-1 map to the full orthogonal group. In all generality, however, *Pin* does not correspond to the double covering group of the orthogonal group (although this is the case for definite signature in dimension greater than 2).

Definition E.1.1. We define $Pin(V,Q)$ as the group constitued by the following elements of the Clifford algebra:

$$Pin(V,Q) = \{x \in \mathcal{C}\ell(V,Q) \mid x \cdot x^* = 1 \text{ and } x \cdot V \cdot x^* \subset V\} \tag{E.1.17}$$

Similarly, with this definition, we can also define the $Spin(V,Q)$ group as follows:

$$Spin(V,Q) = \{x \in \mathcal{C}\ell^+(V,Q) \mid x \cdot x^* = 1 \text{ and } x \cdot V \cdot x^* \subset V\} \tag{E.1.18}$$

Remark E.1.3. This definition might appear to some as being somewhat convoluted and confusing at first. By examining it a little bit closer, we quickly realize that it is not really that much of a mystery. Since the Clifford algebra is generated by V, we can write $x = v_1 \cdots v_{2n}$ for a generic element $x \in \mathcal{C}\ell(V,Q)$. The first condition is simply asking that the elements x be invertible, with inverse x^*. The second condition imposes that $x \cdot v \cdot x^*$ for any $v \in V$ be a linear homomorphism, which makes complete sense. Our last remark is that we may write the *Pin* group to be the invertible elements of $\mathcal{C}\ell(V,Q)$ and $Spin = Pin \cap \mathcal{C}\ell^+$.

Proposition E.1.1. For $x \in Spin(V,Q)$, $\eta(x) \in SO(Q)$. The map $\eta : Spin(V,Q) \to SO(V,Q)$ is a homomorphism, making $Spin(V,Q)$ a connected two-sheet covering of $SO(Q)$. Moreover, $\ker \eta = \{\pm 1\}$.

Proof. We will not give a full proof of this proposition. This can be found in [9]. Instead, we will focus on the main geometrical arguments which can be of importance to the understanding of the reader of what exactly the Clifford algebra is and why how exactly it relates to the orthogonal group. Let us start by considering the *Pin* group and define a similar morphism, that we will by abuse of notation denote also with $\eta : Pin(V,Q) \to O(V,Q)$, which will be defined in the following way:

$$\eta(x)(v) = \alpha(x) \cdot v \cdot x^* \tag{E.1.19}$$

We must show that this morphism does indeed take values in $O(V,Q)$. To do this, it is enough to show that the morphism preserves the quadratic form Q. Notice that, for any element $v \in V$, we have $Q(v) = v^2 = -v \cdot v^*$. By taking the evaluating the quadratic form on a transformed element $\eta(x)(v)$, we get the desired result:

$$\begin{aligned} Q(\eta(x)(v)) \quad &= (\eta(x)(v))^2 = -(\eta(x)(v)) \cdot (\eta(x)(v))^* \\ &= -(\alpha(x) \cdot v \cdot x^*) \cdot (\alpha(x) \cdot v \cdot x^*)^* \\ &= -\alpha(x) \cdot v \cdot \underbrace{x^* \cdot x}_{=1} \cdot v^* \cdot \alpha(x^*) \\ &= -\alpha(x) \cdot \underbrace{v \cdot v^*}_{=-Q(v)} \cdot \alpha(x^*) \\ &= Q(v)\alpha(x) \cdot \alpha(x^*) = Q(v)\alpha(x \cdot x^*) = Q(v) \end{aligned} \tag{E.1.20}$$

We must now prove that the map is surjective, for which we will use the Cartan-Dieudonné theorem, *i.e.* that the orthogonal group is generated by reflections [28]. Assuming this, let us consider to what kind of linear transformation an element $w \in V$ gets mapped under η (we must take $\|w\| = 1$ since $w \cdot w^* = 1$.) Let $v \in V$, then:

$$\eta(w)(v) = \alpha(w) \cdot v \cdot w^* = w \cdot v \cdot w = 2\langle v|w\rangle w + v \cdot w^2 = v + 2\langle v|w\rangle w \tag{E.1.21}$$

In particular, if $v = \lambda w + \mu w^\perp$, we can easily see that:

$$\eta(w)(v) = -\lambda w + \mu w^\perp \tag{E.1.22}$$

Which allows us to instantly recognize that this is nothing other than reflection with respect to the orthogonal hyperplane to w. This means that via the mapping η, we have a correspondence of reflections and vectors in the space. It is in this sense that the reader should understand the *Pin* group. Once this conceptual difficulty has been breached, these groups become much more tractable.

Indeed, via the Cartan-Dieudonné theorem, for every element $R \in O(V,Q)$ such that $R = R_{w_1} \cdots R_{w_r}$, where R_{w_i} is the reflection with respect to the plane orthogonal to the vector w_i, we get two corresponding elements in *Spin* which correspond to R, namely: $\pm w_1 \cdots w_r$. With this, we can describe the *Spin* group by examining the corresponding subgroup $SO(V,Q) \subset O(V,Q)$. In particular, we may write any element of $SO(V,Q)$ as an *even* product of reflections. Thus yielding the desired correspondence:

$$Spin(V,Q) = Pin(V,Q) \cap \mathcal{C}\ell^+(V,Q) = \eta^{-1}(SO(V,Q)) \tag{E.1.23}$$

■

In the case of the construction of the real representations of $Spin_n(\mathbb{R})$ of positive definite signature, very little changes in the construction. In that case, we use the real Clifford algebra $\mathcal{C}\ell(\mathbb{R}^n, Q)$ associated to the quadratic form $Q = -Q_s$ where Q_s is the standard quadratic form on $\mathbb{R}^n$ and follow an analogous treatment.

On the other hand, we also know that over the field of real numbers, all quadratic forms are equivalent up to their signature. This means that we may associate to the group $SO(p,q)$ a corresponding double cover $Spin^+(p,q)$ which lives in the Clifford algebra determined by $\mathcal{C}\ell_{(p,q)}(\mathbb{R})$. In particular, as physicists, we are interested in the case where $p = 1$ and $q = 3$, which corresponds to the Lorentz group. While the construction does not vary greatly from the one presented above, it is important to note that an important difference is that $SO(3,1)$ is no longer a compact group, which gives rise to some difficulties, but we may still use most of the results previously presented.

E.1.2 Matrix Representations

At this point, we have provided an abstract picture of the *Spin* group. We would now like to complete the second step we talked about in our discussion of Clifford algebras and actually embed these in a matrix algebra the reader is perhaps more familiar with. In doing so, we will finally be able to determine the space in which the Dirac spinors actually live.

The Case of 3D Rotations and Reversals

It is perhaps useful to start with a simple example, which we will later generalize to higher dimensions. This is so that the reader understands the geometric origin of the construction of the Clifford modules. Let us start by considering the case where dim $V = 3$, on which we can consider a quadratic form (over $\mathbb{C}$) which we shall take to be:

$$Q = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{E.1.24}$$

The quotient of the orthogonal group by its center, $PSO_n(\mathbb{C})$, may be realized as the group of motions in the projective space $\mathbb{P}(V)$ which map the quadratic hypersuface corresponding to $x^i x_i = 0$ onto itself, *i.e.* which leaves the isotropic spaces of the quadratic form Q invariant. This is simply because of the fact that rotations preserve the norm of the vector induced by the quadratic form Q. In our case of dim $V = 3$, the orthogonal group $PSO_3(\mathbb{C})$ is nothing but the group of motions over $\mathbb{P}(V)$ carrying the conic:

$$\mathcal{C} = \left\{ [x : y : z] \in \mathbb{P}(V) \mid x^2 + y^2 + z^2 = 0 \right\} \tag{E.1.25}$$

into itself. On the projective plane, it is possible to construct an bijective correspondence between any line on $\mathbb{P}(V)$ and the conic $\mathcal{C}$. Explicitly, without loss of generality, we may do this by choosing a projective frame which arranges us. We start by choosing a point on the conic to have coordinates $[1 : i : 0]$ and choosing two other points which don't lie on the conic such that the set of three points is not colinear. We choose the homogeneous coordinates of these two points to be $[0 : 1 : 0]$ and $[0 : 0 : 1]$ respectively. In so doing, we have that the equation of the line joining this two points is given by $x = 0$. It is then easy to see that the homography we are looking for is nothing other than the map determined by the intersection of the line passing through $[1 : i : 0]$ a point $[0 : \xi : \zeta]$ on the line and an additional point on the conic $\mathcal{C}$. In the case where the this line is tangent to the conic, we choose the point $[1 : i : 0]$ to be the one specified by the mapping. This map is an isomorphism is due to the fact that there exists a bijective correspondence between the pencil passing through $[1 : i : 0]$ and the line $x = 0$, in a similar way, there is also a bijective correspondence between this pencil and the conic $\mathcal{C}$. Performing an explicit calculation, we can express any point in the conic as a point in the line by the equation:

$$\begin{aligned} x &= \xi^2 - \zeta^2 & \qquad \xi^2 &= \frac{1}{2}(x - iy) \\ y &= i(\xi^2 + \zeta^2) & \qquad \zeta^2 &= -\frac{1}{2}(x + iy) \\ z &= -2\xi\zeta & \qquad \xi\zeta &= -\frac{1}{2}z \end{aligned} \tag{E.1.26}$$

Furthermore, the ratio ξ/ζ undergoes a homographic transformation under any rotation or reversal. This is simply because we can see ξ/ζ as a parameter of a generator of the isotropic cone. Since rotations and reversals preserve the cross-ratio of any four such generators, it follows that ξ/ζ must undergo a homographic transformation. The opposite is also true, that is, given a homographic transformation of the line $x = 0$, we can find the associated rotation to it. Indeed, consider a vector $v \in V$, then we can find two orthogonal isotropic directions corresponding to this vector v, which we will denote w_1 and w_2, which, in the projective plane, lie in the conic $\mathcal{C}$. We may associate to the latter points on the line $x = 0$ via our construction above. Applying a homographic transformation to these points, which will yield two different points on the conic w_1', w_2', which are orthogonal to the vector v' which corresponds to the mapping of the vector $v \mapsto v'$ under the rotation or reversal corresponding to the homographic transformation. The latter may be found so we indeed have a bijection between the homographic transformations of $[0 : \xi : \zeta]$ and the rotations and reversals on V.

Remark that these $[0 : \xi : \zeta]$'s are exactly the spinors we were looking for. Indeed, the fact that that we may associate a homography to each rotation and reversal is exactly what we set out to do when we were trying to find. The group associated to these homographic transformations is $PGL_2(\mathbb{C})$, which gives us the well-known isomorphism $\mathfrak{so}_3(\mathbb{C}) \cong \mathfrak{sl}_2(\mathbb{C})$.

All of this is consistent with the fact that the equations of the correspondence between the isotropic vector $[x : y : z]$ and the coordinates on the line $x = 0$, $[0 : \xi : \zeta]$, may be put under the form:

$$\begin{pmatrix} z & x - iy \\ x + iy & -z \end{pmatrix} \begin{pmatrix} \xi \\ \zeta \end{pmatrix} = 0 \tag{E.1.27}$$

where we recognize the X's to be generated by the Pauli matrices. We may then extend this into a bijection between traceless unitary matrices and vectors (x, y, z) via the at this point obvious mapping:

$$(x, y, z) \longmapsto X := \begin{pmatrix} z & x - iy \\ x + iy & -z \end{pmatrix} \tag{E.1.28}$$

It is then possible to check explicitly that these matrices satisfy $X^2 = (x^2+y^2+z^2)\,\mathrm{id}_{2\times 2}$, which means that they obey the Clifford relations. By the universal property of Clifford algebras, we have successfully found a matrix representation of the algebra we were looking for, since the latter is generated by vectors in V.

It is helpful to conclude this section with a couple of remarks will be of help in the generalization that is to come.

Remark E.1.4. We have seen that isotropic spaces played a fundamental role in our determination of what the spinor spaces look like. This holds in higher dimensions as well. This is because the isotropic space will always determine a conic hypersurface in the projective space $\mathbb{P}(V)$ which is the stabilized by the orthogonal group. We can treat in a similar way the cases of dimensions 5,6 and 7 [9]. We will treat the case of dimension 4 in section 4.1.3. The main take away and hint from this treatment is that we should consider the spinor space to be of the form $S = \bigwedge W$, where W is one of the maximally isotropic subspaces of the quadratic form Q.

Remark E.1.5. The same ideas of projective geometry can be used in order to find all the generators of Pythagoran triples over $\mathbb{Q}$ if we consider the conic $x^2 + y^2 - z^2 = 0$ instead.

The General Case

As mentionned above, the isotropic subspaces of the quadratic form Q are fundamental to finding the matrix representations of the Clifford algebra, and by extension, of the spin representations themselves. With the help of our previous example and with our newly acquired geometrical intuition, we now focus on the decomposition of V into maximally isotropic subspaces. To see exactly how our space V decomposes into maximally isotropic subspaces, the following handwavy argument might help the reader get a sense of what is going on. For the sake of illustration, let us place ourselves in 3D and in 4D. Since on $\mathbb{C}$ all quadratic forms are equivalent, we may choose the following expressions for the quadratic forms:

$$Q_{3D} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \quad \text{and} \quad Q_{4D} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \tag{E.1.29}$$

From this identification, we can clearly see the isotropic subspaces of the quadratic forms. Indeed, in the 3D case, we can see that $Q(e_1) = Q(e_3) = 0$ for the quadratic form Q and where $\{e_i\}_{i\in I}$ is the standard basis of $\mathbb{C}^n$. Similarly, in the case of 4D, we can easily see that the subspace spanned by e_1 and e_2 will form one maximally isotropic subspace and analogously, so will be the subspace spanned by e_3 and e_4.

With this illustration in mind, it becomes then clear that:

1. For $\dim V = 2k$, we get the splitting of the vector space V into two maximally isotropic subspaces, *i.e.* $V = W \oplus W'$;

2. For $\dim V = 2k+1$, we have that V breaks into two maximally isotropic subspaces of dimension k and a space U of dimension 1, *i.e.* $V = W \oplus U \oplus W'$.

At this point, it is useful to treat the even and the odd case separately. In fact, the difference between these groups and algebras is so profound, that they have completely different Dynkin diagrams (*cf.* section A.2.1 for an in depth discussion of Dynkin diagrams). We start by stating our first non-trivial lemma:

Lemma E.1.2. In accordance to the above, we have two cases:

1. Let $\dim V = 2k$, then the decomposition $V = W \oplus W'$ determines an isomorphism of algebras $\mathcal{C}\ell(V,Q) \cong \mathrm{End}(S)$, where $S := \bigwedge W = \Lambda^0 W \oplus \Lambda^1 W \oplus \cdots \oplus \Lambda^k W$.

2. Let $\dim V = 2k+1$, the decomposition $V = W \oplus U \oplus W'$ determines an isomorphism of algebras $\mathcal{C}\ell(V,Q) \cong \mathrm{End}(S)$.

Proof. We won't detail much of the proof here, however, more details can be found in [9]. It can be helpful to give the reader an intuition as to what is going on. What we need to do is find a linear action of V on $\bigwedge W$ such that the Clifford relations remain preserved. The mappings defined by this action $*$ then constitute a subset of $\mathrm{End}(\bigwedge W)$, which will then factorize through the Clifford algebra because of its universal property, since $*$ respects the Clifford relations. Diagramatically, the universal property for this particular case is:

$$\begin{tikzcd} V \arrow[r, "i"] \arrow[dr, "*"'] & \mathcal{C}\ell(V,Q) \arrow[d, "\Phi"] \\ & \mathrm{End}(\bigwedge W) \end{tikzcd}$$

where Φ is the isomorphism seeked in the lemma. In this way, we thus obtain the result we seek.

The next natural step is to give such an action $*$. By the decomposition $V = W \oplus W'$, we may write any vector $v = w + w'$, where $w \in W$ and $w' \in W'$. For any $\psi \in \bigwedge W$ the action of V on $\bigwedge W$ we seek and which respects the Clifford relations is given by:

$$v * \psi = \sqrt{2}\,(w \wedge \psi + \iota_{w'}\psi) \tag{E.1.30}$$

Here, $\iota_{w'}$ denotes the interior multiplication. For the reader unfamiliar with this operation, it is an antiderivation which can be taken as an analogous operation acting in the opposite way as d, in the sense that:

$$\iota_X : \Lambda^k W \to \Lambda^{k-1} W \tag{E.1.31}$$

Explicitly, we have that $\iota_X(\omega)(X_1, \cdots, X_{p-1}) = \omega(X, X_1, \cdots, X_{p-1})$. Furthermore, if β is a p-form and γ is a q-form, then we have:

$$\iota_X(\beta \wedge \gamma) = (\iota_X \beta) \wedge \gamma + (-1)^p \beta \wedge (\iota_X \gamma) \tag{E.1.32}$$

It is in fact this antiderivation propery which makes the whole thing work. In the odd case, it is also necessary to specify what the action of the unit vector $u \in U$ is on the spinor ψ. In order for this map to be an algebra homomorphism, it is necessary to impose:

$$u * \psi = \begin{cases} +\mathrm{id} & \text{if } \psi \in \Lambda^{\text{even}} W \\ -\mathrm{id} & \text{if } \psi \in \Lambda^{\text{odd}} W \end{cases} \tag{E.1.33}$$

One can then check that this action respects the Clifford relations and check what happens on a basis of V in order to see that the map is indeed an isomorphism. ∎

It turns out that these spaces $\mathrm{End}(\bigwedge V)$ are the irreducible spinorial representations of $\mathfrak{so}_{2k+1}(\mathbb{C})$. However, in the even case, the representation provided by S splits into two parts, one with forms of even degree and the other one with odd degree. We thus have the following theorem:

Theorem E.1.3. *Let $S^+ := \Lambda^{even} W$ and $S^- := \Lambda^{odd} W$, then the following holds:*

1. *If $\dim V = 2k$, the representations S^+ and S^- are the irreducible spinorial representations of $\mathfrak{so}_{2k}(\mathbb{C})$;*
2. *If $\dim V = 2k+1$, the representation S is a spinorial irreducible representation of $\mathfrak{so}_{2k+1}(\mathbb{C})$*

This construction at first glance seems very abstract and not very useful, in the next section the construction will be clarified by example. The main takeaway from the above is that we have realized the Clifford algebra as a matrix algebra by explicitly constructing a representation. At this point, however a couple of remarks are in order:

Remark E.1.6. For the groups in low dimension, we have accidental isomorphisms which help us identify this matrix algebra in an easy way, this will be illustrated in what will follow in more detail.

Remark E.1.7. While handy, the realization of the Clifford algebra inside a matrix algebra is not necessary to get most basic results. While it certainly makes calculations more explicit, we have seen that we can show most results at the level of the Clifford algebra level. Furthermore, staying at the Clifford algebra level has the advantage to keep the geometrical intuition on our side. This geometrical intuition can easily be lost if we only consider the matrix structure of the representation. Sometimes, the understanding of these spin representations will be unavoidable, such as when we derive the Dirac equation as we will do in section 4.1.2.

Remark E.1.8. The elements of S are where the Dirac spinors live (more precisely, they are *sections* of associated bundle of representation S to a *Spin* principal bundle).

E.1.3 Accidental Isomorphisms

Illustration of $Spin(3) \cong SU(2)$

The discussion we had about matrix representations of the *Spin* groups could've come across as too abstract to the reader. In order to clarify some aspects of it, it is perhaps helpful to look at a simple example of the realization of the matrix representation of the Clifford algebra, *i.e.* $\mathrm{End}(\bigwedge W)$.

We start by recognizing that in the case of $Spin(3)$, our 3-dimensional space V decomposes as : $V = W \oplus U \oplus W'$ as per our previous discussion. In dimension 3, $\dim W = \dim W' = \dim U = 1$. This means that $\bigwedge W = \Lambda^0 W \oplus \Lambda^1 W = \mathbb{C} \oplus W \cong \mathbb{C}^2$. The matrix algebra is then $\mathrm{End}(\mathbb{C}^2) \cong \mathbb{C}^2 \otimes \mathbb{C}^2$, which indeed corresponds to the 2×2 matrices. So far, at least dimensionally, everything is consistent with what we know. The next step is to assign to every vector $(x, y, z) \in V$ an element of $\mathrm{End}(\bigwedge W)$ such that the Clifford relations are satisfied. We do this by considering the action we mentioned in the proof of lemma E.1.2.

For any vector $v \in V$, we have that we can write $v = a_1 w + a_2 w' + a_3 u$ with $a_i \in \mathbb{C}$. It follows that we need only check the action of v on a basis of $\bigwedge V$, which in this case is simply given by two vectors $\{e, w\}$. It follows that:

$$v * e \sim a_1 w \wedge e + \cancel{\iota_{a_2 w'} e} + a_3 e = a_3 e + a_1 w \tag{E.1.34}$$

$$v * w \sim \cancel{a_1 w \wedge w} + \iota_{a_2 w'} w - a_3 w = a_2 e - a_3 w \tag{E.1.35}$$

Now, we realize that with respect to the standard basis, we may take $w = \frac{1}{2}\begin{pmatrix}1 & i & 0\end{pmatrix}$, $w' = \frac{1}{2}\begin{pmatrix}1 & -i & 0\end{pmatrix}$ and $u = \begin{pmatrix}0 & 0 & 1\end{pmatrix}$ since we have that $\langle w|w \rangle = 0$ with respect to the standard quadratic form on $\mathbb{C}^3$. It follows then that $W = \mathbb{C}w$. This means that we have that if the vector $v = x\widehat{x} + y\widehat{y} + z\widehat{z}$, under a change of basis from $\widehat{x}, \widehat{y}, \widehat{z} \mapsto w, w', u$, we have that the vector v has components:

$$v = \begin{pmatrix} 1/2 & 1/2 & 0 \\ i/2 & -i/2 & 0 \\ 0 & 0 & 1 \end{pmatrix}^{-1} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x - iy \\ x + iy \\ z \end{pmatrix} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \tag{E.1.36}$$

which means that the endomorphism associated to the action of v on $\bigwedge W$ can be written as a matrix as:

$$v* = \begin{pmatrix} z & x - iy \\ x + iy & -z \end{pmatrix} \tag{E.1.37}$$

As previously illustrated, this is indeed a matrix representation of the Clifford algebra and thus by extension of $\mathfrak{so}(3)$.

Accidental Isomorphism Classification

It is often useful to identify the spin groups to some classical semisimple Lie groups. To do the classification of these so-called accidental or exceptional isomorphisms, we will use Dynkin diagrams, which provide a straightforward graphical way of visualizing these isomorphisms (*cf.* section A.2.1).

Let us start with the two main cases that will concern us, namely the case of $Spin(3)$ and $Spin(4)$. The Lie algebras are given by $\mathfrak{so}(3)$ and $\mathfrak{so}(4)$ accordingly. It is trivial to see then, according to the classification of semisimple Lie algebras, that:

$$Spin(3) \cong SU(2) \quad \text{and} \quad Spin(4) \cong SU(2) \times SU(2) \tag{E.1.38}$$

We may see these isomorphisms as coming from the correspondeing Dynkin diagram structure of the groups considered. For the first case, we have that $\bullet$ is the Dynkin diagram corresponding to both $Spin(3)$ and $SU(2)$ and similarly, for $Spin(4)$ and $SU(2) \times SU(2)$ the corresponding diagrams are: $\overset{\alpha}{\bullet} \quad \overset{\beta}{\bullet} \cong \overset{\alpha}{\bullet} \times \overset{\beta}{\bullet}$. With the help of Dynkin diagrams, it is easy to see that

Table E.1: Some accidental isomorphisms of the real Lie algebras of some rotation groups.

Euclidean signature	Minkowskian Signature	Other Signature
$\mathfrak{so}(2) \cong \mathfrak{u}(1)$	$\mathfrak{so}(1,1) \cong \mathbb{R}$	
$\mathfrak{so}(3) \cong \mathfrak{sp}(1)$	$\mathfrak{so}(2,1) \cong \mathfrak{sl}(2,\mathbb{R})$	
$\mathfrak{so}(4) \cong \mathfrak{sp}(1) \oplus \mathfrak{sp}(1)$	$\mathfrak{so}(3,1) \cong \mathfrak{sl}(2,\mathbb{C})$	$\mathfrak{so}(2,2) \cong \mathfrak{sl}(2,\mathbb{R}) \oplus \mathfrak{sl}(2,\mathbb{R})$
$\mathfrak{so}(5) \cong \mathfrak{sp}(2)$	$\mathfrak{so}(4,1) \cong \mathfrak{sp}(1,1)$	$\mathfrak{so}(3,2) \cong \mathfrak{sp}(4,\mathbb{R})$

the problem of finding these isomorphisms is reduced to nothing other than a graph theory problem (and a pretty simple one at that). We can now immediately see that the following isomorphisms exist:

$$A_1 \cong C_1 \cong B_1 \quad \text{or} \quad \mathfrak{sl}_2(\mathbb{C}) \cong \mathfrak{sp}_1(\mathbb{C}) \cong \mathfrak{so}_3(\mathbb{C}) \tag{E.1.39}$$
$$B_2 \cong C_2 \quad \text{or} \quad \mathfrak{so}_5(\mathbb{C}) \cong \mathfrak{sp}_2(\mathbb{C}) \tag{E.1.40}$$
$$D_2 \cong A_1 \times A_1 \quad \text{or} \quad \mathfrak{so}_4(\mathbb{C}) \cong \mathfrak{sl}_2(\mathbb{C}) \oplus \mathfrak{sl}_2(\mathbb{C}) \tag{E.1.41}$$
$$A_3 \cong D_3 \quad \text{or} \quad \mathfrak{sl}_4(\mathbb{C}) \cong \mathfrak{so}_6(\mathbb{C}) \tag{E.1.42}$$
$$A_4 \cong E_4 \quad \text{or} \quad \mathfrak{sl}_5(\mathbb{C}) \cong \mathfrak{e}_4(\mathbb{C}) \tag{E.1.43}$$
$$D_5 \cong E_5 \quad \text{or} \quad \mathfrak{so}_{10}(\mathbb{C}) \cong \mathfrak{e}_5(\mathbb{C}) \tag{E.1.44}$$

The associated Dynkin diagrams relating to the isomorphisms above are shown in figure E.1. Notice that we have done this in the case of the base field being complex. A similar analysis

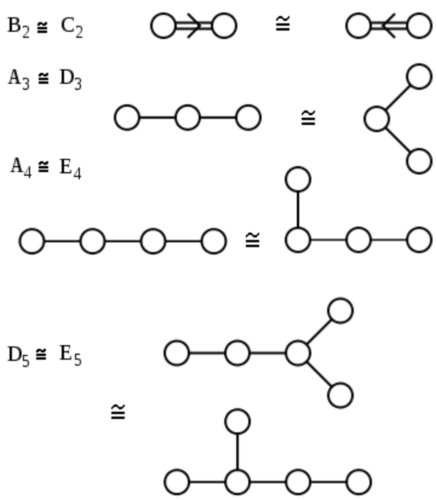


Figure E.1: The Dynkin diagrams of the accidental isomorphisms.

is possible can be performed in the case of the base field being $\mathbb{R}$, up to the consideration of different signatures. This means that we will have extra isomorphisms to consider. We give some accidental isomorphisms relating to the *Spin* groups in table E.1.

E.2 Applications in Computations in Physics

Now that we have explored the construction of spin and understand a bit better its relation with Clifford algebras and rotations, we are ready to see how we can apply all these concepts in physics. In this section we will attack some of the common computations one is forced to do in Quantum Field Theory. We will proceed in three steps. First, we show some typical results about γ-matrices, which can be obtained trivially without the need for any explicit computations. Second, we will discuss a bit about taking traces of γ-matrices and the geometrical meaning of the trace. Third, we will give combinatorial arguments and some results obtained for these traces by considering our geometrical picture.

E.2.1 Some Standard Results

The embedding of $\mathfrak{so}(V,Q)$ inside the Clifford algebra shown previously shown implies most of the typical results that can be shown via nasty computations if one picks a given representation of $\mathfrak{so}(V,Q)$. The advantage of the methods hereby introduced lie in the fact that we understand these objects geometrically, which helps ease or completely skip the actual computations, while providing us with a deeper intuition of what is actually going on.

Dimension of the Clifford Algebra

The first result which we can obtain trivially is the dimension of the Clifford algebra. In our previous discussion, we mentionned that the dimension of the Clifford algebra is $2^{\dim V}$. We also know that the Clifford algebra is generated by elements $v \in V$. In particular, we may pick an orthonormal basis for V, which we will suggestively note $\{\gamma^\mu\}_{\mu\in I}$. In this case, we have that any element in the Clifford algebra can be written in terms of the following spanning set:

$$1, \quad \gamma^\mu, \quad \gamma^{\mu_1}\gamma^{\mu_2}, \quad \cdots \quad , \gamma^{\mu_1}\cdots\gamma^{\mu_{\dim V}} \tag{E.2.45}$$

where we have taken $\mu_1 < \mu_2 < \cdots < \mu_{\dim V} \in I$. There are multiple ways of checking linear independence, for example, notice simply that if $Q = 0$, then $\mathcal{C}\ell(V,Q) = \bigwedge V$, whose dimension is also $2^{\dim V}$. We can then consider a filtration of the Clifford algebra by subpsaces F_k consisting of those elements which may be written as sums of at most k products of elements in V. This filtration is inherited from the natural one present in $T(V)$. It follows that $\bigoplus_k F_{k+1}/F_k \cong \bigwedge V$ and thus the dimensions of the two spaces must be the same.

Generators of $\mathfrak{so}(V,Q)$

We have already shown another classical result in our previous discussion. However, it is perhaps helpful to explicit it again using the suggestive notation $\{\gamma^\mu\}_{\mu\in I}$, which we just employed for the dimensional result. We previously embedded $\mathfrak{so}(V,Q)$ inside the Clifford algebra. Using what we previously found about this embedding, we have the standard result that the generators of $\mathfrak{so}(V,Q)$ can be mapped via the morphisms previously explicited to an element of the Clifford algebra, namely: $\varphi_{\gamma^\mu\wedge\gamma^\nu} \mapsto \gamma^\mu \wedge \gamma^\nu \mapsto \frac{1}{2}[\gamma^\mu,\gamma^\nu]$. If follows that the generators of $\mathfrak{so}(V,Q)$ are:

$$\sigma^{\mu\nu} = \frac{1}{2}[\gamma^\mu,\gamma^\nu] \tag{E.2.46}$$

for $\mu < \nu$ and where the Lie bracket here now denotes the Clifford commutator. However, we now see that this is nothing other than an easy consequence of what we had previously seen and that this result is totally independent from the Dirac equation itself and indeed of any matrix representation of the Clifford algebra itself.

Remark E.2.1. We can say a bit more about the morphisms we explicited earlier; for $\chi : \Lambda^2 V \to \mathcal{C}\ell(V,Q)$ explicited in equation E.1.12, we have that:

$$\mathrm{Im}(\chi) = \mathcal{C}\ell^2(V,Q) \cap \ker(\mathrm{Tr}) \tag{E.2.47}$$

where Tr denotes the trace (the trace of an element of $\mathcal{C}\ell(V,Q)$ is the trace of left Clifford multiplication by that element on $\mathcal{C}\ell(V,Q)$).

Remark E.2.2. We can also see this trace operation as simply being the projection of any element of $\mathcal{C}\ell(V,Q)$ onto $\mathbb{C}$, recalling that we may decompose: $\mathcal{C}\ell(V,Q) = \mathbb{C} \oplus V \oplus \mathcal{C}\ell^2(V) \oplus \cdots \oplus \mathcal{C}\ell^n(V)$. Both definitions are totally consistent with one another and the point of view taken is left up to case-by-case convinience of use.

E.2.2 Geometrical Picture of γ-Matrices

In Quantum Field Theory, calculations often imply the computation of traces of products of γ-matrices. In our case, we have seen that it is possible to regard these γ-matrices as honest to goodness vectors (in the mathematical sense, *i.e.* elements of $\mathbb{C}^n$ or $\mathbb{R}^n$ equipped with some quadratic form η), which live in a Clifford algebra. This point of view allows us to give a geometrical interpretation to computations we encounter and to gain some insight about some of the formulas typically given in QFT.

In order to gain some insight on what products of γ matrices represent, it is useful to go back to the sketch of proof on the dimension of $\mathcal{C}\ell(V,Q)$ we previously gave, in which we proceeded to identify the Clifford algebra with the exterior one. We do this because the homogeneous elements of the exterior algebra can really be understood as vectors, surfaces, volumes, *etc.* We can do the same in the case of the Clifford algebra, up to some small subtlety. To illustrate this, consider the product:

$$v \cdot w = \underbrace{\frac{1}{2}(vw + wv)}_{\langle v|w\rangle} + \underbrace{\frac{1}{2}(vw - wv)}_{\chi(v\wedge w)} \tag{E.2.48}$$

We see that we have a component in $\mathbb{C}$ and another one which can be readily identified with $v \wedge w$. In the particular case where $\langle v|w\rangle = 0$, we have that $v \cdot w$ can really be interpreted as the oriented surface element spanned by v and w. This implies that if we return back to our orthonormal basis $\{\gamma^\mu\}_{\mu\in I}$, we have that $\gamma^{\mu_1}\cdots\gamma^{\mu_n} \mapsto \gamma^{\mu_1}\wedge\cdots\wedge\gamma^{\mu_n}$, which means that we may identify the product of n γ-matrices with the volume element spanned by these vectors γ^{μ_i}. This geometrical interpretation of the wedge product as oriented volume elements is illustrated in figure E.2. In order to generalize this illustration to products of more than 2 γ-matrices, we must first give the explicit extension of the χ morphism mentioned in remark E.1.1. We proceed by antisymmetrization:

$$\chi : v_1 \wedge \cdots \wedge v_k \longmapsto \frac{1}{k!}\sum_{\sigma\in S_n} \mathrm{sgn}(\sigma)\, v_{\sigma(1)}\cdots v_{\sigma(n)} \tag{E.2.49}$$

With this morphism of vector spaces, in what will follow, we will systematically omit to write it explicitly by abuse of notation.

At this point, we may be tempted to decompose the product $v_1\cdots v_k$ in the Clifford algebra as we did before. Due to the $\mathbb{Z}_2$ gradation of the Clifford algebra, we have that it is necessary to consider two cases, one for when $k \in 2\mathbb{Z}$ and the other one when $k \in 2\mathbb{Z}+1$. So, we have the following proposition.

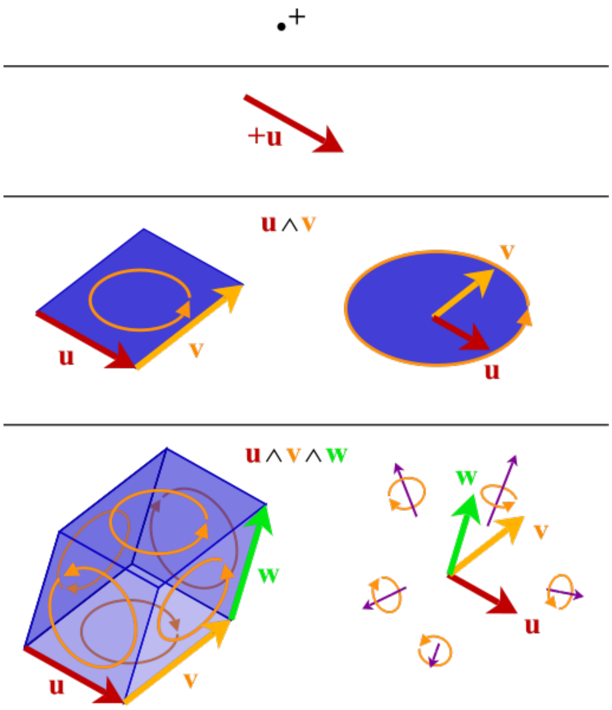


Figure E.2: The geometrical interpretation of the homogeneous elements of the exterior algebra (*i.e.* elements in $\Lambda^k V$). For $k = 0$ we have a signed point, $k = 1$ corresponds to a directed line segment, $k = 2$ is an orientated surface element and $k = 3$ corresponds to an orientated volume element.

Proposition E.2.1. Let $\dim V = n$, then we have the following decomposition of the Clifford algebra:

$$\mathcal{C}\ell^{+}(V,Q) = \mathbb{C} \oplus \Lambda^2 V \oplus \cdots \oplus \Lambda^{2\lfloor \frac{n}{2} \rfloor} V \tag{E.2.50}$$

$$\mathcal{C}\ell^{-}(V,Q) = V \oplus \Lambda^3 V \oplus \cdots \oplus \Lambda^{2\lceil \frac{n}{2} \rceil - 1} V \tag{E.2.51}$$

This means that any homogeneous element $x = v_1 \cdots v_k$ can be decomposed as above.

Proof. The statement follows from the $\mathbb{Z}_2$-gradation of Clifford algebras and the identification we previously performed between $\mathcal{C}\ell$ and the exterior algebra. Consider an orthonormal basis. Then we can indeed see that the spanning elements will indeed decompose as above, that is that:

$$e_{i_1} \cdots e_{i_k} \mapsto e_{i_1} \wedge \cdots \wedge e_{i_k} \tag{E.2.52}$$

via this isomorphisms of vector spaces, we may always find such a decomposition. ∎

The above proposition motivates the following definition, which become one of the point of views that we will have for the trace.

Definition E.2.1. We call the trace of the product $v_1 \cdots v_k$ and denote it $\mathrm{Tr}(v_1 \cdots v_k)$ the projection onto $\mathbb{C}$ of the decomposition of $v_1 \cdots v_k$ provided by proposition E.2.1.

Remark E.2.3. This way of defining the trace is complitely consistent with its usual definition as a linear functional on $\mathrm{End}(\mathcal{C}\ell(V,Q)) \cong \mathrm{End}(\bigwedge V)$ as vector spaces. This is because for any $k > 0$, we always have that:

$$\mathrm{Tr}(v_1 \wedge \cdots \wedge v_k) = 0 \tag{E.2.53}$$

by the antisymmetry of $\wedge$ and the cyclicity of the usual trace. In this sense, because of proposition E.2.1 it is clear to see that the two definitions are exactly equivalent. It is important to note that, since in physics we normally use a matrix representation, this implies rescaling of the trace in the way we have defined it over the Clifford product. This can be easily seen by considering the fact that for us $\mathrm{Tr}(1) = 1$ as the trace is seen as a simple projection operator. However, $1 \mapsto \mathrm{id}_D$ under any realization of the Clifford algebra as a matrix representation of dimension D. In this case, the consistent operator Tr as seen as a linear functional over such a matrix representation yields the result $\mathrm{Tr}(\mathrm{id}_D) = D$. Once this scaling factor is fixed by the multiplicative identity of the Clifford algebra, it is determined for all other elements.

Remark E.2.4. It is trivially true that $\mathrm{Tr}(v) = 0$ in this case, again because of $\mathbb{Z}_2$-gradation.

Proposition E.2.1 can be a little bit unsatisfying. After all, we did not give the explicit decomposition of a generic product in terms of its composing vectors. In what will follow, we will approach this problem and give the combinatorial term-by-term expression for the decomposition. We will then tackle the problem of finding the combinatorial decomposition of the trace, which we will then be able to easily generalize to the other parts of the decomposition.

E.2.3 Traces of γ-Matrices

In order to tackle the problem of the decomposition of the trace, we develop a graphical method, which will greatly ease the calculation. It is first useful to prove the following proposition:

Proposition E.2.2 (Wick's theorem for traces)**.** Let $v_1 \cdots v_k \in V$ and $k \in 2\mathbb{Z}$ (the odd case being trivial). Consider the product of these elements. Then,

$$\mathrm{Tr}(v_1 \cdots v_k) = \underbrace{\sum \prod}_{\text{Pairings}} \mathrm{sgn}(\sigma) \langle v_i | v_j \rangle \tag{E.2.54}$$

where $\mathrm{sgn}(\sigma)$ is the sign of a permutation σ which assigns to $(v_1, \cdots, v_k)$ its corresponding pairing in the product.

Proof. We prove the result by induction. Consider the base case $k = 2$. In that case, we simply have that:

$$v_1 v_2 = \underbrace{\langle v_1 | v_2 \rangle}_{=\mathrm{Tr}(v_1 v_2)} + v_1 \wedge v_2 \tag{E.2.55}$$

which indeed corresponds to a (trivial) sum of products of parings. Suppose the result is true for k, we now show that it is also true for $k + 2$.

We need now only consider $\mathrm{Tr}(v_1 \cdots v_k v_{k+1} v_{k+2})$. The strategy now is to recognize that due to the cyclicity of the trace, if we can obtain a term by anticommuting things through which goes like $\mathrm{Tr}(v_2 \cdots v_{k+2} v_1)$, this will be identical to our original term. In so doing, the anticommutation will yield an inner product, together with a product of k vectors, at which point we can apply the induction hypothesis and we will be done. Symbolically, we first start by defining a set of permutations:

$$K_{i,j} := \left\{ \prod_{k=0}^{\ell} (i, j-k), \quad \ell \in \{1, \cdots, j-i\} \right\} \tag{E.2.56}$$

for $i < j$. In this case, we may write the terms yielded by this anticommutation by considering:

$$\operatorname{Tr}(v_1 \cdots v_{k+2}) = \sum_{\sigma \in K_{1,k+2}} 2\,\operatorname{sgn}(\sigma)\langle v_1 | v_{\sigma(2)}\rangle \operatorname{Tr}(v_{\sigma(3)} \cdots v_{\sigma(k+2)}) + (-1)^{k+1} \operatorname{Tr}(v_2 \cdots v_{k+2} v_1) \tag{E.2.57}$$

recognizing the term we seeked at the end of the sum and putting it at the other side taking into account that $(-1)^{k+1} = -1$, we simply obtain that:

$$\operatorname{Tr}(v_1 \cdots v_{k+2}) = \sum_{\sigma \in K_{1,k+2}} \operatorname{sgn}(\sigma)\langle v_1 | v_{\sigma(2)}\rangle \operatorname{Tr}(\underbrace{v_{\sigma(3)} \cdots v_{\sigma(k+2)}}_{k \text{ terms}}) \tag{E.2.58}$$

Finally, we notice that we may use the induction hypothesis on this last Tr term since it has k terms. Thus, this indeed yields a sum over all possible parings, which concludes the proof of the theorem. ∎

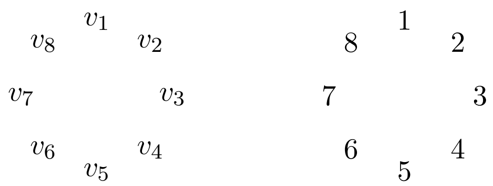
Having proved this result, it is now convenient to compute the number of terms which we expect in general in the decomposition of the trace. By proposition E.2.2, this corresponds to counting how many different products of inner products of pairs of vectors can be chosen among $2n$ vectors. This can be found by considering:

$$\#\text{ terms} = \underbrace{\frac{1}{n!}}_{\substack{\text{Order of}\\ \text{product}\\ \text{irrelevant}}} \underbrace{\prod_{k=1}^{n} \binom{2k}{2}}_{\substack{\text{Inductively}\\ \text{choose 2}\\ \text{vectors}\\ \text{among } 2k}} = \underbrace{\binom{2n}{n}}_{\substack{\text{Choose } n\\ \text{vectors}}} \underbrace{n!}_{\substack{\text{Assign}\\ \text{remaining}\\ \text{vectors to}\\ \text{chosen ones}}} \underbrace{\frac{1}{(2!)^n}}_{\substack{\text{Order of}\\ \text{pairs}\\ \text{irrelevant}}} \tag{E.2.59}$$

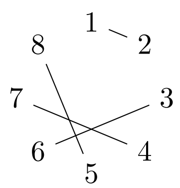
This is totally consistent with a combinatorial analysis of the indices that must be present in the η's. Now, the advantage of the geometrical combinatorial decomposition is that it also hints at a systematic way of generating the terms.

In fact, we can do much better than this as we will see later on. Indeed, we can actually give an expression for every part of the decomposition of any product $v_1 \cdots v_n$.

With this said, before we start considering the most general case, let us formulate the graphical method for the trace. Consider $v_1 \cdots v_n$, the combinatorial expression we found earlier gives a strong hint of how to generate these terms automatically. Indeed, for the sake of illustration suppose we are considering the product of 8 vectors. Then we may generate the terms in the sum by considering the following equivalent diagrams (we simply choose to label the vertices in terms of the indices directly, for simplicity):

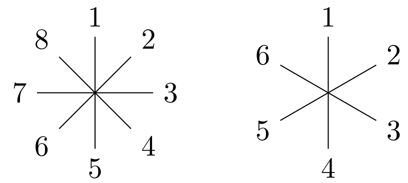


This arrangement allows us to generate all pairings possible in a straight forward manner by simply connecting two paired vertices together. In this way, for example the diagram:



Represents the term: $\langle v_1|v_2\rangle\langle v_3|v_6\rangle\langle v_4|v_7\rangle\langle v_5|v_8\rangle$. The only thing we need to determine now is the sign in front of this term. Using the proof of proposition E.2.2, it is possible to link the number of transpositions of the form $(i, i+k)$ needed to use in order to generate the permutation to get the pairing described by the diagram to the number of line crossings present in the diagram. Notice that these $(i, i+k)$ are nothing other than generators of the set $K_{i,j}$ mentioned in the proof of the proposition. Thus, in the case of the diagram above, we should put a minus sign in front, since it contains 3 crossings. The full diagram can then be interpreted as reading $-\langle v_1|v_2\rangle\langle v_3|v_6\rangle\langle v_4|v_7\rangle\langle v_5|v_8\rangle$.

Remark E.2.5. The crossings should be counted with multiplicity. For example, in the case of the diagrams:



we should count 6 crossings instead of just 1 for the first diagram and , similarly, for the second diagram, 3 crossings instead of 1. The presence of the multiplicity as well as its value becomes clear by deforming the topology of the circle into a line (while keeping the vertices ordered in the clockwise or counterclockwise orientation) and then connecting the corresponding vertices along the same half-plane.

Remark E.2.6. That $\mathrm{Tr}(\gamma^{\mu_1}\cdots\gamma^{\mu_n}) = \mathrm{Tr}(\gamma^{\mu_n}\cdots\gamma^{\mu_1})$ is trivial in this context, as we can clearly see that the number of crossings as well as the diagrams possible are exactly the same up to relabelling of the vertices. It thus doesn't matter if we decide to order our vertices clockwise or counterclockwise, it is nothing but a matter of preference. In fact, this is true for any cyclic permutation, which reflects the cyclicity of the trace.

Proposition E.2.3. The number of crossings of edges in the diagrams of the kind specified above correspond exactly to the number of transpositions of the kind $(i, i+k)$ needed in order to generate the pairing specified by the diagram. Moreover, the sign associated with the diagram is $(-1)^c$, where c is the number of crossings, counted with multiplicity.

Proof. We will proceed with reductive induction in order to prove the result.

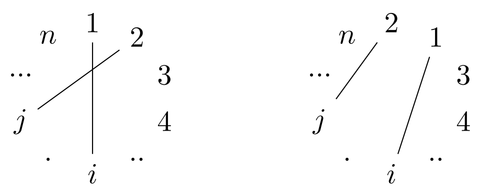
First, we start by realizing that there is a maximum number of crossings that can occur. To see the exact number, we may consider lines on the projective plane in order to count the maximum number of points a set of n lines can intersect. We first draw two secant lines, then we consider adding a third line. The fourth line can be added in such a way that it is parallel to the third line (which means these two must intersect at infinity). We may continue to add lines parallel to the third and fourth lines while keeping track that each time that they will intersect the 3^{rd}, 4^{th},$\cdots$ lines at infinity by simply adding the $n-3$ points to the line of intersection to the line itself. In this way, we clearly see that the result must be a triangular number, which is:

$$\sup\{\text{crossings}\} = \frac{n(n-1)}{2} \tag{E.2.60}$$

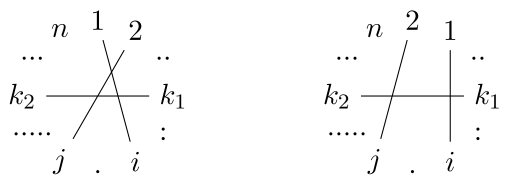
With this upper bound, it is enough to prove that if we can reduce k crossings to $k-1$ crossings using a well-defined algorithm involving a single permutation of the form $(i, i+k)$, the algorithm will terminate and we will have proved the result.

Without loss of generality, we may take out the vertices which are paired adjacently to one another. This is simply because anticommutation of the adjacent vertices will yield the corresponding inner product, while preserving the sign unchanged, *e.g.* $abc = 2\langle a|b\rangle c - bac$. The first step in our algorithm will thus be to take out these adjacently paired vertices, if they are all adjacently paired, we are done and there is nothing to do and the algorithm has terminated. This first step thus yields a diagram of n vertices with k crossings without any adjacently paired vertices.

Next, we pick two vertices which are adjacent to one another whose pairing edges intersect. Such two adjacent elements must exist, since we have taken out all paired adjacent elements and this is the only case for which we expect no intersections to occur. Up to relabeling or rotation of the diagram we may take these two elements to be $\{1, 2\}$. Now, take the permutation $(1, 2)$, which exchanges both of these vertices. Clearly, this eliminates the crossing between the pairing edges of 1 and 2. Diagramatically:



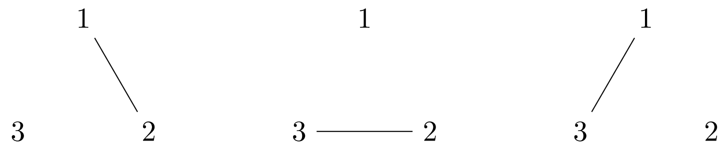
so this much is clear. We have only to verify that this only undoes one crossing. However, this is also clear, since the only crossings influenced by the exchange of 1 with 2 are the crossings which originally are the intersection of pairing edges which intersect edges $1 \to i$ and $2 \to j$. Since we are counting the intersections with multiplicities, it is enough to consider a single one of these intersections and see that the number of crossings is indeed left invariant. Since 1 and 2 are adjacent vertices, any pairing edge $k_1 \to k_2$ intersecting the pairing edge of 1 must also intersect the pairing edge of 2, which implies that the number of crossings generated by edge $k_1 \to k_2$ remains invariant under this construct. This is simply because the set $\{1, 2, i, j\}$ forms a partition of the vertices of the circle. Any edges contained between i and j which are paired together will not be cross the pairing edges of 1 and 2. The only possibility for such a pairing edge $k_1 \to k_2$ crossing the pairing edge of 1 or of 2 is that the vertex k_1 must be between 2 and i and k_2 must be between j and 1. Diagramatically, we have the following situation:



Thus, the number of such crossings is conserved and we have not generated any new crossings with this permutation. Having done this, we go back to the first step and iterate recursively. Through this algorithm, we get a descending chain which eventually terminates when the number of crossings is zero, at which point we will have retrieved all the pairs in the original diagram in the form of adjacent pairs. It follows from this that the sign of the total permutation is given by the number of crossings in the diagram, since at each step of crossing reduction we used a single transposition. ■

Now that we have established a combinatorial way to get the result for the trace, a couple of remarks are in order:

Remark E.2.7. Notice that in the odd case, we can perform a very similar study as we have done before in the case of the trace to find the part of the decomposition fitting into V, with the exception that we will have a single vector which is not paired, this is a reflection of the Z_2-grading of the algebra. In fact, this diagramatic technique extends without any difficulty to this case. An interesting simple case which the reader can check by hand is the case where we consider the product of 3 vectors, the diagrams are :



which yields the expression:

$$\langle v_1|v_2\rangle v_3 + \langle v_2|v_3\rangle v_1 + \langle v_1|v_3\rangle v_2 \tag{E.2.61}$$

This result for the odd case will serve as inspiration of how we can generalize what we found in the case of the trace to higher orders in $\wedge$.

Remark E.2.8. Notice that the upper bound formula we found in the proof of proposition E.2.3 is also a formula which gives us the multiplicity of crossings between multiple lines. This ends up giving us a straightforward way of computing these multiplicities we previously mentionned as follows: if there are n lines at a certain crossing, the multiplicity of the crossing is:

$$\frac{n(n-1)}{2} \tag{E.2.62}$$

E.2.4 Combinatorial Expression of the Full Decomposition

Finally, we are ready to tackle the generalization of what we found for the trace to all of the other terms. It turns out that everything boils down to computing traces. This is simply because the trace induces an inner product on $\mathrm{End}(\bigwedge V)$ (or alternatively induces an inner product on $\mathcal{C}\ell(V,Q)$) by simply considering:

$$\langle x, y\rangle := \mathrm{Tr}(x^* y) \tag{E.2.63}$$

where x^* denotes the conjugation morphism we previously defined. This new interpretation of the trace allows for powerful geometrical arguments, which we will use extensively in the proof of the following theorem.

Theorem E.2.1. *We may extent the graphical method we developped for the trace to find the combinatorial expression at all orders in $\wedge$. Consider the product $v_1 \cdots v_k$, the following rules yield the term going with $v_{i_1} \wedge \cdots \wedge v_{i_q}$ provided that it is not zero:*

1. *Compute the sum:*

$$s \equiv \left\lfloor \frac{q}{2} \right\rfloor + \sum_{j=1}^{\lfloor q/2 \rfloor} i_{2j} - i_{2j-1} \mod 2\,; \tag{E.2.64}$$

2. *The combinatorial expression of the term $v_{i_1} \wedge \cdots \wedge v_{i_q}$ is simply given by:*

$$(-1)^s\, Tr\,(v_1 \cdots \widehat{v}_{i_p} \cdots v_k)\, v_{i_1} \wedge \cdots \wedge v_{i_q} \tag{E.2.65}$$

where $\widehat{v}_{i_p}$ denotes that we have excluded from the product all v_{i_p}'s. To compute this trace, it is sufficient to exclude the i_p vertices in the graphical method developped for the trace and express the sum of all corresponding diagrams.

Proof. Without loss of generality, we will consider only normalized vectors in our product, as the only consequence of not doing so will be an overall constant out front. As we explained before, the Tr operator induces an inner product of $\mathcal{C}\ell(V, Q)$ as a vector space, which itself is an isomorphic to $\bigwedge V$ as vector spaces. In particular, this means that we may define the concept of projection using this inner product as we did above. In so doing, to find the expression of the term $v_{i_1} \wedge \cdots \wedge v_{i_q}$ (provided that this expression, is not zero, in which case the result of the theorem is trivial), it is sufficient to consider its normalized projection. This means that we can express this term as simply being:

$$\frac{\operatorname{Tr}\left[(v_1 \cdots v_k)^*(v_{i_1} \wedge \cdots \wedge v_{i_q})\right]}{\operatorname{Tr}\left[(v_{i_1} \wedge \cdots \wedge v_{i_q})^*(v_{i_1} \wedge \cdots \wedge v_{i_q})\right]}\, v_{i_1} \wedge \cdots \wedge v_{i_q} \tag{E.2.66}$$

Our normalization condition implies simply that the denominator can be taken to be 1. We are thus only left with the numerator of the expression being of importance. While performing a direct inductive proof on the expression given above would be ideal, this procedure can be quite tedious and won't give us much geometrical perspective on the problem. Instead, we start by simplifying the problem a bit. We claim that, in fact:

$$\operatorname{Tr}\left[(v_1 \cdots v_k)^*(v_{i_1} \wedge \cdots \wedge v_{i_q})\right] = \overbrace{\frac{1}{q!}\sum_{\sigma \in S_q} \operatorname{sgn}(\sigma) \operatorname{Tr}\left[(v_1 \cdots v_{\sigma(i_1)} \cdots v_{\sigma(i_q)} \cdots v_k)^*(v_{i_1} \wedge \cdots \wedge v_{i_q})\right]}^{=:\,N} \tag{E.2.67}$$

This statement is obvious if we consider that we may always permute the elements spanning the volume form with any permutation in S_q provided we take the sign of the permutation into account. This is just simply invariance under exchange of the vectors that describe a hyperparallepiped, up to its orientation. Symbolically,

$$v_{i_1} \wedge \cdots \wedge v_{i_q} = \operatorname{sgn}(\sigma)\, v_{\sigma(i_1)} \wedge \cdots \wedge v_{\sigma(i_q)} \quad \forall \sigma \in S_q \tag{E.2.68}$$

Using this identity in the expression of N, we obtain simply that symbolically, all the elements of the sum look the same symbolically. Thus up to relabeling, all the terms are identical and we may sum up the series. With this reasoning, we get:

$$\begin{aligned} N &= \frac{1}{q!}\sum_{\sigma \in S_q} \operatorname{sgn}(\sigma)^2 \operatorname{Tr}\left[(v_1 \cdots v_{\sigma(i_1)} \cdots v_{\sigma(i_q)} \cdots v_k)^*(v_{\sigma(i_1)} \wedge \cdots \wedge v_{\sigma(i_q)})\right] && \text{(E.2.69)} \\ &= \frac{1}{\cancel{q!}}\cancel{q!} \operatorname{Tr}\left[(v_1 \cdots v_k)^*(v_{i_1} \wedge \cdots \wedge v_{i_q})\right] && \text{(E.2.70)} \end{aligned}$$

thus proving our claim that equation E.2.66 holds. Thus, it is sufficient to find the term within

$$\frac{1}{q!}\sum_{\sigma \in S_q} v_1 \cdots v_{\sigma(i_1)} \cdots v_{\sigma(i_q)} \cdots v_k \tag{E.2.71}$$

proportional to $v_{i_1} \wedge \cdots \wedge v_{i_q}$. With this said, the idea now is to put all the $v_{\sigma(i_p)}$'s on the same side by anticommutation. About this anticommutation, it is helpful to make some remarks before we dwell into the procedure:

1. The terms containing inner products of the form $\langle v_{i_p} | a \rangle$ produced by the anticommutation of terms can be discarded in this particular context since the it is impossible that one of these terms is proportional to $v_{i_1} \wedge \cdots \wedge v_{i_q}$ (since the vector v_{i_p} is already in the inner product);

2. Discarding these terms, we also notice that anticommuting through an even number of terms at once does **not** change the overall sign of the product, on the other hand for an odd number of terms, we get a (-1) for every anticommutation.

With these considerations, we have that after anticommuting all the terms through and discarding the inner product part for the reasons above, we obtain:

$$\frac{1}{q!}\sum_{\sigma\in S_q}\operatorname{sgn}(\sigma)\, v_1\cdots v_{\sigma(i_1)}\cdots v_{\sigma(i_q)}\cdots v_k \longmapsto (-1)^s\frac{1}{q!}\sum_{\sigma\in S_q}\operatorname{sgn}(\sigma)\, v_1\cdots v_k\cdot v_{\sigma(i_1)}\cdots v_{\sigma(i_q)} \tag{E.2.72}$$

where this s has to count the number of anticommutations done by an odd number of terms. This turns out to be of the form:

$$\underbrace{(i_2-i_1+1)}_{\substack{\text{Commuting } i_1\\ \text{to } i_2}}+\underbrace{(i_4-i_3+1)}_{\substack{\text{Commuting } i_1, i_2\\ \text{and } i_3 \text{ to } i_4}}+\underbrace{(i_6-i_5+1+\cdots)}_{\text{etc.}} \tag{E.2.73}$$

So we see that we may express this as:

$$s=\sum_{j=1}^{\lfloor\frac{q}{2}\rfloor} i_{2j}-i_{2j-1}+1=\left\lfloor\frac{q}{2}\right\rfloor+\sum_{j=1}^{\lfloor\frac{q}{2}\rfloor} i_{2j}-i_{2j-1} \tag{E.2.74}$$

Notice that this works even in the odd case because at no point have we used the fact that k should be in $2\mathbb{Z}$, so this algorithm is fully general. Furthermore, to ease the computation, we make take this sum mod 2, since in the end we care only about the parity of the result. Finally, we notice a couple of things :

$$(-1)^s\,\overbrace{v_1\cdots\widehat{v}_{\sigma(i_p)}\cdots v_k}^{\substack{\text{No terms con-}\\ \text{taining } v_{\sigma(i_p)}}}\cdot\underbrace{\frac{1}{q!}\sum_{\sigma\in S_q}\operatorname{sgn}(\sigma)\, v_{\sigma(i_1)}\cdots v_{\sigma(i_q)}}_{\text{Volume form}} \tag{E.2.75}$$

Finally, we recognize that for both odd and even products, the product of $k-q\in 2\mathbb{Z}$ elements with which we are left decomposes by virtue of proposition E.2.1 into $\mathrm{Tr}\oplus\bigwedge$. The only term which concerns us the the part of the product which contains the trace of these elements, the rest of the terms in $\bigwedge$ will not contribute. In order to see this, we use once again the trick of using the Tr as a projector, indeed, we can see that the elements going like $v_1\wedge\cdots\wedge\widehat{v}_{i_p}\wedge\cdots\wedge v_k$ containing $k-q-\ell$ terms yield identically zero:

$$\mathrm{Tr}[\underbrace{(v_{i_1}\wedge\cdots\wedge v_{i_q})}_{q \text{ elements}}{}^*\underbrace{(v_1\wedge\cdots\wedge\widehat{v}_{i_p}\wedge\cdots\wedge v_k)}_{k-q-\ell \text{ elements}}{}^*\underbrace{(v_{i_1}\wedge\cdots\wedge v_{i_q})}_{q \text{ elements}}] \tag{E.2.76}$$

We recall that we assume that $v_{i_1}\wedge\cdots\wedge v_{i_q}$ is not zero, which implies this set of q vectors is linearly independent. Now, we have three cases to consider. First, the case where the set $\{v_1,\cdots,v_k\}$ of k vectors is linearly independent. In this case, the trace above is clearly zero, since we have that we are projecting an element of lower degree of the Grassmannian onto the full volume element. Second, the case where the set $\{v_1,\cdots,v_k\}\setminus\{v_{i_1},\cdots v_{i_q}\}$ is linearly independent but $\{v_1,\cdots,v_k\}$ is linearly dependent and $\{v_{i_1},\cdots v_{i_q}\}$ is not a full basis for the space, in this case, the trace also yields zero, since we are projecting different elements of the Grassmannian onto each other. Finally, we need only check the case where we have that both $\{v_{i_1}, \cdots v_{i_q}\}$ and $\{v_1, \cdots, v_k\} \setminus \{v_{i_1}, \cdots v_{i_q}\}$ as being bases of the same space. In this case, we have a change of basis map we may specify with a matrix A. It holds then that $\mathrm{Tr}((\text{basis } 1)^*(\text{basis } 2)^*(\text{basis } 1)) = \det(A)\,\mathrm{Tr}(\text{basis } 1) = 0$. In every case, we see that we get zero contribution from the wedge terms, which finishes the proof of the theorem. ■

Finally, we give some examples for the method. Suppose, for example we want to compute the term going like $v_3 \wedge v_5 \wedge v_7 \wedge v_8$ in a product of 10 elements v_i. We first calculate $s \mod 2$, which yields:

$$2 + (8 - 7) + (5 - 3) = 1 \mod 2 \tag{E.2.77}$$

which yields a total minus sign out front. Then we simply need to compute all diagrams excluding vectors v_3, v_5, v_7, v_8, which can be put under the form:

$$\begin{array}{ccc} & 1 & \\ 10 & & 2 \\ & & \\ 9 & & 4 \\ & 6 & \end{array}$$

Bibliography

[1] R. W. Sharpe, "Differential Geometry: Cartan's Generalization of Klein's Erlangen Program," Springer, New York, (1997).

[2] M. Spivak, "A Comprehensive Introduction to Differential Geometry (Volume I)," Publish or Perish, Houston, TX, (1999).

[3] M. Spivak, "A Comprehensive Introduction to Differential Geometry (Volume II)," Publish or Perish, Houston, TX, (1999).

[4] S. Kobayashi, K. Nomizu, "Foundations of Differential Geometry (Volume I)," Wiley Classics, New York, (1963).

[5] H. Weyl, "Raum · Zeit · Materie.," Verlag von Julius Springer, Berlin, (1919).

[6] F. Klein, "Vergleichende Betrachtungen über neuere geometrische Forschungen," Mathematische Annalen, 43 (1893), pp.63-100.

[7] C. Ehresmann, "Les connections infinitésimales dans un espace fibré différentiable," Colloq. de Topol., CBRM, Bruxelles (1950), pp. 29-55.

[8] S. S. Chern, "The Geometry of G-structures," Bull. Amer. Math. Soc. 72, New York (1966), pp.167-219.

[9] W. Fulton and J. Harris, "Representation Theory: A First Course," Springer, (2004).

[10] É. Cartan, "The Theory of Spinors," Hermann, Paris (1966).

[11] A. Haelfliger, 'Feuilletages sur les variétés ouvertes," Topology. an International Journal of Mathematics, 9 (2): pp.183-194, (1970).

[12] J. Attard, J. François, S. Lazzarini, T. Masson, "The Dressing Field Method of Gauge Symmetry Reduction, A Review With Examples," arXiv:1702.02753, (2017).

[13] J. François, "Reduction of Gauge Symmetries: A New Geometrical Approach," Aix-Marseille University, www.theses.fr/2014AIXM4037.pdf, (2014).

[14] C. Lorcé, "Geometrical Approach to the Proton Spin Decomposition," Phys. Rev. D 87, 034031 (2013), doi:10.1103/PhysRevD.87.034031

[15] P. A. M. Dirac, "Gauge-Invariant Formulation of Quantum Electrodynamics," Canadian Journal of Physics, 33(11): pp.650-660, (1955).

[16] S. Mandelstam, "Feynman Rules for Electromagnetic and Yang-Mills Fields from the Gauge-Independent Field-Theoretic Formalism," Phys. Rev. 175, p.1580, (1968).

[17] O. Steinmann, "Gauge Invariant Fields in Nonabelian Gauge Theories," Helv. Phys. Acta 58, p.995, (1985).

[18] C. Lorcé, "Gauge-Covariant Canonical Formalism Revisited With Application to the Proton Spin Decomposition," Phys. Rev. D 88, 044037, (2013).

[19] É. Cartan, "Sur une gnralisation de la notion de courbure de Riemann et les espaces torsion," C. R. Acad. Sci. (Paris) 174, 593-595 (1922).

[20] É. Cartan, "Sur les varits connexion affine et la thorie de la relativit gnralise," Part I: Ann. c. Norm. 40, 325-412 (1923) and ibid. 41, 1-25 (1924); Part II: ibid. 42, 17-88 (1925).

[21] D. W. Sciama, "The Physical Structure of General Relativity," Rev. Mod. Phys. 36, 463 (1964), doi:10.1103/RevModPhys.36.463

[22] T. W. B. Kibble, "Lorentz Invariance and the Gravitational Field," Journal of Mathematical Physics 2, 212 (1961), doi:10.1063/1.1703702

[23] F. W. Hehl, *et al.* "General Relativity with Spin and Torsion: Foundations and Prospects," Rev. Mod. Phys. 48, 393 (1976), doi:10.1103/RevModPhys.48.393

[24] O. Stern, "Ein Weg zur experimentellen Prüfung der Richtungsquantelung im Magnetfeld," Zeitschrift für Physik, Volume 7, Issue 1, pp.249-253 (1921),doi:10.1007/BF01332793

[25] O. Stern, W. Gerlach, "Der experimentelle Nachweis des magnetischen Moments des Silberatoms," Zeitschrift für Physik, 8, pp.110-111 (1921).

[26] É. Cartan, "Les groupes projectifs qui ne laissent invariante aucune multiplicit plane," Bul. Soc. Math. France, 41: pp.53-96 (1913).

[27] G. Farmelo, "The Strangest Man: The Hidden Life of Paul Dirac, Quantum Genius," Faber & Faber, ISBN 978-0-571-22286-5, p. 430 (2009).

[28] J.H. Gallier, "Geometric Methods and Applications," Springer-Verlag, ISBN 0-387-95044-3, p.38 (2001).

[29] A. Haefliger, "Sur l'extension du groupe structural d'un espace fibré," C. R. Acad. Sci. Paris. 243: pp. 558-560 (1956).