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Internship Report

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Title:

Introduction to the Critical Properties of the ϕ^4-model

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Contents

Chapter 1

Introduction

In this report, we will give an introduction to the critical properties of the ϕ^4-model. To do this, we will begin by motivating the study of ϕ^4-theories by examining an example of application in Condensed Matter physics of this model to the study of the critical phenomena relating to spontaneous magnetization, which will be studied with the Ising Model. After this brief motivation of the study of ϕ^4-theories, we will present a systematic way of approaching the problem using Perturbation Theory, as we will note that no exact analytical solution has been found to date.
We will introduce various concepts along the way, such as the Feynman integrals, which naturally arise from the perturbation expansion and which can be represented diagramatically. In so doing, we will also provide the reader a clear introduction to Feynman diagrams, so that wherever possible the manuscript is self-contained. For more calculatory results, which don't necessarily provide the reader with extra intuition, the reader will be referred to Kleinert's book on ϕ^4 theories[2] for a more detailed proof of the results given.
We will quickly realize that with the perturbative approach taken, divergences arising at all orders in the perturbation expansion will have to be dealt with. This will motivate us to introduce renormalization process as a way of systematically absorbing these divergences in order to be able to make finite physical predictions. We will indeed present 3 different approaches to renormalization, which for the avid reader are: the conventional approach, the counter-term method and finally the more modern and perhaps most efficient of the above, the Bogoliubov-Parasiuk-Hepp-Zimmermann (BPHZ) method using the Zimmermann Forest Formula.
Finally, we will retrieve the β-function from the renormalization group and we will calculate the η critical exponent presented in Chapter 2.
This internship was done at the Laboratoire de Physique Théorique et Hautes Énergies (LPTHE), which is a Joint Research Unit (Unit Mixte de Recherche) of the Université Pierre et Marie Curie and of the CNRS. It is composed of 19 researchers and 10 professors-researchers. The scienfitic activity of the laboratory is centered around Quantum Field Theory (QFT), both in its very theoretical aspects as well as in its applications. It posseses four main fields of research, namely: mathematical physics, string, brane and field theory, condensed matter and statistical physics, and particle physics and cosmology.

Chapter 2

ϕ^4-model as an effective field theory

2.1 Modelization of spin networks

One of the notable applications of field theories is the modelization of spin networks. Consider a spin network in D-dimensional space. We can model this system by considering the following Hamiltonian:

$$\mathcal{H} = -J \sum_{\langle i,j \rangle} \vec{S}_i \cdot \vec{S}_j \,, \tag{2.1.1}$$

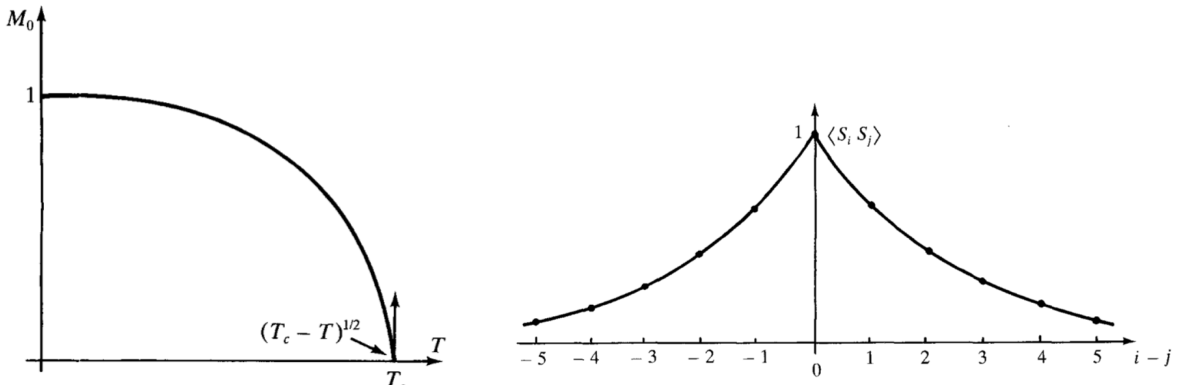
where $\langle i, j \rangle$ denotes the sum over the closest neighbours, J is a coupling constant and $\vec{S}_i$ are vectors in $\mathbb{R}^3$. This is known as the Heisenberg model and it is an incredibly challenging problem for which no analytic solution is known to date and numerical treatment of the problem can also be quite delicate. We can take the simplification one step further by only considering the projection S_z of the spins, which can take values ± 1 (which is valid for a system in which there exists an anisotropy). In this way, the Hamiltonian then becomes:

$$\mathcal{H} = -J \sum_{\langle i,j \rangle} S_{i,z} S_{j,z} \,. \tag{2.1.2}$$

We call this simplification of the problem the Ising Model. This Hamiltonian clearly exhibits $\mathbb{Z}/2\mathbb{Z}$ symmetry. Notice that the ground state is doubly degenerate having two configurations that minimize the energy: all spins down or all spins up. Remarkably, at low temperatures, the system exhibits spontaneous magnetization, and the symmetry of the Hamiltonian is broken, which means that the average magnetization is either positive or negative. This model has an exact solution in dimension 2, which was found by Onsager in 1944. For higher dimensions, since no exact solution is known, we will need approximate methods in order to calculate the thermodynamic properties of this system. In order to define the magnetization $\langle M \rangle$, we first need to introduce the partition function , $\mathcal{Z}$, conventionally used in statistical mechanics:

$$\mathcal{Z} := \sum_{\text{Configs}} \mathrm{e}^{-\beta \mathcal{H}}, \qquad \langle M \rangle := \frac{1}{\mathcal{Z}} \sum_{i \in \text{ Network}} S_i \mathrm{e}^{-\beta \mathcal{H}} \tag{2.1.3}$$

A depiction of the behaviour of the magnetization as a function of temperature is described in Figure 2.1a. As we can see, there is a phase transition at the critical temperature T_c as indicated on the Figure, and we can appreciate that there is spontaneous magnetization of this system below this critical temperature. This constitutes what we call a second order phase



(a) Magnetization of the system as a function of temperature. T_c is the critical temperature, at which the $\mathbb{Z}/2\mathbb{Z}$ symmetry of the Hamiltonian is broken.

(b) Correlation function $G(\mathbf{x})$ of the $D = 1$ Ising model, as we can see, there is an exponential decay as we get away from the 0th spin.

Figure 2.1: Measurable parameters of the system

transition[1], which means that this transition happens continuously. It turns out that in a neighbourhood of T_c, the system is characterized by a set of *critical exponents* β, ν, η and ξ and various quantities in the system depend asymptotically on these exponents. Some examples of these asymptotic relations are are:

$$\langle M \rangle \sim |T - T_c|^{\beta} \tag{2.1.4}$$

$$G(\mathbf{x}) \sim \frac{1}{|\mathbf{x}|^{D-2+\eta}} \mathrm{e}^{-|\mathbf{x}|/\xi} \tag{2.1.5}$$

$$\xi \sim |T - T_c|^{-\nu} \tag{2.1.6}$$

Here, $G(\mathbf{x}) := \langle \mathbf{M}(\mathbf{x})\mathbf{M}(\mathbf{0}) \rangle$ a correlation function, from which we are able to deduce the typical length scale of the phase transition ξ at which the spins are correlated (ξ is the correlation length). Notice that at the critical temperature $T = T_c$, $\xi \to \infty$ and there is an algebraic decay in the correlation function. This means that there is no characteristic scale for the system at the critical temperature.

It turns out that these critical exponents are universal quantities which do not depend on the microscopic features of the system, *e.g.* the crystal structure in which the spin system is embedded. Instead they depend on the symmetries of the Hamiltonian at hand. In particular, this means that there are so-called *classes of universality*, which means that every Hamiltonian that can be associated to a particular symmetry group will share critical exponents with other Hamiltonians belonging to the same class, despite the fact that the actual physical systems can be very different in general. It is important to note that T_c is *not* a universal quantity. This motivates the finding of these critical exponents, which require solving in some way the Ising Model. Seeing as though we have no analytical way of doing so for dimensions $D > 2$, we require a new way to study this problem. In the following, we will aim at calculating the critical exponent η, however the methods we will use can also be used to compute the other exponents as well.

[1] There are also first order transitions

2.2 From the Ising model to the ϕ^4-model

We motivate the transition from the Ising Model lattice to a field theoretic ϕ^4-model by considering two things. On one hand, a model exhibiting reasonably accurate behaviour mimicking the Ising Model must posses the inherent symmetries present in the Ising Model, *i.e.* the $\mathbb{Z}/2\mathbb{Z}$ symmetry we previously discussed.

On the other hand, it is practical to go back to the D-dimensional Ising Model to understand how exactly we can formulate such a model. Since we are interested only in what happens at a macroscopic scale, keeping track of small-scale features of the system is redundant. Furthermore, if the length scale at which a correlation between the spins exists is much greater than the space between each of the spins, this motivates identifying S_i in the Ising Hamiltonian with a continuous scalar field $\phi(\mathbf{x}_i)$. Following Wilson [1], let us consider a generalization of the Ising model:

$$\mathcal{Z}(K,h;r,\lambda) = \prod_{n\in\text{Network}} \int_{-\infty}^{\infty} \mathrm{d}\phi_n \exp\left[\sum_{n\in\text{Network}} -r\phi_n^2 - \lambda\phi_n^4 + K\sum_i \phi_{n+i}\phi_n + h\phi_n\right] \tag{2.2.7}$$

where we allow the fields ϕ_i to vary between $\pm\infty$. We can then consider two cases:

1. For $\lambda = r = 0$, which we will refer to as the Gaussian model, we have that:

$$\sum_{\langle i,j\rangle} S_iS_j \sim \sum_{\langle i,j\rangle} \phi(\mathbf{x}_i)\phi(\mathbf{x}_j) = \sum_{i,a_i} \phi(\mathbf{x}_i)\phi(\mathbf{x}_i+\mathbf{a}_i)\,. \tag{2.2.8}$$

   Where a_i is simply displacing the ith coordinate over to its closest neighbours. Since we assume that ϕ is analytic, we may Taylor expand $\phi(\mathbf{x}_i+\mathbf{a}_i)$ thus obtaining:

$$\sum_{i,a_i} \phi(\mathbf{x}_i)\phi(\mathbf{x}_i+\mathbf{a}_i) = \sum_i\sum_{a_i} \phi(\mathbf{x}_i)\left(\phi(\mathbf{x}_i) + \cancel{\mathbf{a}_i\cdot\nabla\phi(\mathbf{x}_i)} + \frac{\mathbf{a}_i^2}{2}\nabla^2\phi(\mathbf{x}_i) + \mathrm{O}(\mathbf{a}_i^3)\right)\,. \tag{2.2.9}$$

   Where the term was cancelled because it does not exhibit $\mathbb{Z}/2\mathbb{Z}$ symmetry like the original Hamiltonian, which was one of our requirements. And so, performing the sum over the closest neighbours and going to the continuum limit, we simply obtain:

$$\sum_i \phi^2(\mathbf{x}_i) + \frac{a^2}{2}\nabla^2\phi(\mathbf{x}_i) + \mathrm{O}(a^3) \sim \int \mathrm{d}^Dx\,\left(\phi^2(\mathbf{x}) - \frac{1}{2}(\nabla\phi(\mathbf{x}))^2\right)\,. \tag{2.2.10}$$

2. For $r = -2\lambda$ and $\lambda \to +\infty$, we retrieve the Ising Model.

   To see this more clearly, recall that in the Ising Hamiltonian, the intrinsic values of $S_i = \pm 1$. In particular, this means that the intrinsic probability density function for the spins is of the form of two δ-functions centered at ±1 with the appropriate normalization. Now, consider the more general model we previously introduced and further consider $r = -2\lambda$. We then have that the intrinsic probability function for the variable ϕ is given by:

$$P(\phi) = \mathrm{e}^{-\lambda(\phi^2-1)^2}\,. \tag{2.2.11}$$

   The behaviour of this function with varying values of λ is plotted in Figure 2.2. As we can see, as $\lambda \to \infty$ we retrieve the desired result of δ-functions around ±1. Furthermore, this results in the introduction of the term in ϕ^4 as desired (we notice that the term of the gradient comes from the interaction between spins that was considered in the Gaussian model).

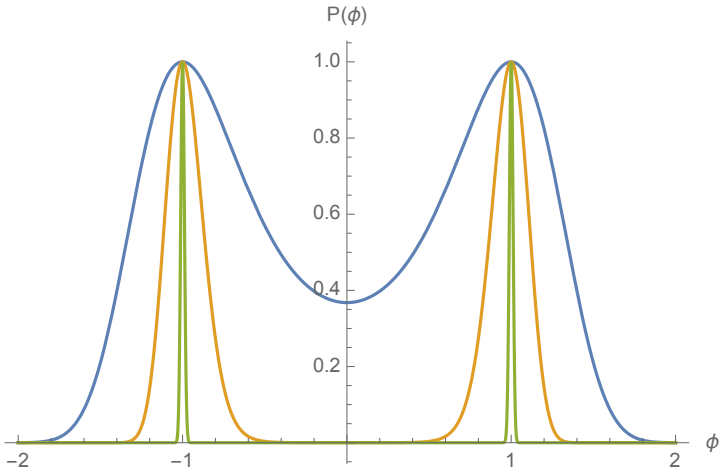


Figure 2.2: Behaviour of Equation 2.2.11 with varying λ

It is noteworthy that this reasoning is not unique. We could've also reasoned using a Landau-Ginzburg approach, in which we could've simply justified the extension from Ising to the field theory by simply keeping all the terms that preserve the $\mathbb{Z}/2\mathbb{Z}$ symmetry

At this point, to retrieve the ϕ^4 model, it is necessary to shrink our network spacing down to zero, thus really retrieving a continuous function ϕ. We shall thus make this procedure more explicit by giving a very handwavy intuitive explanation: let's start by going back to the case where we consider a finite, or at least countable product, which is well defined we shall thus consider a box with N particles as in Wilson's generalized Ising Model. We then have:

$$\mathcal{Z} = \sum_{\text{configs}} \mathrm{e}^{-\beta\mathcal{H}} = \sum_{S_1} \cdots \sum_{S_N} \mathrm{e}^{-\beta\mathcal{H}(S_1,\cdots,S_N)} \sim \int \mathrm{d}\phi(\mathbf{x}_1)\cdots\mathrm{d}\phi(\mathbf{x}_N)\mathrm{e}^{-E[\phi]}\,, \tag{2.2.12}$$

where $E[\phi]$ is taken to be dimensionless (we have set $k_BT = 1$). However, because we have gone to the continuum limit, we have also implicitly made the number of particles N larger, thus we must take the product over all $\mathbf{x}$'s possible in the space (because our cell-size is going to zero). Thus we have:

$$\mathcal{Z} = \int \mathcal{D}\phi\, \mathrm{e}^{-E[\phi(\mathbf{x})]} := \mathcal{N}\prod_{\mathbf{x}} \int \mathrm{d}\phi(\mathbf{x})\, \mathrm{e}^{-E[\phi(\mathbf{x})]}\,. \tag{2.2.13}$$

Where $\mathcal{N}$ is a constant that will absorb all the infinities coming out of the infinite non-countable product of the measures. We have thus defined the partition function in the continuum case, we may thus write it as:

$$\mathcal{Z} = \int \mathcal{D}\phi(\mathbf{x})\, \mathrm{e}^{-E[\phi(\mathbf{x})]}, \qquad E[\phi] = \int \mathrm{d}^D\mathbf{x} \left(\frac{1}{2}(\nabla\phi(\mathbf{x}))^2 + \frac{1}{2}m^2\phi^2(\mathbf{x}) + \frac{\lambda}{4!}\phi^4(\mathbf{x})\right)\,. \tag{2.2.14}$$

Thus, we have arrived at the ϕ^4-model succesfully. Of course, there are other applications of this model in other areas of physics, but this at least motivates our continued study of this model in subsequent chapters. Notice, however, that the $\phi \mapsto -\phi$ map still leaves the energy functional invariant, which is closely related to the $\mathbb{Z}/2\mathbb{Z}$ original symmetry that the Ising model exhibited. More generally, we may consider ϕ as a field having N components, *i.e.* $\phi = (\phi_1, \cdots, \phi_N)$. In this case, the ϕ^4 model will have $O(N)$ symmetry and this more general model has extensive applications. This corresponds to systems exhibiting different universality classes, which depend only on their symmetries. For example:

1. $N = 0$ Self-avoiding walks: polymers [15],
2. $N = 1$ Ising universality class: liquid-vapour transitions and uniaxial magnets,
3. $N = 2$ XY universality class: superfluid λ-transition of He [16],
4. $N = 3$ Heisenberg universality class: isotropic ferromagnets,
5. $N = 4$ Finite temperature QCD with two light flavours [17]

However, our modest goal will be to focus only on the $N = 1$ Ising universality class case.

Chapter 3

Perturbative methods

In the last chapter, we gave a physical motivation to study the ϕ^4-model. In order to do so, it will be necessary to introduce some terminology and give some definitions. Let us write the energy functional in Equation 2.2.14 as:

$$E[\phi] = E_0[\phi] + V[\phi] \tag{3.0.1}$$

Where we define:

$$E_0[\phi] = \int \mathrm{d}^D\mathbf{x}\, \frac{1}{2}(\nabla\phi(\mathbf{x}))^2 + \frac{1}{2}m^2\phi^2(\mathbf{x})\,, \tag{3.0.2}$$

$$V[\phi] = \frac{\lambda}{4!}\int \mathrm{d}^D\mathbf{x}\, \phi^4(\mathbf{x})\,. \tag{3.0.3}$$

We call $E_0[\phi]$ the free field theory (or Gaussian model) and we call $V[\phi]$ the interaction term. In the following, all the quantities with a subscript 0 will refer to the free field theory and all others without subscript or with a different subscript will be terms due to the interaction.

Let us start by defining the n-point correlation function:

$$G^{(n)}(\mathbf{x}_1, \cdots, \mathbf{x}_n) := \langle\phi(\mathbf{x}_1)\cdots\phi(\mathbf{x}_n)\rangle = \frac{1}{\mathcal{Z}}\int \mathcal{D}\phi\, \phi(\mathbf{x}_1)\cdots\phi(\mathbf{x}_n)\, \mathrm{e}^{-E[\phi]} \tag{3.0.4}$$

We could similarly define $G_0^{(n)}$ by simply taking $E_0[\phi]$ as the functional in the exponential term. From this expression, it can be quite challenging to compute the correlation function due to the term in ϕ^4, for the case of the perturbed functional $E[\phi]$. It is useful at this point to use the conventional probability theoretic approach and introduce a moment generating functional $\mathcal{Z}[j]$. In order to get this generating functional, we make E a functional of a new field j which we will call the current field (this will be equivalent to adding a source to the Lagrangian). The expression for $E[\phi, j]$ is given by:

$$E[\phi, j] := E[\phi] - \int \mathrm{d}^D\mathbf{x}\, \phi(\mathbf{x})j(\mathbf{x})\,. \tag{3.0.5}$$

We can then make $\mathcal{Z}$ a functional depending on $j(\mathbf{x})$ by simply replacing the $E[\phi]$ in the expression of $\mathcal{Z}$ by a $E[\phi, j]$. In doing so, we obtain the generating function needed for our problem. The correlation function with n points is thus given by:

$$G^{(n)}(\mathbf{x}_1, \cdots, \mathbf{x}_n) = \frac{1}{\mathcal{Z}}\left[\frac{\delta}{\delta j(\mathbf{x}_1)}\cdots\frac{\delta}{\delta j(\mathbf{x}_n)}\mathcal{Z}[j]\right]_{j=0}\,, \tag{3.0.6}$$

where $\frac{\delta}{\delta j(\mathbf{x}_i)}$ is the functional derivative with respect to $j(\mathbf{x}_i)$.
In this chapter, we will provide an introduction to the perturbative study of the ϕ^4-model. We will show that the Gaussian model is an exactly solvable model and how we can treat the ϕ^4 term using Perturbation Theory. This is similar to Quantum Mechanics, with the exception that when we are dealing with fields, we deal with an infinite number of degrees of freedom, namely $\phi(\mathbf{x})$ with $\mathbf{x} \in \mathbb{R}^4$.

3.1 Free field theory

With the help of the above, we now proceed to solve the Gaussian model, which we can do exactly, since the energy functional is quadratic in ϕ. Let us then simply start by rewriting the free field action as:

$$E_0[\phi] = \frac{1}{2}\int \mathrm{d}^D\mathbf{x}\,(\nabla\phi(\mathbf{x}))^2 + m^2\phi(\mathbf{x})^2 \tag{3.1.7}$$

$$= \frac{1}{2}\int \mathrm{d}^D\mathbf{x}\,\phi(\mathbf{x})(-\nabla^2 + m^2)\phi(\mathbf{x}) \tag{3.1.8}$$

$$= \frac{1}{2}\int \mathrm{d}^D\mathbf{x}_1\,\mathrm{d}^D\mathbf{x}_2\ \phi(\mathbf{x}_1)D(\mathbf{x}_1,\mathbf{x}_2)\phi(\mathbf{x}_2) \tag{3.1.9}$$

where $D(\mathbf{x}_1,\mathbf{x}_2) := \delta^{(D)}(\mathbf{x}_1-\mathbf{x}_2)(-\nabla^2+m^2)$ is a symmetric matrix, we can thus diagonalize it and express it in terms of its eigenvalues λ_i. We can then calculate the partition function as follows for the diagonalized D:

$$\mathcal{Z}_0 = \int \mathcal{D}\phi\,\mathrm{e}^{-E_0[\phi]} \tag{3.1.10}$$

$$= \mathcal{N}\prod_{\mathbf{x}}\left[\int \mathrm{d}\phi(\mathbf{x})\exp\left(-\frac{1}{2}\int \mathrm{d}^D\mathbf{x}_1\,\mathrm{d}^D\mathbf{x}_2\,\phi(\mathbf{x}_1)D(\mathbf{x}_1,\mathbf{x}_2)\phi(\mathbf{x}_2)\right)\right] \tag{3.1.11}$$

$$= \mathcal{N}\prod_i \int \mathrm{d}\phi_i\,\mathrm{e}^{-\frac{1}{2}\lambda_i\phi_i^2} = \frac{1}{\sqrt{\prod_i \lambda_i}} = \frac{1}{\sqrt{\det D}} = \mathrm{e}^{-\frac{1}{2}\mathrm{Tr}(\log D)} \tag{3.1.12}$$

Where we define the determinant of the operator acting on the corresponding Hilbert space to be defined by the identity: $\det D := \mathrm{e}^{\mathrm{Tr}(\log D)}$ (since the trace can be defined using integration and D is a compact, self-adjoint operator, which guarantees the existence of $\log D$). Using this equivalency, we obtain the last equality in the above equation. Like before, we may also define a generating function for the correlation functions by simply introducing a field of current, we thus have that the generating function is given by (see Kleinert [2] for proof):

$$\mathcal{Z}_0[j] = \mathrm{e}^{\frac{1}{2}\int \mathrm{d}^D\mathbf{x}\,\mathrm{d}^D\mathbf{x}'\,j(\mathbf{x})D^{-1}(\mathbf{x},\mathbf{x}')j(\mathbf{x}')} \tag{3.1.13}$$

Thus we can use this as a generating function for the correlation functions. Explicitly this gives us that:

$$G_0^{(n)}(\mathbf{x}_1,\cdots,\mathbf{x}_n) = \left[\frac{\delta}{\delta j(\mathbf{x}_1)}\cdots\frac{\delta}{\delta j(\mathbf{x}_n)}\,\mathcal{Z}_0[j]\right]_{j=0} \tag{3.1.14}$$

$$= \left[\frac{\delta}{\delta j(\mathbf{x}_1)}\cdots\frac{\delta}{\delta j(\mathbf{x}_n)}\,\mathrm{e}^{\frac{1}{2}\int \mathrm{d}^D\mathbf{x}\,\mathrm{d}^D\mathbf{x}'\,j(\mathbf{x})D^{-1}(\mathbf{x},\mathbf{x}')j(\mathbf{x}')}\right]_{j=0} \tag{3.1.15}$$

First, we notice that only for even n do we have a correlation function which is not identically zero. We can perform the n functional derivatives on the exponential, realizing each time we

will have the derivative of the argument of the exponential times the exponential, when we set $j = 0$ at the end, we thus simply retrieve:

$$G_0^{(n)}(\mathbf{x}_1, \cdots, \mathbf{x}_n) = \frac{1}{2^{n/2}(n/2)!}\left[\frac{\delta}{\delta j(\mathbf{x}_1)}\cdots\frac{\delta}{\delta j(\mathbf{x}_n)}\left(\int \mathrm{d}^D\mathbf{x}\,\mathrm{d}^D\mathbf{x}'\, j(\mathbf{x})D^{-1}(\mathbf{x},\mathbf{x}')j(\mathbf{x}')\right)^{n/2}\right] \tag{3.1.16}$$

In particular, for the 2-point correlation function we have that:

$$\begin{aligned} G_0^{(2)}(\mathbf{x}_1,\mathbf{x}_2) &= \frac{1}{2}\left[\frac{\delta}{\delta j(\mathbf{x}_1)}\frac{\delta}{\delta j(\mathbf{x}_2)}\int \mathrm{d}^D\mathbf{x}\,\mathrm{d}^D\mathbf{x}'\, j(\mathbf{x})D^{-1}(\mathbf{x},\mathbf{x}')j(\mathbf{x}')\right]_{j=0} && (3.1.17)\\ &= \frac{1}{2}(D^{-1}(\mathbf{x}_1,\mathbf{x}_2) + D^{-1}(\mathbf{x}_2,\mathbf{x}_1)) && (3.1.18)\\ &= D^{-1}(\mathbf{x}_1,\mathbf{x}_2) && (3.1.19)\end{aligned}$$

Where the last equality yields due to the symmetry in the arguments of the operator D. This result means that the propagator (*i.e.* the 2-point correlation function) is the inverse of the operator D we found previously characterizing the quadratic form we started with.

Theorem 1. *(Wick-Isserlis Theorem) Let $\langle\phi(\mathbf{x}_1)\cdots\phi(\mathbf{x}_n)\rangle$ denote the nth correlation function given by the expectation value of the fields at $\mathbf{x}_1\cdots\mathbf{x}_n$. Then:*

$$\begin{aligned}\langle\phi(\mathbf{x}_1)\cdots\phi(\mathbf{x}_{2n})\rangle &= \sum\prod\langle\phi(\mathbf{x}_i)\phi(\mathbf{x}_j)\rangle\,, && (3.1.20)\\ \langle\phi(\mathbf{x}_1)\cdots\phi(\mathbf{x}_{2n+1}))\rangle &= 0\,, && (3.1.21)\end{aligned}$$

Where $\sum\prod$ denotes a sum over all distinct ways of partitioning $\phi(\mathbf{x}_1)\cdots\phi(\mathbf{x}_{2n})$ into pairs. Moreover, this sum contains $\frac{(2n)!}{2^n n!}$ terms. Taking into account the symmetry in the arguments of G and the commutativity of multiplication, we get a sum of $(2n)!/[2^n n!] := (2n-1)!!$ different terms.

Proof. Indeed, by looking at the expression of the kth correlation function found previously in terms of the generating function, we can see that for any odd number $k = 2n+1$, we will have that the correlation function will be identically zero. On the other hand, for any even $k = 2n$, we have that after performing the functional derivatives we arrive at:

$$G_0^{(2n)}(\mathbf{x}_1\cdots\mathbf{x}_{2n}) = \frac{1}{2^n n!}\left\{\frac{\delta}{\delta j(\mathbf{x}_1)}\cdots\frac{\delta}{\delta j(\mathbf{x}_{2n})}\left[\int \mathrm{d}^D\mathbf{x}\,\mathrm{d}^D\mathbf{x}'\, j(\mathbf{x})D^{-1}(\mathbf{x},\mathbf{x}')j(\mathbf{x}')\right]^n\right\}_{j=0}. \tag{3.1.22}$$

We then proceed by induction on even $k = 2n$. For $n = 1$ the statement is trivial, and $G_0^{(2)}(\mathbf{x}_1,\mathbf{x}_2) = D^{-1}(\mathbf{x}_1,\mathbf{x}_2)$. We then assume that the statement is true for $n > 1$. We then have that (for simplicity and to simplify the notation we will note $j(\mathbf{x}_i) := j_i$ and we will note

the integral $I := \int \mathrm{d}^D\mathbf{x}\,\mathrm{d}^D\mathbf{x}'\, j(\mathbf{x})D^{-1}(\mathbf{x},\mathbf{x}')j(\mathbf{x}'))$

$$\begin{aligned}
\langle\phi_1\cdots\phi_{2n+2}\rangle &= \frac{1}{2^{n+1}(n+1)!}\left\{\frac{\delta}{\delta j_1}\cdots\frac{\delta}{\delta j_{2n}}\left(\frac{\delta}{\delta j_{2n+1}}\frac{\delta}{\delta j_{2n+2}}I^{n+1}\right)\right\}_{j=0}\\
&= \frac{1}{2^{n+1}\cancel{(n+1)}!}\left\{\frac{\delta}{\delta j_1}\cdots\frac{\delta}{\delta j_{2n}}\left(\frac{\delta}{\delta j_{2n+1}}\cancel{(n+1)}\frac{\delta I}{\delta j_{2n+2}}I^{n}\right)\right\}_{j=0}\\
&= \frac{1}{2^{n+1}n!}\left\{\frac{\delta}{\delta j_1}\cdots\frac{\delta}{\delta j_{2n}}\left(\underbrace{\frac{\delta^2 I}{\delta j_{2n+1}\delta j_{2n+2}}}_{2\langle\phi_{2n+1}\phi_{2n+2}\rangle}I^n + \frac{\delta I^n}{\delta j_{2n+1}}\frac{\delta I}{\delta j_{2n+2}}\right)\right\}_{j=0}\\
&= \frac{1}{2^{n+1}n!}\left\{2\langle\phi_{2n+1}\phi_{2n+2}\rangle\frac{\delta^{2n}I^n}{\delta j_1\cdots\delta j_{2n}} + \frac{\delta^{2n}}{\delta j_1\cdots\delta j_{2n}}\left(\frac{\delta I^n}{\delta j_{2n+1}}\frac{\delta I}{\delta j_{2n+2}}\right)\right\}_{j=0}.
\end{aligned}$$

By the induction hypothesis, we can declare the first term to be the sum over all distinct ways of partitioning $\phi_1\cdots\phi_{2n}$ into pairs. Thus we have:

$$\langle\phi_1\cdots\phi_{2n+2}\rangle = \sum_{i,j\in\{1,\cdots,2n\}}\prod_{i\neq j}\langle\phi_i\phi_j\rangle\langle\phi_{2n+1}\phi_{2n+2}\rangle + \frac{1}{2^{n+1}n!}\left\{\frac{\delta^{2n}}{\delta j_1\cdots\delta j_{2n}}\left(\frac{\delta I^n}{\delta j_{2n+1}}\frac{\delta I}{\delta j_{2n+2}}\right)\right\}_{j=0}. \tag{3.1.23}$$

We then focus our attention on the second term and we use the generalized Leibniz property (which is of course exhibited by the functional derivative):

$$\frac{\delta^{2n}}{\delta j_1\cdots\delta j_{2n}}\left(\frac{\delta I^n}{\delta j_{2n+1}}\frac{\delta I}{\delta j_{2n+2}}\right) = \sum_{S\in\mathcal{P}\{1,\cdots,2n\}}\left(\frac{\delta^{|S|}}{\prod_{i\in S}\delta j_i}\frac{\delta I^n}{\delta j_{2n+1}}\right)\left(\frac{\delta^{2n-|S|}}{\prod_{i\notin S}\delta j_i}\frac{\delta I}{\delta j_{2n+2}}\right). \tag{3.1.24}$$

From this, we can see that the term on the right will vanish if $|S|\neq 2n-1$, thus we impose the condition that $|S|=2n-1$. We thus obtain (using the induction hypothesis):

$$\frac{\delta^{2n}}{\delta j_1\cdots\delta j_{2n}}\left(\frac{\delta I^n}{\delta j_{2n+1}}\frac{\delta I}{\delta j_{2n+2}}\right) = \sum_{\substack{S\in\mathcal{P}\{1,\cdots,2n\}\\|S|=2n-1}}\underbrace{\left(\frac{\delta^{2n-1}}{\prod_{i\in S}\delta j_i}\frac{\delta I^n}{\delta j_{2n+1}}\right)}_{\sum\prod\langle\phi_i\phi_l\rangle}\underbrace{\left(\frac{\delta}{\prod_{k\notin S}\delta j_k}\frac{\delta I}{\delta j_{2n+2}}\right)}_{2\langle\phi_k\phi_{2n+2}\rangle}. \tag{3.1.25}$$

Finally, going back to the original equation we obtain the desired result:

$$\langle\phi_1\cdots\phi_{2n+2}\rangle = \sum_{i,j\in\{1,\cdots,2n\}}\prod\langle\phi_i\phi_j\rangle\langle\phi_{2n+1}\phi_{2n+2}\rangle + \sum_{i,j\in\{1,\cdots,2n+1\}}\prod\langle\phi_i\phi_j\rangle + \sum_{i,j\in\{1,\cdots 2n,2n+2\}}\prod\langle\phi_i\phi_j\rangle. \tag{3.1.26}$$

Thus, recognizing this is a sum over all pairings and thus finishing the induction process, we finally have our desired result namely:

$$\langle\phi(\mathbf{x}_1)\cdots\phi(\mathbf{x}_{2n})\rangle = \sum\prod\langle\phi(\mathbf{x}_i)\phi(\mathbf{x}_j)\rangle. \tag{3.1.27}$$

□

Thus, we can express all correlation functions in terms of the 2-point correlation function. Furthermore, given the translational invariance of the problem, we have that the 2-point correlation function is independent of global position, thus we can write:

$$G_0^{(2)}(\mathbf{x},\mathbf{x}') = G_0^{(2)}(\mathbf{x}',\mathbf{x}) = G_0(\mathbf{x}-\mathbf{x}') \tag{3.1.28}$$

3.2 Perturbative expansion

We start by defining a useful normalization of the peturbed partition function $\mathcal{Z}$. Indeed, following Kleinert [2], it is convenient to normalize $\mathcal{Z}$ with $\mathcal{Z}_0$ and setting $\mathcal{Z}_0$ to unity:

$$\mathcal{Z} = \frac{1}{\mathcal{Z}_0}\int \mathcal{D}\phi\, \mathrm{e}^{-E[\phi]} \tag{3.2.29}$$

On the other hand, we have already calculated the unperturbed partitition function in Equation 3.1.12 to be $\mathrm{e}^{-\frac{1}{2}\mathrm{Tr}(\log(D))}$. Thus by placing this quantity in the numerator, we can redefine our measure such that:

$$\int \mathcal{D}\phi(\mathbf{x})' := \int \mathcal{D}\phi\ \mathrm{e}^{\frac{1}{2}\mathrm{Tr}(\log(D))} \tag{3.2.30}$$

We recall that our perturbed $\mathcal{Z}[j]$ was written in terms of an exponential of an integral over a term dependent on ϕ^4. However, due to our inability to treat this problem analytically and the fact we are capable of only dealing with the Gaussian integral, we will take a perturbative approach. We start off by writing $\mathcal{Z}[j]$ in a series expansion in terms of λ. More precisely we have that:

$$\mathcal{Z}[j] = \mathcal{Z}_0[j] + \sum_{p=1}^{\infty}\frac{\lambda^p}{p!}\left[\frac{\mathrm{d}^p}{\mathrm{d}\lambda^p}\mathcal{Z}[j]\right]_{\lambda=0}, \tag{3.2.31}$$

Which corresponds to an expansion of the exponential term in ϕ^4. Writing the expression for $\mathcal{Z}[j]$ and simplifying we obtain:

$$\begin{aligned}\mathcal{Z}[j] &= \mathcal{Z}_0[j] + \sum_{p=1}^{\infty}\frac{\lambda^p}{p!}\left[\frac{\mathrm{d}^p}{\mathrm{d}\lambda^p}\int \mathcal{D}\phi' \mathrm{e}^{-E_0[\phi,j]+\int \mathrm{d}^D\mathbf{x}\,\frac{\lambda}{4!}\phi(\mathbf{x})}\right]_{\lambda=0}\\ &= \mathcal{Z}_0[j] + \sum_{p=1}^{\infty}\frac{\lambda^p}{(4!)^p p!}\underbrace{\int \mathcal{D}\phi' \int \mathrm{d}^D\mathbf{z}_1\cdots\mathrm{d}^D\mathbf{z}_p\,\phi^4(\mathbf{z}_1)\cdots\phi^4(\mathbf{z}_p)\mathrm{e}^{-E[\phi,j]}}_{:=\mathcal{Z}_p}.\end{aligned}$$

With this expression for the partition function, it is then possible to find the approximation of the n-point correlation function perturbatively as:

$$G^{(n)}(\mathbf{x}_1,\cdots,\mathbf{x}_n) = \mathcal{Z}^{-1}\left[\frac{\delta^n}{\delta j(\mathbf{x}_1)\cdots\delta j(\mathbf{x}_n)}\mathcal{Z}[j]\right]_{j=0} = \mathcal{Z}^{-1}\left[\frac{\delta^n}{\delta j(\mathbf{x}_1)\cdots\delta j(\mathbf{x}_n)}\left(\mathcal{Z}_0[j]+\sum_{p=1}^{\infty}\mathcal{Z}_p[j]\right)\right]_{j=0} \tag{3.2.32}$$

$$= \mathcal{Z}^{-1}\left[G_0^{(n)}(\mathbf{x}_1,\cdots,\mathbf{x}_n) + \sum_{p=1}^{\infty}\underbrace{\frac{(-\lambda)^p}{(4!)^p p!}\int \mathrm{d}^D\mathbf{z}_1\cdots\mathrm{d}^D\mathbf{z}_p\int \mathcal{D}\phi'\,\phi(\mathbf{x}_1)\cdots\phi(\mathbf{x}_n)\phi^4(\mathbf{z}_1)\cdots\phi^4(\mathbf{z}_p)\,\mathrm{e}^{E_0[\phi,j]}}_{G_p^{(n)}(\mathbf{x}_1,\cdots,\mathbf{x}_n)}\right]_{j=0}. \tag{3.2.33}$$

Note that the $G_p^{(n)}$'s are defined without taking into account the final division by $\mathcal{Z}$. Finally, we may express $G_p^{(n)}(\mathbf{x}_1,\cdots,\mathbf{x}_n)$ in terms of the correlation functions of the free field theory. Indeed we have that:

$$G_p^{(n)}(\mathbf{x}_1,\cdots,\mathbf{x}_n) = \frac{(-\lambda)^p}{(4!)^p p!}\int \mathrm{d}^D\mathbf{z}_1\cdots\mathrm{d}^D\mathbf{z}_p\, G_0^{(n+4p)}(\mathbf{z}_1,\mathbf{z}_1,\mathbf{z}_1,\mathbf{z}_1,\cdots,\mathbf{z}_p,\mathbf{z}_p,\mathbf{z}_p,\mathbf{z}_p,\mathbf{x}_1,\cdots,\mathbf{x}_n). \tag{3.2.34}$$

This last expression may be Wick expanded according to Wick's theorem previously proved, hence understanding the two point correlation function is fundamental.

3.3 Feynman diagrams

As we have seen, while we have been able to retrieve the perturbation expansion it is composed of some quite tedious expressions. To somehow remediate to this, we will introduce the graphical representation of these integrals. It is useful to start by recalling the expression of the $G_p^{(n)}$'s and then proceed to its Wick expansion. We have:

$$G_p^{(n)}(\mathbf{x}_1, \cdots, \mathbf{x}_n) = \frac{(-\lambda)^p}{(4!)^p p!} \int \mathrm{d}^D\mathbf{z}_1 \cdots \mathrm{d}^D\mathbf{z}_p \, \langle \phi(\mathbf{z}_1)^4 \cdots \phi(\mathbf{z}_p)^4 \phi(\mathbf{x}_1) \cdots \phi(\mathbf{x}_n) \rangle \quad (3.3.35)$$

$$= \frac{(-\lambda)^p}{(4!)^p p!} \sum \prod \langle \phi_{\pi_i} \phi_{\pi_j} \rangle \, . \quad (3.3.36)$$

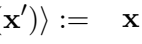
As seen previously, the Wick expansion gives rise to exactly $(4p + n - 1)!!$ terms, each term is a product of $\frac{1}{2}(4p + n)$ free correlation functions. Here, the π_i's denote all different distinct permutations of the $\mathbf{z}$'s and $\mathbf{x}$'s that arise from Wick expanding the expression. Because of the integrations on the $\mathbf{z}_i$'s, some of the terms will be equivalent. We are thus interested in finding how many and which terms are independent from each other. In order to understand how to do this, it is easier to illustrate an example and then proceed to generalize to higher orders.

Consider for example, finding the independent products for $G_1^{(2)}$ as well as their multiplicity. At the first order, the naive Wick expansion will have 15 terms that are *a priori* not equal. However, only two are independent after integration, since the integrals are all equivalent up to their labelling, because of this we shall just write ϕ when referring to $\phi(\mathbf{z}_i)$ for some i, we will note ϕ_i the field $\phi(\mathbf{x}_i)$.

1. In the first case, we can have a product of the form $\langle \phi\phi_1 \rangle \langle \phi\phi_2 \rangle \langle \phi\phi \rangle$.

2. In the second case, instead of coupling a field ϕ_i to some ϕ the only way to make it different from the product of the previous form is to couple ϕ_1 and ϕ_2 together. Thus the second possible independent form of product is $\langle \phi_1\phi_2 \rangle \langle \phi\phi \rangle \langle \phi\phi \rangle$. It is clear that these two possibilities are exhaustive because of the equivalence of the $\mathbf{z}_i$'s.

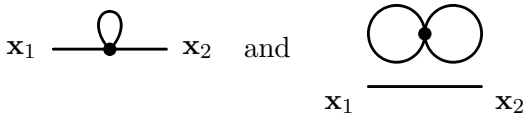
At this point, it is natural to introduce the Feynman rules to represent all of this mathematical notation graphically. We shall note:

$G_0^{(2)}(\mathbf{x}, \mathbf{x}') = \langle \phi(\mathbf{x})\phi(\mathbf{x}') \rangle :=$ 

, $-\lambda \int \mathrm{d}^D\mathbf{z} :=$ 

(3.3.37)

This allows us to simply represent all the possible diagrams up to first order (which we have already found with the logic we have used above). The two diagrams are:



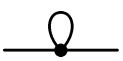
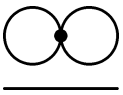
(3.3.38)

The first remark we can make is that the first diagram is connected, while the other one isn't, this will turn out to be important as we will find that all disconnected diagrams cancel out. However, we have yet to find the multiplicity of these integrals. It is a good pedagogical exercise to find the multiplicity of at least one diagram by hand. Let us take for example the disconnected graph above. We first start by realizing that what characterizes this diagram is that it is disjoint, and the vertex $\mathbf{x}_1$ must be linked to $\mathbf{x}_2$. So we are left with the integration vertex  . There are four edges to this vertex, so choose any edge of the vertex. We have 3

choices of edges to connect it to, once this choice is made, we have no more freedom and the two other edges interconnect, thus retrieving the original graph. This means that the multiplicity of the graph is 3, since this way of counting is equivalent to ways of rearranging the ϕ's in $\langle\phi_1\phi_2\rangle\langle\phi\phi\rangle\langle\phi\phi\rangle$. Similarly, for the connected graph, to which we shall refer to as the *tadpole*. We start off by considering the connections possible from the initial vertex $\mathbf{x}_1$: there are 4 edges avalaible at the middle vertex, so we have 4 choices. Then, we must choose an additional edge from the middle vertex to connect to the $\mathbf{x}_2$ vertex, which gives us 3 choices. Thus the multiplicity of the graph is $3 \times 4 = 12$. Finally, having this multiplicity, we can include the numerical prefactor in Equation 3.3.36 for $G_p^{(n)}$ and give a weight W_G to each diagram, namely:

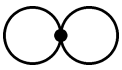
$$W_G = \frac{M_G}{4!^p p!} \tag{3.3.39}$$

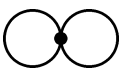
Thus, we have reduced the problem of computing the $G_p^{(n)}$ function to finding all the independent diagram and computing their respective multiplicities. In what will follow, we will give out the weight factors without further explanation, however Kleinert [2] gives a thorough explanation of how to compute this multiplicity in general. It is then possible to simply express the 2-point correlations function up to first order as:

$$G_1^{(2)}(\mathbf{x}_1, \mathbf{x}_2) = \frac{1}{2}$$  $$+ \frac{1}{8}$$  (3.3.40)

3.4 Diagrammatic expansion

As we have seen, it is thus possible to give a diagrammatical representation to the perturbation series obtained in powers of λ. It turns out that all of the non-connected diagrams cancel out with the expansion of the denominator of Equation 3.0.6. Indeed, we may verify this up to first order with the $n = 2$:

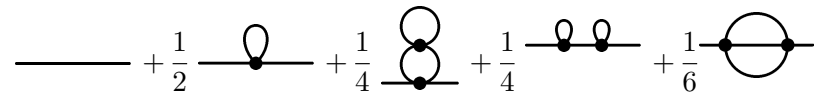
$$G^{(2)}(\mathbf{x}_1, \mathbf{x}_2) = \frac{1}{1 + \frac{1}{8}}$$  $$+ \mathrm{O}(\lambda^2)$$

$$= \left(1 - \frac{1}{8}\right.$$  $$\left. + \mathrm{O}(\lambda^2)\right)$$

$$= \quad + \frac{1}{2}$$  $$+ \mathrm{O}(\lambda^2)$$

Thus we can see that the non-connected diagrams do indeed cancel each other out (and it turns out this holds for all orders). On the other hand, they can be important when computing $\mathcal{Z}$, which can be used to calculate the Helmholtz free energy of the system F. For our purposes, however, we can then allow ourselves to calculate the corrections up to second order by simply looking at the connected diagrams. For this, it is simply necessary to find all the connected graphs with two vertices and to calculate their weights as stipulated above. We can see that,

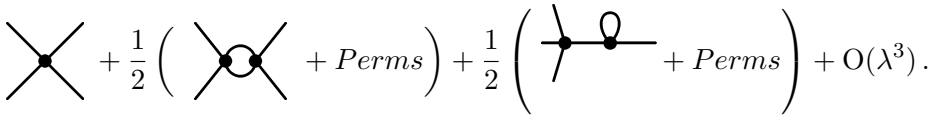
diagramatically and at relatively low orders, climbing in order is not too complicated. Doing so, we simply obtain:

$G(\mathbf{x}_1, \mathbf{x}_2) =$  $+ \mathrm{O}(\lambda^3)$ (3.4.41)

Here, it is useful to introduce the notion of one particle irreducible (1PI) diagrams.

Definition 3.4.1. A graph G is said to be *one particle irreducible* if it does not separate into two different pieces if you cut one edge. Furthermore, if we take away the external edges, we say we *amputate* the external edges (or legs) of the diagram.

We thus notice that in the expansion above, only the fourth diagram is not a 1PI diagram. Similarly, we could compute the 4-point correlation function by considering the diagrams with 4 external vertices which are connected and their multiplicities to obtain:

$G(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_4) =$  $+ \mathrm{O}(\lambda^3)\,.$ (3.4.42)

3.4.1 Fourier transformation

It is often easier to calculate the integrals that we are interested in by first taking the Fourier transform and then performing the integration. We will use for our convention that the Fourier transform and its inverse are given by:

$$F(\mathbf{k}) = \int \mathrm{d}^D\mathbf{x}\, \mathrm{e}^{-i\mathbf{k}\cdot\mathbf{x}} F(\mathbf{x}) \tag{3.4.43}$$

$$F(\mathbf{x}) = \frac{1}{(2\pi)^D} \int \mathrm{d}^D\mathbf{k} \mathrm{e}^{i\mathbf{k}\cdot\mathbf{x}} F(\mathbf{k}) \tag{3.4.44}$$

We are now interested in finding the expression for $G_0^{(2)}(\mathbf{k}) = G_0(\mathbf{k})$ in Fourier space. To do this, we start by considering the transformation of the two point correlation function in real space namely:

$$\begin{aligned} G(\mathbf{k}, \mathbf{k}') &= \int \mathrm{d}^D\mathbf{x}'\mathrm{d}^D\mathbf{x}\ \mathrm{e}^{-i(\mathbf{k}\cdot\mathbf{x}+\mathbf{k}'\cdot\mathbf{x}')} G_0(\mathbf{x}, \mathbf{x}') \\ &= \int \mathrm{d}^D\mathbf{x}'\, \mathrm{e}^{-i(\mathbf{k}+\mathbf{k}')\cdot\mathbf{x}'} \int \mathrm{d}^D\mathbf{x}\ \mathrm{e}^{-i\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}')} G_0(\mathbf{x}-\mathbf{x}') \\ G(\mathbf{k}, \mathbf{k}') &= (2\pi)^D \delta^{(D)}(\mathbf{k}+\mathbf{k}') G_0(\mathbf{k}) \end{aligned} \tag{3.4.45}$$

Thus we can express the correlation function again by simply taking the inverse Fourier transform to obtain:

$$G_0(\mathbf{x}-\mathbf{x}') = \frac{1}{(2\pi)^D} \int \mathrm{d}^D\mathbf{k}\, \mathrm{e}^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}')} G_0(\mathbf{k}) \tag{3.4.46}$$

Furthermore, recall that we had defined the operator $D(\mathbf{x}, \mathbf{x}') = \delta^{(D)}(\mathbf{x}-\mathbf{x}')(-\nabla^2 + m^2)$ while we were talking about Free Field Theories. It turns out that the propagator is the inverse of this operator, thus we have the identity:

$$\int \mathrm{d}^D\mathbf{x}'\, D(\mathbf{x}, \mathbf{x}') G_0(\mathbf{x}', \mathbf{x}'') = \delta^{(D)}(\mathbf{x}-\mathbf{x}'') \tag{3.4.47}$$

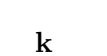
Inserting the expression of Equation 3.4.46 into Equation 3.4.47 we simply find that:

$$\begin{aligned}\delta^{(D)}(\mathbf{x}-\mathbf{x}'') &= \int \mathrm{d}^D\mathbf{x}'\,\delta^{(D)}(\mathbf{x}-\mathbf{x}')(-\nabla^2+m^2)\int \frac{\mathrm{d}^D\mathbf{k}}{(2\pi)^D}\,\mathrm{e}^{i\mathbf{k}\cdot(\mathbf{x}'-\mathbf{x}'')}G_0(\mathbf{k}) \\ &= \int \frac{\mathrm{d}^D\mathbf{k}}{(2\pi)^D}(\mathbf{k}^2+m^2)G_0(\mathbf{k})\,\mathrm{e}^{i\mathbf{k}\cdot(\mathbf{x}'-\mathbf{x}'')}\end{aligned}$$

Thus we obtain the expression for the propagator in Fourier space, namely:

$$G_0(\mathbf{k}) = \frac{1}{\mathbf{k}^2+m^2} \tag{3.4.48}$$

In a similar way, we can take the Fourier transformation of each of the diagrams we found previously (keeping in mind they represent integrals), in this way, we discover a duality in the sense of the Fourier transform that is present within the graphs. The end result is that the duality consists in turning an integral which was originally over the vertices of the diagrams into an integration over the edges. With this duality, it is then possible to give the Feynman rules that can be associated to the diagrams. We note however, that due to the presence of the $\delta^{(D)}(\mathbf{k}+\mathbf{k}')$ in Equation 3.4.45, the flow of momentum $\mathbf{k}$ must be conserved at all vertices. Keeping the last remark in mind, we thus retrieve the following rules:

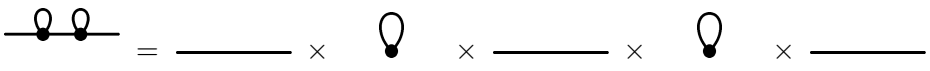
$$G_0(\mathbf{k}) = \int \frac{\mathrm{d}^D\mathbf{k}}{(2\pi)^D}\frac{1}{\mathbf{k}^2+m^2} :=$$ 

$$, \qquad -\lambda :=$$  (3.4.49)

We note however that we retrieve exactly the same diagrams at all orders, the only thing that we change are the Feynman rules.

3.5 Self-energy

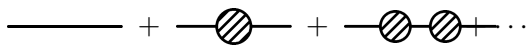
Using the notion of 1PI diagrams, it is possible to simplify the expression of $G(\mathbf{k})$. This is motivated by the fact that many diagrams can be seen as a product (in Fourier space) of 1PI diagrams. For example:

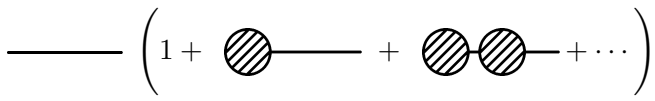
 (3.5.50)

We clearly see that the fundamental piece of this diagram is  . Intuitively using this simple example, it is possible to extrapolate and further simplify the expression of $G(\mathbf{k})$, we consider noting all 1PI diagrams with a $\Sigma(\mathbf{k}) :=$  . We thus have that:

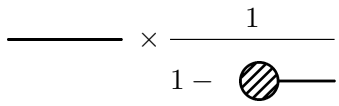
$$\Sigma(\mathbf{k}) = \frac{1}{2}$$  $$+\frac{1}{4}$$  $$+\frac{1}{6}$$  $$+\cdots$$ (3.5.51)

Notice that here, we have only considered amputated diagrams. Given this definition of Σ, which we will call the *self-energy*, we have:

$G(\mathbf{k})$ =  (3.5.52)

=  (3.5.53)

$$= G_0(\mathbf{k}) \sum_{n=0}^{\infty} (\Sigma(\mathbf{k}) G_0(\mathbf{k}))^n \tag{3.5.54}$$

=  (3.5.55)

$$= \frac{G_0(\mathbf{k})}{1 - \Sigma(\mathbf{k}) G_0(\mathbf{k})} \,. \tag{3.5.56}$$

We notice that by introducing this self-energy, we have reduced the problem of the computation of Feynman integrals that arise from the perturbative expansion down to identifying the 1PI diagrams and computing their corresponding integrals. This can be done consistently up to $\mathrm{O}(\lambda^n)$ in Perturbation Theory using Equation 3.5.56 by computing this self-energy up to $\mathrm{O}(\lambda^n)$. In our case, we have that, up to $\mathrm{O}(\lambda^3)$:

$$\Sigma(\mathbf{k}) = \Sigma^{(1)}(\mathbf{k}) + \Sigma^{(2)}(\mathbf{k}) + \mathrm{O}(\lambda^3) \tag{3.5.57}$$

$\Sigma^{(1)} = \frac{1}{2}$  (3.5.58)

$\Sigma^{(2)} = \frac{1}{4}$  $+ \frac{1}{6}$  (3.5.59)

However, it turns out that even with naive dimensional analysis, we can guess that for $D = 4$, there will be a quadratic divergence in the ultraviolet regime, *e.g.* when $\mathbf{k} \to +\infty$.

3.6 Vertex Functions

In the spirit of generalization, there is no reason to limit ourselves to the concept of self-energy that we encountered in the calculation of the 2-point correlation function $G(\mathbf{k})$. Indeed, we may define an analogous concept for the n-point correlation function. We may thus define the *vertex function*. For our purposes, it is sufficient to explicit this concept for the 4-point correlation function only, however, a further extension of the generalization is presented by Kleinert [2].

Definition 3.6.1. The n-th vertex function $\Gamma^{(n)}(\mathbf{k}_1, \cdots, \mathbf{k}_n)$ is the weighted (by the multiplicity factors of the diagrams) sum of all the amputated diagrams that are 1PI having n external vertices. We may graphically represent $\Gamma^{(n)}$ by: .

In Fourier space, we may indeed write the connected part of the n-point correlation function as:

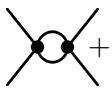
$$G^{(n)}(\mathbf{k}_1,\cdots,\mathbf{k}_n) = -\underbrace{(2\pi)^D\delta^{(D)}\left(\sum_{i=1}^{n}\mathbf{k}_i\right)}_{\text{Translation invariance}}\overbrace{\prod_{j=1}^{n}G_0(\mathbf{k}_j)}^{\text{Amputated}}\underbrace{\Gamma^{(n)}(\mathbf{k}_1,\cdots,\mathbf{k}_n)}_{\text{Vertex function}} \tag{3.6.60}$$

In the same way we explicited $\Sigma(\mathbf{k})$ previously, we will explicit $\Gamma^{(4)}$. From a graphical point of view, we may thus write:

$\Gamma^{(4)}(\mathbf{k}_1,\cdots,\mathbf{k}_4) =$  (3.6.61)

In our perturbative approach, our goal is thus to calculate $\Gamma^{(4)}$ consistently up to different orders. We will note $\Gamma_i^{(n)}$ the terms up to $\mathrm{O}(\lambda^i)$ in the expression of $\Gamma^{(n)}$. In this fashion, we obtain:

$\Gamma_1^{(4)} = -$  $= +\lambda$ (3.6.62)

$\Gamma_2^{(4)} = -\frac{1}{2}\Big($  $+ Perms\Big)$ (3.6.63)

Notice here that the translational invariance is indeed translated into the conservation of incoming momenta from the 4 incoming vertices. This invariance characterized by $\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3+\mathbf{k}_4 = 0$ allows us to define the Mandelstam variables in Fourier Space as:

$$s := (\mathbf{k}_1+\mathbf{k}_2)^2 = (\mathbf{k}_3+\mathbf{k}_4)^2 \tag{3.6.64}$$
$$t := (\mathbf{k}_1+\mathbf{k}_3)^2 = (\mathbf{k}_2+\mathbf{k}_4)^2 \tag{3.6.65}$$
$$u := (\mathbf{k}_1+\mathbf{k}_4)^2 = (\mathbf{k}_2+\mathbf{k}_3)^2 \tag{3.6.66}$$

We note that up to order $\mathrm{O}(\lambda^2)$, we have a symmetry between all permutations of the diagrams in Equation 3.6.63, which means that we can change the prefactor of $\frac{1}{2}$ to $\frac{3}{2}$ and only compute one integral. Once again, however, it turns out that this integral diverges (although this time logarithmically) in $D = 4$.

3.7 Feynman Integral Regularization

We have seen thus far how to describe the perturbation series obtained for the n-point correlation function G of a ϕ^4-theory. However, we have not yet spoken about how to actually *calculate* these integrals. We start off by recalling and proving a number of mathematical facts.

Definition 3.7.1. The Gamma function $\Gamma(z)$ is an extension of the factorial function with the property: $\Gamma(z+1) = z\Gamma(z)$. For $\Re(z) \geq 0$ it is defined by the improper integral:

$$\Gamma(z) := \int_0^\infty \mathrm{d}t\, t^{z-1}\mathrm{e}^{-t} \tag{3.7.67}$$

and $1/\Gamma(z)$ is an entire function.

Definition 3.7.2. The Beta function $B(\alpha,\gamma)$ is defined as, and characterized by:

$$B(\alpha,\gamma) := \frac{\Gamma(\alpha)\Gamma(\gamma)}{\Gamma(\alpha+\gamma)} = \int_0^\infty \mathrm{d}y\, y^{\alpha-1}(1+y)^{-\alpha-\gamma} \tag{3.7.68}$$

Lemma 2. The surface of a unit D-sphere is given by

$$S_D = \frac{2\pi^{D/2}}{\Gamma(D/2)}\,. \tag{3.7.69}$$

Furthermore, we may extend this result analytically for non-integer D since $1/\Gamma(z)$ is entire.

Proof. Consider the Gaussian integral in D-dimensions given by:

$$\int \left(\prod_{i=1}^{D} \mathrm{d}x_i\right) \mathrm{e}^{-\sum_{i=1}^{D} x_i^2} = \prod_{i=1}^{D} \left(\int \mathrm{d}x\, \mathrm{e}^{-x^2}\right) = \pi^{D/2}\,. \tag{3.7.70}$$

We can also calculate it by considering hyperspherical coordinates. In this case, by performing the change in variables, the Jacobian will be equal to $S_D r^{D-1}$. Thus we have that:

$$\begin{aligned}\int \prod_{i=1}^{D} \mathrm{d}_i^x\, \mathrm{e}^{-\sum_{i=1}^{D} x_i^2} &= \int_0^\infty S_D r^{D-1} \mathrm{d}r\, e^{-r^2} = \int_0^\infty S_D (r^2)^{\frac{D-1}{2}} \mathrm{e}^{-r^2} \\ &= \frac{1}{2} S_D \Gamma(D/2) = \pi^{D/2}\,.\end{aligned}$$

Thus we finally obtain the desired result:

$$S_D = \frac{2\pi^{D/2}}{\Gamma(D/2)}\,. \tag{3.7.71}$$

□

3.7.1 Calculation of $\Sigma^{(1)}(\mathbf{k})$

With these mathematical facts in our tool belt, we start off by considering the massive tadpole diagram integral, which is the first non-trivial integral we need to calculate in our perturbation expansion. Expressing the integral explicitly we have:

$$= \int \frac{\mathrm{d}^D \mathbf{k}}{(2\pi)^D} \frac{1}{\mathbf{k}^2 + m^2} \tag{3.7.72}$$

We then proceed to compute the integral by exploiting its symmetry and switching to hyperspherical coordinates:

$$= -\lambda \frac{S_D}{(2\pi)^D} \int_0^\infty \mathrm{d}k\, \frac{k^{D-1}}{k^2 + m^2} \tag{3.7.73}$$

$$= -\lambda \frac{S_D}{2(2\pi)^D} (m^2)^{D/2-1} \int_0^\infty \mathrm{d}y\, \frac{y^{D/2-1}}{1+y}\,,\ y = \frac{p^2}{m^2} \tag{3.7.74}$$

$$= -\lambda \frac{(m^2)^{D/2-1}}{(4\pi)^{D/2}} \frac{\Gamma(1 - D/2)}{\Gamma(D/2)} \tag{3.7.75}$$

Since the Γ-function has poles at all negative integers, we notice in particular that in $D = 4$ this integral diverges. We thus need to find a way of regularizing this integral. There are multiple ways of doing this, amongst others:

1. The introduction of a hard cut-off Λ: this is notably applied in condensed matter applications. In general, however, this hard cut-off will not preserve the symmetries of the system, so another method is usually preferred.

2. Analytic regularization: we replace the propagator by $(\mathbf{k}^2+m^2)^{-z}$ for $z \in \mathbb{C}$ with real part big enough such that the integral converges and we then extend the result analytically to the case $z = 1$.

3. Dimensional regularization: we place ourselves in dimension $D = 4 - 2\varepsilon$ and introduce a scale of renormalization $\mu^{2\varepsilon}$ in the measure of the integral. This will lead to the appeareance of poles, which can then be removed from the Laurent series of the Feynamn Integrals to eliminate their divergent part as $\varepsilon \to 0$.

We will, however, focus only on the last technique, dimensional regularization. We place ourselves in dimension $D = 4 - 2\varepsilon$. We can write the measure in the integral as: $\mathrm{d}^4\mathbf{k} = \mu^{2\varepsilon}\mathrm{d}^{4-2\varepsilon}\mathbf{k}$. Furthermore, we generalize the integral we are considering (this will be useful for the computation of other diagrams later on) to:

$$\int \frac{\mathrm{d}^D\mathbf{k}}{(\mathbf{k}^2+m^2)^\alpha} = \frac{(m^2)^{D/2-\alpha}}{(4\pi)^{D/2}} \frac{\Gamma(\alpha - D/2)}{\Gamma(\alpha)} \,. \tag{3.7.76}$$

We note that from this formula, we conclude that the non-massive ($m = 0$) tadpole integral is identically zero $\forall\alpha$. Next, we consider the first correction to the self-energy:

$$\begin{aligned} \Sigma^{(1)}(\mathbf{k}) \quad &= -\frac{\lambda}{2}\mu^{2\varepsilon} \int \frac{\mathrm{d}^D\mathbf{k}}{(2\pi)^D} \frac{1}{\mathbf{k}^2+m^2} && (3.7.77)\\ &= -\frac{\lambda}{2}\mu^{2\varepsilon} \frac{(m^2)^{1-\varepsilon}}{(4\pi)^{2-\varepsilon}} \frac{\Gamma(\varepsilon-1)}{\Gamma(1)} && (3.7.78)\\ &= -\frac{1}{2} \underbrace{\frac{\lambda}{16\pi^2}}_{:=\bar\lambda} \left(\frac{4\pi\mu^2}{m^2}\right)^\varepsilon \frac{\Gamma(1+\varepsilon)}{\varepsilon(1-\varepsilon)} m^2 \,. && (3.7.79) \end{aligned}$$

At this point, it is necessary to specify a couple of technicalities. We will place ourselves in the modified minimal substraction ($\overline{\mathrm{MS}}$) renormalization scheme. That is, we will simply take $\bar\mu = 4\pi\mu^2\mathrm{e}^{-\gamma_E}$ for our renormalization scale, whereas μ is typically used in the conventional minimal substraction (MS) scheme. Where γ_E is simply the Euler-Mascheroni constant. Under this convention, we can express $\Sigma^{(1)}(\mathbf{k})$ as:

$$\Sigma^{(1)}(\mathbf{k}) = \frac{m^2}{2}\,\bar\lambda\,\left(\frac{\bar\mu}{m}\right)^{2\varepsilon} \frac{\mathrm{e}^{\gamma_E}\Gamma(1+\varepsilon)}{\varepsilon(1-\varepsilon)} \,. \tag{3.7.80}$$

To proceed further, we consider the Laurent series expansion of the Γ-function, that is:

$$\Gamma(1+\varepsilon) = \exp\left(-\gamma_E\varepsilon + \sum_{n=2}^{\infty} \zeta(n)\frac{\varepsilon^n}{n!}\right) \,, \tag{3.7.81}$$

where $\zeta(s)$ is the Riemann Zeta function. Up to $\mathrm{O}(\varepsilon^0)$, the correction is:

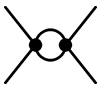
$$\Sigma^{(1)}(\mathbf{k}) = \frac{m^2}{2}\bar\lambda\left(\frac{1}{\varepsilon} + 1 - \log\left(\frac{m^2}{\bar\mu^2}\right) + \mathrm{O}(\varepsilon)\right) \,. \tag{3.7.82}$$

This expression explicits the pole present as $\varepsilon \to 0$ in the ultraviolet regime.

3.7.2 Calculation of $\Gamma_2^{(4)}(\{\mathbf{k}\})$

In this case, we can calculate it by simply considering Equation 3.6.63. Expliciting the integral we have:

$$\Gamma_2^{(4)} = \frac{3}{2} \tag{3.7.83}$$



$$= (-\lambda)^2 \int \frac{\mathrm{d}^D\mathbf{q}}{(2\pi)^D} \frac{1}{\mathbf{q}^2+m^2} \frac{1}{(\mathbf{k}+\mathbf{q})^2+m^2}\,. \tag{3.7.84}$$

At this point we use dimensional regularization as in the calculation of $\Sigma^{(1)}$ by performing the integral in dimension $D = 4 - 2\varepsilon$. Furthermore, to simplify, we will use the infrared rearrangement (IRR) (which means we aim to calculate the integral in the UV regime, *i.e.* $m \to 0$). The result of the full calculation with $m \neq 0$ is explicited by Kleinert [2] and will be given later on, but for the purpose of illustrating how to compute these integrals, using IRR is enough and less tedious. Taking the above into consideration, the integral then becomes

$$\Gamma_2^{(4)} = \frac{3}{2}\lambda^2\mu^{2\varepsilon} \int \frac{\mathrm{d}^D\mathbf{q}}{(2\pi)^D} \frac{1}{\mathbf{q}^2(\mathbf{k}+\mathbf{q})^2}\,. \tag{3.7.85}$$

It is helpful to generalize the integral first before evaluating it (also for the evaluation of other integrals later on). We thus consider the integral:

$$\int \frac{\mathrm{d}^D\mathbf{q}}{(2\pi)^D} \frac{1}{\mathbf{q}^{2\alpha}(\mathbf{k}+\mathbf{q})^{2\beta}} = \frac{(\mathbf{k}^2)^{D/2-\alpha-\beta}}{(4\pi)^{D/2}} G(\alpha,\beta) \tag{3.7.86}$$

$$G(\alpha,\beta) := \frac{a(\alpha)a(\beta)}{a(\alpha+\beta-D/2)} \tag{3.7.87}$$

$$a(\alpha) := \frac{\Gamma(D/2-\alpha)}{\Gamma(\alpha)}\,. \tag{3.7.88}$$

We can then express using these definition the expression of the $\Gamma_2^{(4)}$ up to $\mathrm{O}(1/\varepsilon)$.

$$\Gamma_2^{(4)} = \frac{3}{2}\lambda^2\mu^{2\varepsilon} \int \frac{\mathrm{d}^D\mathbf{q}}{(2\pi)^D} \frac{1}{\mathbf{q}^2(\mathbf{k}+\mathbf{q})^2} \tag{3.7.89}$$

$$= \frac{3}{2}\lambda^2\mu^{2\varepsilon} \frac{(k^2)^{-\varepsilon}}{(4\pi)^{2-\varepsilon}} G(1,1) \tag{3.7.90}$$

$$= \frac{3}{2}\left(\frac{\lambda^2}{16\pi^2}\right)\left(\frac{4\pi\mu^2}{k^2}\right)^{\varepsilon} \frac{\Gamma(1-\varepsilon)\Gamma(1-\varepsilon)\Gamma(\varepsilon)}{\Gamma(2-2\varepsilon)} \tag{3.7.91}$$

$$= \frac{3}{2}\left(\frac{\lambda^2}{16\pi^2}\right)\left(\frac{4\pi\mu^2}{k^2}\right)^{\varepsilon} \underbrace{\frac{\Gamma(1-\varepsilon)\Gamma(1-\varepsilon)\Gamma(1+\varepsilon)}{\varepsilon(1-2\varepsilon)\Gamma(1-2\varepsilon)}}_{\text{Using } \Gamma(1+z)=z\Gamma(z)} \tag{3.7.92}$$

$$= \frac{3}{2}\left(\frac{\lambda^2}{16\pi^2}\right)\frac{1}{\varepsilon}\underbrace{\left(\frac{4\pi\mu^2}{k^2}\right)^{\varepsilon} \frac{\Gamma(1-\varepsilon)\Gamma(1-\varepsilon)\Gamma(1+\varepsilon)}{(1-2\varepsilon)\Gamma(1-2\varepsilon)}}_{1 \text{ up to } \mathrm{O}(\varepsilon^0)} \tag{3.7.93}$$

$$= \frac{3\bar{\lambda}^2}{2\varepsilon} + \mathrm{O}(\varepsilon^0)\,. \tag{3.7.94}$$

3.7.3 Calculation of $\Sigma^{(2)}(\mathbf{k})$

At this point with the results we have obtained, we can calculate the second order correction to the self-energy by considering the second order diagrams. Indeed diagramatically we had found an expression for $\Sigma^{(2)}(\mathbf{k})$ in Equation 3.5.59. We start off by considering:

$$= (-\lambda)^2 \int \frac{\mathrm{d}^D\mathbf{q}}{(2\pi)^D} \frac{1}{\mathbf{q}^2+m^2} \int \frac{\mathrm{d}^D\mathbf{p}}{(2\pi)^D} \frac{1}{(\mathbf{p}^2+m^2)^2}\,. \tag{3.7.95}$$

We notice that this Feynman integral is separable into two different integrals. We then place ourselves in the usual $D = 4 - 2\varepsilon$ scheme and change the measure accordingly thus obtaining:

$$= (-\lambda)^2 (\mu^{2\varepsilon})^2 \underbrace{\int \frac{\mathrm{d}^D\mathbf{q}}{(2\pi)^D} \frac{1}{\mathbf{q}^2+m^2}}_{\text{One loop integral}} \underbrace{\int \frac{\mathrm{d}^D\mathbf{p}}{(2\pi)^D} \frac{1}{(\mathbf{p}^2+m^2)^2}}_{\text{One loop integral}}\,. \tag{3.7.96}$$

A helpful formula to consider here is:

$$\int \frac{\mathrm{d}^D\mathbf{k}}{(2\pi)^D} \frac{1}{(\mathbf{k}^2+m^2)^\alpha} = \frac{(m^2)^{D/2-\alpha}}{(4\pi)^{D/2}} \frac{\Gamma(\alpha - D/2)}{\Gamma(\alpha)}\,. \tag{3.7.97}$$

With this under our toolbelt, the computation of  becomes trivial, indeed we see:

$$= (-\lambda\mu^{2\varepsilon})^2 \frac{(m^2)}{(4\pi)^D} \underbrace{\Gamma(1-D/2)\Gamma(2-D/2)}_{\text{UV Divergences}} \tag{3.7.98}$$

$$= -m^2\bar{\lambda}^2 \left(\frac{\bar{\mu}^2}{m^2}\right)^{2\varepsilon} \mathrm{e}^{2\gamma_E\varepsilon} \frac{\Gamma(1+\varepsilon)^2}{\varepsilon^2(1-\varepsilon)} \tag{3.7.99}$$

$$= -m^2\bar{\lambda}^2 \left(\frac{1}{\varepsilon^2} + \frac{1}{\varepsilon} - \frac{2}{\varepsilon} L_m + 2L_m\left(L_m - 1\right) + \zeta(2) + 1 + \mathrm{O}(\varepsilon)\right)\,, \tag{3.7.100}$$

where $L_m = \log\left(\frac{m^2}{\bar{\mu}^2}\right)$. We notice a couple of things. First, this diagram has only a dependence on m^2, this means that as we will see later, it will only contribute to the renormalization of the mass. Second, the non-polynomial mass dependent singularity in $\log(m)/\varepsilon$ and the second order pole come from divergent subdiagrams. Furthermore, in the $\overline{\text{MS}}$-scheme, all singular terms should not depend on mass and should be local in momentum. We must have that all of the singularities of $\mathrm{O}(\log(m)/\varepsilon)$ must cancel out when we sum for each order (this result has been proved to all orders) for the self-energy.
Next, we consider the sunset integral. This integral is a lot more challenging than the previous one. Indeed, where in the previous case we could split the at first glance two-loop integral into one-loop integrals, in this case, this is not possible.

$$= (-\lambda)^2 (\mu^{2\varepsilon})^2 \int \frac{\mathrm{d}^D\mathbf{p}_1}{(2\pi)^D} \int \frac{\mathrm{d}^D\mathbf{p}_2}{(2\pi)^D} \frac{1}{(\mathbf{p}_1^2+m^2)(\mathbf{p}_2^2+m^2)((\mathbf{k}+\mathbf{p}_1+\mathbf{p}_2)^2+m^2)} \tag{3.7.101}$$

Massaging a bit this term, we can make a couple of remarks:

$$= (-\lambda)^2(\mu^{2\varepsilon})^2 \underbrace{\int \frac{\mathrm{d}^D\mathbf{p}_1}{(2\pi)^D}\frac{1}{(\mathbf{p}_1^2+m^2)} \underbrace{\int \frac{\mathrm{d}^D\mathbf{p}_2}{(2\pi)^D}\frac{1}{(\mathbf{p}_2^2+m^2)((\mathbf{k}+\mathbf{p}_1+\mathbf{p}_2)^2+m^2)}}_{\text{One loop, massive propagator}}}_{\text{One loop, massive propagator}} \tag{3.7.102}$$

Here we can see that in both cases, we are dealing with one loop integrals and both are of the massive propagator type. However, we notice that there is an overlap in the evaluation of the integrals, which is why this the sunset is considerably more challenging than  . We thus have an *overlapping singularity*. Kleinert [2] gives a clear proof of the result of the computation of this integral, which we will not show for the sake of brevity. The result is:

$$= -m^2\bar{\lambda}^2\left(\frac{3}{2\varepsilon^2}+\frac{3}{\varepsilon}\left(\frac{3}{2}-L_m\right)+\mathrm{O}(\varepsilon)\right)-\mathbf{k}^2\bar{\lambda}^2\left(\frac{1}{4\varepsilon}+\mathrm{O}(1)\right) \tag{3.7.103}$$

Putting both of the results together we obtain the second order correction:

$$\Sigma^{(2)}(\mathbf{k}) \quad = \frac{1}{4} \;  \; + \frac{1}{6} \;  \tag{3.7.104}$$

$$= -m^2\bar{\lambda}^2\left(\frac{1}{2\varepsilon^2}+\frac{1}{\varepsilon}+\mathrm{O}(1)\right)-\mathbf{k}^2\bar{\lambda}^2\left(\frac{1}{24\varepsilon}+\mathrm{O}(1)\right) \tag{3.7.105}$$

We notice that all non-polynomial mass-dependent singular terms have consistently cancelled out. Now that we have these calculatory results, we can move on to the next step, which is the renormalization of the parameters of the theory, which will allow us to get rid and give a meaning to these divergences.

Chapter 4

Renormalization

In the last chapter, we looked at the perturbative methods used in order to estimate the correlation functions G. In so doing, we found that most of the integrals coming out of the perturbative expansion actually diverge. This motivates finding a way to get rid of these divergences in a process we will call *renormalization*. The idea is to redefine the parameters of the ϕ^4-model, *i.e.* m, λ as well as the field itself ϕ in order to obtain finite values that can be associated to physical predictions.

4.1 Dimensional analysis

By placing ourselves in the context of having natural units ($c = \hbar = 1$), we can measure everything in terms of units of energy. Indeed, we can express all quantities of length (L), time (T) and mass (M) can be expressed dimensionally using speed ($[L]$), action ($[S]$) and energy ($[E]$) as follows:

$$[L] = [S][v]/[E], \quad [T] = [S]/[E], \quad [M] = [E]/[v]^2 \,. \tag{4.1.1}$$

However since we have set $c = \hbar = 1$ and the latter have units of speed and action accordingly, the only dimensional quantity left is energy. Using this, we can express all units in terms of powers of energy. For example, via the relations in Equation 4.1.1, we find that $[L] = -1$, $[M] = 1$ and $[T] = -1$ in energy units.
Using this system, we aim to calculate the energy dimension of the different parameters in the ϕ^4-model we have thus far considered explicited by the action functional:

$$E[\phi] = \int \mathrm{d}^D\mathbf{x} \; \frac{1}{2}(\partial\phi)^2 + \frac{m^2}{2}\phi^2 + \frac{\lambda}{4!}\phi^4 \,, \tag{4.1.2}$$

where $E[\phi]$ is dimensionless, since it has the units of an action. We can start by considering the units of the field ϕ, by looking at the first term in Equation 4.1.2:

$$-D + 2 + 2[\phi] = 0 \iff [\phi] = \frac{D-2}{2} \,. \tag{4.1.3}$$

From this, we can check that the parameter m has indeed units of mass (or energy) by using the second term of the action:

$$-D + 2[m] + 2[\phi] = 0 \iff [m] = 1 \,. \tag{4.1.4}$$

Finally, we can get the dimensions of the coupling constant λ using the last term:

$$-D + [\lambda] + 4[\phi] = 0 \iff [\lambda] = 4 - D \,. \tag{4.1.5}$$

From here we can make a couple of remarks. The first is that in dimension $D = 4$, the coupling constant λ is dimensionless. In $D > 4$, it is has the dimensions of the inverse power of a mass and in $D < 4$ it has the mass of a mass.

Definition 4.1.1. We can classify theories in terms of the energy dimension associated to the coupling constant. In particular, if:

- $[\lambda] = 0$, the theory is said to be *renormalizable*, this is the case of this is the case of the ϕ^4-theory in dimension $D = 4$.
- $[\lambda] < 0$, the theory is said to be *non-renormalizable*, this is the case of the ϕ^4 in dimension $D > 4$.
- $[\lambda] > 0$, the theory is said to be *superrenormalizable*, this is the case of ϕ^4 in dimension $D < 4$.

In particular, renormalizable theories are characterized by the fact that they posses a finite number of divergent *structures* and superrenormalizable theories only have a finite number of divergent *diagrams*. The exact sense of this will become clearer with the next definition. By contrast, non-renormalizable theories posses an infinite number of divergent structures. In order to localize the UV-divergence of a diagram, we can simply use power counting as a first approach.

Definition 4.1.2. The *superficial degree of divergence (SDD)*, $\omega(G)$ of a Feynman diagram, G, having L loops, N external edges, P internal edges and V vertices is given by:

$$\omega(G) = DL - 2P\,. \tag{4.1.6}$$

Indeed, for each loop there will be an integration, thus there will be DL powers of momentum the numerator coming from the integrations. On the other hand, we have $2P$ powers of momentum in the numerator due to the internal edges, which are related to the propagators.

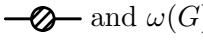
This expression can further be massaged by taking into account graph theoretical and topological considerations of the graphs (for example, all vertices must have 4 edges for a ϕ^4-theory, etc.) This imposes some extra conditions (and so equations for L, P, V and N), which we can then use to express:

$$\omega(G) = D + (D-4)V - \frac{D-2}{2}N\,. \tag{4.1.7}$$

We remark a couple of things:

- *Renormalizable case* ($D = 4$): $\omega(G)$ does not depend on V and thus for a given number of external lines, this determines the SDD up to all orders in perturbation theory;
- *Non-renormalizable case* ($D > 4$): when V increases, so does the $\omega(G)$, which means that for a fixed number of external lines, we have an increase in the number of divergence as we climb in powers of λ. This means the number of divergent structures is actually infinite;
- *Superrenormalizable case* ($D < 4$): $\omega(G)$ decreases as V increases.

From here on, we will focus on the case of the renormalizable ϕ^4-theory,*i.e.* $D = 4$. In this case, the expression for $\omega(G)$ can be simplified to be $\omega(G) = 4 - N$. We see that for all orders in perturbation theory, there are only a finite number of structures that are divergent, namely:

- $N = 0$:  , the diagrams of the vacuum ($\mathcal{Z}$), for which $\omega(G) = 4$
- $N = 1$: $\langle\phi\rangle = 0$ by the $\mathbb{Z}/2\mathbb{Z}$ symmetry, we can extend this to say that for $N = 2k + 1$ are all identically zero
- $N = 2$:  and $\omega(G) = 2$
- $N = 4$:  and $\omega(G) = 0$ which gives a logarithmic divergence.

After $N = 4$, there are no more divergent structures, which means there are only two we must focus on, namely $G^{(2)}$ and $G^{(4)}$. It is still unclear how we treat these divergences. There are multiple approaches that are possible, but we will focus on three only: the so-called conventional method, which will provide us with the intuition necessary, we will then move on to the counter-term method, and finally the Zimmermann forest formula (BPHZ Method), which is much more efficient in practice and more systematic.

4.2 Conventional approach

We start by considering the perturbed propagator, which can be written as:

$$G(\mathbf{k}) = \frac{1}{\mathbf{k}^2 + m^2 - \Sigma(\mathbf{k})}\,. \tag{4.2.8}$$

We may parametrize $\Sigma(\mathbf{k})$ as:

$$\Sigma(\mathbf{k}) = m^2\Sigma_{m^2}(\mathbf{k}) + \mathbf{k}^2\Sigma_{\mathbf{k}}(\mathbf{k})\,. \tag{4.2.9}$$

This way, we have that $\Sigma_m, \Sigma_{\mathbf{k}}$ are divergent, but dimensionless. Using this parametrization, it is possible to write:

$$G(\mathbf{k}) = \frac{1}{1-\Sigma_{\mathbf{k}}} \quad \frac{1}{\mathbf{k}^2 + m^2\dfrac{1-\Sigma_{m^2}}{1-\Sigma_{\mathbf{k}}}}\,. \tag{4.2.10}$$

This way, Σ_{m^2} and $\Sigma_{\mathbf{k}}$ can diverge, but the parameters and the physical fields have to be finite in the $E[\phi]$ action functional of the start. It is thus helpful at this point to introduce a new notation to help us differentiate the diverging fields and parameters from the physical ones. We will note with a subscript B the bare fields and parameters, *e.g.* ϕ_B is the bare field, and we will index an R subscript for the renormalized quantities, ϕ_R is the renormalized field. We must thus find what the link between the two is. We postulate the following scheme:

$$\phi_B = Z_\phi^{1/2}\phi_R \tag{4.2.11}$$

$$m_B^2 = Z_{m^2}Z_\phi^{-1}m_R^2 \tag{4.2.12}$$

$$\lambda_B = Z_\lambda Z_\phi^{-2}\mu^\varepsilon\lambda_R\,. \tag{4.2.13}$$

Going back to the expression of the propagator we realize that:

$$G_B(\mathbf{k}) = \langle\phi_B\phi_B\rangle = Z_\phi\langle\phi_R\phi_R\rangle = Z_\phi G_R(\mathbf{k}) \tag{4.2.14}$$

$$= \frac{(1-\Sigma_{\mathbf{k}})^{-1}}{\mathbf{k}^2 + m_R^2\dfrac{Z_{m^2}}{Z_\phi}\dfrac{1-\Sigma_{m^2}}{1-\Sigma_{\mathbf{k}}}}\,. \tag{4.2.15}$$

From which we extract:

$$G_R(\mathbf{k}) = \frac{1}{Z_\phi(1-\Sigma_\mathbf{k})} \quad \frac{1}{\mathbf{k}^2 + m_R^2 \frac{Z_{m^2}}{Z_\phi}\frac{1-\Sigma_{m^2}}{1-\Sigma_\mathbf{k}}} \tag{4.2.16}$$

We thus see that to obtain finite results we must impose the conditions:

$$Z_\phi(1-\Sigma_\mathbf{k}) = \text{ finite} \tag{4.2.17}$$

$$\frac{Z_{m^2}}{Z_\phi}\frac{1-\Sigma_{m^2}}{1-\Sigma_\mathbf{k}} = \text{ finite}\,. \tag{4.2.18}$$

In our renormalization scheme (modified minimal substraction), we will set these "finite" quantities to 1. Furthermore, we may express the Z coefficients as a formal Laurent series in ε, in this way we have that:

$$Z_x = 1 + \sum_{l=1}^{L} \lambda^l \sum_{j=1}^{l} \frac{Z_x^{(l,j)}}{\varepsilon^j}\,. \tag{4.2.19}$$

In particular, we note that $Z_x^{(l,1)}$ is the residue of the Laurent series. We note also that the Z_x's depend only on λ and ε, but not on m and $\mathbf{k}$, they also depend on μ but only through $\lambda(\mu)$. At this point it becomes helpful to introduce the following definition:

Definition 4.2.1. The operator $\mathcal{K}$ acts on a formal Laurent series by picking up the negative power part of the Laurent series, namely:

$$\mathcal{K}\left(\sum_{n\in\mathbb{Z}} c_n \varepsilon^n\right) = \sum_{n<0} c_n \varepsilon^n\,. \tag{4.2.20}$$

For illustrative purposes, we may consistently calculate the Z_ϕ up to two-loops:

$$Z_\phi = \frac{1}{1-\Sigma_\mathbf{k}(\mathbf{k})} = \frac{1}{1-\Sigma_\mathbf{k}^{(1)}(\mathbf{k}) - \Sigma_\mathbf{k}^{(2)}(\mathbf{k}) - \mathrm{O}(\lambda^3)}\,. \tag{4.2.21}$$

At first order, there is no correction in $\Sigma_\mathbf{k}$ and so $Z_k^{(1)} = 0 \leftarrow Z_\phi^{(1)} = 0$. Taking into account the corrections up to second order, we have that using Equation 3.7.105:

$$Z_\phi \quad = 1 + \mathcal{K}(\Sigma_\mathbf{k}^{(2)}(\mathbf{k})) + \mathrm{O}(\bar{\lambda}^3) \tag{4.2.22}$$

$$= 1 - \frac{\bar{\lambda}_R^2}{24\varepsilon} + \mathrm{O}(\bar{\lambda}_R^3)\,. \tag{4.2.23}$$

We now do the same thing for Z_{m^2} up to one-loop:

$$\frac{Z_{m^2}^{(1)}}{Z_\phi^{(1)}}\frac{1-\mathcal{K}(\Sigma_{m^2}^{(1)}(\mathbf{k}))}{1-\underbrace{\mathcal{K}(\Sigma_\mathbf{k}^{(1)}(\mathbf{k}))}_{=0}} = 1 \iff Z_{m^2}^{(1)} = 1 + \mathcal{K}(\Sigma_{m^2}^{(1)}(\mathbf{k}))\,, \tag{4.2.24}$$

which gives us the result consistently up to first order:

$$Z_{m^2} = 1 + \frac{\bar{\lambda}}{2\varepsilon} + \mathrm{O}(\bar{\lambda}^2)\,. \tag{4.2.25}$$

Finally, we want to obtain Z_λ. Whereas before we used the propagator and the self-energy as a source of the corrections, we use the vertex function $\Gamma^{(4)}$. The result is:

$$Z_\lambda = 1 + \frac{3\bar{\lambda}_R}{2\varepsilon} + \mathrm{O}(\bar{\lambda}^2)\,. \tag{4.2.26}$$

4.3 Counter-term method (BPH method)

We go back to the original action of the ϕ^4-model (Equation 4.1.2) and retake the same scheme as for the previous method (Equations 4.2.12, 4.2.13, 4.2.13). By plugging the scheme into the action we get:

$$E[\phi] = \int \mathrm{d}^D\mathbf{x} \left(\frac{1}{2} Z_\phi (\partial\phi_R)^2 + Z_{m^2} \frac{1}{2} m_R^2 \phi_R^2 + Z_\lambda \frac{\lambda_R}{4!} \mu^\varepsilon \phi_R^4 \right) . \tag{4.3.27}$$

This formula makes a bit clearer the scheme we chose earlier, which could've appeared somewhat obscure at first glance. For reasons that will also become apparent, let us allow ourselves to rewrite the Z_x variables previously defined to be:

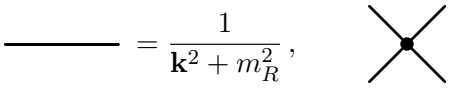
$$Z_x = 1 + \underbrace{\sum_{l=1}^{\infty} \sum_{j=1}^{l} \frac{Z_x^{(l,j)} \bar{\lambda}^l}{\varepsilon^j}}_{\delta Z_x} = 1 + \delta Z_x \, . \tag{4.3.28}$$

In this fashion, we can rewrite the action functional to be:

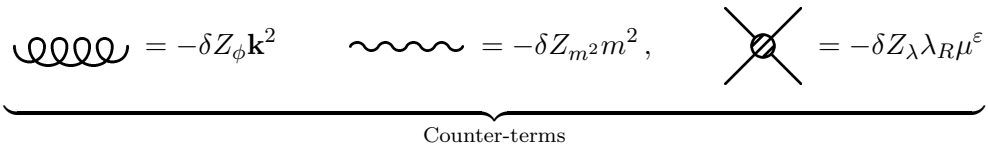
$$E[\phi] = \underbrace{\int \mathrm{d}^D\mathbf{x} \left(\frac{1}{2} (\partial\phi_R)^2 + \frac{1}{2} m_R^2 \phi_R^2 \right)}_{\text{Gaussian model } E_0[\phi]} + \underbrace{\int \mathrm{d}^D\mathbf{x} \left(\frac{\delta Z_\phi}{2} (\partial\phi_R)^2 + \frac{\delta Z_{m^2} m_R^2}{2} + (1 + \delta Z_\lambda) \frac{\lambda_R \mu^\varepsilon}{4!} \phi_R^4 \right)}_{\text{Interaction } V[\phi]} . \tag{4.3.29}$$

To simplify the notation, we will drop the R subscript, any variable without a subscript will be implicitly the renormalized variable. We will still always specify whether the variable is bare. We now simply apply renormalized perturbation theory (noting that we have 4 different terms in the interaction term!).

We can now simply give the Feynman Rules associated and repeat the logic we had previously developped in Chapter 3.



(4.3.30)



(4.3.31)

The role of these counterterms introduced directly into the action functional is to cancel out and absorb all the divergences. We shall illustrate this with the simple calculation of $\Sigma^{(1)}(\mathbf{k})$ in renormalized perturbation theory:

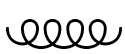
$$\Sigma^{(1)}(\mathbf{k}) = \frac{1}{2}$$

(4.3.32)

We recall the calculated values of the Feynman diagrams and proceed to perform the calculation:

$\frac{1}{2}$  $= \frac{m^2}{2}\bar{\lambda}\left(\frac{1}{\varepsilon} + 1 - L_m + \mathrm{O}(\varepsilon)\right)$ (4.3.33)

 $= -m^2 \delta Z^{(1)}_{m^2}$ (4.3.34)

 $= -\mathbf{k}^2 \delta Z^{(1)}_{\phi}\,.$ (4.3.35)

Thus plugging this back we have:

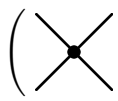
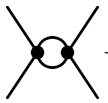
$$\Sigma^{(1)}(\mathbf{k}) = m^2 \underbrace{\left(\frac{\bar{\lambda}}{2\varepsilon} - \delta Z^{(1)}_{m^2}\right)}_{=0} - \mathbf{k}^2 \underbrace{\delta Z^{(1)}_{\phi}}_{=0} + \frac{m^2}{2}\bar{\lambda}(1 - L_m + \mathrm{O}(\varepsilon))\,. \tag{4.3.36}$$

From this equation, we *deduce* the δZ_x's up to first order consistently in this manner. The result is:

$$\delta Z^{(1)}_{m^2} = \frac{\bar{\lambda}}{2\varepsilon} \quad \Longleftrightarrow \; Z_{m^2} = 1 + \frac{\bar{\lambda}}{2\varepsilon} + \mathrm{O}(\bar{\lambda}^2) \tag{4.3.37}$$

$$\delta Z^{(1)}_{\phi} = 0 \quad \Longleftrightarrow \; Z_{\phi} = 1 + \mathrm{O}(\bar{\lambda}^2)\,. \tag{4.3.38}$$

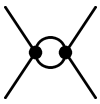
We can do a similar calculation with $\Gamma^{(4)}$ using:

$\Gamma^{(4)}(\{\mathbf{k}\}) = -\Big($  $+ \frac{3}{2}$  $+$  $+ \mathrm{O}(\bar{\lambda}^3)\Big)\,.$ (4.3.39)

We start by recalling that:

 $= -\lambda_R$ (4.3.40)

 $= -\lambda_R \mu^{\varepsilon} \delta Z_{\lambda}$ (4.3.41)

 $= \lambda_R \bar{\lambda}_R (\frac{1}{\varepsilon} - L_m + \mathrm{O}(\varepsilon))\,.$ (4.3.42)

With this we can then simply write consistently up to first order:

$$\Gamma^{(4)}(\{\mathbf{k}\}) = -\left(-\lambda_R + \frac{3}{2}\lambda_R\bar{\lambda}_R\left(\frac{1}{\varepsilon} - L_m\right) - \lambda_R \delta Z^{(1)}_{\lambda} + \mathrm{O}(\varepsilon)\right) \tag{4.3.43}$$

$$= -\underbrace{\left(\frac{3\lambda_R\bar{\lambda}_R}{2\varepsilon} - \lambda_R \delta Z^{(1)}_{\lambda}\right)}_{=0} + \mathrm{O}(\varepsilon^0) \tag{4.3.44}$$

$$\Longleftrightarrow \; \delta Z^{(1)}_{\lambda} = \frac{3\bar{\lambda}_R}{2\varepsilon}\,. \tag{4.3.45}$$

As we had previously found. We can further extend this to second order. The calculation of $\Sigma^{(2)}(\mathbf{k})$ can be done diagramatically as previously illustrated using the diagrammatic equation:

$$\Sigma^{(2)}(\mathbf{k}) = \frac{1}{4}$$  $$+\frac{1}{6}$$  $$+\frac{1}{2}$$  $$+\frac{1}{2}$$  . (4.3.46)

We immediately see that this method is much more efficient. However, it poses the problem of generating the right counter-terms, which for low orders is relatively easy intuitively, however, for higher orders, it is necessary to find a systematic way of generating these terms.

4.4 Zimmermann forest formula (BPHZ method)

In order to find this systematic way of finding the counter-terms, we must first give a couple of results. In the following, G will denote a Feynman diagram.

Theorem 3. *(Weinberg [11]): A graph G converges (absolutely) if its superficial degree of divergence, $\omega(G)$, is negative and if the superficial degrees of divergence of all its subgraphs are also negative.*

Theorem 4. *(Collins [3]): For a graph G of superficial degree of divergence $\omega(G)$, the counter-term associated can be expressed as a polynomial in the masses and/or the external momenta. Moreover, the degree of this polynomial is at most $\omega(G)$.*

Collins' theorem ensures that the counter-terms remain local, as they are polynomials of $\mathbf{k}$ and m. Knowing this, we must specify what are the possible subdivergences that can occur in a given graph. There are two kinds:

- Nested divergences.
- Overlapping divergences. An example of an overlapping divergence arose when we examined the case of the  diagram.

The renormalization process is a technique which eliminates all the divergences. In particular the BPH approach we previously saw aims to do so recursively. Zimmermann found a solution to this recursion with the forest formula which we will now explicit:

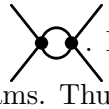
$$\mathcal{R}G = (1 - \mathcal{K})\mathcal{R}'G\,, \tag{4.4.47}$$

where $\mathcal{R}$ is the Bogoliubov operator, $\mathcal{R}'$ is the incomplete Bogoliubov operator and $\mathcal{K}$ is the operator which extracts the strictly negative part of formal Laurent Series. In this way, we have that $\mathcal{R}G$ is the renormalized graph (which is therefore finite), and $\mathcal{R}'$ takes away all of the subdivergences of the diagram. For any graph G, $\mathcal{K}\mathcal{R}'G$ gives the counterterm Z_G. The action of the operator $\mathcal{R}'$ is defined as:

$$\mathcal{R}'G := G + \sum_{\bar{\Gamma}\neq\emptyset} \prod_{\gamma\in\bar{\Gamma}} -\mathcal{K}(\mathcal{R}'\gamma) * G/\bar{\Gamma} \tag{4.4.48}$$

$G/\bar{\Gamma}$ is the quotient graph, we may retrieve it graphically by reducing the $\gamma \in \bar{\Gamma}$ subgraphs to a point. Here, the set $\bar{\Gamma}$ is defined as:

$$\bar{\Gamma} = \{\gamma \subseteq G \mid \gamma \text{ is disjoint or nested (not overlapping)}\} \,. \tag{4.4.49}$$

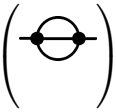
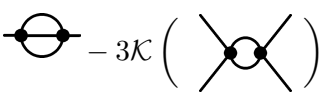
We call $\bar{\Gamma}$ a *normal forest*. By including G, we say that we have a *full forest*.
In order to better understand what this formula is actually doing, it is good to understand it on a conceptual level. First, we notice that in the previous counter-term method, we always had subdivergent diagrams being added in renormalized perturbation theory. The Zimmermann formula basically allows us to generate all of these diagrams in a systematic way. Indeed, we can see that the formula for the incomplete Bogoliubov operator aims to find all non-overlapping divergences. Finally, we can understand the form of the forest formula by simply examining it: the first term, $\mathcal{R}'G$ gets subtracted by the divergent part of the subdiverging graphs, as well as the divergent part of the entire graph itself, thus only the finite result is left. This is all best illustrated by some examples. A first somewhat trivial example is to consider 1 loop diagams,  and  . In this case, the normal forests are all empty, because there are no subdivergent diagrams. Thus we simply have for these two diagrams that:

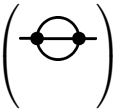
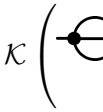
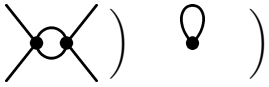
$$\mathcal{R}'G = G \text{ and } \underbrace{\mathcal{R}G = G - \mathcal{K}G}_{\text{Expected result}} \,. \tag{4.4.50}$$

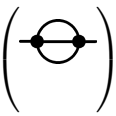
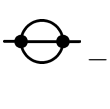
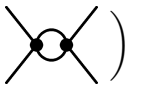
Indeed, since there are no subdivergent diagrams up to first order, we must only take the divergent part of the Laurent Series away from the original graph to retrieve our renormalized graph. Two loops is perhaps a better example to consider to get the intuition behind how the formula works. Let us consider the  diagram. In this case, there are three subdivergent diagrams, the top half loop, the bottom half-loop and the whole loop. Naming these γ_i, we have that the normal forest we consider is:

$$\bar{\Gamma} = \{\{\emptyset\}, \{\gamma_1\}, \{\gamma_2\}, \{\gamma_3\}\} \,. \tag{4.4.51}$$

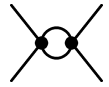
Thus we simply have that:

$\mathcal{R}'$  $=$  (4.4.52)

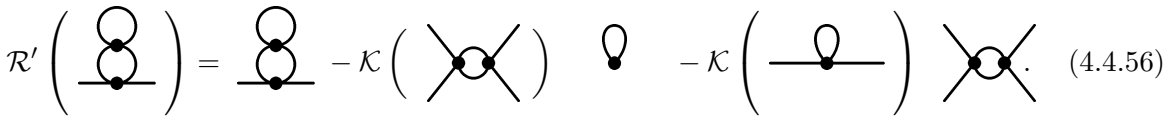
$\mathcal{K}\mathcal{R}'$  $= \mathcal{K}$  $- 3\mathcal{K}\Big($  $\Big)$ (4.4.53)

$\mathcal{R}$  $=$  $- \mathcal{K}$ $- 3\mathcal{K}\Big(\mathcal{K}\Big($  $\Big)$  $\Big)$. (4.4.54)

Next, we can consider the  diagram. In this case, the forest is simply:

$$\bar{\Gamma} = \Bigg\{\{\emptyset\}, \Big\{ \text{  } \Big\}, \Big\{ \text{} \Big\}\Bigg\} \,. \tag{4.4.55}$$

We may thus simply calculate:



(4.4.56)

We immediately notice the power of this method. By simply computing the diagrams needed and taking their corresponding products and divergent parts, we can imediagely get the result for the Z_x constants consistently up to second order (not forgetting the weights of each diagram). In this fashion, we obtain:

$$Z_{m^2} = 1 + \frac{\bar{\lambda}_R}{2\varepsilon} + \bar{\lambda}_R \left(\frac{1}{2\varepsilon^2} - \frac{1}{4\varepsilon} \right) + \mathrm{O}(\bar{\lambda}^2) \tag{4.4.57}$$

$$Z_\phi = 1 - \frac{\bar{\lambda}_R^2}{24\varepsilon} + \mathrm{O}(\bar{\lambda}^3)\,. \tag{4.4.58}$$

To conclude: the BPHZ method works on a diagram-by-diagram basis. For a given graph G, the divergences are eliminated from G by defining $\mathcal{R}G$, the renormalized diagram. The singular part of each diagrams allows us to compute the renormalization constants ($\mathcal{K}\mathcal{R}'G$). In the next chapter, we shall give a meaning to the latter by relating them to the so-called renormalization group functions, which we can eventually relate to critical exponents.

Chapter 5

Renormalization group

In Chapter 4 we focused mainly on the treatment of the divergences that arise in the calculations of the correlation functions. We thus managed to renormalize the theory and regularize it in the limit $\varepsilon \to 0$. The two main results from the previous Chapter were thus the introduction of renormalized (finite) parameters and fields and the introduction of renormalization constants which are divergent. At this point, we were left with the renormalized correlation functions and parameters that still depend on an arbitrary mass-scale μ, which was the renormalization scale. Its physical interpretation is simple: it is roughly the scale of energy at which the described phenomena are observed. This dependence of μ therefore arises from the fact that physics is not scale invariant, and we normally observe different kinds of events at different scales of energy. However, it is important to note that the Gaussian Model in $D = 4$ for $m = 0$ is scale invariant, which means that the interactive term in ϕ^4 is the one that breaks this invariance under scale. Our objective in this chapter is obtain an expression of the correlation functions in terms of the parameter μ. This will require introducing renormalization group functions related to the Z_x's and from which critical exponents can be deduced.

5.1 Callan-Symanzik equations

We can obtain the correlation function with the help of equations that arise from the renormalization group (which, by the way is not a group in the mathematical sense, the name is purely historical). These equations take the form of partial differential equations (PDEs) having coefficients which are parameters of the renormalization group. The derivation of these equations, called the Callan-Symanzik equations is quite simple. Indeed, let us consider:

$$\underbrace{G_B(\mathbf{k}, \bar{\lambda}_B, m_B, \varepsilon)}_{\text{Independent of } \mu} = \underbrace{Z_\phi(\bar{\lambda}_R(\mu))\; G_R(\mathbf{k}, \bar{\lambda}_R(\mu), m_R(\mu), \mu, \varepsilon)}_{\text{Dependent of } \mu} . \tag{5.1.1}$$

The Callan-Symanzik equations then arise from the simple fact that since the bare parameters do not depend on μ, the derivative of the left-hand side of the equation must be identically zero:

$$\mu \frac{\mathrm{d}}{\mathrm{d}\mu} G_B(\mathbf{k}, \bar{\lambda}_B, m_B, \varepsilon) = 0 . \tag{5.1.2}$$

This means in particular that the full-derivative of the right hand side of Equation 5.1.1 must also be identically zero, yielding the Callan-Symanzik equations we desired. A couple of re-

minders are helpful before expliciting them out. Recall that:

$$\phi_R(\mu) = Z_\phi^{-1/2}(\bar{\lambda}_R(\mu))\phi_B \tag{5.1.3}$$

$$m_R^2(\mu) = m_B^2 \frac{Z_\phi(\bar{\lambda}_R(\mu))}{Z_{m^2}(\bar{\lambda}_R(\mu))} \tag{5.1.4}$$

$$\bar{\lambda}_R(\mu) = \mu^{2\varepsilon} \frac{Z_\phi^2}{Z_\lambda} \bar{\lambda}_B \,. \tag{5.1.5}$$

Expressing the Callan-Symanzik condition, we have that:

$$\mu \frac{\mathrm{d}Z_\phi}{\mathrm{d}\mu} G_R + Z_\phi \mu \frac{\mathrm{d}G_R}{\mathrm{d}\mu} = 0 \tag{5.1.6}$$

By carrying the appropriate differentiations, we get:

$$\left(2 \underbrace{\mu \frac{\mathrm{d}\log Z_\phi(\bar{\lambda}_R)}{\mathrm{d}\mu}}_{:=\gamma(\bar{\lambda}_R)} + \mu \frac{\partial}{\partial \mu} + \underbrace{\mu \frac{\partial \bar{\lambda}_R}{\partial \mu}}_{:=\beta(\bar{\lambda}_R)} \frac{\partial}{\partial \bar{\lambda}_R} + \underbrace{\frac{\mu}{m_R} \frac{\partial m_R}{\partial \mu}}_{:=\gamma_m(\bar{\lambda})} \frac{\partial}{\partial m_R} \right) G_R = 0 \,, \tag{5.1.7}$$

where $\gamma(\bar{\lambda}_R)$ and $\gamma_m(\bar{\lambda}_R)$ are the anomalous dimensions of the field and $\beta(\bar{\lambda}_R)$ is the β-function (not to be confused with the mathematical B-function defined in terms of Γ-functions). Finally, by replacing by these newly defined functions, we finally obtain the Callan-Symanzik equation in familiar form:

$$\left(\mu \frac{\partial}{\partial \mu} + \beta(\bar{\lambda}_R) \frac{\partial}{\partial \bar{\lambda}_R} + \gamma_m(\bar{\lambda}_R) \frac{\partial}{\partial m_R} + 2\gamma(\bar{\lambda}_R) \right) G_R = 0 \,. \tag{5.1.8}$$

An important remark is that all of the functions of the renormalization group, β, γ and γ_m are all regular as $\varepsilon \to 0$. This is crucial for the renormalization process. In order to solve this PDE, it is customary to use the method of characteristics. However, for our purposes, it is sufficient to consider the particular case of the critical point. Indeed, at the critical point, we have $m_R = 0 \iff m_R \sim |T - T_c|$, and so at the critical temperature T_c, we must have: $\beta(\bar{\lambda}_R^*) = 0$, which implies physics is invariant of scale.

Definition 5.1.1. The point $\bar{\lambda}_R^*$, which satisfies:

$$\beta(\bar{\lambda}_R^*) = 0 \,, \tag{5.1.9}$$

will be called a fixed point of the renormalization group.

With this simplification, the Callan-Symanzik equation simply becomes:

$$\mu \frac{\mathrm{d}G_R}{\mathrm{d}\mu} = -2\gamma(\bar{\lambda}_R^*) G_R \,. \tag{5.1.10}$$

For which we postulate the Ansatz:

$$G_R(\mathbf{k}, \bar{\lambda}_R, \mu, \varepsilon) = \frac{1}{\mathbf{k}^2} \tilde{G}_R \left(\frac{\mathbf{k}^2}{\bar{\mu}^2}, \bar{\lambda}_R, m_R, \varepsilon \right) . \tag{5.1.11}$$

By solving the differential equation we then just simply get:

$$G_R(\mathbf{k}, \bar{\lambda}_R, m_R, \varepsilon) = \tilde{G}_R(1) \frac{1}{\mathbf{k}^2} \left(\frac{\bar{\mu}^2}{\mathbf{k}^2} \right)^{-\gamma(\bar{\lambda}_R^*)} . \tag{5.1.12}$$

We can then see that the effective dependence is asymptotically given by:

$$G_R \sim \left(\frac{1}{\mathbf{k}^2}\right)^{1-\gamma^*} . \tag{5.1.13}$$

We remark that our original naive dimensional analysis for the dimensions of the propagator is now modified by the interaction present in the theory, which is noticeable due to the present of γ^* correction in the exponent. We can thus finally come back to our original talk on critical exponents. Recall that in Chapter 2 we had talked about the asymptotic dependence of the 2-point correlation function to be:

$$G(\mathbf{k}) = \frac{1}{\mathbf{k}^{2-\eta}} . \tag{5.1.14}$$

Then, by looking at the previous equation, we simply get that the η correction can be obtained from the renormalization group and is simply

$$\eta = 2\gamma^* . \tag{5.1.15}$$

We can also appreciate why γ is called the anomalous dimension, since as an exponent, it modifies the expected result that we get from dimensional analysis. We can, in a similar way, obtain the corresponding Callan-Symanzik equations for the n-point vertex. Indeed:

$$\Gamma_B^{(n)}(\mathbf{k}_1, \cdots, \mathbf{k}_n; \bar{\lambda}_B, m_B) = Z_\phi^{-1/2}(\bar{\lambda}_R(\mu))\ \Gamma_R^{(n)}(\mathbf{k}_1, \cdots, \mathbf{k}_n; \bar{\lambda}_R, m_R, \mu, \varepsilon) . \tag{5.1.16}$$

Repeating the procedure from before, the corresponding Callan-Symanzik PDE is:

$$\left(\mu\frac{\partial}{\partial\mu} + \beta(\bar{\lambda}_R)\frac{\partial}{\partial\bar{\lambda}} + m_R\gamma_m(\bar{\lambda}_R)\frac{\partial}{\partial m_R} - n\gamma(\bar{\lambda})\right)\Gamma_R^{(n)} = 0 . \tag{5.1.17}$$

5.2 Calculation of the renormalization group functions

The procedure to calculate the critical exponents can be put into 3 simple steps:

1. Calculate the Z_x functions
2. Calculate the renormalization group up to some order of ε
3. Examine the asymptotic behaviour of G and other functions using the Callan-Symanzik equations

Let us calculate the β-function. We have that:

$$\beta(\bar{\lambda}_R) = \mu\frac{\partial\bar{\lambda}_R}{\partial\mu} \tag{5.2.18}$$

$$\bar{\lambda}_R(\mu) = \mu^{-2\varepsilon}\frac{Z_\phi^2(\bar{\lambda}_R(\mu))}{Z_\lambda(\bar{\lambda}_R(\mu))}\bar{\lambda}_B . \tag{5.2.19}$$

Therefore, we simply have that:

$$\beta(\bar{\lambda}_R) = \mu\frac{\partial}{\partial\mu}\left(\mu^{-2\varepsilon}\frac{Z_\phi^2}{Z_\lambda}\bar{\lambda}_B\right) \tag{5.2.20}$$

$$= -2\varepsilon\bar{\lambda}_R - \bar{\lambda}_R\mu\frac{\partial\log(\tilde{Z}_\lambda)}{\partial\mu} \tag{5.2.21}$$

$$= -2\varepsilon\bar{\lambda}_R - \bar{\lambda}_R\beta(\bar{\lambda}_R)\frac{\mathrm{d}\log(\tilde{Z}_\lambda)}{\mathrm{d}\bar{\lambda}_R} \tag{5.2.22}$$

$$= \frac{-2\varepsilon\bar{\lambda}_R}{1+\bar{\lambda}_R\dfrac{\mathrm{d}\log\tilde{Z}_\lambda}{\mathrm{d}\bar{\lambda}_R}}\,. \tag{5.2.23}$$

Recall that we previously expressed the Z_x functions as $Z_x = 1+\delta Z_x$, where $\delta Z_x = \sum_{l=1}^{\infty}\sum_{j=1}^{l}\frac{Z_x^{(l,j)}\bar{\lambda}_R^l}{\varepsilon^j}$. Noting that β must be finite as $\varepsilon\to 0$, we can develop β as a formal series in terms of δZ_λ. Doing so, we obtain:

$$\beta(\bar{\lambda}_R) = -2\varepsilon\bar{\lambda}_R\sum_{n=0}^{\infty}(-1)^n\left[-\bar{\lambda}_R\frac{\mathrm{d}}{\mathrm{d}\bar{\lambda}}\sum_{m=1}^{\infty}\frac{(-\delta\tilde{Z}_\lambda)^m}{m}\right]^n\,. \tag{5.2.24}$$

The regularity of β implies that as $\varepsilon\to 0$ imposes: $n=1, m=1$ and $j=1, n=0$. This yields:

$$\beta(\bar{\lambda}_R) = -2\varepsilon\bar{\lambda}_R + \sum_{l=1}^{\infty}\underbrace{2l\,\tilde{Z}_\lambda^{(l,1)}}_{:=\beta_l}\bar{\lambda}^{l+1}\,. \tag{5.2.25}$$

We remark a couple of things:

- The β-function is exclusively determined by the coefficients of the simple poles. This behaviour turns out to be the same for γ and γ_m;
- The terms in $\mathrm{O}(1/\varepsilon^2)$ are identically zero and constrain the $Z^{(l,j)}$'s. It is actually possible to show that all of the $Z^{(l,j>1)}$ are determined exclusively by: $Z^{(l,j=1)}$. Intuitively, this shouldn't be a surprise, as it is completely explicable with the Zimmermann forest formula, indeed all higher order poles come from subdivergent diagrams, thus the simple poles of these building-block diagrams must generate all of the others.

A similar calculation for the γ function will yield:

$$\gamma(\bar{\lambda}_R) = -\sum_{l=1}^{\infty}\underbrace{lZ_\phi^{(l,1)}}_{:=\gamma_l}\bar{\lambda}_R^l \tag{5.2.26}$$

Plugging in the values of the expansion, we obtain:

$$\gamma(\bar{\lambda}_R) = \frac{\bar{\lambda}_R^2}{12} + \mathrm{O}(\bar{\lambda}_R^3) \tag{5.2.27}$$

$$\beta(\bar{\lambda}_R) = -2\varepsilon\bar{\lambda}_R + 3\bar{\lambda}_R^2 + \mathrm{O}(\bar{\lambda}_R^3)\,. \tag{5.2.28}$$

For dimension $D=4$, this yields $\beta(\bar{\lambda}_R) = 3\bar{\lambda}_R^2 + \mathrm{O}(\bar{\lambda}_R^3)$. Because this quantity is positive, this implies that the renormalized constant $\bar{\lambda}_R$ grows as μ increases (the coupling constant grows with increasing energy).

5.3 Fixed points

We find the fixed points of the theory. There is evidently one at $\bar{\lambda}^* = 0$, which is a fixed point stable in the infrared regime. This $\bar{\lambda}^*$ corresponds to exact mean field (gaussian) theory and is the Gaussian fixed point. On the other hand, going back to dimension $D = 4 - 2\varepsilon$ and taking $\varepsilon > 0$, we have that:

$$\beta(\bar{\lambda}) = 3\bar{\lambda}(\bar{\lambda} - \frac{2\varepsilon}{3}) \,. \tag{5.3.29}$$

This yields an extra fixed, non-trivial point $\bar{\lambda}^* = 2\varepsilon/3$. In this case, the coupling constant diverges in the infrared regime, as $\bar{\lambda} \to 0$. On the other hand, it is stable in the ultraviolet regime. The existence in $D < 4$ of this extra non-trivial (Wilson-Fisher) critical point characterizes the critical properties of the ϕ^4-model as being different from the Gaussian theory. Indeed, recall that the critical exponent η depends on $\gamma(\bar{\lambda}^*)$. We can thus calculate the critical exponent η to be:

$$\eta = 2\gamma(\bar{\lambda}^*) = \frac{\bar{\lambda}^*}{6} + \mathrm{O}((\bar{\lambda}^*)^2) \,. \tag{5.3.30}$$

In the case of the Gaussian theory, $\eta = 0$ and it is trivial. However for $D = 3$ ($\varepsilon = \frac{1}{2}$), there is the non-trivial fixed point which is infrared stable and we have:

$$\bar{\lambda}^*_{\frac{1}{2}} = \frac{1}{3} \iff \eta \sim \frac{1}{2.27} = 0.019 \,. \tag{5.3.31}$$

Finally, we remark that the sign of the β-function is crucial. Indeed, for a positive sign like the one we obtained in the case of the ϕ^4-model, we have that the coupling constant increases with increasing energy. On the other hand, if we had a situation where the β-function was negative, we would have the opposite effect and the coupling constant would increase with *decreasing energy*. This would imply that at low energies, a perturbative approach might cease to work, which is an *a priori* unintuitive result. Thus, the behaviour of the β-function is crucial to the understanding of the theory we have at hand. Indeed, for example, in the case of Quantum Electrodynamics, the β-function is positive and we have a situation similar to the one we just observed, which makes sense because this describes a situation of screening at low energies for electrodynamics. On the other hand for Quantum Chromodynamics (QCD), the β-function is negative, which means that in some way we have anti-screening at low energies, which is one of the reasons why low-energy calculations for QCD are challenging (since the perturbative approach is no longer valid). To give a sense of comparison, the exact result for the Ising Model in $D = 2$ found by Osanger in 1944 yields $\eta = \frac{1}{4}$. Furthermore, a recent 6-loop ϕ^4 calculation yielded: $\eta \sim 0.0362(6)$, thus at $\mathrm{O}(\bar{\lambda}^2)$, our result gives the correct order of magnitude.

Chapter 6

Conclusion

We started off by introducing the ϕ^4-model as a way to study the Ising Model which arose from studying spin networks in condensed matter physics. We then solved the free field theory in order to then be able to expand the ϕ^4-model in a perturbation series and calculate different physical quantities of interest, amongst others the perturbed propagator G (or the 2-point correlation function), the 4-point correlation function. This was motivated by our interest in calculating critical exponents and the β-function, which describes the behaviour of the renormalized coupling constant as a function of the scale in energy. We introduced Feynman integrals and diagrams as well as expliciting the calculation of the diagrams at low orders. After noticing the divergences in the perturbation expansion at all orders, we introduced the process of renormalization and the renormalization group. In so doing, we discovered a non-trivial fixed point of the theory as well as the importance of the β-function in the behaviour of the theory in the infrared and the ultraviolet range. We successfully computed the critical exponent η to be $\eta \sim 0.019$, which yields the same order of magnitude as the recent 6-loop result with resummation.

On a personal level, this internship has given me the opportunity to learn a lot about QFT and about the ϕ^4-model which has a variety of applications in physics. In particular it confirmed that I want to pursue further studies in QFT. In particular, I have become interest in further studying the perturbative series that are present in physics. Indeed, the mathematical structure of these perturbative series has proved time and time again to be very rich. For example, there are interesting algebraic interpretations of the BPHZ method as having a Hopf Algebra underlying structure and I would like to study it further in the future. Some recent results were presented at the Multi-loop 2017 Workshop, organized at the UPMC and in which I participated.

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