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Moduli of real algebraic surfaces, and their superanalogues. Differentials, spinors, and Jacobians of real curves

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Dedicated to my parents

Abstract. The survey is devoted to various aspects of the theory of real algebraic curves. The involution defined by complex conjugation induces an antiholomorphic involution $\tau\colon P \to P$ on the complexification P of a real curve. This involution acts on all structures related to the Riemann surface P, namely, on vector bundles, Jacobians, Prymians, and so on. The greater part of the survey is devoted to finding topological invariants and studying the corresponding moduli spaces. Statements of these problems were inspired by applications of the theory of real curves to problems in mathematical physics (theory of solitons, string theory, and so on).

Contents

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Introduction

According to standard definitions, a *real algebraic curve* is a pair (P,τ), where P is a complex algebraic curve (that is, a compact Riemann surface) and $\tau\colon P\to P$ is an antiholomorphic involution. The category of real algebraic curves is isomorphic to the category of Klein surfaces [1], [35]. Investigations of real algebraic curves were started by Klein [25] and Weichold [51]. For a long time thereafter researchers studied only plane algebraic curves, that is, real curves embedded in $\mathbb{RP}^2$. The systematic study of "general" real algebraic curves was renewed only in the seventies [1], [16], [20], [31]–[33], [48]. The method of algebraic-geometric integration of equations of mathematical physics, which was discovered in the seventies in the works by S. P. Novikov and his school, posed a number of new problems in the theory of real curves and significantly stimulated the development of this theory [10], [12]–[14], [34], [37], [42]. Conformal field theory and, in particular, string theory [9], [23], [24], [49] has become another area of applications of real curves.

The antiholomorphic involution τ acts on all structures related to the Riemann surface P, namely, on vector bundles, Jacobians, Prymians, and so on. The greater part of this survey is devoted to finding topological invariants and describing the moduli spaces corresponding to any set of such invariants.

In § 1 we describe topological invariants of real algebraic curves following Weichold [51]. The set of these invariants forms the topological type of a curve. In § 2 we associate with a real algebraic curve a special type of group of isometries of the Lobachevskii plane (real Fuchsian groups). Applying this relationship and the parametrization of Fuchsian groups described in [33] and [47], (§§ 1–4), we prove that to each topological type there corresponds a connected component that is homeomorphic to $\mathbb{R}^m/\operatorname{Mod}$, where Mod is a discrete group.

In § 3 the Arf functions equivalent to θ-characteristics [2], [30] appear in the survey for the first time. In contrast to the complex situation, many topological invariants are connected with these functions in the real case. In § 4 a correspondence is established between Arf functions and representations $\pi_1(P/\langle\tau\rangle)\to GL(2,\mathbb{R})$ that generate real Fuchsian groups. These representations are used in § 5 to describe real spinors on (P,τ). The properties of real spinors enable one to describe non-trivial topological properties of real holomorphic differentials in § 6. In § 7 we show that the simplest meromorphic tensors of arbitrary weight on real curves of arbitrary genus behave just like classical trigonometric functions. Here we use the apparatus developed for complex curves in the papers by Krichever and Novikov in connection with conformal field theory [27]. For lack of space we do not include the classification of meromorphic functions on real algebraic curves of arbitrary genus [40].

In § 8 we pass to a description of Jacobians of real curves, and, in particular, real and imaginary tori of the Jacobian. The results of § 6 enable one to find all such tori disjoint from the θ-divisor. In § 9 the analogous problem is solved for Prymians of real curves with a symmetry. The results in §§ 8 and 9 play the key role in singling out the non-singular real solutions of important equations in mathematical physics [13], [14], [34]. In § 10 we described Bobenko's approach to the calculation of Jacobians of real curves by means of Schottky groups and Poincaré series [5], [6]. Like the parametrization in § 2, this approach uses the parametrization of Fuchsian groups [33], [47], (§§ 1–4). A similar method of describing the Prymians is contained in [34].

In § 11, we return to spinors and describe the moduli space of spinor bundles. It turns out that its components are determined by the topological invariants of the Arf functions introduced in § 3. In § 11 we also describe the topological structure of the connected components of the moduli space of spinor bundles.

The last three sections are devoted to real algebraic supercurves. The complex and real supercurves form the central object of the theory of superstrings that relates the unified quantum field theory with integrals over the moduli space of algebraic supercurves [4], [9], [18]. We define real supercurves via uniformizing groups as is done for complex curves in [4], [29]. In § 12 we describe the moduli space of $N = 1$ real algebraic supercurves. The numerical part (the body) of this superspace coincides with the moduli space of spinor bundles. The connected components correspond to topological types of the real Arf functions, and each of the components is of the form $\mathbb{R}^{(n|m)}/\operatorname{Mod}$, where $\mathbb{R}^{(n|m)}$ is a linear superspace and Mod is a discrete group. In § 13 the system of topological invariants of $N = 2$ real algebraic supercurves is described. As is shown in § 14, these invariants describe the connected components of the moduli space of the supercurves. As in the case $N = 1$, each of the components can be represented in the form $\mathbb{R}^{(n|m)}/\operatorname{Mod}$.

The present survey is a natural continuation of [47] and is based on the results presented there. The topological description of the connected components of (super) real curves and spinor bundles is based, in particular, upon the special description of the connected components of the spaces of (super) Riemann surfaces constructed in [33], [39], and [47]. The topological invariants of (super) real curves include those of (super) Riemann surfaces. However, the total system of topological invariants is much more complicated and diverse.

In this survey the results of the author over several years are presented in a unified style. Some of these topics arose as a result of discussions with V. I. Arnol'd, É. B. Vinberg, and S. P. Novikov, and the author is sincerely indebted to them.

§ 1. Topological type of real algebraic curves

1. By a (*non-singular*) *real algebraic curve* we mean a pair $X = (P, \tau)$, where $P = X(\mathbb{C})$ is a compact Riemann surface (called a *complexification* of the curve X) and $\tau = \tau_X \colon P \to P$ is an antiholomorphic involution (the so-called *involution of complex conjugation*). The fixed points $X(\mathbb{R}) = P^\tau$ of this involution form the set of real points of the curve. For instance, to a non-singular plane real algebraic curve $F(x,y) = 0$ there corresponds a pair (P, τ), where P is the normalization and compactification of the surface $\{(x,y) \in \mathbb{C}^2 \mid F(x,y) = 0\}$ and τ is generated by the involution $(x,y) \mapsto (\overline{x}, \overline{y})$.

Real algebraic curves $X_1 = (P_1, \tau_1)$ and $X_2 = (P_2, \tau_2)$ are taken to be the same if there is a biholomorphic map $\psi \colon P_1 \to P_2$ such that $\psi\tau_1 = \tau_2\psi$.

A curve X is said to be *separating* (type I in the Klein classification) if the set $X(\mathbb{C}) \setminus X(\mathbb{R})$ is disconnected. Otherwise the curve is said to be *non-separating* (type II in the Klein classification).

By the *topological type of a real algebraic curve* X we mean the triple (g, k, ε), where $g = g(X)$ is the genus of the curve, that is, the genus of the surface $X(\mathbb{C})$,

$k = k(X)$ is the number of connected components of the set $X(\mathbb{R})$ of real points, and

$$\varepsilon = \varepsilon(X) = \begin{cases} 0 & \text{if the curve } X \text{ is non-separating,} \\ 1 & \text{if the curve } X \text{ is separating.} \end{cases}$$

In what follows, we often use the fact that every Riemann surface P is biholomorphically equivalent to a surface of the form H/Γ, where H is the Riemann sphere $\overline{\mathbb{C}}$, the complex plane $\mathbb{C}$, or the upper half-plane $\Lambda = \{z \in \mathbb{C} \mid \operatorname{Im} z > 0\}$, and Γ is a discrete group that acts without fixed points. The standard metric of constant curvature on H induces a metric of constant curvature on $P = H/\Gamma$.

Let us present two examples of real algebraic curves.

Example 1.1. Let P be a surface of genus $\widetilde{g}$ with k holes. Let us endow P with the structure of a Riemann surface P^+ and consider an atlas of holomorphic charts

$$\{(U_i, z_i)\}, \quad P^+ = \bigcup U_i, \quad z_i \colon U_i \to \mathbb{C}.$$

The atlas $\{(U_i, \overline{z}_i)\}$ endows P with the structure of another Riemann surface P^-. The natural map $\alpha\colon P^+ \to P \to P^-$ is antiholomorphic. The complex structure of P^+ and P^- generates on these surfaces some metrics of constant curvature with respect to which α is an isometry. Let us surround each of the holes of the surface P^+ by a geodesic. The geodesics cut out a compact surface $\widetilde{P}^+ \subset P^+$ with boundary $\partial\widetilde{P}^+$. We set $\widetilde{P}^- = \alpha\widetilde{P}^+$.

Let us identify the boundaries $\partial\widetilde{P}^+$ and $\partial\widetilde{P}^-$ by means of α. As the result, we obtain a compact Riemann surface $P_{\widetilde{g},k}$ of genus $2\widetilde{g} + k - 1$ on which the map α induces an antiholomorphic involution $\tau_{\widetilde{g},k}\colon P_{\widetilde{g},k} \to P_{\widetilde{g},k}$. Thus, $X_{\widetilde{g},k} = (P_{\widetilde{g},k}, \tau_{\widetilde{g},k})$ is a real algebraic curve, and $X_{\widetilde{g},k}(\mathbb{R}) = \partial\widetilde{P}^+ = \partial\widetilde{P}^-$. Hence, $X_{\widetilde{g},m}$ is a real algebraic curve of type $(2\widetilde{g} + k - 1, k, 1)$.

Example 1.2. Repeating the construction of Example 1.1, we take the Riemann surface with boundary $\widetilde{P}^+, \widetilde{P}^-$ and the antiholomorphic map $\alpha\colon \widetilde{P}^+ \to \widetilde{P}^-$. The boundary $\partial\widetilde{P}^+$ consists of contours $c_1, \dots, c_k$. Let us consider fixed-point-free isometries $\alpha_i\colon c_i \to c_i$ such that $\alpha_i^2 = 1$. Let $0 \leqslant m < k$. For $i \leqslant m$, we identify the contours c_i and αc_i by means of the map α. For $i > m$, we identify the contours c_i and αc_i by means of the map $\alpha\alpha_i$. We again obtain a real curve $Y^m_{\widetilde{g},k} = (P^m_{\widetilde{g},k}, \tau^m_{\widetilde{g},k})$ of the same genus; however, in this case $Y^m_{\widetilde{g},k}(\mathbb{R}) = \bigcup_{i=1}^m c_i$, and hence $Y^m_{\widetilde{g},k}$ is a curve of topological type $(2\widetilde{g} + k - 1, m, 0)$.

2. Real curves (P_1, τ_1) and (P_2, τ_2) are said to be *topologically equivalent* if there is a homeomorphism $\varphi\colon P_1 \to P_2$ such that $\tau_2\varphi = \varphi\tau_1$.

Our immediate goal is to show that any real algebraic curve is topologically equivalent to one of the curves in Examples 1.1 and 1.2.

Lemma 1.1. *The set $X(\mathbb{R})$ of real points of a real algebraic curve $X = (P, \tau)$ decomposes into pairwise disjoint simple closed smooth contours* (*called ovals*).

Proof. The complex structure of the surface P induces a metric of constant curvature, and τ is an isometry with respect to this metric. If $x \in X(\mathbb{R})$, then the involution $d\tau_x\colon T_x \to T_x$ of the tangent plane T_x is the reflection with respect to a line $v \in T_x$. We denote by $\ell \subset P$ the geodesic that passes through x in the direction of the line v. All its points are fixed under τ, and in a small neighbourhood of x there are no other fixed points of τ. Thus, each of the points $x \in X(\mathbb{R})$ belongs to exactly one maximal geodesic $\ell \subset X(\mathbb{R})$ without self-intersections. Since P is compact, it follows that each of these geodesics is a closed smooth contour.

Theorem 1.1. *Let (P,τ) be a real curve of type $(g,k,1)$. Then $1 \leqslant k \leqslant g+1$, $k \equiv g+1 \pmod 2$, and (P,τ) is topologically equivalent to the curve $(P_{\widetilde{g},k}, \tau_{\widetilde{g},k})$ of Example* 1.1, *where $\widetilde{g} = \frac{1}{2}(g+1-k)$.*

Proof. By Lemma 1.1, the set $P \setminus P^\tau$ decomposes into two surfaces P_1 and P_2 of genus $\widetilde{g}$ with k holes. Hence, $g = 2\widetilde{g} + k - 1$, and therefore $k \leqslant g+1$ and $k \equiv g+1 \pmod 2$. Let us consider a homeomorphism $\varphi_1\colon (P_1 \cup P^\tau) \to \widetilde{P}^+$. We set

$$\varphi(x) = \begin{cases} \varphi_1(x) & \text{for } x \in P_1 \cup P^\tau, \\ \tau_{\widetilde{g},k}\varphi_1\tau(x) & \text{for } x \in P_2. \end{cases}$$

We can readily see that φ realizes the desired topological equivalence.

3. Let us now study curves of non-separating type. Up to the end of the section, Q stands for a Riemann surface of genus g with n holes and $\beta\colon Q \to Q$ is an antiholomorphic involution without fixed points.

A simple closed contour $a \subset Q$ is said to be *invariant* if $\beta a = a$.

A system $A = (a_1, \dots, a_m)$ of pairwise disjoint invariant contours is said to be *complete* if the set $Q \setminus A$ is disconnected. Obviously, $Q \setminus A$ then consists of two surfaces Q' and Q'' of genus $\frac{1}{2}(g-m+1)$ with $m + \frac{1}{2}n$ holes, and $\beta Q' = Q''$.

Lemma 1.2. a) *There is at least one invariant contour $a \subset Q$.* b) *If $g > 0$, then there is an invariant contour $b \subset Q$ such that $Q \setminus b$ is connected.* c) *There is a complete system formed by $g+1$ invariant contours.* d) *If $A = (a_1, \dots, a_m) \subset Q$ is a complete system of invariant contours and if $m > 2$, then there is an element $b \subset Q$ such that $(a_1, \dots, a_{m-3}, b)$ is also a complete system of invariant contours.*

Proof. a) Without loss of generality we may assume that $n > 2$. We consider the function $f(x) = \rho(x, \beta x)$ on Q, where ρ is the distance in the standard metric of constant negative curvature on Q. The function f attains its minimum $f(z) = c > 0$. If ℓ is a minimal geodesic joining z and βz, then $a = \ell \cup \beta\ell$ is an invariant contour.

b) Let $a \subset Q$ be the contour constructed in item a) and let $Q \setminus a$ be disconnected. Then $Q \setminus a = Q' \cup Q''$, where Q' and Q'' are surfaces of positive genus, and $\beta Q' = Q''$. Let us join points $x \in a$ and τx by a curve $\ell \subset Q'$ without self-intersections and such that $Q' \setminus \ell$ is connected (see Fig. 1.1). Then $b = \ell \cup \tau\ell$ is an invariant contour, and $Q \setminus b$ is connected.

c) Let b be the contour constructed in item b). The surface $Q \setminus b$ is of genus $g-1$, and if $g-1 > 0$, then we can again apply the assertion in item b). For $g = 0$, we apply item a).

d) The set $Q \setminus A$ decomposes into the surfaces Q' and Q'' (see Fig. 1.2).

Let us complete these surfaces by boundary contours. Corresponding to a contour $a_i \subset A$ are contours $a'_i \subset Q'$ and $a''_i \subset Q''$. Let q_1, q_2, q_3 be points of the contours a_{m-2}, a_{m-1}, a_m and let q'_i be the corresponding points of the contours a'_{m-3+i}.

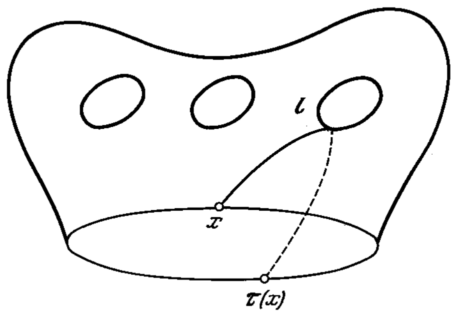

Figure 1.1

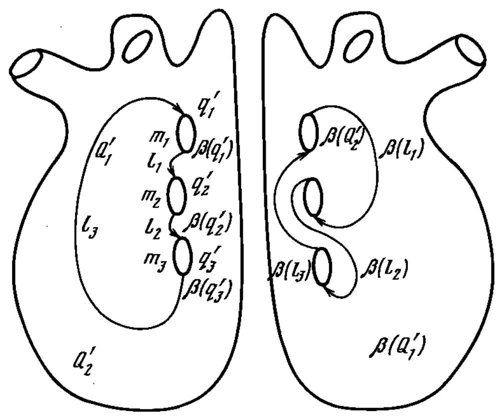

Figure 1.2

We denote by m_i one of the two arcs into which the points q_i' and $\beta(q_i')$ divide the contour a_i'. Let us join the points $\beta(q_1')$ and q_2' by a curve $\ell_1 \subset Q'$ and the points $\beta(q_2')$ and q_3' by a curve $\ell_2 \subset Q'$ so that ℓ_1 and ℓ_2 are disjoint, have no self-intersections, and do not intersect $\partial Q'$ (except for the endpoints). Let us join the points $\beta(q_3')$ and q_1' by a curve ℓ_3 without self-intersections which is homotopic to the curve $(m_1\ell_1m_2\ell_2m_3)^{-1}$ and has no points in common with the latter curve and with $\partial Q'$, except for the endpoints (this can always be done because the set $Q' \setminus (\ell_1 \cup \ell_2)$ is connected). The closed contour $\ell_3m_1\ell_1m_2\ell_2m_3$ without self-intersections decomposes the surface Q' into two parts, Q_1' and Q_2'. We consider now the invariant contour $b = \ell_1\beta(\ell_2)\ell_3\beta(\ell_1)\ell_2\beta(\ell_3) \subset Q$. Then $Q \setminus (b, a_1, \dots, a_{m-3})$ decomposes into the surfaces $Q_1' \cup \beta(Q_2')$ and $Q_2' \cup \beta(Q_1')$.

Theorem 1.2. *Let (P,τ) be a real algebraic curve of topological type $(g,m,0)$. Then for any $m < k \leqslant g+1$ with $k \equiv g+1 \pmod 2$ the curve (P,τ) is topologically equivalent to the curve $(P^m_{\widetilde{g},k}, \tau^m_{\widetilde{g},k})$ in* Example 1.2, *where $\widetilde{g} = \frac{1}{2}(g+1-k)$.*

Proof. According to Lemma 1.2, there is a complete set A of contours on the surface $P \setminus P^\tau$ that are invariant with respect to τ, $A = (a_{m+1}, \dots, a_k)$. The surface $P \setminus (P^\tau \cup A)$ decomposes into two surfaces P_1 and P_2 of genus $\widetilde{g}$

with k holes. Let us consider now a homeomorphism $\varphi_1\colon (P_1 \cup P^\tau \cup A) \to \widetilde{P}^+$ such that $\varphi_1(P^\tau) = (c_1, \dots, c_k)$. We set

$$\varphi(x) = \begin{cases} \varphi_1(x) & \text{for } x \in P_1 \cup P^\tau \cup A, \\ \tau^m_{\widetilde{g},k}\varphi_1\tau(x) & \text{for } x \in P_2. \end{cases}$$

We can readily see that φ defines a topological equivalence.

Examples 1.1 and 1.2 and Theorems 1.1 and 1.2 imply the following.

Corollary 1.1 [51]. *Real algebraic curves are topologically equivalent if and only if they have the same topological type. A set (g,k,ε) is a topological type of a real algebraic curve if and only if either $\varepsilon = 1$, $1 \leqslant k \leqslant g+1$, and $k \equiv g+1 \pmod 2$ or $\varepsilon = 0$ and $0 \leqslant k \leqslant g$.*

Remark. For plane real curves, the inequality $k \leqslant g+1$ was first proved by Harnack [21] and bears his name.

§ 2. Moduli of real algebraic curves

1. In what follows, we need some definitions and notation from [47], §§ 1–5. Each hyperbolic automorphism $C \in \operatorname{Aut}(\Lambda)$ of the Lobachevskii plane $\Lambda = \{z \in \mathbb{C} \mid \operatorname{Im} z > 0\}$ is of the form

$$C(z) = \frac{(\lambda\alpha - \beta)z + (1-\lambda)\alpha\beta}{(\lambda - 1)z + (\alpha - \lambda\beta)},$$

where $\alpha \neq \beta \in \mathbb{R} \cup \infty$ and $\lambda > 1$. We denote by $\ell(C) \subset \Lambda$ the geodesic (in the Lobachevskii metric) that joins α and β and is oriented from β to α. The automorphism C preserves the line $\ell(C)$ while shifting it in the direction of the orientation.

A triple of hyperbolic automorphisms (C_1, C_2, C_3) is said to be *sequential of type* $(0,3)$ if $(C_1 \cdot C_2 \cdot C_3) = 1$ and, for some $D \in \operatorname{Aut}(\Lambda)$, the curves $\ell(DC_iD^{-1})$ are placed as in Fig. 2.1.

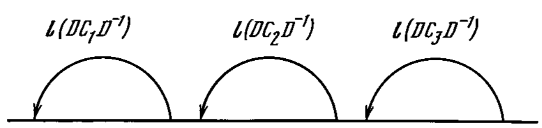

Figure 2.1

An n-tuple of hyperbolic automorphisms $(C_1, \dots, C_n)$ is said to be *sequential of type* $(0,n)$ if, for any j, the triple $(C_1 \cdots C_{j-1}, C_j, C_{j+1} \cdots C_n)$ is sequential of type $(0,3)$.

A set

$$\{A_i, B_i \ (i = 1, \dots, g),\ C_i \ (i = 1, \dots, k)\}$$

is said to be *sequential of type* (g,k) if the tuple

$$(A_1, B_1A_1^{-1}B_1^{-1}, \dots, A_g, B_gA_g^{-1}B_g^{-1}, C_1, \dots, C_k)$$

is sequential of type $(0, 2g+k)$.

By the classical Fricke–Klein theorem [17], [47], a moduli space of complex algebraic curves (that is, of compact Riemann surfaces) can be represented as T/Mod, where T is a linear space and Mod is a discrete group. Our immediate goal is to prove a similar theorem for real algebraic curves.

For T, we take the Fricke–Klein–Teichmüller space $T_{g,k}$ constructed in [33] and in [47], § 4. This space can be defined as follows. Let $n = g + k$ and let $v_{g,n} = \{a_i, b_i\ (i = 1, \dots, g),\ c_i\ (i = g+1, \dots, n)\}$ be a system of generators of a free group $\gamma_{g,n}$ of rank $2g + n - 1$ with the defining relation

$$\prod_{i=1}^{g}[a_i, b_i] \prod_{i=g+1}^{n} c_i = 1.$$

Let us consider the set $\widetilde{T}_{g,k}$ of all monomorphisms $\psi\colon \gamma_{g,n} \to \operatorname{Aut}(\Lambda)$ such that $\{\psi(a_i), \psi(b_i)\ (i = 1, \dots, g),\ \psi(c_i)\ (i = g+1, \dots, n)\}$ is a sequential set of type (g, k). The group $\operatorname{Aut}(\Lambda)$ acts on $\widetilde{T}_{g,k}$ by conjugations $\psi \mapsto C\psi C^{-1}$. By [33] and [47], § 4, the space $T_{g,k} = \widetilde{T}_{g,k}/\operatorname{Aut}(\Lambda)$ is homeomorphic to $\mathbb{R}^{6g+3k-6}$. Moreover, the correspondence

$$\psi \mapsto \Lambda/\psi(\gamma_{g,n})$$

generates a homeomorphism

$$\Psi_{g,n} \colon T_{g,k}/\operatorname{Mod}_{g,k} \to M_{g,k}$$

onto the moduli space $M_{g,k}$ of Riemann surfaces of genus g with k holes. Here $\operatorname{Mod}_{g,k}$ is a discrete group that consists of the classes $\widetilde{\operatorname{Mod}}_{g,k}/\operatorname{Int}(\gamma_{g,n})$, where $\widetilde{\operatorname{Mod}}_{g,k} \subset \operatorname{Aut}(\gamma_{g,n})$ is the group of automorphisms that send monomorphisms in the set $\widetilde{T}_{g,k}$ to monomorphisms in $\widetilde{T}_{g,k}$.

2. In what follows, we consider curves of genus $g > 1$ only. The cases $g \leqslant 1$ are much simpler but need different approaches.

Real algebraic curves of genus $g > 1$ are can be uniformized by discrete groups of isometries of the metric $\dfrac{|dz|}{\operatorname{Im} z}$ of the Lobachevskii plane $\Lambda = \{z \in \mathbb{C} \mid \operatorname{Im} z > 0\}$. The full group $\widetilde{\operatorname{Aut}}(\Lambda)$ of isometries consists of the holomorphic automorphisms that form the group $\operatorname{Aut}(\Lambda)$ and of antiholomorphic ones.

The discrete subgroups $\Gamma \subset \widetilde{\operatorname{Aut}}(\Lambda)$ are called *non-Euclidean crystallographic groups* (*NEC-groups*) [28]. In what follows, we need only NEC-groups $\widetilde{\Gamma}$ for which $\Gamma = \widetilde{\Gamma} \cap \operatorname{Aut}(\Lambda)$ is a Fuchsian group that consists of hyperbolic automorphisms, $\Gamma \neq \widetilde{\Gamma}$, and $P = \Lambda/\Gamma$ is a compact surface. These groups $\widetilde{\Gamma}$ will be called *real Fuchsian groups*. In this case $\widetilde{\Gamma} \setminus \Gamma$ induces an antiholomorphic involution $\tau = \Phi(\widetilde{\Gamma} \setminus \Gamma)\Phi^{-1}\colon P \to P$ (where $\Phi\colon \Lambda \to P$ is the natural projection). Thus, a real Fuchsian group $\widetilde{\Gamma}$ generates a real algebraic curve $(P, \tau) = [\widetilde{\Gamma}]$.

Lemma 2.1. *Every real algebraic curve is generated by some real Fuchsian group.*

Proof. Let $\Gamma \subset \operatorname{Aut}(\Lambda)$ be a Fuchsian group uniformizing the Riemann surface P, and let $\Phi\colon \Lambda \to P$ be the natural projection (see, for instance, [47], § 2). Since Λ

is simply connected, there is an element $\sigma \in \widetilde{\mathrm{Aut}}(\Lambda) \setminus \mathrm{Aut}(\Lambda)$ such that $\Phi\sigma = \tau\Phi$. Let $\widetilde{\Gamma}$ be the group generated by σ and Γ. Then $(P,\tau) = [\widetilde{\Gamma}]$.

3. Let $M_{g,k,\varepsilon}$ be the moduli space of real algebraic curves of type (g,k,ε). Our immediate object is to construct a natural map $\Psi^k_{\widetilde{g},k}\colon \widetilde{T}_{\widetilde{g},k} \to M_{g,k,1}$, where $\widetilde{g} = \frac{1}{2}(g+1-k)$.

Let $n = \widetilde{g} + k$, $\psi \in \widetilde{T}_{\widetilde{g},k}$, and $\{A_i, B_i \ (i = 1,\dots,\widetilde{g}),\ C_i \ (i = 1,\dots,k)\} = \{\psi(a_i), \psi(b_i)(i = 1,\dots,\widetilde{g}),\ \psi(c_i) \ (i = \widetilde{g}+1,\dots,n)\}$. Denote by $\overline{C}_i \in \widetilde{\mathrm{Aut}}(\Lambda) \setminus \mathrm{Aut}(\Lambda)$ the reflection (in the sense of Lobachevskian geometry) with respect to the geodesic $\ell(C_i)$. Let $\Gamma_\psi = \psi(\gamma_{\widetilde{g},n})$ and let Γ^k_ψ be the group generated by Γ_ψ and the elements $\overline{C}_1,\dots,\overline{C}_k$.

Lemma 2.2. *Γ^k_ψ is a real Fuchsian group, and $[\Gamma^k_\psi] \in M_{g,k,1}$.*

Proof. Let $\{\widetilde{a}_i, \widetilde{b}_i \ (i = 1,\dots,g)\}$ be the generators of the group $\gamma_{g,0}$ with the defining relation $\prod_{i=1}^{g}[\widetilde{a}_i, \widetilde{b}_i] = 1$. We set

$$\begin{aligned}
&\widetilde{\psi}(\widetilde{a}_i) = \overline{C}_n B_{\widetilde{g}+1-i}\overline{C}_n, && \widetilde{\psi}(\widetilde{b}_i) = \overline{C}_n A_{\widetilde{g}+1-i}\overline{C}_n && (i = 1,\dots,\widetilde{g}),\\
&\widetilde{\psi}(\widetilde{a}_i) = A_{i-\widetilde{g}}, && \widetilde{\psi}(\widetilde{b}_i) = B_{i-\widetilde{g}} && (i = \widetilde{g}+1,\dots,2\widetilde{g}),\\
&\widetilde{\psi}(\widetilde{a}_i) = W_i C_i W_i, && \widetilde{\psi}(\widetilde{b}_i) = W_i D_i W_i^{-1} && (i = 2\widetilde{g}+1,\dots,2\widetilde{g}+k),
\end{aligned}$$

where $D_i = \overline{C}_n\overline{C}_i$ and $W_i = \prod_{j=i-1}^{1} D_j C_j D_j^{-1}$ (see Fig. 2.2).

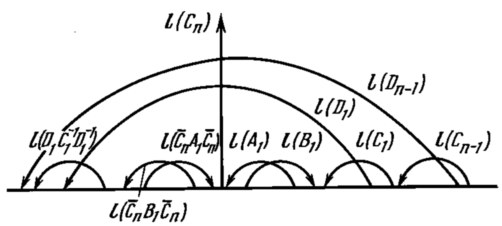

Figure 2.2

Then

$$\prod_{i=1}^{g}[\widetilde{\psi}(\widetilde{a}_i), \widetilde{\psi}(\widetilde{b}_i)] = \overline{C}_n \prod_{i=\widetilde{g}}^{1}[B_i, A_i]\overline{C}_n \prod_{i=1}^{\widetilde{g}}[A_i, B_i] \prod_{i=1}^{k} C_i \prod_{i=k}^{1} \overline{C}_n C_i^{-1}\overline{C}_n = 1$$

because $\prod_{i=1}^{\widetilde{g}}[A_i, B_i]\prod_{i=1}^{k} C_i = 1$. Moreover,

$$\bigl(\widetilde{\psi}(\widetilde{a}_1), \widetilde{\psi}(\widetilde{b}_1\widetilde{a}_1^{-1}\widetilde{b}_1^{-1}), \dots, \widetilde{\psi}(\widetilde{a}_g), \widetilde{\psi}(\widetilde{b}_g\widetilde{a}_g^{-1}\widetilde{b}_g^{-1})\bigr)$$

is a sequential set of type $(0,2g)$ (see Fig. 2.2). Thus, $\widetilde{\psi} \in \widetilde{T}_{g,0}$, and hence $P = \Lambda/\widetilde{\psi}(\gamma_{g,0}) \in M_{g,0}$. The group Γ^k_ψ is generated by the group $\widetilde{\psi}(\gamma_{g,0})$ together with the involutions $\overline{C}_i$, and $\overline{C}_i\widetilde{\psi}(\gamma_{g,0})\overline{C}_i = \widetilde{\psi}(\gamma_{g,0})$. Hence, Γ^k_ψ is a real Fuchsian group, and the images $\ell(C_i)$ form ovals of the curve $[\Gamma^k_\psi]$. By construction, these contours form the boundary of a surface of genus $\widetilde{g}$.

Thus, the correspondence $\psi \mapsto [\Gamma^k_\psi]$ defines a map $\Psi^k_{\widetilde{g},k}\colon \widetilde{T}_{\widetilde{g},k} \to M_{g,k,1}$.

Lemma 2.3. $\Psi^k_{\widetilde{g},k}(\widetilde{T}_{\widetilde{g},k}) = M_{g,k,1}$.

Proof. Let $(P,\tau) \in M_{g,k,1}$. By Lemma 2.1, $(P,\tau) = [\widetilde{\Gamma}]$ for some real Fuchsian group $\widetilde{\Gamma}$. Let $\Gamma \in \widetilde{\Gamma} \cap \operatorname{Aut}(\Lambda)$, let $\Phi\colon \Lambda \to \Lambda/\Gamma = P$ be the natural projection, let $\Phi(q) = p$, and let $\Phi_q\colon \gamma \to \pi_1(P,p)$ be an isomorphism that sends $h \in \Gamma$ into the image $\Phi(\ell)$ of the segment $\ell = [q, hq] \subset \Lambda$. The ovals of P^τ decompose P into two surfaces P_1 and P_2. Let $p \in P_1$ and let $v = \{a_i, b_i\ (i = 1,\dots,\widetilde{g}),\ c_i\ (i = \widetilde{g}+1,\dots,n)\}$ be a standard system of generators of the group $\pi(P_1,p)$ in the sense of [47], § 2. By [47], Theorem 2.1, $V = \Phi_q^{-1}(v)$ is a sequential set of type $(\widetilde{g},k)$, that is, $V = \psi(v)$, where $\psi \in \widetilde{T}_{\widetilde{g},k}$. Then $[\Gamma^k_\psi] = (P,\tau)$.

We recall that

$$T_{\widetilde{g},k} = \widetilde{T}_{\widetilde{g},k}/\operatorname{Aut}(\Lambda) \cong \mathbb{R}^{6\widetilde{g}+3k-6} = \mathbb{R}^{3g-3}$$

[33], [47], § 4.

For clear reasons, the map

$$\Psi^k_{\widetilde{g},k}\colon \widetilde{T}_{\widetilde{g},k} \to M_{g,k,1}$$

induces the map

$$\Psi^k_{\widetilde{g},k}\colon T_{\widetilde{g},k} \to M_{g,k,1}.$$

We also need the map

$$\alpha\colon T_{\widetilde{g},k} \to T_{\widetilde{g},k}$$

determined by the relations

$$\begin{aligned}
&\alpha\psi(a_i) = \beta\psi(b_{\widetilde{g}+1-i})\beta, \\
&\alpha\psi(b_i) = \beta\psi(a^{-1}_{\widetilde{g}+1-i})\beta \qquad && (i = 1,\dots,\widetilde{g}), \\
&\alpha\psi(c_i) = w\beta\psi(c^{-1}_{\widetilde{g}+k+1-i})\beta w^{-1} && (i = \widetilde{g}+1,\dots,\widetilde{g}+k),
\end{aligned}$$

where $\beta(z) = -\overline{z}$ and $w = \alpha\psi\big(\prod_{i=1}^{\widetilde{g}}[a_i,b_i]\big)$. Let $\operatorname{Mod}^k_{\widetilde{g},k}$ be the group of automorphisms of $T_{\widetilde{g},k}$ generated by $\operatorname{Mod}_{\widetilde{g},k}$ and α. Then $\operatorname{ind}(\operatorname{Mod}_{\widetilde{g},k} : \operatorname{Mod}^k_{\widetilde{g},k}) = 2$. Moreover, we can readily see from the construction that $[\Gamma^k_\psi] = [\Gamma^k_{\psi'}]$ if and only if $\psi' = \gamma\psi$, where $\gamma \in \operatorname{Mod}^k_{\widetilde{g},k}$. Thus, Lemmas 2.2 and 2.3 imply the following result.

Theorem 2.1 ([31]–[33]). $M_{g,k,1} = T_{\widetilde{g},k}/\operatorname{Mod}^k_{\widetilde{g},k}$, *where the action of* $\operatorname{Mod}^k_{\widetilde{g},k}$ *is discrete.*

4. Let us pass now to a description of the space $M_{g,m,0}$. To this end, we construct a map

$$\Psi^m_{\widetilde{g},k}\colon \widetilde{T}_{\widetilde{g},k} \to M/T_{g,m,0},$$

where $m < k$, $k \equiv g+1 \pmod 2$, and $\widetilde{g} = \frac{1}{2}(g+1-k)$.

As above, to a monomorphism $\psi \in \widetilde{T}_{\widetilde{g},k}$ there corresponds the sequential set

$$\begin{aligned}
&\{A_i, B_i\ (i = 1,\dots,\widetilde{g}),\ C_i\ (i = 1,\dots,k)\} \\
&\qquad = \{\psi(a_i), \psi(b_i)\ (i = 1,\dots,\widetilde{g}),\ \psi(c_i)\ (i = \widetilde{g}+1,\dots,n)\}.
\end{aligned}$$

We write $\widetilde{C}_i = \overline{C}_i\sqrt{C_i}$, where $\sqrt{C_i}$ is a hyperbolic automorphism such that $(\sqrt{C_i}\,)^2 = C_i$. Let $\Gamma_\psi = \psi(\gamma_{\widetilde{g},n})$ and let $\Gamma^m_{\psi,k}$ be the group generated by Γ_ψ, $\overline{C}_1,\dots,\overline{C}_m$ together with $\widetilde{C}_{m+1},\dots,\widetilde{C}_k$.

Lemma 2.4. *$\Gamma^m_{\psi,k}$ is a real Fuchsian group and $[\Gamma^m_{\psi,k}] \in M_{g,m,0}$.*

Proof. The proof repeats that of Lemma 2.2 almost literally. The only difference is that the images $\ell(C_i)$ form ovals if and only if $i \leqslant m$, and hence $[\Gamma^m_{\psi,k}]$ is a non-separating curve.

Thus, the correspondence $\psi \mapsto [\Gamma^m_{\psi,k}]$ defines a map

$$\Psi^m_{\widetilde{g},k} \colon \widetilde{T}_{\widetilde{g},k} \to T_{g,m,0}.$$

Lemma 2.5. $\Psi^m_{\widetilde{g},k}(\widetilde{T}_{\widetilde{g},k}) = M_{g,m,0}$.

Proof. Let $(P,\tau) \in M_{g,m,0}$. By Theorem 1.2, there is a set of invariant contours $A = (a_{m+1},\dots,a_k) \subset P \setminus P^\tau$ such that $P \setminus (P^\tau \cup A)$ decomposes into two surfaces P_1 and P_2 of genus $\widetilde{g} = \frac{1}{2}(g+1-k)$. The rest of the proof is just like that of Lemma 2.3.

Theorem 2.2 ([31]–[33]). *We have $M_{g,k,0} = T_{\widetilde{g},k}/\operatorname{Mod}^m_{\widetilde{g},k}$, where $\operatorname{Mod}^m_{\widetilde{g},k}$ acts discretely and $\operatorname{ind}(\operatorname{Mod}^m_{\widetilde{g},k} \cap \operatorname{Mod}^k_{\widetilde{g},k} : \operatorname{Mod}^k_{\widetilde{g},k}) = \binom{k}{m}$.*

Proof. The map $\Psi^m_{g,k} \colon \widetilde{T}_{\widetilde{g},k} \to M_{g,m,0}$ induces a map $\Psi^m_{g,k} \colon T_{\widetilde{g},k} \to M_{g,m,0}$ in an obvious way. Let $\Psi^m_{\widetilde{g},k}(\psi) = \Psi^m_{\widetilde{g},k}(\psi')$. This means that $(P,\tau) = [\Gamma^m_{\psi,k}] = [\Gamma^m_{\psi',k}] = (P',\tau')$. Let us consider the monomorphisms $\psi, \psi' \in T_{g,0}$. We have $\widetilde{\psi}' = \widetilde{\psi}\gamma$, where γ belongs to the group $\operatorname{Mod}^m_{\widetilde{g},k}$ generated by the group $\{\gamma \in \operatorname{Mod}_{g,0} \mid \gamma\tau = \tau\gamma\}$ together with α, so that $\Psi^m_{\widetilde{g},k}(\psi\gamma) = \Psi^m_{\widetilde{g},k}(\psi)$ for any $\gamma \in \operatorname{Mod}^m_{\widetilde{g},k}$. Let us now consider the subgroup $\operatorname{Mod}^m_{\widetilde{g},k} \cap \operatorname{Mod}^k_{\widetilde{g},k}$ that consists of the automorphisms of $\operatorname{Mod}^k_{g,k}$ preserving the set c_i $(i = 1,\dots,m)$. We can readily see that the index of this subgroup in $\operatorname{Mod}^k_{\widetilde{g},k}$ is equal to $\binom{k}{m}$.

Comparing Theorems 2.1 and 2.2 with [47], § 4, we obtain the following.

Corollary 2.1 ([31]–[33]). *The moduli space of real algebraic curves of genus $g > 1$ decomposes into the connected components $M_{g,k,\varepsilon}$, where (g,k,ε) is an arbitrary topological type of a real algebraic curve. Each of the components is diffeomorphic to $\mathbb{R}^{3g-3}/\operatorname{Mod}_{g,k,\varepsilon}$, where $\operatorname{Mod}_{g,k,\varepsilon}$ is a discrete group of diffeomorphisms.*

Remark. The assertion of Corollary 2.1 concerning the topological structure of the connected components of the space of real algebraic curves was first presented in [16]. The proof given in [16] used the theory of quasiconformal maps and was based upon a theorem in [26], which turned out later to be wrong. A correct proof based on the theory of quasiconformal maps was obtained in [48].

§ 3. Arf functions on real algebraic curves

1. In the study of spinor bundles and super Riemann surfaces, the Arf functions play an important role [15], [47], §§ 7–15. Special Arf functions are connected with real algebraic curves and we pass to their description.

Let P be a surface of genus $g = g(P)$ with k holes. A basis $v = \{a_i, b_i$ $(i = 1,\dots,g), c_i$ $(i = g+1,\dots,g+k)\}$ of the group $H_1(P,\mathbb{Z}_2)$ (where $\mathbb{Z}_2 = \mathbb{Z}/2\mathbb{Z} = \{0,1\}$) is said to be *standard* if the generators c_i correspond to the holes of the surface P, $(a_i, a_j) = (b_i, b_j) = 0$, and $(a_i, b_j) = \delta_{ij}$, where $(\,\cdot\,,\cdot\,) \in \mathbb{Z}_2$ is the homology intersection number for $H_1(P, \mathbb{Z}_2)$.

By an *Arf function* on P we mean a function $\omega\colon H_1(P, \mathbb{Z}_2) \to \mathbb{Z}_2$ such that $\omega(a+b) = \omega(a) + \omega(b) + (a, b)$. We say that an Arf function ω is *even* and set $\delta = \delta(P, \omega) = 0$ if there is a standard basis v such that

$$\sum_{i=1}^{g} \omega(a_i)\omega(b_i) \equiv 0 \pmod 2.$$

Otherwise we set $\delta = \delta(P, \omega) = 1$ and say that ω is *odd.* By $k_\alpha = k_\alpha(P, \omega)$ $(\alpha = 0, 1)$ we denote the cardinality of the set of elements c_i of a standard basis v such that $\omega(c_i) = \alpha$. The triple (g, δ, k_α) is called the *topological type* of the Arf function ω.

By [47], § 8, a triple (g, δ, k_α) is the topological type of an Arf function if and only if $k_1 \equiv 0 \pmod 2$ and $\delta = 0$ for $k_1 > 0$. Moreover, there is a standard basis v such that $\omega(a_i) = \omega(b_i) = 0$ for $i > 1$ and $\omega(a_1) = \omega(b_1) = \delta$.

Two Arf functions ω_1 and ω_2 on P are said to be *topologically equivalent* if there is a homeomorphism $\psi\colon P \to P$ that induces an automorphism $\widetilde{\psi}\colon H_1(P, \mathbb{Z}_2) \to H_1(P, \mathbb{Z}_2)$ satisfying the relation $\omega_1 = \omega_2\widetilde{\psi}$.

By [47], § 8, Arf functions are topologically equivalent if and only if they have the same topological type.

2. Let (P, τ) be a real algebraic curve. It what follows, we denote a simple contour and the homology class of this contour in $H_1(P, \mathbb{Z}_2)$ by the same symbol. The involution $H_1(P, \mathbb{Z}_2) \to H_1(P, \mathbb{Z}_2)$ induced by the involution $\tau\colon P \to P$ will also be denoted by the same letter τ.

By an *Arf function on a real algebraic curve* (P, τ) (or simply a *real Arf function*) we mean an Arf function $\omega\colon H_1(P, \mathbb{Z}_2) \to \mathbb{Z}_2$ such that $\omega\tau = \omega$.

Lemma 3.1. *Let (P, τ) be a real curve, let $c_1, c_2 \subset P$ be simple closed contours such that $\tau(c_i) = c_i$, $c_i \cap P^\tau = \varnothing$, and $c_1 \cap c_2 = \varnothing$, and let ω be an arbitrary Arf function on (P, τ). Then $\omega(c_1) = \omega(c_2)$.*

Proof. By Theorem 1.2, there is a set of pairwise disjoint simple contours $c_3, \dots, c_r$ belonging to $P \setminus (c_1 \cup c_2)$ and such that $\tau(c_i) = c_i$ and the set $P \setminus \bigcup_{i=1}^{r} c_i$ decomposes into surfaces P_1 and P_2 with $\tau P_1 = P_2$. Let us join the contours c_1 and c_2 by a curve $\ell \subset P_1$ without self-intersections. Let d be a simple closed contour of the form

$$d = \ell \cup f_1 \cup \tau\ell \cup f_2,$$

where $f_i \subset c_i$ is a segment joining the points $\ell \cap c_i$ and $\tau\ell \cap c_i$ (see Fig. 3.1). Then $\tau(d) = d + c_1 + c_2$, and hence

$$\omega(d) = \omega(d) + \omega(c_1) + \omega(c_2).$$

An Arf function ω on (P, τ) is said to be *singular* if there is a simple closed contour c such that $\tau(c) = c$, $c \cap P^\tau = \varnothing$, and $\omega(c) = 0$.

Lemma 3.2. *If $P^\tau \neq \varnothing$, then any real Arf function on (P, τ) is non-singular.*

Proof. Let $c \subset P$ be a simple contour such that $\tau(c) = c$ and $c \cap P^\tau = \varnothing$. Let $c' \subset P^\tau$ be an oval of the real curve (P, τ). By Theorem 1.2, there is a set of

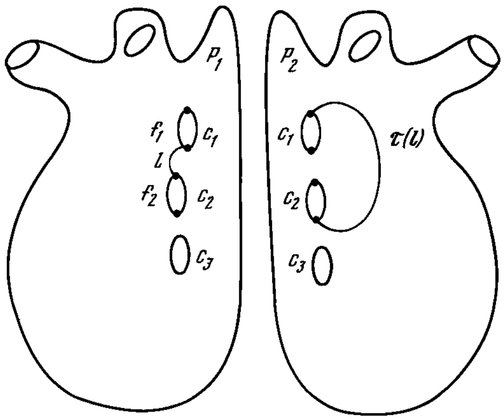

Figure 3.1

simple, pairwise disjoint contours $c_1,\dots,c_r \in P \setminus (c \cup c')$ such that $\tau(c_i) = c_i$ and the difference $P \setminus \bigl(c \cup c' \cup \bigcup_{i=1}^r c_i\bigr)$ decomposes into surfaces P_1 and P_2 with $\tau P_1 = P_2$. Let us join the contours c and c' by a curve $\ell \subset P_1$ without self-intersections. Let d be a simple closed contour of the form $d = \ell \cup f \cup \tau\ell$, where $f \subset c$. Then $\tau(d) = d + c$, and hence $\omega(d) = \omega(d + c) = \omega(d) + \omega(c) + 1$.

Lemma 3.3. *A singular real Arf function vanishes on all invariant contours.*

Proof. Let ω be a singular Arf function on a real algebraic curve (P,τ). Suppose that there is a contour $c \in P$ such that $\tau c = c$ and $\omega(c) = 1$. By Lemmas 1.2 and 3.2, there is a complete system of invariant contours $c, c_1,\dots,c_g$ that decompose P into spheres P_1 and P_2 with holes. Let us join the contour c_i to the contour c by a segment $\ell_i \subset P_1$ and set

$$d_i = \ell_i \cup \tau\ell_i \cup r_i \cup r,$$

where $r_i \subset c_i$ ($r \subset c$) are arcs joining the points $p_i = \ell_i \cap c_i$ and τp_i (the points $p = \ell_i \cap c$ and τp, respectively). Let us consider a disc $D_1 \subset P_1$. We identify the boundary contours of the surface $P \setminus (D_1 \cup \tau D_1)$ by means of the involution τ. On the surface $\widetilde{P}$ thus obtained, the involution τ induces an involution with oval $\widetilde{c} = \partial D_1$. Let us join the contour $\widetilde{c}$ to the contour c by a segment $\widetilde{\ell} \subset P_1$ and set $\widetilde{d} = \widetilde{\ell} \cup \tau\widetilde{\ell} \cup \widetilde{r}$, where $\widetilde{r} \subset c$ is an arc joining the points $\widetilde{p} = \widetilde{\ell} \cap c$ and $\tau\widetilde{p}$. The contours $\{c_i, d_i\ (i = 1,\dots,g),\ \widetilde{c},\ \widetilde{d}\}$ form a basis of $H_1(\widetilde{P},\mathbb{Z}_2)$. Let us consider the Arf function $\widetilde{\omega}$ on $\widetilde{P}$ such that $\widetilde{\omega}(c_i) = \omega(c_i)$, $\widetilde{\omega}(d_i) = \omega(d_i)$, and $\widetilde{\omega}(\widetilde{c}) = \widetilde{\omega}(\widetilde{d}) = 0$. Then $\widetilde{\omega}(c) = \sum_{i=1}^g \widetilde{\omega}(c_i) = \sum_{i=1}^g \omega(c_i) = \omega(c)$ and $\widetilde{\omega}(\tau\widetilde{d}) = \widetilde{\omega}(\widetilde{d} + c) = \widetilde{\omega}(\widetilde{d}) + \widetilde{\omega}(c) + 1 = \widetilde{\omega}(\widetilde{d})$, and hence $\widetilde{\omega}$ is a real Arf function. By Lemma 3.2, this proves that $\widetilde{\omega}$ is equal to one on all contours c' of the surface $P \setminus \widetilde{c}$ such that $\tau c' = c'$. However, on these contours, ω and $\widetilde{\omega}$ must coincide, and hence ω is non-singular. The contradiction thus obtained shows that $\omega(c) = 0$.

Theorem 3.1 [41]**.** *A singular Arf function on a real curve (P,τ) of type (g,k,ε) exists if and only if $k = \varepsilon = 0$. In this case, there are 2^g real Arf functions, and all of them are even.*

Proof. The condition $k=\varepsilon=0$ for singular Arf functions follows from Lemma 3.2. Suppose that $k=\varepsilon=0$. Let us consider the standard basis $\{c_i, d_i\ (i=1,\dots,g)\}\subset H_1(P,\mathbb{Z}_2)$ with $\tau c_i = c_i$ and $\tau d_i = d_i + c_i + \sum_{i=1}^{g} c_i$ that was constructed in the proof of Lemma 3.3. We set $\omega(c_i)=0$ for all i, assign to $\omega(d_i)$ $(i=1,\dots,g)$ arbitrary values in $\mathbb{Z}_2$, and extend ω to $H_1(P,\mathbb{Z}_2)$ by setting $\omega(a+b) = \omega(a)+\omega(b)+(a,b)$. Then $\omega(\tau d_i)=\omega(d_i)$, and hence ω is a singular even real Arf function. By Lemma 3.3, this construction gives all singular Arf functions on (P,τ).

3. By the *topological type* of a non-singular Arf function ω on a real curve (P,τ) of type $(g,k,0)$ we mean the triple (g,δ,k_α), where $\delta=\delta(P,\omega)$ and k_α $(\alpha=0,1)$ is the number of ovals $c_i\in P^\tau$ such that $\omega(c_i)=\alpha$.

Theorem 3.2 [41]**.** *A triple (g,δ,k_α) is the topological type of a non-singular Arf function on a real curve of type $(g,k,0)$ if and only if $k=k_0+k_1\leqslant g$ and $k_0=g+1$* $\pmod 2$. *In this case, there are $\binom{k}{k_0}\cdot 2^{g-1}$ such functions.*

Proof. Let (P,τ) be a real curve of type $(g,k,0)$. By Theorem 1.2, there is a set $(c_1,\dots,c_{g+1})$ of pairwise disjoint simple contours such that $P^\tau=\bigcup_{i=1}^{k} c_i$ and $\tau(c_i)=c_i$. This set decomposes P into two spheres P_1 and P_2 with $g+1$ holes, and $\tau P_1 = P_2$. Let ω be a non-singular Arf function on (P,τ). Then, by [47], § 8, the Arf function $\omega\big|_{P_1}$ takes the value 1 on evenly many holes. Hence, if ω is non-singular, then $k_1+(g+1-k)\equiv 0 \pmod 2$, that is, $k_0\equiv g+1 \pmod 2$.

We assume now that (g,δ,k_α) is an arbitrary triple such that $k_0+k_1\leqslant g$ and $k_0\equiv g+1 \pmod 2$. Let us join the contours c_i and c_{g+1} by a segment $\ell_i\subset P_i$ and consider a simple contour $d_i=\ell_i\cup\tau\ell_i\cup r_i\cup r_{g+1}$, where $r_j\subset c_j$. Then $\tau(d_i)=d_i+c_{g+1}+\alpha_i c_i$, where $\alpha_i=0$ for $i\leqslant k$ and $\alpha_i=1$ for $i>k$.

We now set $\omega(c_i)=0$ for an arbitrary k_0-tuple of contours from among $c_1,\dots,c_k$. We set $\omega(c_i)=1$ on the other contours in $\{c_1,\dots,c_g\}$. Since $k_0\equiv g+1 \pmod 2$, it follows that such contours do exist. Let c_r be one of them, that is, let $\omega(c_r)=1$. We assign arbitrary values to $\omega(d_i)$, $i\neq r$, and let

$$\omega(d_r)=\delta-\sum_{i\neq r}\omega(c_i)\omega(d_i).$$

Let us extend ω to the whole of $H_1(P,\mathbb{Z}_2)$ by setting

$$\omega(a+b)=\omega(a)+\omega(b)+(a,b).$$

We can readily see that this construction gives all real non-singular Arf functions of type (g,δ,k_α).

4. Arf functions on curves of separating type (all such functions are automatically non-singular) have additional topological invariants.

Let (P,τ) be a real curve of separating type and let $P_1\cup P_2 = P\setminus P^\tau$. Let us join ovals $c_i, c_j\in P^\tau$ by a segment $\ell_{ij}\subset P_1$ and consider the contour $d_{ij}=\ell_{ij}\cup\tau\ell_{ij}$. Ovals c_i and c_j are said to be *similar with respect to an Arf function* ω on (P,τ) if $\omega(d_{ij})=0$.

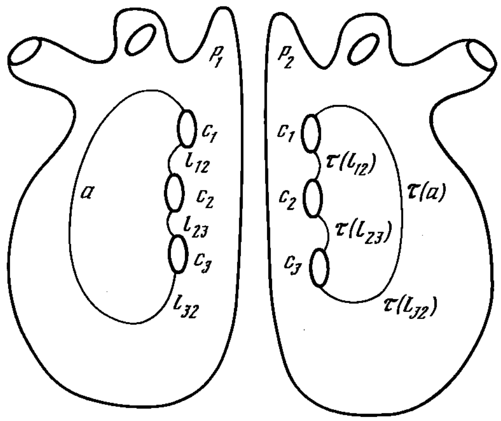

Figure 3.2

Theorem 3.3. *The similarity relation is well defined, and it partitions the ovals into at most two equivalence classes.*

Proof. Let $\widetilde{\ell}_{ij} \subset P_1$ be another segment joining c_i and c_j, let $\widetilde{d}_{ij} = \widetilde{\ell}_{ij} \cup \tau\widetilde{\ell}_{ij}$, and let $b \subset P_1 \cup P^\tau$ be a closed contour composed of ℓ_{ij}, $\widetilde{\ell}_{ij}$, and parts of the ovals c_i and c_j. Then $\omega(d_{ij} + \widetilde{d}_{ij}) = \omega(b + \tau b) = 2\omega(b) = 0$, and hence $\omega(d_{ij}) = \omega(d_{ij} + \widetilde{d}_{ij}) + \omega(\widetilde{d}_{ij}) = \omega(\widetilde{d}_{ij})$. Thus, the definition of similarity does not depend on the choice of the segment ℓ_{ij}. Suppose now that $a \subset P_1 \cup P^\tau$ is a closed contour formed by the segments ℓ_{ij}, ℓ_{jk}, and ℓ_{ki} and by parts of the ovals c_i, c_j, and c_k (see Fig. 3.2). Then $\omega(d_{ij} + d_{jk} + d_{ki}) = \omega(a + \tau a) = 2\omega(a) = 0$. Hence, $\omega(d_{ij}) = \omega(d_{ik}) + \omega(d_{kj})$. Thus, if c_i is (not) similar to c_k and c_k is (not) similar to c_j, then c_i is similar to c_j.

Let us choose some oval $c \in P^\tau$. Let B_c be the set of ovals similar to c. By $k^0_\alpha = k^0_\alpha(P, \tau, \omega)$ (by $k^1_\alpha = k^1_\alpha(P, \tau, \omega)$, respectively) we denote the number of ovals c_i in the set B_c (in $P^\tau \setminus B_c$, respectively) such that $\omega(c_i) = \alpha$. The set of numbers k^γ_α $(\alpha, \gamma \in \{0, 1\})$ is defined up to the simultaneous substituion $k^\gamma_\alpha \mapsto k^{1-\gamma}_\alpha$ related to the choice of the contour c.

By the *topological type of an Arf function* ω on a real curve (P, τ) of type $(g, k, 1)$ we mean the triple $(g, \widetilde{\delta}, k^\gamma_\alpha)$, where $k^\gamma_\alpha = k^\gamma_\alpha(P, \tau, \omega)$, $\widetilde{\delta} = \delta(P_1, \omega|_{P_1})$, and $P_1 \cup P_2 = P \setminus P^\tau$.

Theorem 3.4 [41]. *A triple $(g, \widetilde{\delta}, k^\gamma_\alpha)$ is the topological type of an Arf function on a real curve (P, τ) of type $(g, k, 1)$ if and only if $(\widetilde{g}, \widetilde{\delta}, k^0_\alpha + k^1_\alpha)$ is the topological type of an Arf function on a surface of genus $\widetilde{g} = \frac{1}{2}(g + 1 - k)$ with k holes. In this case the number of such Arf functions is*

$$\binom{k}{k_0} \cdot \binom{k_0}{k^0_0} \cdot \binom{k_1}{k^0_1} \cdot 2^{\widetilde{g}-2} \cdot (2^{\widetilde{g}} + m),$$

where $m = 2^{\widetilde{g}}$ for $k_1 > 0$, $m = 1$ for $\widetilde{\delta} = 0$, and $m = -1$ for $k_1 = 0$ and $\widetilde{\delta} = 1$. The parity of the Arf function coincides with that of k^0_1.

Proof. If $(g, \widetilde{\delta}, k_\alpha^\gamma)$ is the topological type of an Arf function on a real curve (P, τ) of type $(g, k, 1)$, then the set $(\widetilde{g}, \widetilde{\delta}, k_\alpha^0 + k_\alpha^1)$ is the topological type of an Arf function of the form $\omega\big|_{P_1}\colon H_1(P_1, \mathbb{Z}_2) \to \mathbb{Z}_2$, where $P \setminus P^\tau = P_1 \cup P_2$. Let (P, τ) be a real curve of type $(g, k, 1)$, let $P \setminus P^\tau = P_1 \cup P_2$, let $\widetilde{\omega}\colon H_1(P_1, \mathbb{Z}_2) \to \mathbb{Z}_2$ be an Arf function on P_1 of type $(\widetilde{g}, \widetilde{\delta}, k_\alpha^0 + k_\alpha^1)$, and let $v = \{a_i, b_i \ (i = 1, \dots, \widetilde{g}),\ c_j \ (j = \widetilde{g}+1, \dots, \widetilde{g}+k)\} \subset H_1(P_1, \mathbb{Z}_2)$ be a standard basis. Let us partition the ovals c_i arbitrarily into groups A_0^0, A_0^1, A_1^0, A_1^1, where A_α^γ contains k_α^γ contours. Let us join the ovals c_i and c_k by segments $\ell_i \subset P_1$ and set $d_i = \ell_i \cup \tau\ell_i$. We assume that $\omega(c_i) = \alpha$ if $c_i \in A_\alpha^0 \cup A_\alpha^1$, and $\omega(d_i) = 0$ if c_i and c_k belong to the same set of the form $A_0^\alpha \cup A_1^\alpha$. Otherwise we set $\omega(d_i) = 1$. Finally, we set $\omega(\tau a_i) = \omega(a_i)$ and $\omega(\tau b_i) = \omega(b_i)$ $(i = 1, \dots, \widetilde{g})$. The relation

$$\omega(a + b) = \omega(a) + \omega(b) + (a, b)$$

enables one to extend ω uniquely to an Arf function on (P, τ). We can readily see that ω is of type $(g, \widetilde{\delta}, k_\alpha^\gamma)$, and that the construction gives all Arf functions of this type. The function ω is even for $k_1 = 0$ and, for $k_1 > 0$, its parity coincides with that of the number of contours in A_1^0 (recall that $k_1^0 + k_1^1$ is even).

§ 4. Lifting of real Fuchsian groups

1. By

$$J\colon SL(2, \mathbb{R}) \to PSL(2, \mathbb{R}) = \mathrm{Aut}(\Lambda)$$

we denote the natural projection. Let

$$\Gamma \subset \mathrm{Aut}(\Lambda)$$

be a Fuchsian group that consists of hyperbolic automorphisms. A subgroup $\Gamma^* \subset SL(2, \mathbb{R})$ is called a lifting of Γ if $J(\Gamma^*) = \Gamma$ and $J\big|_{\Gamma^*}\colon \Gamma^* \to \Gamma$ is an isomorphism.

By [47], § 7, to the lifting Γ^* there corresponds an Arf function

$$\omega_{\Gamma^*}\colon H_1(\Lambda/\Gamma, \mathbb{Z}_2) \to \mathbb{Z}_2,$$

which can be defined as follows. Let $a' \in \Gamma$ and let $a \in H_1(\Lambda/\Gamma, \mathbb{Z}_2)$ be the image of a' under the natural projection $P_r\colon \Gamma \to \pi_1(\Lambda/\Gamma) \to H_1(\Lambda/\Gamma, \mathbb{Z}_2)$. Let

$$A = J^{-1}(a') \cap \Gamma^*$$

and let $\mathrm{Tr}(A)$ be the trace of the matrix $A \in SL(2, \mathbb{R})$. We set

$$\omega_{\Gamma^*}(a) = \begin{cases} 0 & \text{for } \mathrm{Tr}(A) < 0, \\ 1 & \text{for } \mathrm{Tr}(A) > 0. \end{cases}$$

By [47], Theorem 7.2, the correspondence $\Gamma^* \mapsto \omega_{\Gamma^*}$ between the liftings of the group Γ and the Arf functions on $P = \Lambda/\Gamma$ is one-to-one.

2. We consider now the group

$$SL_\pm(2, \mathbb{R}) = \{A \in GL(2, \mathbb{R}) \mid \det A = \pm 1\}.$$

We extend the projection J to a homomorphism $J\colon SL_{\pm}(2,\mathbb{R}) \to \widetilde{\mathrm{Aut}}(\Lambda)$ by setting

$$J(A) = \frac{a\overline{z}+b}{c\overline{z}+d} \quad \text{for} \quad A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad \text{and} \quad \det A = -1.$$

Let $\widetilde{\Gamma}$ be a real Fuchsian group. A subgroup $\widetilde{\Gamma}^* \subset SL_{\pm}(2,\mathbb{R})$ is called a lifting of $\widetilde{\Gamma}$ if $J(\widetilde{\Gamma}^*) = \widetilde{\Gamma}$ and $J\big|_{\widetilde{\Gamma}^*}\colon \widetilde{\Gamma}^* \to \widetilde{\Gamma}$ is an isomorphism. It is clear that a lifting $\widetilde{\Gamma}^*$ of the group $\widetilde{\Gamma}$ induces a lifting $\Gamma^* = \widetilde{\Gamma}^* \cap SL(2,\mathbb{R})$ of the group $\Gamma = \widetilde{\Gamma} \cap \mathrm{Aut}(\Lambda)$, and hence an Arf function $\omega_{\widetilde{\Gamma}^*} = \omega_{\Gamma^*}\colon H_1(\Lambda/\Gamma, \mathbb{Z}_2) \to \mathbb{Z}_2$.

Lemma 4.1. *The Arf function $\omega_{\widetilde{\Gamma}^*}$ is a non-singular Arf function on the real curve $[\widetilde{\Gamma}]$.*

Proof. The Arf function $\omega_{\widetilde{\Gamma}^*}$ is real because, for any $\alpha \in \widetilde{\Gamma}^* \setminus \Gamma^*$, $a' \in \Gamma$, and $a = P_r(a')$, we have

$$\omega_{\widetilde{\Gamma}^*}(\tau a) = \mathrm{Tr}\big(\alpha(J^{-1}(a') \cap \Gamma^*)\alpha^{-1}\big) = \mathrm{Tr}\big(J^{-1}(a') \cap \Gamma^*\big) = \omega_{\widetilde{\Gamma}^*}(a).$$

Let us prove that $\omega_{\widetilde{\Gamma}^*}$ is non-singular. Let $c \subset P \setminus P^{\tau}$ be a simple contour such that $\tau c = c$ and let $C \subset \Gamma$ be its image under the natural isomorphism $\pi_1(\Lambda/\Gamma, p) \to \Gamma$. Let

$$\widetilde{C}^* = J^{-1}(\widetilde{C}) \cap \widetilde{\Gamma}^* = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

where $\widetilde{C} = \overline{C}\sqrt{C}$ (see § 2.4). Then

$$J^{-1}(C) \cap \Gamma^* = (\widetilde{C}^*)^2 = \begin{pmatrix} a & b \\ c & d \end{pmatrix}^2.$$

Hence,

$$\mathrm{Tr}\big(J^{-1}(C) \cap \Gamma^*\big) > 0, \quad \text{and} \quad \omega(c) = 1.$$

Liftings $\widetilde{\Gamma}_1^*$ and $\widetilde{\Gamma}_2^*$ of a real Fuchsian group $\widetilde{\Gamma}$ are said to be *similar* if $(\widetilde{\Gamma}_1^* \setminus \Gamma^*) = -(\widetilde{\Gamma}_2^* \setminus \Gamma^*)$.

Lemma 4.2. *Let ω be a non-singular Arf function on $[\widetilde{\Gamma}]$. Then there are exactly two liftings $\widetilde{\Gamma}^*$ of the group $\widetilde{\Gamma}$ for which $\omega_{\widetilde{\Gamma}^*} = \omega$, and these liftings are similar.*

Proof. By [47], § 7, there is a unique lifting $\Gamma^* \subset SL(2,\mathbb{R})$ of the group $\Gamma = \widetilde{\Gamma} \cap \mathrm{Aut}(\Lambda)$ with $\omega_{\Gamma^*} = \omega$. Therefore, any lifting $\widetilde{\Gamma}^*$ of the group $\widetilde{\Gamma}$ with $\omega_{\widetilde{\Gamma}^*} = \omega$ is generated by Γ^* and a matrix α such that $J(\alpha) \in \widetilde{\Gamma} \setminus \Gamma$. If $(J(\alpha))(z) = \dfrac{a\overline{z}+b}{c\overline{z}+d}$, then $\alpha = \pm \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Since the Arf function ω is real, we have $\mathrm{Tr}(\alpha A \alpha^{-1}) = \mathrm{Tr}(A)$ for $A \in \Gamma^*$, and hence $\alpha \Gamma^* \alpha^{-1} = \Gamma^*$. Thus, the group $\widetilde{\Gamma}^*$ generated by Γ^* and α is a lifting of the group $\widetilde{\Gamma}$.

Lemmas 4.1 and 4.2 imply the following assertion.

Theorem 4.1 [44]**.** *The correspondence* $\widetilde{\Gamma}^* \mapsto \omega_{\widetilde{\Gamma}^*}$ *between similarity classes of liftings of a real Fuchsian group* $\widetilde{\Gamma}$ *and non-singular Arf functions on a real curve* $[\widetilde{\Gamma}]$ *is one-to-one.*

3. The natural isomorphism $\pi_1(\Lambda/\Gamma, p) \to \Gamma$ sends each free homotopy class of a contour $c \in P = \Lambda/\Gamma$ to a conjugacy class $\Gamma_c \subset \Gamma$ that does not depend on the choice of p. Thus, to each simple geodesic contour $c \in P$ there corresponds a set $\Gamma_c \subset \Gamma$, and $\Phi(\ell(C)) = c$ if $C \in \Gamma_c$ and $\Phi\colon \Lambda \to P$ is the natural projection.

We assume now that $\widetilde{\Gamma}$ is a real Fuchsian group and c is an oval of a curve $(P, \tau) = [\widetilde{\Gamma}]$. Let us consider $C \in \Gamma_c$. Replacing the group $\widetilde{\Gamma}$ by a conjugate group, we may assume that $\ell(C) = I = \{z \in \Lambda \mid \operatorname{Re} z = 0\}$. Then $\widetilde{\Gamma}$ contains the involution $\beta(z) = -\overline{z}$. A lifting $\widetilde{\Gamma} \to \widetilde{\Gamma}^*$ maps β into a matrix of the form $\sigma \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$, where $\sigma = \pm 1$. Let us endow the half-line I with the orientation in which $\operatorname{Im} z$ increases for $\sigma = 1$ and with the opposite orientation for $\sigma = -1$. The projection Φ transfers the orientation to the contour $c = \Phi(I)$. The latter's orientation is completely determined by the lifting $\widetilde{\Gamma}^*$ and is called the *orientation generated on the oval by the lifting* $\widetilde{\Gamma}^*$.

Lemma 4.3 [36]**.** *Let* $\widetilde{\Gamma}^*$ *be a lifting of a real Fuchsian group* $\widetilde{\Gamma}$, *let* $(P, \tau) = [\widetilde{\Gamma}]$, *let* c_1 *and* c_2 *be ovals of the involution* τ *endowed with the orientation generated by* $\widetilde{\Gamma}^*$, *and let* $a \subset P$ *be a simple oriented contour intersecting* c_1 *and* c_2 *and such that* $\tau a = -a$. *Then* a *has the same intersection numbers with* c_1 *and* c_2 *if and only if* $\omega_{\widetilde{\Gamma}^*}(a) = 1$.

Proof. Replacing the group $\widetilde{\Gamma}$ by a conjugate group, we may assume that $\Gamma_a \supset A$, where $A(z) = \lambda z$ and $\lambda > 1$.

In this case we have $\Gamma_{c_i} \supset C_i$ (because $\tau a = -a$, $c_i \cap a \neq \varnothing$, and $c_i \subset P^\tau$), where

$$C_i = \frac{\alpha_i(\lambda_i + 1)z + \alpha_i^2(\lambda_i - 1)}{(\lambda_i - 1)z + \alpha_i(\lambda_i + 1)}, \quad \lambda_i > 1, \quad \overline{C}_i = \frac{\alpha_i^2}{\overline{z}},$$

and $A = \overline{C}_1 \overline{C}_2$ (see Fig. 4.1). We set $A^* = J^{-1}(A) \cap \widetilde{\Gamma}^*$, $C_i^* = J^{-1}(C_i) \cap \widetilde{\Gamma}^*$, and $\overline{C}_i^* = J^{-1}(\overline{C}_i) \cap \widetilde{\Gamma}^*$. Then, by the definition of the orientation generated by $\widetilde{\Gamma}^*$, we obtain $\overline{C}_i^* = -\begin{pmatrix} 0 & \alpha_i \\ \alpha_i^{-1} & 0 \end{pmatrix}$, and hence

$$A^* = \overline{C}_1^* \overline{C}_2^* = -\begin{pmatrix} \alpha_1 \alpha_2^{-1} & 0 \\ 0 & \alpha_1^{-1} \alpha_2 \end{pmatrix}.$$

On the other hand, the intersection numbers of the contour a with the ovals c_1 and c_2 coincide if and only if the attracting fixed points α_1 and α_2 have the same sign. This is equivalent to the condition $\operatorname{Tr}(A^*) > 0$, or, which is the same, $\omega_{\widetilde{\Gamma}^*}(a) = 1$.

4. We assume now that $c \subset P$ is an invariant contour of a curve $(P, \tau) = [\widetilde{\Gamma}]$ such that $c \cap P^\tau = \varnothing$. Let us consider $C \in \Gamma_c$. As above, replacing $\widetilde{\Gamma}$ by a conjugate group,

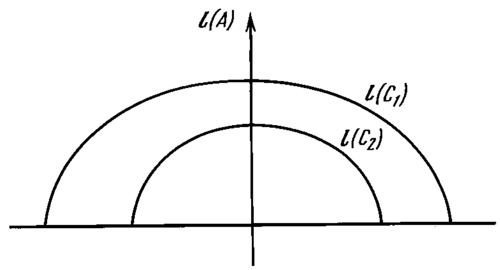

Figure 4.1

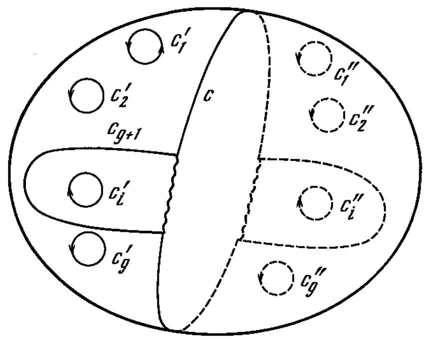

Figure 4.2

we may assume that $l(C) = I$. Hence, the group $\widetilde{\Gamma}$ contains a map of the form $\beta(z) = -\lambda\bar{z}$, where $\lambda > 0$. A lifting $\widetilde{\Gamma} \to \widetilde{\Gamma}^*$ sends β into a matrix of the form

$$\sigma\begin{pmatrix} -\lambda^{\frac{1}{2}} & 0 \\ 0 & \lambda^{-\frac{1}{2}} \end{pmatrix},$$

where $\sigma = \pm 1$. As above, we endow I with the orientation in which $\operatorname{Im} z$ increases for $\sigma = 1$, and with the opposite orientation for $\sigma = -1$. The projection Φ transfers the orientation to the contour $c = \Phi(I)$. The latter's orientation depends only on the lifting $\widetilde{\Gamma}^*$ and is called the *orientation generated on the invariant contour by the lifting* $\widetilde{\Gamma}^*$.

Theorem 4.2. *Let $\widetilde{\Gamma}^*$ be a lifting of a real Fuchsian group $\widetilde{\Gamma}$ and let $(P, \tau) = [\widetilde{\Gamma}]$ be a real algebraic curve of type $(g, k, 0)$. Let $(c_1, \dots, c_g)$ be a set of pairwise disjoint simple contours such that $P^\tau = \bigcup_{i=1}^k c_i$ and $\tau(c_i) = c_i$. Then there is an invariant contour c_{g+1} that is disjoint from the above contours and that, together with the contours $(c_1, \dots, c_g)$, decomposes the surface P into spheres P_1 and P_2 with holes so that the orientation of $c_1, \dots, c_g$ generated by $\widetilde{\Gamma}^*$ coincides with their orientation as parts of the boundary of one of the surfaces P_i.*

Proof. By Lemma 1.2, there is a set of pairwise disjoint invariant contours $c_1, \dots,$ c_g, c belonging to P and such that $P^\tau = \bigcup_{i=1}^k c_i$ and the set $P \setminus \left(\bigcup_{i=1}^g c_i \cup c\right)$ decomposes into two spheres with holes. Let us endow the contours $c_1, \dots, c_g$

with the orientation generated by the lifting $\widetilde{\Gamma}^*$. Their images on the surface $\widetilde{P} = P \setminus \bigcup_{i=1}^{g} c_i$ are represented by pairs of contours c_i' and c_i'' of opposite orientation, where c_i' and c_j' belong to the same connected components of the surface $\widetilde{P} \setminus c$ (see Fig. 4.2). Symmetrically modifying the contour c as shown in Fig. 4.2, we can pass from c to a symmetric contour c_{g+1} that separates the contours of different orientation.

§ 5. Rank-one spinors on real algebraic curves

1. We recall that a linear bundle $e\colon E \to P$ is said to be a *spinor bundle of rank one* if the tensor square of this bundle coincides with the cotangent bundle. In what follows, unless otherwise stated, a spinor bundle is understood to be a rank-one spinor bundle over a Riemann surface P.

In [47], § 10, a one-to-one correspondence is established between the liftings Γ^* of a Fuchsian group Γ and the spinor bundles on $P = \Lambda/\Gamma$. A spinor bundle e_{Γ^*} corresponding to Γ^* is of the form

$$(\Lambda \times \mathbb{C})/\Gamma^* \to \Lambda/\Gamma,$$

where Γ^* acts on $(\Lambda \times \mathbb{C})$ by the rule

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} (z, x) = \left(\frac{az+b}{cz+d}, \, (cz+d)x \right).$$

Thus, the correspondence $e_{\Gamma^*} \to \Gamma^* \to \omega_{\Gamma^*}$ established between spinor bundles and Arf functions on $P = \Lambda/\Gamma$ by the map $e \to \omega_e$ is one-to-one.

2. By a *spinor bundle on a real curve* (P, τ) we mean a pair (e, β), where $e\colon E \to P$ is a spinor bundle and $\beta\colon E \to E$ is an antiholomorphic involution such that $e\beta = \tau e$. Two spinor bundles (e_1, β_1) and (e_2, β_2) on curves (P_1, τ_1) and (P_2, τ_2), respectively, are assumed to be isomorphic if there are biholomorphic maps $\varphi_E\colon E_1 \to E_2$ and $\varphi_P\colon P_1 \to P_2$ such that

$$e_2 \varphi_E = \varphi_P e_1, \quad \beta_2 \varphi_E = \varphi_E \beta_1, \quad \tau_2 \varphi_P = \varphi_P \tau_1.$$

As usual, we do not distinguish between isomorphic bundles.

With any lifting $\widetilde{\Gamma}^*$ of a real Fuchsian group $\widetilde{\Gamma}$ we associate a spinor bundle $e_{\widetilde{\Gamma}^*}$ on the real curve $(P, \tau) = [\widetilde{\Gamma}]$. By definition, the bundle $e_{\widetilde{\Gamma}^*}$ is of the form $(e_{\Gamma^*}, \beta_{\widetilde{\Gamma}^*})$, where $\beta_{\widetilde{\Gamma}^*}\colon (\Lambda \times \mathbb{C})/\Gamma^* \to (\Lambda \times \mathbb{C})/\Gamma^*$ is generated by the map

$$(z, x) \mapsto \left(\frac{a\overline{z}+b}{c\overline{z}+d}, \, (c\overline{z}+d)\overline{x} \right), \qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \widetilde{\Gamma}^* \setminus \widetilde{\Gamma}.$$

Lemma 5.1. *The correspondence $\widetilde{\Gamma}^* \mapsto e_{\widetilde{\Gamma}^*}$ between similarity classes of liftings $\widetilde{\Gamma}^*$ of a real Fuchsian group $\widetilde{\Gamma}$ and spinor bundles on $(P, \tau) = [\widetilde{\Gamma}]$ is one-to-one.*

Proof. Let (e, β) be an arbitrary spinor bundle on (P, τ). By [47], § 10, there is a unique lifting Γ^* of the group $\Gamma = \widetilde{\Gamma} \cap \operatorname{Aut}(\Lambda)$ such that

$$e\colon (\Lambda \times \mathbb{C})/\Gamma^* \to \Lambda/\Gamma.$$

By replacing the group $\widetilde{\Gamma}$ by a conjugate group, we may assume that $\widetilde{\Gamma}$ contains a map of the form

$$z \mapsto -\mu\overline{z},$$

where $\mu \geqslant 1$. Let μ_* be the minimal value of all these μ's. We set $\nu = \sqrt{\mu_*}$. Then the group Γ^* and the matrices $\pm\begin{pmatrix} -\nu^{-1} & 0 \\ 0 & \nu \end{pmatrix}$ generate some liftings $\widetilde{\Gamma}^*_+$ and $\widetilde{\Gamma}^*_-$ of the group $\widetilde{\Gamma}$. These are the only liftings of $\widetilde{\Gamma}$ that contain Γ^*. Moreover, $e_{\widetilde{\Gamma}^*_\pm} = e$, and an isomorphism between $e_{\widetilde{\Gamma}^*_+}$ and $e_{\widetilde{\Gamma}^*_-}$ is generated by the involution $(z,x) \mapsto (z,-x)$.

By Lemma 2.1 and Theorem 4.1, this immediately yields the following assertion.

Theorem 5.1 ([36], [44]). *The correspondence $e \mapsto \omega_e$ between spinor bundles and non-singular Arf functions on a real curve (P,τ) is one-to-one.*

Let (e,β) be a spinor bundle on a real curve (P,τ). Applying Lemmas 2.1 and 5.1, we construct an isomorphism

$$(e,\beta) \to (e_{\widetilde{\Gamma}^*}, \beta_{\widetilde{\Gamma}^*}),$$

where $\widetilde{\Gamma}^*$ is a lifting of a real Fuchsian group $\widetilde{\Gamma}$ and $(P,\tau) = [\widetilde{\Gamma}]$. Let us endow the ovals and the invariant contours of $[\widetilde{\Gamma}]$ disjoint from them with the orientation induced by $\widetilde{\Gamma}^*$ (see § 4). Thus, a spinor bundle (e,β) on a real curve (P,τ) generates an orientation on the ovals and the invariant contours of (P,τ) disjoint from them. This orientation is defined up to its simultaneous reversal on all ovals and invariant contours.

3. A holomorphic section $\eta\colon P \to E$ of a spinor bundle $e\colon E \to P$ is called a *spinor*. A section η of an arbitrary spinor bundle (e,β) on a real curve (P,τ) is called a *real spinor* if $\beta\eta = \eta\tau$. Let $\{\widetilde{\Gamma}^*_1, \widetilde{\Gamma}^*_2\}$ be a similarity class that corresponds to the bundle (e,β) by Lemma 5.1. Then the spinor η can be regarded as a section of the spinor bundle e_{Γ^*}, where $\Gamma^* = \widetilde{\Gamma}^*_1 \cap \widetilde{\Gamma}^*_2$. Moreover, η is invariant with respect to one of the involutions $\beta_{\widetilde{\Gamma}^*_i}$ and is anti-invariant with respect to the other. To be definite, let $\beta_{\widetilde{\Gamma}^*_1}\eta = \eta\tau$. The orientation generated by the lifting $\widetilde{\Gamma}^*_1$ on the ovals and invariant contours of (P,τ) is called the *orientation generated by* η.

A local chart $u\colon U \to \mathbb{C}$ in a neighbourhood of a real point $p_0 \in P^\tau$ is said to be *real* if $\tau U = U$ and $u(\tau p) = \overline{u(p)}$. In this case $u(U \cap P^\tau) \subset \mathbb{R}$. We say that the local chart u *agrees with the spinor* η if the spinor generates an orientation of the oval ap_0 that passes under the action of u into the orientation of increasing real values on $\mathbb{R} \subset \mathbb{C}$.

A local chart on a Riemann surface defines a local trivialization of the cotangent bundle, and hence a local trivialization of the spinor bundle. Thus, in the local chart u, a complex function $f(u)$ corresponds to the spinor.

Lemma 5.2. *Let (e,β) be a spinor bundle on a real curve (P,τ) and let η be a real spinor of this bundle. Then in any real chart $u\colon U \to \mathbb{C}$ that agrees with η, the spinor η is described by a function $f(u)$ such that $f(u\tau) = \overline{f(u)}$.*

Proof. We set $i_e\colon (z,x)\mapsto(iz,x)$. By Lemmas 2.1 and 5.1, we may assume that

$$(P,\tau)=[\widetilde{\Gamma}],\quad e\colon(\Lambda\times C)/\Gamma^*\to[\Gamma],\quad \begin{pmatrix}-1&0\\0&1\end{pmatrix}\in\widetilde{\Gamma}^*\setminus\Gamma^*,$$

and that $ei_e\colon(-i\Lambda\times 0)\to P$ generates a real chart u that agrees with η. In this chart, the relation $\beta\eta=\eta\tau$ becomes

$$(u\tau,\overline{f(u)}\,)=\beta_{\widetilde{\Gamma}^*}(u,f(u))=(u\tau,f(u\tau)),$$

and hence $\overline{f(u)}=f(u\tau)$. A passage to any other real chart that agrees with η preserves this relation.

Theorem 5.2 ([42], [44])**.** *Let (e,β) be a spinor bundle on a real curve (P,τ), let η be a real spinor of this bundle, and let a be an oval of the curve (P,τ). Then the parity of the number of zeros of η on a is opposite to the parity of $\omega_e(a)$.*

Proof. By Lemmas 2.1 and 5.1, we may assume that

$$(P,\tau)=[\widetilde{\Gamma}],\quad e\colon(\Lambda\times\mathbb{C})/\Gamma^*\to[\Gamma],\quad \begin{pmatrix}-1&0\\0&1\end{pmatrix}\in\widetilde{\Gamma}^*\setminus\Gamma^*,$$
$$a=I/\Gamma,\quad\text{where}\quad I=\{z\in\Lambda\mid \operatorname{Re}z=0\}.$$

In the local chart u generated by the projection $e\colon(\Lambda\times 0)\to P$, the spinor η is represented in the form $(u,f(u))$, where $u\in\Lambda$ and $f(u)$ is a holomorphic function such that

$$f\left(\frac{\alpha u+\beta}{\gamma u+\delta}\right)=f(u)(\gamma u+\delta)$$

for any element $\begin{pmatrix}\alpha&\beta\\\gamma&\delta\end{pmatrix}\in\Gamma^*$.

Corresponding to the contour a is the matrix

$$A=\sigma(a)\begin{pmatrix}\lambda&0\\0&\lambda^{-1}\end{pmatrix}\in\Gamma^*,$$

where

$$\sigma(a)=\begin{cases}\ \ 1 & \text{for } \omega(a)=1,\\ -1 & \text{for } \omega(a)=0.\end{cases}$$

Thus, $f(\lambda^2u)=\sigma(a)f(u)$. Moreover, the natural projection $\Lambda\to\Lambda/\Gamma$ establishes a one-to-one correspondence between the interval $(v,\lambda^2v]\in I$ and the contour a. Hence, the number of zeros of the spinor η on a is equal to that of the function $f(u)$ on the interval $(v,\lambda^2v]\in I$. On the other hand, the map $e\colon(\Lambda\times 0)\to P$ generates a real chart in a neighbourhood of each point of the oval a, and hence, by Lemma 5.2, $f(u)$ is real on $(v,\lambda^2v]\in I$. Thus, the number of zeros of f in $(v,\lambda^2v]\in I$ is even for $\sigma(a)=1$ and odd for $\sigma(a)=-1$.

4. Theorem 5.3 ([42], [44])**.** *Let $c_1,\dots,c_k$ be oriented ovals of a real algebraic curve (P,τ) of type $(g,k,0)$. Let $0\leqslant m\leqslant k$, $\alpha_1,\dots,\alpha_k\in\mathbb{Z}_2$, and let $\sum_{i=1}^k\alpha_i\equiv g+1 (\mathrm{mod}\ 2)$. Then there is a real spinor η on (P,τ) such that* 1) *the orientation of the oval c_i generated by η coincides with the original orientation if and only if $i\leqslant m$,* 2) *the parity of the number of zeros of the spinor η on the oval c_i is equal to α_i.*

Proof. By Theorem 1.2, there is a set of pairwise disjoint and τ-invariant contours $c_1,\dots,c_{g+1}$ that decompose P into spheres P_1 and P_2 with holes. The orientation of P_1 generates a new orientation on $\partial P_1=\{c_1,\dots,c_{g+1}\}$. Without loss of generality, we may assume that the new orientation coincides on c_1 with the original one. Let us join the contour c_{g+1} with c_i by a segment $\ell_i\subset P_1$ and consider the simple contour $d_i=\ell_i\cup\tau\ell_i\cup r_i\cup r_{g+1}$, where $r_j\subset c_j$. We set $\omega(c_i)=1-\alpha_i$ for $i\leqslant k$ and $\omega(c_i)=1$ for $k<i\leqslant g$. For $1\leqslant i\leqslant m$ we set $\omega(d_i)=0$ if and only if the orientation generated by P_1 coincides with the original orientation of c_i. For $m<i\leqslant k$ we set $\omega(d_i)=0$ if and only if the orientation generated by P_1 is opposite to the original orientation of c_i. For $k<i\leqslant g$ we set $\omega(d_i)=0$. The function ω can be uniquely extended to an Arf function $\omega\colon H_1(P,\mathbb{Z}_2)\to\mathbb{Z}_2$, and $\omega(c_{g+1})=1$ because $\sum_{i=1}^k\alpha_i\equiv g+1\ (\mathrm{mod}\,2)$. Moreover, $\tau d_i=-d_i+c_{g+1}+\widetilde{c}_i$, where

$$\widetilde{c}_i=\begin{cases}0 & \text{for } i\leqslant k,\\ c_i & \text{for } i>k.\end{cases}$$

Thus, $\omega(\tau d_i)=\omega(-d_i+c_{g+1}+\widetilde{c}_i)=\omega(d_i)$, and hence ω is a real Arf function. By Lemma 3.3, it is non-singular. By Lemma 2.1, $(P,\tau)=[\widetilde{\Gamma}]$, where $\widetilde{\Gamma}$ is a real Fuchsian group. In view of Lemma 4.2 we have $\omega=\omega_{\widetilde{\Gamma}^*}$, where $\widetilde{\Gamma}^*$ is a lifting of the group $\widetilde{\Gamma}$. By definition, $\omega_e=\omega$, where $(e,\beta)=(e_{\Gamma^*},\beta_{\widetilde{\Gamma}^*})$.

Along with ω, we consider a real Arf function ω' such that $\omega'(c_i)=\omega(c_i)$ and $\omega'(d_i)=1-\omega(d_i)$. Corresponding to this function is a real spinor bundle (e',β') such that $\omega_{e'}=\omega'$. Moreover,

$$\delta(\omega)+\delta(\omega')=\sum_{i=1}^{g}\omega(c_i)=1$$

because $\sum_{i=1}^k\alpha_i\equiv g+1\ (\mathrm{mod}\,2)$. Hence, either $\delta(\omega)=1$ or $\delta(\omega')=1$. To be definite, let $\delta(\omega_e)=\delta(\omega)=1$. By [2] and [30], this implies that the bundle e has a holomorphic section ξ. Then one of the sections $\eta=\xi+\beta\xi$ and $\eta=i(\xi-\beta\xi)$ is a non-zero real section of the bundle (e,β). By Lemmas 4.3 and Theorem 5.2, this section has the properties indicated in Theorem 5.3.

Theorem 5.4 ([42], [44])**.** *Let (P,τ) be a real algebraic curve of type $(g,k,1)$. Let its ovals $c_1,\dots,c_k$ be orietned as parts of the boundary of a connected component P_1 of the set $P\setminus P^\tau$. Consider a set $\alpha_1,\dots,\alpha_k\in\mathbb{Z}_2$ that has evenly many zeros and for which $\alpha_1=\alpha_k=0$. Let $1\leqslant m<k$ and let $\sum_{i=1}^m\alpha_i\equiv m+1\ (\mathrm{mod}\,2)$. Then there is a real spinor η on (P,τ) such that* 1) *the orientation generated on the oval c_i by η coincides with the original one if and only if $i\leqslant m$,* 2) *the parity of the number of zeros of η on c_i is equal to α_i.*

Proof. Let us join the ovals c_i and c_k by a segment $\ell_i\subset P_1$ and set $d_i=\ell_i\cup\tau\ell_i$ $(i=1,\dots,k-1)$. Let us consider an arbitrary Arf function ω_1 on P_1 such that

$\omega_1(c_i) = 1 - \alpha_i$. (Such a function exists by [47], Lemma 8.1.) Let us extend it to an Arf function ω on P by assuming that $\omega(\tau w) = \omega(w)$ for $w \in H_1(P_1, \mathbb{Z}_2)$ and that $\omega(d_i) = 1$ if and only if $i \leqslant m$. Then $\delta(\omega) = 1$. The rest of the proof coincides with the corresponding part of the proof of Theorem 5.3.

§ 6. Holomorphic differentials on real algebraic curves

1. In this section we assume that the ovals of a real algebraic curve $X = (P, \tau)$ are endowed with an orientation. This orientation is induced by an orientation of one of the connected components of the set $P \setminus P^\tau$ if $\varepsilon(X) = 1$. We say that a real chart $u\colon U \to \mathbb{C}$ *agrees* with the orientation of the set P^τ if u sends an oriented segment $\ell = U \cap P^\tau$ into the segment $u(\ell) \subset \mathbb{R}$ oriented by increasing order of the reals.

We recall that a *holomorphic differential* on a Riemann surface P is defined to be a section $\xi\colon P \to T^*$ of the cotangent bundle $t\colon T^* \to P$. We assume now that (P, τ) is a real algebraic curve. The involution τ induces the antiholomorphic involution $\tau^*\colon T^* \to T^*$ such that $t\tau^* = \tau t$. A differential ξ is said to be *real* if $\tau^*\xi = \xi\tau$. In a real chart, it becomes $\xi = f(u)du$, where $f(\overline{u}) = \overline{f(u)}$. In particular, $f(u(p)) \in \mathbb{R}$ for $p \in P^\tau$. The sign of the number $f(u(p)) \in \mathbb{R}$ is the same for all real charts that agree with the orientation of the set P^τ, and it is called the *sign of the differential* ξ *at the point* $p \in P^\tau$.

We say that a real differential ξ is *positive* (*non-negative*, *non-positive*, *negative*, respectively) on an oval $a \subset P^\tau$ if it is positive (non-negative, non-positive, and negative, respectively) at any point of the oval.

Lemma 6.1. *Let η be a real spinor on the curve (P, τ). Then $\xi = \eta^2$ is a real differential that is non-negative on the oval $a \subset P^\tau$ if the orientation generated by η coincides with the original orientation, and non-positive on a if the orientation generated by η is opposite to the original one.*

Proof. If the spinor η is described by a function $f(u)$ in a real chart $u\colon U \to \mathbb{C}$ that agrees with the orientation of P^τ, then $\xi = f^2(u)du$. If, in addition, the orientation of the oval a is generated by η, then it follows from Lemma 5.2 that $f(u\tau) = \overline{f(u)}$ and f^2 is non-negative on a. A change of orientation of the oval changes the sign of f^2.

Theorem 6.1 ([34], [44])**.** *Let (P, τ) be a real algebraic curve of type (g, k, ε) with ovals $c_1, \dots, c_k$, where $k = k_+ + k_- + k_0$, $k_0 < g$, and let $k_+ \cdot k_- \neq 0$ for $\varepsilon = 1$. Then there is a real differential on (P, τ) that is non-negative on c_i for $i \leqslant k_+$, non-positive on c_i for $k_+ < i \leqslant k_+ + k_-$, and has zeros on c_i for $i > k_+ + k_-$.*

Proof. By Theorems 5.3 and 5.4, there is a real spinor η that has zeros on $c_{k_+ + k_- + 1}$, $\dots$, c_k and generates on any other oval c_i an orientation that coincides with that of P^τ for $i \leqslant k_+$ and is opposite to the orientation of P^τ for $k_+ < i \leqslant k_+ + k_-$. Then by Lemma 6.1, the differential $\xi = \eta^2$ has the desired properties.

2. Let us consider in more detail the real *M-curves*, that is, curves of type $(g, g+1, 1)$.

Lemma 6.2. *Let $c_1,\dots,c_{g+1}$ be ovals of an M-curve of genus g and let $1 \leqslant \alpha \leqslant n < \beta \leqslant g+1$. Then there is a real differential ξ_1 that is positive on c_α, non-negative on $c_1,\dots,c_n$, negative on c_β, and non-positive on $c_{n+1},\dots,c_{g+1}$.*

Proof. By Theorem 5.4, there is a real spinor η that generates on $c_1,\dots,c_n$ the original orientation, generates on $c_{n+1},\dots,c_{g+1}$ the orientation opposite to the original one, and has zeros on the ovals c_i with $i \neq \alpha,\beta$. However, the total number of zeros of the spinor is $g-1$ [30]. Hence, η has no zeros on c_α and c_β. Thus, by Lemma 6.1, the real differential $\xi = \eta^2$ satisfies all hypotheses of the lemma.

This immediately yields the following assertion.

Lemma 6.3. *Let $c_1,\dots,c_{g+1}$ be the ovals of an M-curve of genus g and let $1 \leqslant n < g+1$. Then there is a real differential ξ that is positive on $c_1,\dots,c_n$ and negative on $c_{n+1},\dots,c_{g+1}$.*

Lemma 6.4. *Let $\alpha_1 < \dots < \alpha_{2g+2}$ be real numbers, let $h(x) = \prod_{i=1}^{2g+2}(x-\alpha_i)$, let P be the Riemann surface of the algebraic curve $y^2 = h(x)$, and let $\tau\colon P \to P$ be the antiholomorphic involution generated by the correspondence $(x,y) \mapsto (\overline{x},\overline{y})$. Then (P,τ) is a real M-curve of genus g each of whose real differentials is positive on one of the ovals.*

Proof. The ovals of the curve (P,τ) correspond to the segments $[\alpha_{2i-1},\alpha_{2i}]$. Any real differential on (P,τ) is of the form

$$\xi_f = \frac{f(x)\,dx}{\sqrt{h(x)}}\,,$$

where f is a polynomial with real coefficients and of degree at most $g-1$. If $f(x) > 0$, then the differential has opposite signs on the ovals corresponding to neighbouring segments. Therefore, if on any oval the differential ξ_f is not positive, then f has more than $g-1$ zeros. This is impossible because $\deg f \leqslant g-1$.

Theorem 6.2 [34]**.** *For any real differential on an M-curve, there is an oval on which this differential is positive and an oval on which it is negative.*

Proof. Let $\widetilde{M}$ be the set of all M-curves of genus g with an ordered set of ovals $c_1,\dots,c_{g+1}$. Let us consider a bundle $\widetilde{e}\colon \widetilde{E} \to \widetilde{M}$ with fibre $\widetilde{e}^{-1}(P,\tau)$ that consists of all real differentials on (P,τ). We take a basis of $\widetilde{e}^{-1}(P,\tau)$ that is formed by differentials $\xi_i = \xi_i(P,\tau)$ such that $\oint_{c_i}\xi_j = \delta_{ij}$ $(i,j \leqslant g)$. The correspondence $\xi_i(P,\tau) \mapsto \xi_i(P',\tau')$ defines a connection F on $\widetilde{e}$.

A real differential is called a *differential of type A* (*of type B*) if each of the ovals contains points at which the differential is non-positive (negative, respectively). Let M^A (M^B) be the set of M-curves that admit a differential of type A (of type B, respectively). Then M^A is a closed set. Using the connection F, we can readily prove that M^B is an open set. Moreover, $M^A \supset M^B$. Let us prove that $M^A \subset M^B$. Let $(P,\tau) \subset M^A$ and let ξ be a differential of type A on (P,τ). Since

$$\sum_{i=1}^{g+1}\int_{c_i}\xi = 0,$$

it follows that the differential is negative at least at one point. Let c be an oval containing such a point. By Lemma 6.3, there is a real differential γ that is positive on c and negative on the other ovals. Then the differential $\xi + \alpha\gamma$ is of type B for a sufficiently small α. Thus, $M^A = M^B$ is an open and closed set in $\widetilde{M}$.

However, by Theorem 2.1, $\widetilde{M}$ is a connected set, and hence if $M^A \neq \varnothing$, then $M^A = \widetilde{M}$. The latter relation contradicts Lemma 6.4, according to which the set $\widetilde{M} \setminus M^A$ contains hyperelliptic curves. Thus, $M^A = \varnothing$, that is, any real differential on an M-curve is positive on one of the ovals. We can prove similarly that it is also negative on one of the ovals.

Theorem 6.3 [34]**.** *Let $1 \leqslant k \leqslant g+1$, $k \equiv g+1 \pmod 2$, and $m \geqslant k - \left[\frac{k}{2}\right]$. Then there is a real algebraic curve of type $(g,k,1)$ with ovals $c_1, \dots, c_k$ and such that on this curve any real differential without zeros on $c_1, \dots, c_m$ must be positive on one of the ovals and negative on another.*

Proof. Let us consider the Riemann surface P of the curve

$$\begin{aligned} y^4 - 2y^2[(x-\beta_1)\cdots(x-\beta_m) - (x-\alpha_1)\cdots(x-\alpha_n)] \\ + [(x-\beta_1)\cdots(x-\beta_m) + (x-\alpha_1)\cdots(x-\alpha_n)]^2 = 0, \end{aligned}$$

where $\alpha_1 < \dots < \alpha_n \leqslant \beta_1 < \dots < \beta_m \in \mathbb{R}$, $n > 0$, and $n, m \equiv 0 \pmod 2$. This surface is obtained by resolution of singularities from the set

$$\Bigl((x,y) \in \overline{\mathbb{C}}^2 \mid y = \pm\sqrt{(x-\alpha_1)\cdots(x-\alpha_n)} \pm \sqrt{-(x-\beta_1)\cdots(x-\beta_m)}\Bigr).$$

The correspondences

$$\begin{aligned} \tau\colon (x,y) &\mapsto (\overline{x}, \overline{y}), \\ \tau_\alpha\colon \Bigl(x, \pm\sqrt{-(x-\alpha_1)\cdots(x-\alpha_n)} \pm \sqrt{(x-\beta_1)\cdots(x-\beta_m)}\Bigr) \\ \mapsto \Bigl(x, \mp\sqrt{(x-\alpha_1)\cdots(x-\alpha_n)} \pm \sqrt{-(x-\beta_1)\cdots(x-\beta_m)}\Bigr) \end{aligned}$$

and

$$\begin{aligned} \tau_\beta\colon \Bigl(x, \pm\sqrt{-(x-\alpha_1)\cdots(x-\alpha_n)} \pm \sqrt{(x-\beta_1)\cdots(x-\beta_m)}\Bigr) \\ \mapsto \Bigl(x, \pm\sqrt{(x-\alpha_1)\cdots(x-\alpha_n)} \mp \sqrt{-(x-\beta_1)\cdots(x-\beta_m)}\Bigr) \end{aligned}$$

define commuting involutions on P.

We can readily see that (P, τ) is a real algebraic curve of type $(g,k,1)$, where

$$g = \begin{cases} n+m-1 & \text{for } \alpha_n < \beta_1, \\ n+m-2 & \text{for } \alpha_n = \beta_1 \end{cases}$$

and

$$k = \begin{cases} n & \text{for } \alpha_n < \beta_1, \\ n-1 & \text{for } \alpha_n = \beta_1. \end{cases}$$

The involution τ_α preserves each of the ovals, and the involution τ_β pairwise transposes the ovals for $\alpha_n < \beta_1$ and preserves exactly one oval for $\alpha_n = \beta_1$. Let us number the ovals $c_1, \dots, c_k$ so that $\tau c_i = c_{k+1-i}$. We assume that there is a real differential ξ that is positive on the ovals $c_1, \dots, c_{n/2}$ and is not negative on the other ovals. Then the differential $\xi + \xi\beta$ is negative on no oval. The involution τ induces an antiholomorphic involution $\widetilde{\tau}\colon \widetilde{P} \to \widetilde{P}$ on the surface $\widetilde{P} = P/\langle\beta\rangle$. We can readily see that $(\widetilde{P}, \widetilde{\tau})$ is an M-curve of genus $(n/2)-1$. The differential $\xi+\xi\beta$ induces a real differential on the curve $(\widetilde{P}, \widetilde{\tau})$ that is negative on no oval. This contradicts Theorem 6.2, and thus shows that there is no such differential ξ.

§ 7. Analogues of Fourier series, and the Sturm–Hurwitz theorem on real algebraic curves of arbitrary genus

1. The simplest real algebraic curve is the Riemann sphere $\overline{\mathbb{C}} = \mathbb{C} \cup \infty$ with the antiholomorphic involution $\tau_{\mathbb{C}}\colon z \mapsto 1/\overline{z}$. The curve $(\overline{\mathbb{C}}, \tau_{\mathbb{C}})$ has a unique oval, namely,

$$c = \{z \in \mathbb{C} \mid |z| = 1\} = \{e^{i\psi} \mid \psi \in \mathbb{R}\}.$$

We consider meromorphic functions $f\colon \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ such that $f(\tau_{\mathbb{C}} z) = \overline{f(z)}$. The simplest functions of this form are holomorphic away from 0 and ∞. They can be represented by Fourier series

$$f(z) = \sum_{n=0}^{\infty} \bigl(a_n c_n(z) + b_n s_n(z)\bigr),$$

where

$$c_n(z) = \frac{1}{2}(z^n + z^{-n})$$

and

$$s_n(z) = \frac{1}{2i}(z^n - z^{-n}).$$

The restrictions of s_n and c_n to c are the classical trigonometric functions

$$s_n(e^{i\psi}) = \sin n\psi, \qquad c_n(e^{i\psi}) = \cos n\psi.$$

2. We assume now that (P, τ) is a real algebraic curve of type $(g, k, 1)$ with generic points $p_+, p_- \in P \setminus P^\tau$ such that $\tau p_+ = p_-$. Instead of functions, we consider tensors of integer and half-integer weight λ, that is, sections of the line bundle $E^{\otimes 2\lambda} \to \widetilde{P}$, where (E, β) is the real spinor bundle on $(\widetilde{P}, \tau)$ and $2\lambda \in \mathbb{Z}$, that are meromorphic on P and holomorphic on $\widetilde{P} = P \setminus (p_+ \cup p_-)$. Let M_λ be the space of such tensors. According to [27], if $\lambda \neq 0, 1$ or $|n| > \frac{g}{2}$, then for any integer $n + (g/2)$ there is a unique tensor $f_n^\lambda \in M_\lambda$ with the asymptotic behavior

$$f_n^\lambda = z_\pm^{\pm n - s}\bigl(1 + O(z_\pm)\bigr)(dz_\pm)^\lambda,$$

where $z_\pm$ belongs to a neighbourhood of the corresponding point $p_\pm$ and $s = s(\lambda, g) = \frac{g}{2} - \lambda(g-1)$. The involutions β and τ induce involutions $\beta_\lambda \colon E^{\otimes 2\lambda} \to E^{\otimes 2\lambda}$ and $\tau_\lambda \colon M_\lambda \to M_\lambda$, where $\tau_\lambda f(p) = \beta_\lambda f(\tau p)$. We can readily see that $\tau_\lambda f_n^\lambda = f_{-n}^\lambda$. A tensor $\xi \in M_\lambda$ is said to be *real* if $\tau_\lambda \xi = \xi$. In a real local chart, this tensor takes real values on P^τ.

The analogues of the functions $\cos nx$ and $\sin nx$ are the real tensors

$$c_n^\lambda = \frac{1}{2}(f_n^\lambda + f_{-n}^\lambda) \quad \text{and} \quad s_n^\lambda = \frac{1}{2i}(f_n^\lambda - f_{-n}^\lambda),$$

where $n \geqslant 0$. The corresponding analogue of the addition theorem for trigonometric functions is as follows.

Theorem 7.1. *Let $\lambda_1, \lambda_2, \lambda_1 + \lambda_2 \neq 0, 1$ or let $n_1 + n_2 > g$. Then*

$$c_{n_1}^{\lambda_1} c_{n_2}^{\lambda_2} - s_{n_1}^{\lambda_1} s_{n_2}^{\lambda_2} = \sum_{n=-\frac{g}{2}}^{\frac{g}{2}} \delta_n c_{n_1+n_2-n}^{\lambda_1+\lambda_2}, \qquad c_{n_1}^{\lambda_1} s_{n_2}^{\lambda_2} - c_{n_2}^{\lambda_2} s_{n_1}^{\lambda_1} = \sum_{n=-\frac{g}{2}}^{\frac{g}{2}} \eta_n s_{n_1+n_2-n}^{\lambda_1+\lambda_2},$$

where $\delta_n, \eta_n \in \mathbb{R}$.

Proof. By [27],

$$f_n^\lambda f_m^\mu = \sum_{k=-\frac{g}{2}}^{\frac{g}{2}} Q_{n,m}^{\lambda,\mu,k} f_{n+m-k}^{\lambda+\mu}.$$

The relation $\tau_\lambda f_n^\lambda = f_{-n}^\lambda$ implies

$$Q_{-n,-m}^{\lambda,\mu,k} = \overline{Q}_{n,m}^{\lambda,\mu,-k}.$$

Thus,

$$\begin{aligned}
c_{n_1}^{\lambda_1} c_{n_2}^{\lambda_2} - s_{n_1}^{\lambda_1} s_{n_2}^{\lambda_2} &= \frac{1}{4}(f_{n_1}^{\lambda_1} + f_{-n_1}^{\lambda_1})(f_{n_2}^{\lambda_2} + f_{-n_2}^{\lambda_2}) + \frac{1}{4}(f_{n_1}^{\lambda_1} - f_{-n_1}^{\lambda_1})(f_{n_2}^{\lambda_2} - f_{-n_2}^{\lambda_2}) \\
&= \frac{1}{2}(f_{n_1}^{\lambda_1} f_{n_2}^{\lambda_2} + f_{-n_1}^{\lambda_1} f_{-n_2}^{\lambda_2}) \\
&= \frac{1}{2} \sum_{n=-\frac{g}{2}}^{\frac{g}{2}} (Q_{-n_1,-n_2}^{\lambda_1,\lambda_2,n} + \overline{Q}_{n_1,n_2}^{\lambda_1,\lambda_2,n})(f_{n_1+n_2-n}^{\lambda_1+\lambda_2} + f_{-n_1-n_2+n}^{\lambda_1+\lambda_2}) \\
&= \sum_{n=-\frac{g}{2}}^{\frac{g}{2}} \delta_n c_{n_1+n_2-n}^{\lambda_1+\lambda_2},
\end{aligned}$$

where $\delta_n = 2 \operatorname{Re} Q_{n_1,n_2}^{\lambda_1,\lambda_2,n}$. The other relation can be proved similarly.

The corresponding analogue of Fourier's theorem is as follows.

Theorem 7.2 [43]. *Each real tensor f^λ of weight $\lambda \neq 0, 1$ can be uniquely represented in the form*

$$f^\lambda = \sum_{k=0}^{\infty} (a_k c_k^\lambda + b_k s_k^\lambda),$$

where $a_k, b_k \in \mathbb{R}$.

Proof. According to [27], we have

$$f^\lambda = \sum_{n=0}^{\infty} (\alpha_n f_n^\lambda + \beta_n f_{-n}^\lambda) = \sum_{n=0}^{\infty} (a_n c_n^\lambda + b_n s_n^\lambda),$$

where $a_n = \alpha_n + \beta_n$ and $b_n = i(\alpha_n - \beta_n)$. The relation $\tau f^\lambda = f^\lambda$ yields $\beta_n = \overline{\alpha}_n$, and hence $a_n, b_n \in \mathbb{R}$.

The following assertion is an analogue of the classical Sturm–Hurwitz theorem [22].

Theorem 7.3 [43]. *Let $\lambda \neq 0, 1$ or $n > g/2$. Then the real tensor*

$$F = \sum_{k=n}^{\infty} (a_k c_k^\lambda + b_k s_k^\lambda)$$

has at least $2n - g$ zeros on the ovals of P^τ.

Proof. Let D be a divisor of the tensor c_n^λ. It is of the form $D = D_1 + D_0 + D_2$, where $D_0 \in P^\tau$, $P \setminus P^\tau = P_1 \cup P_2$, $D_i \in P_i$, and $\tau D_1 = D_2$. Let $p_+ \in P_1$, let n_0 be the degree of D_0, and let n_1 be the degree of D_1. We set $G = \sum_{k=n}^{\infty} \alpha_k f_k^\lambda$, where $\alpha_k = \frac{1}{2}(a_k - ib_k)$. Let us consider a system of pairwise disjoint arcs and contours $L \subset P_1$ such that $Q = P_1 \setminus L$ is a simply connected domain (see Fig. 7.1). Let $c \subset Q$ be a simple contour on $Q - p_+$ that is not homotopic to zero. In the domain bounded by c, the function $f = G/c_n^\lambda$ has a zero at p_+ of multiplicity $2n$, and at most n_1 poles. Therefore, the contour $f(c)$ goes around 0 at least $2n - n_1$ times, and hence intersects $\operatorname{Im}\mathbb{C} = \{z \in \mathbb{C} \mid \operatorname{Re} z = 0\}$ at least $2(2n - n_1)$ times. As c tends to the boundary of the domain Q, we see that $f(P^\tau)$ intersects $\operatorname{Im}\mathbb{C}$ at least $4n - (2n_1 + n_0) = 2n - g$ times. However, if $p \in P^\tau$ and $f(p) \in \operatorname{Im}\mathbb{C}$, then

$$F(p) = G(p) + (\tau G)(p) = c_1^\lambda(p)\bigl(f(p) + \overline{f(\tau(p))}\,\bigr) = 0.$$

Hence, F has at least $2n - g$ zeros on P^τ.

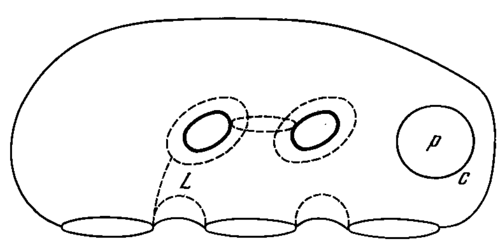

Figure 7.1

Remark. In the case of $g = \lambda = 0$, Theorem 7.3 was proved by Hurwitz [22]. In this case it has been re-proved more than once by various methods in connection with important applications in singularity theory. The above proof is similar to the original Hurwitz proof for the case in which $g = \lambda = 0$ in the interpretation of Arnol'd.

§ 8. Jacobians and θ-functions of real algebraic curves

1. We recall some facts from the classical theory of Riemann surfaces [19]. Let P be a compact Riemann surface of genus g. A homology basis

$$\{a_i, b_i \ (i = 1, \dots, g)\} \in H_1(P, \mathbb{Z})$$

is said to be *symplectic* if the intersection numbers of the cycles are of the form

$$(a_i, a_j) = (b_i, b_j) = 0, \qquad (a_i, b_j) = \delta_{ij}.$$

We say that a basis $\xi_1, \dots, \xi_g$ of the space of holomorphic differentials on P is *generated* by a symplectic basis $\{a_i, b_i\}$ if $\oint_{a_k} \xi_j = 2\pi i \delta_{kj}$. In this case, the matrix $B = (B_{kj})$ given by $B_{kj} = \oint_{b_k} \xi_j$ is symmetric and has negative-definite real part $\operatorname{Re} B = (\operatorname{Re} B_{ij})$. This enables one to define a θ-function $\theta\colon \mathbb{C}^g \to \mathbb{C}$ by

$$\theta(z) = \theta(z \mid B) = \sum_{N \in \mathbb{Z}^g} \exp\left\{\frac{1}{2}\langle BN, N\rangle + \langle N, z\rangle\right\},$$

where

$$\langle (x_1, \dots, x_g), (y_1, \dots, y_g)\rangle = \sum_{i=1}^{g} x_i y_i.$$

Let G be the group generated by the vectors

$$\ell_k = 2\pi i(\delta_{k1}, \dots, \delta_{kg}) \quad \text{and} \quad h_k = (B_{k1}, \dots, B_{kg}) \quad (k = 1, \dots, g).$$

The complex torus $J = J(P) = \mathbb{C}^g/G$ is called the *Jacobian of the surface* P. Let $\Phi\colon \mathbb{C}^g \to J$ be the natural projection.

A set of k points of P is called a (*positive*) *divisor* of degree k. Let S_k be the set of all positive divisors of degree k. Let us choose a point q on P. With a divisor $D = \sum_{i=1}^{k} p_i$ we associate the point form

$$A_q(D) = \Phi\left(\int_q^D \xi_1, \dots, \int_q^D \xi_g\right) = \Phi\left(\sum_{i=1}^{k}\left(\int_q^{p_i} \xi_1, \dots, \int_q^{p_i} \xi_g\right)\right)$$

of the Jacobian. Then $A_q(S_g) = J$ and the *Abel map* A_q is invertible at a generic point. The image K_q in J of the vector $(K_q^1, \dots, K_q^g)$ with components

$$K_q^j = \frac{2\pi i + B_{jj}}{2} - \frac{1}{2\pi i}\sum_{\ell \neq j}\int_{a_\ell}\left(\omega_\ell(p)\int_q^p \omega_j\right)$$

is called the *vector of Riemann constants*. We also have $2K_q = -A_q(D_\xi)$, where D_ξ is the divisor of zeros of an arbitrary (holomorphic) differential ξ on P. The set

$$(\theta) = A_q(S_{g-1}) + K_q \subset J$$

coincides with the image in J of the set of zeros of the θ-function and is called the *θ-divisor*. A subset $\Sigma \subset J$ is said to be *singular* if $\Sigma \cap A_q(S_{g-1}) \neq \varnothing$. In this case, the set $\Sigma + K_q$ contains a zero of the θ-function.

2. We assume now that (P,τ) is a real algebraic curve. In this subsection and the next two we consider only curves with real points. Let $q \in P^\tau$ be such a point. We need a symplectic basis that agrees with τ,

$$\{a_i, b_i\ (i=1,\dots,g)\} \subset H_1(P,\mathbb{Z}),$$

which is called a *real* homology basis. For curves of type $(g,k,0)$ this is a basis with the following properties: 1) $\tau(a_i) = a_i$ $(i=1,\dots,g)$, $\tau(b_i) = -b_i$ $(i=1,\dots,k-1)$, and $\tau(b_i) = -b_i + a_i$ $(i=k,\dots,g)$, 2) the oval containing the point q is homologous to $\sum_{i=1}^{g} a_i$. For curves of type $(g,k,1)$, this is a basis with the following properties: 1) $\tau(a_i) = a_i$, $\tau(b_i) = -b_i$ $(i=1,\dots,k-1)$, $\tau(a_i) = a_{i+m}$, and $\tau(b_i) = -b_{i+m}$ $(i=k,\dots,k+m-1)$, where $m = \frac{1}{2}(g+1-k)$, 2) the oval containing the point q is homologous to $\sum_{i=1}^{k-1} a_i$.

Lemma 8.1. *A real basis exists.*

Proof. Let (P,τ) be a real curve of type $(g,k,0)$. Then by Lemma 1.2 there is a set of pairwise disjoint contours $a_0, a_1, \dots, a_g$ such that

$$\tau(a_i) = a_i, \qquad P^\tau = \bigcup_{i=0}^{k} a_i,$$

and $P \setminus \bigcup_{i=0}^{g} a_i$ decomposes into two components P_1 and P_2. Let us number the contours so that $q \in a_0$. We set $b_i = c_i \cup \tau c_i \cup r_i$, where $c_i \subset P_1$ joins a_0 and a_i and $r_i \subset a_i$ joins $p_i = c_i \cap a_i$ and τp_i. The case $(g,k,1)$ can be treated similarly.

The next assertion follows directly from the definitions.

Lemma 8.2. *Let $\{a_i, b_i\ (i=1,\dots,g)\}$ be a real basis of an algebraic curve (P,τ) of type (g,k,ε). Then $\overline{h}_j = h_j$ for $j \leqslant k-1$, $\overline{h}_j = h_j - \ell_j$ for $\varepsilon = 0$ and $j = k,\dots,g$, and $\overline{h}_j = h_{j+m}$ for $\varepsilon = 1$ and $j = k,\dots,k+m-1$.*

In the rest of the section we assume that the homology basis is real.

3. Let (P,τ) be a real algebraic curve of type (g,k,ε) and let $J = J(P)$. Let us consider an involution $\widetilde{\tau}\colon \mathbb{C}^g \to \mathbb{C}^g$ that is defined on the basis $(\ell_i, h_i\ (i=1,\dots,g))$ of the space $\mathbb{R}^{2g} = \mathbb{C}^g$ by the linear map $\ell_j \mapsto \ell_j, h_j \mapsto -h_j$ for $j \leqslant k-1$ or for $\varepsilon = 0$, by $\ell_j \mapsto \ell_{j+m}, h_j \mapsto -h_{j+m}$ for $\varepsilon = 1$ and $j = k,\dots,k+m-1$, and by $\ell_j \mapsto \ell_{j-m}, h_j \mapsto -h_{j-m}$ for $\varepsilon = 1$ and $j = k+m,\dots,g$. By Lemma 8.2, the map $\widetilde{\tau}$ induces an involution $\tau_{\mathbb{R}}\colon J \to J$. By the same lemma, the Abel map A_q identifies $\tau_{\mathbb{R}}$ with an involution $S_g \to S_g$ that sends a divisor $D \in S_g$ to the divisor τD.

The fixed points of the involution $\tau_{\mathbb{R}}$ are called the *real points of the Jacobian* of the curve (P,τ). These points form the *real part* $J_{\mathbb{R}}$ of the Jacobian.

Theorem 8.1. *The real part of the Jacobian of a real algebraic curve (P,τ) of type (g,k,ε), where $k>0$, decomposes into 2^{k-1} real tori of the form*

$$\Phi(T_{\mathbb{R}}+\delta),$$

where

$$\delta=\frac{1}{2}\sum_{j=1}^{k-1}\delta_j h_j, \qquad \delta_j\in\{0,1\},$$

$T_{\mathbb{R}}=i\mathbb{R}^g$ *if* $\varepsilon=0$, *and*

$$T_{\mathbb{R}}=\bigl\{(x_1,\dots,x_g)\in\mathbb{C}^g \mid x_j\in i\mathbb{R} \text{ for } j\leqslant k-1,\ \overline{x}_k=-x_{j+m} \text{ for } k\leqslant j\leqslant k+m\bigr\}$$

if $\varepsilon=1$.

Such a torus is non-singular if and only if $\varepsilon=1$, $k=g+1$, *and* $\delta_1=\dots=\delta_g=1$.

Proof. The equations for the real part can be found by direct calculation. If $p\in P$, then

$$\left(\int_q^p\xi_1+\int_q^{\tau p}\xi_1,\dots,\int_q^p\xi_g+\int_q^{\tau p}\xi_g\right)\in T_{\mathbb{R}}.$$

If $p\in a_j$, then

$$\left(\int_q^p\xi_1,\dots,\int_q^p\xi_g\right)=\frac{1}{2}h_j.$$

Therefore, $x\in\Phi(T_{\mathbb{R}}+\delta)$ if and only if $x=A_q(D)$, where $D\in R_\delta=\{D\in S_g \mid \tau D=D$ and the parity of the degree of the divisor $D\cap a_j$ is equal to $\delta_i\}$.

On the other hand, $R_\delta\cap S_{g-1}=\varnothing$ if and only if

$$\sum_{i=1}^{k-1}\delta_i>g-1,$$

that is, if and only if $k=g+1$ and $\delta_1=\dots=\delta_g=1$.

4. Along with the involution $\tau_{\mathbb{R}}$, we consider the involution $\tau_{\mathbb{I}}=-\tau_{\mathbb{R}}\colon J\to J$. The fixed points of this involution form the imaginary part $J_{\mathbb{I}}$ of the Jacobian J.

Theorem 8.2. *The imaginary part of the Jacobian of a real algebraic curve (P,τ) of type (g,k,ε), where $k>0$, decomposes into 2^{k-1} real tori of the form $\Phi(T_{\mathbb{I}}+\delta)$, where $\delta=\pi i(\delta_1,\dots,\delta_{k-1}),\delta_i\in\{0,1\}$, and $T_{\mathbb{I}}=\mathbb{R}^g$ if $\varepsilon=0$ and $T_{\mathbb{I}}=\{(x_1,\dots,x_g)\in\mathbb{C}^g \mid x_j\in\mathbb{R}$ for $j\leqslant k-1$, $\overline{x}_j=-x_{j+m}$ for $k\leqslant j\leqslant k+m\}$ if $\varepsilon=1$. For $\varepsilon=0$ all the tori are singular. For $\varepsilon=1$ there is exactly one non-singular torus among them, corresponding to $\delta_1=\delta_2=\dots=\delta_{k-1}=1$.*

Proof. The equations for the imaginary part can be found by direct calculation. Let us consider the set $I=\{D\in S_g \mid D+\tau D=$ (the divisor of zeros of a meromorphic differential that is holomorphic away from q and has a pole of order 0 or 2 at q)$\}$. Then $A_q(I)-K_q=J_{\mathbb{I}}$ because $\tau_{\mathbb{R}}K_q=K_q$.

By definition, corresponding to a divisor $D\in I$ is a meromorphic differential ξ_D. Let $A_1\cup A_2$ be an arbitrary decomposition of the set of ovals $A=(a_0,\dots,a_{k-1})$.

By $I_{A_1,A_2} = I_{A_2,A_1} \subset I$ we denote the set of all $D \in I$ such that the differential ξ_D or the differential $-\xi_D$ is non-negative on the ovals of A_1 and non-positive on the ovals of A_2. The zeros and the poles of ξ_D that belong to the ovals have even degrees, and hence $I = \bigcup I_{A_1,A_2}$.

By Theorem 6.1, for any decomposition $A = A_1 \cup A_2$ with $A_1 \neq \varnothing$ and $A_2 \neq \varnothing$ we can find a holomorphic real differential ξ that is non-negative on A_1 and non-positive on A_2. By adding the differential $\lambda\xi$ to an arbitrary differential ξ_D, $D \in I$, we can readily prove that $I_{A_1,A_2} \neq \varnothing$. Thus, $I = \bigcup I_{A_1,A_2}$ consists of at least 2^{k-1} connected components. However, as was already proved, the set $J_{\mathbb{I}} = A_q(I) - K_q$ consists of 2^{k-1} connected components. Therefore, each of the sets I_{A_1,A_2} is connected. If $A_1 \neq \varnothing$ and $A_2 \neq \varnothing$ or if $\varepsilon = 0$, then it follows from Theorem 6.1 that there is a differential $D \in I_{A_1,A_2}$ such that ξ_D is holomorphic. In this case $q \in D$ and

$$A_q(D) = A_q(D \setminus q) \in A_q(S_{g-1}),$$

and hence the component I_{A_1,A_2} is singular. If $A_1 = \varnothing$ or $A_2 = \varnothing$, then the condition $A_q(D) \subset A_q(S_{g-1})$ means that the differential ξ_D is holomorphic and has the same signs on all ovals. This is impossible for $\varepsilon = 1$ because $\sum_{i=0}^{k-1} a_i = 0$. Hence, for $\varepsilon = 1$ the components of $A_q(I_{\varnothing,A})$ is non-singular.

Let us find a vector δ to which this component corresponds. We assume first that $k = g+1$ and (P,τ) is a hyperelliptic curve. Then the imaginary part of the Jacobian of (P,τ) coincides with the real part of the Jacobian of the curve $(P,\alpha\tau)$, where $\alpha\colon P \to P$ is the hyperelliptic involution. We assume that $q \in P^\tau \cap P^{\alpha\tau}$ is a fixed point of this involution. It follows from Theorem 8.1 that a non-singular imaginary torus of the Jacobian of (P,τ) (or, which is the same, a non-singular real torus of the Jacobian of $(P,\alpha\tau)$) corresponds to the vector $\delta = \pi i(1,\dots,1)$. This vector remains the same under a continuous deformation of the curve (P,τ). Since the set $M_{g,g+1,1}$ is connected (Theorem 2.1), the same vector corresponds to a non-singular torus of the imaginary part of the Jacobian for any M-curve.

The case $k < g+1$ can be reduced to the case $k = g+1$ as follows. Let us consider a simple contour a on the surface P such that $a \cup \tau a$ cuts out a surface $\widetilde{P}$ of genus $k-1$ with two holes on P. We introduce a continuous deformation of the curve (P,τ) that contracts the contour a to a point. In the course of deformation, the vector corresponding to a non-singular torus of the imaginary part of the Jacobian does not change. In the limit it gives a vector corresponding to the M-curve, that is, $\pi i(1,\dots,1)$.

Remark. The number of real and imaginary tori of the Prymian was first found in [11]. The number of singular and non-singular tori was found in [19] for $\varepsilon = 1$, and in [13] and [34] for $\varepsilon = 0$. This was done in another way in [50].

§ 9. Prymians of real algebraic curves

1. To curves with automorphisms in classical algebraic geometry (see, for example, [19]) there correspond algebraic varieties that are similar to Jacobians but do not coincide with them, namely, the Prymians. We consider only the simplest example of such varieties, which is, however, important in applications.

Let P be a compact Riemann surface of genus $2g$ and let $\alpha\colon P \to P$ be a holomorphic involution with two fixed points q_1 and q_2. A symplectic basis

$\{a_i, b_i\ (i = 1,\dots,2g)\}$ is said to be *symmetric* if $\alpha a_i = -a_{i+g}$ and $\alpha b_i = -b_{i+g}$ $(i = 1,\dots,g)$. The divisor map $D \mapsto \alpha(D)$ induces an involution $\alpha^*\colon S_g \to S_g$. The Abel map A_{q_1} transfers it to $J = J(P)$, and thus generates an involution $\alpha^*\colon J \to J$. The subset

$$Pr = Pr(P,\alpha) = \{x \in J \mid \alpha^* x = -x\}$$

is called the *Prymian* of the surface with involution (P,α). The Prymian is isomorphic to the torus $\mathbb{C}^g/G$, where G is the lattice generated by the vectors ℓ_i and the column vectors ξ_i of the matrix

$$A_{ij} = \int_{b_i} \xi_j + \xi_{j+g} \qquad (i,j = 1,\dots,g)$$

in the notation of § 7.

2. By a real curve with involution (P,τ_1,α) we mean a compact Riemann surface P of genus $2g$ with two commuting involutions one of which, τ_1, is antiholomorphic and the other, α, is holomorphic and has exactly two fixed points q_1 and q_2, with $\tau_1 q_1 = q_2$. We set $\tau_2 = \tau_1\alpha$. We assume that among the ovals of the involution τ_i there are r_i that are invariant with respect to α and $2t_i$ that are pairwise transposed by the involution α. Then

$$(\widetilde{P},\widetilde{\tau}\,) = (P/\langle\alpha\rangle, \tau_i/\langle\alpha\rangle)$$

is a real algebraic curve of type (g,k,ε), where $k = t_1 + r_1 + t_2 + r_2$. Moreover, the pre-image of the set $\widetilde{P}^{\widetilde{\tau}}$ coincides with $P^{\tau_1} \cup P^{\tau_2}$. This pre-image decomposes P into two parts if and only if $\varepsilon = 1$. The set $(g,\varepsilon,t_1,r_1,t_2,r_2)$ is called the *type of the real curve with involution* (P,τ_1,α).

Example 9.1. Let $(\widetilde{P},\widetilde{\tau})$ be a real curve of type $(g,k,1)$ and let $k = t_1+r_1+t_2+r_2$, where $r_1 + r_2 = 1 \pmod 2$. Let us consider a connected component $\widetilde{P}_1$ of the set $\widetilde{P} \setminus \widetilde{P}^{\tau}$ and a two-sheeted covering $\varphi_1\colon P_1 \to \widetilde{P}_1$ with a unique branch point $q_1 \in P_1$, the covering being two-sheeted on the r_1+r_2 contours $c_1,\dots,c_{r_1+r_2} \in \partial P_1$ and one-sheeted on the other boundary contours $c_{r_1+r_2+1},\dots,c_{\widehat{k}}$, where $\widehat{k} = r_1 + r_2 + 2t_1 + 2t_2$. By using the construction of Example 1.1, we form a real algebraic curve $(\widehat{P},\widehat{\tau})$ such that $\widehat{P}^{\widehat{\tau}} = \bigcup_{i=1}^{\widehat{k}} c_i$ decomposes $\widehat{P}$ into P_1 and $P_2 = \widehat{\tau}P_1$. The covering φ_1 induces a two-sheeted covering $\widehat{\varphi}\colon \widehat{P} \to \widetilde{P}$, where $\widehat{\varphi}\widehat{\tau} = \widetilde{\tau}\widehat{\varphi}$. Let $\alpha\colon \widehat{P} \to \widehat{P}$ be the involution defined by transposition of the sheets. This involution commutes with $\widehat{\tau}$ and has exactly two fixed points q_1 and $q_2 = \widehat{\tau}q_1$. Let us cut the surface $\widehat{P}$ along the contours $c_{r_1+1},\dots,c_{r_1+r_2}$ and $c_{r_1+r_2+2t_1+1},\dots,c_{r_1+r_2+2t_1+2t_2}$ and paste together the boundary contours in accordance with the map $\alpha\widehat{\tau}$. On the surface P thus obtained, the involution $\widehat{\tau}$ induces an involution $\tau_1\colon P \to P$ that commutes with α. We set $\tau_2 = \alpha\tau_1$. It can readily be seen that (P,τ_1,α) is a real curve with involution of type $(g,1,t_1,r_1,t_2,r_2)$.

The following lemma is clear.

Lemma 9.1. *The construction of Example* 9.1 *enables one to produce all real curves with involution of type* $(g,1,t_1,r_1,t_2,r_2)$.

Example 9.2. Let $(\widetilde{P}, \widetilde{\tau})$ be a real curve of type $(g, k, 0)$ and let $k = t_1 + r_1 + t_2 + r_2$, where $r_1 + r_2 = 1 \pmod 2$. Using Lemma 1.2, we construct a set of pairwise disjoint contours $\widetilde{c}_1, \dots, \widetilde{c}_{g+1}$ such that $\widetilde{\tau}\widetilde{c}_i = \widetilde{c}_i$ and $\widetilde{P}^{\widetilde{\tau}} = \bigcup_{i=1}^{k} \widetilde{c}_i$. Let us consider a connected component $\widetilde{P}_1$ of the set $\widetilde{P} \setminus \bigcup_{i=1}^{g+1} \widetilde{c}_i$ and a two-sheeted covering $\varphi_1 \colon P_1 \to \widetilde{P}_1$ with a single branch point $q_1 \in P_1$ that is two-sheeted on the contours $c_1, \dots, c_{r_1+r_2}$ and one-sheeted on the other contours $c_{r_1+r_2+1}, \dots, c_v$. Using the construction of Example 1.2, we form a real algebraic curve $(\widehat{P}, \widehat{\tau})$ such that $\widehat{P} \setminus \bigcup_{i=1}^{v} c_i$ decomposes $\widehat{P}$ into P_1 and $P_2 = \widehat{\tau} P_1$, and we have $\widehat{P}^{\widehat{\tau}} = \bigcup_{i=1}^{\widehat{k}} c_i$, where $\widehat{k} = r_1 + r_2 + 2t_1 + 2t_2$. Repeating the cuts and pastings together described in Example 9.1, we obtain a real curve with involution (P, τ_1, α) of type $(g, 0, t_1, r_1, t_2, r_2)$.

Lemma 9.2 ([7], [35]). *The construction of* Example 9.2 *enables one to produce all real curves with involution of type* $(g, 0, t_1, r_1, t_2, r_2)$.

3. Let (P, τ_1, α) be a real curve with involution of type $(g, \varepsilon, t_1, r_1, t_2, r_2)$. The intersection of the Prymian $Pr = Pr(P, \alpha) \subset J(P) = J$ with the real part of the Jacobian of the curve (P, τ_1) is called the *real part of the Prymian* of the real curve with involution (P, τ_1, α). The connected components of this part are called *real tori of the Prymian* of the curve (P, τ_1, α). These tori form the fixed tori of the involution $(\tau_1)_{\mathbb{R}}\big|_{Pr} \colon Pr \to Pr$.

Theorem 9.1. *The real part of the Prymian of a real curve with involution* (P, τ_1, α) *of type* $(g, \varepsilon, t_1, r_1, t_2, r_2)$, *where* $k = t_1 + r_1 + t_2 + r_2 > 0$, *decomposes into* 2^{k-1} *real tori of dimension* g.

Proof. Let us choose a symmetric basis $\Delta = \{a_i, b_i\ (i = 1, \dots, g)\}$ of the pair (P, α) so that the projections of the cycles $\{a_i, b_i\ (i = 1, \dots, g)\}$ give a real basis $\widetilde{\Delta}$ of the real curve $(\widetilde{P}, \widetilde{\tau}) = (P/\langle\alpha\rangle, \tau_1/\langle\alpha\rangle)$ of type (g, k, ε). Let $\{\ell_i, d_i\}$ be the generators of the lattice of the Prymian Pr of a real curve with involution (P, τ_1, α) that corresponds to the basis Δ, and let $\{\widetilde{\ell}_i, \widetilde{h}_i\}$ be the generators of the lattice of the Jacobian $\widetilde{J}$ of the real curve $(\widetilde{P}, \widetilde{\tau})$. In these bases, the involutions $(\tau_1)_{\mathbb{R}}|_{Pr} \colon Pr \to Pr$ and $\widetilde{\tau}_{\mathbb{R}} \colon \widetilde{J} \to \widetilde{J}$ are described by the same formulae, and hence have equally many fixed tori.

4. Let (P, τ_1, α) be a real curve with involution of type $(g, \varepsilon, t_1, r_1, t_2, r_2)$. Let us number the ovals $a_1^j, \dots, a_{2t_j+r_j}^j$ of the involution τ_j so that $\alpha a_i^j = a_{t_j+i}^j$ for $i \leqslant t_j$. We put a divisor $D \subset P$ of degree g in the set Ω if $\tau_1 D = D$ and $\alpha D + D$ is the divisor of zeros of a meromorphic differential ξ_D that is holomorphic away from the fixed points q_1 and q_2 of the involution α and has poles of order 0 or 1 at these points. We say that ξ_D is positive definite on an oval $a = a_i^1$, where $i > 2t_1$, if either 1) ξ_D is non-negative on a or 2) there is a point $p \in a \cap D$ that, together with the point αp, divides the contour a into two open arcs so that the arc on which the differential is positive contains evenly many points of D in a neighbourhood of p. Otherwise we say that ξ_D is negative definite on a. We also say that ξ_D is positive (negative) definite on an oval a_i^2 if it is non-negative (non-positive, respectively) on this oval as a real differential of the curve (P, τ_2).

Let us decompose the set

$$a_{2t_1+1}^1, \dots, a_{2t_1+r_1}^1, a_1^2, \dots, a_{2t_2+r_2}^2$$

into subsets A_1 and A_2. Let

$$\delta = (\delta_1, \dots, \delta_{t_1}) \in \mathbb{Z}_2^{t_1}.$$

We denote by $\Omega(\delta, A_1, A_2)$ the subset of Ω consisting of the divisors $D \in \Omega$ such that ξ_D or $-\xi_D$ is positive definite on A_1 and negative definite on A_2, and the parity of the degree of the divisor $D \cap a_i^1$ coincides for $i \leqslant t_1$ with the parity of δ_i.

Lemma 9.3. *Each of the sets* $\Omega(\delta, A_1, A_2)$ *is non-empty.*

Proof. Let us prove first that, on any real algebraic curve $(\widetilde{P}, \widetilde{\tau})$ with ovals $c_1, \dots, c_k$, where $k = k_+ + k_- + k_0$, and for any pair of points $q_1 \neq q_2$, where $q_2 = \widetilde{\tau} q_1$, there is a meromorphic real differential ξ that is holomorphic away from q_1 and q_2, has poles of degree at most one at these points, is non-negative on c_i for $i \leqslant k_+$, non-positive on c_i for $k_+ < i \leqslant k_+ + k_-$, and has zeros on c_i for $i > k_+ + k_-$. To this end, we take disjoint neighbourhoods $U_i \supset q_i$ of q_1 and q_2 such that $\widetilde{\tau} U_1 = U_2$ and paste together the boundaries of the surface $\widetilde{P} \setminus (U_1 \cup U_2)$ by means of the involution $\widetilde{\tau}$. Then the boundary is mapped to an oval c_0 of a new real algebraic curve (P', τ'). Applying Theorem 6.1 to this curve, we find a real differential ξ' with the desired properties on the ovals $c_1, \dots, c_k$. Degenerating the oval c_0, we obtain the desired differential on the curve $(\widetilde{P}, \widetilde{\tau})$.

Applying the above result to the real curve $(\widetilde{P}, \widetilde{\tau}) = (P/\langle\alpha\rangle, \tau_1/\langle\alpha\rangle)$, we find a meromorphic differential that is holomorphic away from the images $\widetilde{q}_1$ and $\widetilde{q}_2$ of the points q_1 and q_2, has at most simple poles at these points, is non-negative on the images of the ovals of A_1 and non-positive on the images of the ovals in A_2, and has zeros on the other ovals. Its pre-image ξ on P is a meromorphic differential that is holomorphic away from q_1 and q_2, has at most simple poles at these points, and is positive definite on the ovals of A_1 and negative definite on the ovals of A_2. The divisor of zeros of the above differential intersected with $a_i^1 \cup \alpha a_i^1$ $(i \leqslant t_1)$ has positive degree divisible by four and is symmetric with respect to α. Hence, there is a divisor $D \in \Omega$ such that $\xi_D = \xi$, and the parity of the degree of the divisor $D \cap a_i^1$ coincides with the parity of δ_i for $i \leqslant t_1$.

Theorem 9.2 ([34], [42]). *Let* (P, τ, α) *be a real curve with involution of type* $(g, \varepsilon, t_1, r_1, t_2, r_2)$, *where* $k = t_1 + r_1 + t_2 + r_2 > 0$. *Then the following assertions hold*:

1) *for* $\varepsilon = 0$, *all real tori of the Prymian are singular*,
2) *for* $\varepsilon = 1$, *there is at most one non-singular real torus of the Prymian*,
3) *for* $\varepsilon = 1$ *and* $k = g + 1$, *a non-singular real torus of the Prymian always exists*,
4) *for* $\varepsilon = 1$ *and* $t_1 + r_1 \leqslant k/2$, *there are curves* (P, τ_1, α) *of type* $(g, \varepsilon, t_1, r_1, t_2, r_2)$ *such that there is a non-singular torus among the real tori of their Prymians.*

Proof. We can readily see that

$$\Omega(\delta, A_1, A_2) \cap \Omega(\delta', A_1', A_2') \neq \varnothing$$

if and only if $\delta' = \delta$, $A_1' = A_2$, and $A_2' = A_1$, in which case these sets coincide. Thus, the number of disjoint sets of the form $\Omega(\delta, A_1, A_2)$ is equal to 2^{k-1}.

On the other hand, the real part of the Prymian of a curve with involution (P, τ_1, α) coincides with $\bigcup A_{q_1}(\Omega(\delta, A_1, A_2)) - K_{q_1}$ and, by Theorem 9.1, it consists of 2^{k-1} connected components. Thus, by Lemma 9.3, each real torus of the Prymian is of the form

$$A_{q_1}(\Omega(\delta, A_1, A_2)) - K_{q_1}.$$

The torus is singular if and only if there is a $D \in \Omega(\delta, A_1, A_2)$ such that $\alpha D + D$ is the divisor of zeros of a holomorphic differential on P.

1) Let $\varepsilon = 0$ and let $T = A_{q_1}(\Omega(\delta, A_1, A_2)) - K_{q_1}$ be an arbitrary real torus of the Prymian. Let $\widetilde{A}_i$ be the image of the set A_i on the real curve $(\widetilde{P}, \widetilde{\tau}) = (P/\langle\alpha\rangle, \tau_1/\langle\alpha\rangle)$. By Theorem 6.1, there is a holomorphic real differential $\widetilde{\xi}$ on $(\widetilde{P}, \widetilde{\tau})$ that is non-negative on $\widetilde{A}_1$, non-positive on $\widetilde{A}_2$, and has zeros on the other ovals. Its pre-image ξ on P is a holomorphic differential that is positive definite on the ovals of A_1 and negative definite on the ovals of A_2. The divisor of zeros of this differential intersected with $a_i^1 \cup \alpha a_i^1$ $(i \leqslant t_1)$ has positive degree divisible by four and is symmetric with respect to α. Hence, there is a differential $D \in \Omega(\delta, A_1, A_2)$ such that $\xi_D = \xi$ and T is a singular torus.

2) Let $\varepsilon = 1$ and let $T = A_{q_1}(\Omega(\delta, A_1, A_2)) - K_{q_1}$ be a torus that differs from $A_{q_1}(\Omega(\delta, A, \varnothing)) - K_{q_1}$, where $\delta = (1, \dots, 1)$. Then, repeating the arguments used in the case $\varepsilon = 0$, we see that T is a singular torus.

3) Let $\varepsilon = 1$ and $k = g + 1$. We prove that for $\delta = (1, \dots, 1)$ the real torus $A_{q_1}(\Omega(\delta, A_1, \varnothing)) - K_{q_1}$ is non-singular. Indeed, otherwise there must be a real holomorphic differential ξ on P that is positive definite on all ovals of the involutions τ_1 and τ_2 where it has no zeros, and such that $\alpha\xi = \xi$. This differential induces a holomorphic real differential $\widetilde{\xi}$ on the M-curve $(\widetilde{P}, \widetilde{\tau}) = (P/\langle\alpha\rangle, \tau_1/\langle\alpha\rangle)$ that is positive on all ovals on which it has no zeros. However, by Theorem 6.2, there are no such differentials.

4) Let $\varepsilon = 1$ and $t_1 + k_1 \leqslant \frac{k}{2}$. Let T be a real torus of the form $A_{q_1}(\Omega(\delta, A_1, \varnothing)) - K_{q_1}$, where $\delta = (1, \dots, 1)$. If T is singular, then, repeating the reasoning used in the case of $k = g + 1$, we find a differential $\widetilde{\xi}$ on the real curve $(\widetilde{P}, \widetilde{\tau}) = (P/\langle\alpha\rangle, \tau_1/\langle\alpha\rangle)$ that is non-negative on $t_2 + k_2 > \frac{k}{2}$ images of the ovals in A_1 and either has zeros or is positive on the other ovals of the curve. Example 9.1 shows that we can take $(\widetilde{P}, \widetilde{\tau})$ to be any real curve and, in particular, the curve constructed in Theorem 6.3 on which there are no such differentials.

Remark. Under a small deformation of a curve with involution (P, τ_1, α), a non-singular torus is mapped into a non-singular one. Therefore, the curves with involution (P, τ_1, α) that have a non-singular real torus of the Prymian form an open set in the space of all curves of a given type.

§ 10. Uniformization of real algebraic curves by Schottky groups

1. Let $\psi \in \widetilde{T}_{\widetilde{g},k}$, where $k > 0$, and let $\{A_i, B_i\ (i = 1, \dots, \widetilde{g}),\ C_i\ (i = 1, \dots, k)\} = \psi(\gamma_{\widetilde{g},k})$ be the corresponding sequential set of shifts. We set

$$\Delta = \{A_i, B_i\ (i = 1, \dots, \widetilde{g}),\ C_i\ (i = 1, \dots, k-1)\}.$$

The invariant lines $\ell(\Delta)$ of the set Δ are shown in Fig. 10.1.

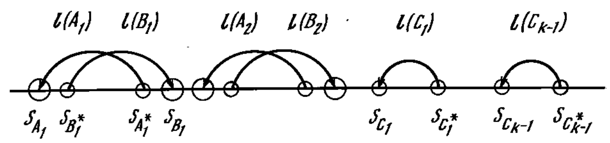

Figure 10.1

By [33] and [47], §§ 1–3, for any $D \in \Delta$ there are discs S_D and S_{D^*} with centres on $\mathbb{R}\cup\infty$ and such that $S_D\cap\ell(\Delta) \subset \ell(D)$, $S_{D^*}\cap\ell(\Delta) \subset \ell(D)$, $D(\ell(\Delta)\setminus\ell(D)) \subset S_D$, and $D^{-1}(\ell(\Delta) \setminus \ell(D)) \subset S_{D^*}$. By the methods described in [33] and [47], §§ 1–3, we can readily show that S_D and S_{D^*} can be chosen so that

$$S_{D_1} \cap S_{D_2} = S_{D_1^*} \cap S_{D_2} = S_{D_1^*} \cap S_{D_2^*} = \varnothing \quad \text{for} \quad D_1 \neq D_2$$

and $D(\partial S_{D^*}) = \partial S_D$. In this case $\Omega = \mathbb{C} \cup \infty \setminus \bigcup_{D\in\Delta}(S_D \cup S_{D^*})$ is a fundamental domain of the Schottky group G generated by Δ. On the quotient surface $P = \Omega/G$ of genus $g = 2\widetilde{g} + k - 1$, the involution $z \mapsto \overline{z}$ induces a separating involution $\tau\colon P \to P$ with k ovals.

To prove that this construction gives all separating real algebraic curves, it suffices to construct for such a curve (P,τ) a system of cuts on a connected component P_1 of the surface $P \setminus P^\tau$ that transforms P_1 into half of a fundamental domain of a Schottky group of the desired form. Such a system of cuts is presented in [5] and shown in Fig. 10.2.

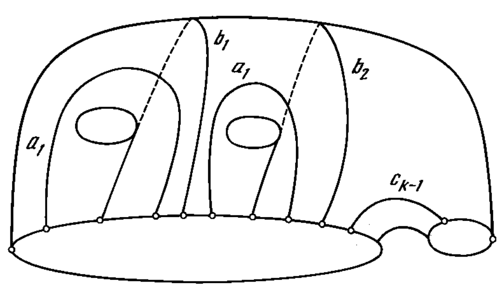

Figure 10.2

Thus (see [5] and [6]), the correspondence $\psi \mapsto (P,\tau)$ defines a map

$$\Psi_k\colon T_{\widetilde{g},k} \to M_{g,k,1},$$

where $\Psi_k(T_{\widetilde{g},k}) = M_{g,k,1}$.

A similar description of non-separating curves with real points can be obtained on replacing the system Δ of generators of the Schottky group G by the set

$$\Delta^* = \{A_i, B_i\ (i = 1,\dots,\widetilde{g}),\ C_i\ (i = 1,\dots,\widetilde{k}),\ C_i^*\ (i = \widetilde{k}+1,\dots,k-1)\},$$

where C_i^* is obtained from

$$C_i(z) = \frac{(\lambda_i\alpha_i - \beta_i)z - (1-\lambda)\alpha_i\beta_i}{(\lambda_i - 1)z + (\alpha_i - \lambda_i\beta_i)}$$

by replacing λ_i by $-\lambda_i$. The system Δ^* generates a Schottky group G^*. On the quotient surface $P^* = \Omega/G^*$ the involution $z \mapsto \overline{z}$ induces a non-separating involution $\tau^* \colon P^* \to P^*$ with $\widetilde{k}+1$ ovals. Thus, the correspondence $\psi \mapsto (P^*, \tau^*)$ generates a map

$$\Psi_{\widetilde{k}+1} \colon T_{\widetilde{g},k} \to M_{g,\widetilde{k}+1,0}.$$

The relation

$$\Psi_{\widetilde{k}+1}(T_{\widetilde{g},k}) = M_{g,\widetilde{k}+1,0}$$

is proved by the scheme used in the case of separating curves. We need only complete the set of ovals of the curve (P^*, τ^*) to form a system of pairwise disjoint invariant contours $c_1, \dots, c_k$ so that the surface $P^* \setminus \bigcup_{i=1}^k c_i$ decomposes into two connected components. Thus, any moduli space $M_{g,\widetilde{k},\varepsilon}$ has a representation of the form

$$M_{g,\widetilde{k},\varepsilon} = \Psi_{\widetilde{k}}(T_{\widetilde{g},k}).$$

This, together with the theorem

$$T_{\widetilde{g},k} \cong \mathbb{R}^{6\widetilde{g}+3k-6}$$

presented in [33] and [47], gives another proof of Corollary 2.1:

$$M_{g,k,\varepsilon} \cong \mathbb{R}^{6g-6} / \operatorname{Mod}_{g,k,\varepsilon}.$$

2. The Schottky uniformization enables one to solve the Schottky problem for real algebraic curves, that is, to find the matrices B_{ij} described in § 8.

We find the matrix corresponding to the system of generators

$$\widetilde{\Delta} = \{A_i, B_i \ (i = 1, \dots, \widetilde{g}),\ \widetilde{C}_i \ (i = 1, \dots, k-1)\} = \{\widetilde{D}_i \ (i = 1, \dots, 2\widetilde{g}+k-1)\}$$

of a Schottky group $\widetilde{G}$ of the above type. Let

$$D_i(z) = \frac{(\lambda_i\alpha_i - \beta_i)z - (1-\lambda)\alpha_i\beta_i}{(\lambda_i - 1)z + (\alpha_i - \lambda_i\beta_i)}.$$

By G_{mn} we denote the subset of the group G that consists of the elements

$$D = D_{i_1}^{j_1} \cdots D_{i_k}^{j_k},$$

where $j_\ell \neq 0$, $i_1 \neq m$, and $i_k \neq n$. We set $\{z_1, z_2, z_3, z_4\} = \dfrac{(z_1 - z_2)(z_2 - z_4)}{(z_1 - z_4)(z_2 - z_3)}$. Then by [3] and [8] the Jacobi matrix (B_{nm}) (of the algebraic curve $\Omega/\widetilde{G}$) corresponding to the generators $\widetilde{\Delta}$ is given by the convergent series

$$B_{nn} = \ln\lambda_n + \sum_{D \in G_{nn}} \ln\{\alpha_n, \beta_n, D\alpha_n, D\beta_n\} \tag{1}$$

and

$$B_{nm} = \sum_{D \in G_{mn}} \ln\{\alpha_m, \beta_m, D\alpha_n, D\beta_n\} \quad \text{for} \quad m \neq n. \tag{2}$$

Thus (1) and (2), together with the explicit description of the space $T_{\widetilde{g},k}$ in the coordinates $\{\alpha_i, \beta_i, \lambda_i \ (i = 1, \dots, 2\widetilde{g}+k)\}$ (see [33] and [47], §§ 3, 4), enable one to find the Jacobians of algebraic real curves and, by means of formulae in § 8, their real and imaginary tori.

A modification of this approach enables one to describe the Prymians of real curves [34].

§ 11. The moduli space of rank-one spinor bundles on real algebraic curves

1. We recall that the Fuchsian groups uniformizing Riemann surfaces of genus 0 are of the form $\psi(\gamma_{0,g+1})$, where $\gamma_{0,g+1}$ is the group with generators $c_1,\dots,c_{g+1}$ that has a single defining relation $c_1\cdots c_{g+1}=1$, and $\psi\colon \gamma_{0,g+1}\to \operatorname{Aut}(\Lambda)$ is a monomorphism belonging to the set $\widetilde{T}_{0,g+1}$ [47], §§ 1, 2. Corresponding to such a monomorphism is the group Γ^k_ψ $(k\leqslant g)$ generated by $\psi(\gamma_{0,g+1})$ and the maps

$$\widehat{C}_i=\begin{cases}\overline{C}_i & \text{for } i\leqslant k,\\ \widetilde{C}_i & \text{for } i>k,\end{cases}$$

where $C_i=\psi(c_i)$. We set $D_i=\widehat{C}_{g+1}\widehat{C}_i$ $(i=1,\dots,g)$. The natural isomorphism $\Gamma^k_\psi\to\pi_1(P,p)$, where $P=\Lambda/\Gamma^k_\psi$, sends $\{C_i,D_i\ (i=1,\dots,g)\}$ into elements $\{c_i,d_i\ (i=1,\dots,g)\}$ of the group $\pi_1(P,p)$ that generate it and satisfy a single defining relation

$$\prod_{i=1}^{g}c_i\prod_{i=g}^{1}d_ic_i^{-1}d_i^{-1}=1.$$

Lemma 11.1. *Let $\widetilde{\Gamma}^*$ be a lifting of a real Fuchsian group $\widetilde{\Gamma}$, where $[\widetilde{\Gamma}]=(P,\tau)$ is a real algebraic curve of type $(g,k,0)$. Then there is a monomorphism $\psi\in\widetilde{T}_{0,g+1}$ such that $\widetilde{\Gamma}=\Gamma^k_\psi$ and $\omega_{\widetilde{\Gamma}^*}(d_i)=\omega_{\widetilde{\Gamma}^*}(d_j)$ for any $i,j\leqslant g$.*

Proof. Let us consider a set of contours $c_1,\dots,c_{g+1}$ that has the properties listed in Theorem 4.2. Let P_1 be a sphere with $g+1$ holes and let the boundary ∂P_1 consist of the contours $\widetilde{c}_1,\dots,\widetilde{c}_{g+1}$ with the orientation generated by $\widetilde{\Gamma}^*$. We consider the standard system of generators $(c_1,\dots,c_{g+1})$ of the group $\pi(P_1,p)$ that is associated with these contours and identify the c_i's with the standard generators of the group $\gamma_{0,g+1}$. Then the natural isomorphism $\pi_1(P,p)\to\Gamma$ induces an element $\psi\in\widetilde{T}_{0,g+1}$. We can readily see that $\Gamma^k_\psi=\widetilde{\Gamma}$.

Let us find $\omega_{\widetilde{\Gamma}^*}(d_i)$. We set $\widehat{C}^*_i=J^{-1}(\widehat{C}_i)\cap\widetilde{\Gamma}^*$. Replacing $\widetilde{\Gamma}$ by a conjugate group, we may assume that

$$\widehat{C}_{g+1}=\sigma_{g+1}\begin{pmatrix}-\mu_{g+1} & 0\\ 0 & \mu_{g+1}^{-1}\end{pmatrix},$$

where $\mu_{g+1}>0$ and $\sigma_{g+1}=\pm1$. The shifts $C_1,\dots,C_{g+1}$ form a sequential set (see § 2), and hence the invariant lines $\ell(C_i)$ are arranged as in Fig. 11.1.

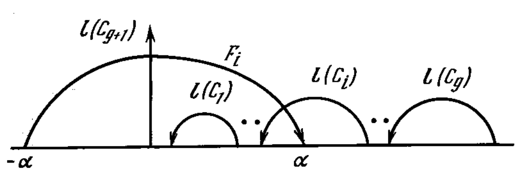

Figure 11.1

Thus,

$$\overline{C}_i = F_i \overline{C}_{g+1} F_i^{-1},$$

where

$$F_i = \frac{(\lambda_i\alpha_i + \alpha_i)z - (1-\lambda_i)\alpha_i^2}{(1-\lambda_i)z + (\alpha_i + \lambda_i\alpha_i)} = \frac{\alpha_i(\lambda_i+1)z + (\lambda_i - 1)\alpha_i^2}{(1-\lambda_i)z + \alpha_i(\lambda_i+1)}.$$

We set

$$F_i^* = \begin{pmatrix} \alpha_i(\lambda_i+1) & (\lambda_i-1) \\ (1-\lambda_i) & \alpha_i(\lambda_i+1) \end{pmatrix}.$$

Then

$$\widehat{C}_i^* = \sigma_i F_i^* \begin{pmatrix} -\mu_i & 0 \\ 0 & \mu_i^{-1} \end{pmatrix} (F_i^*)^{-1},$$

where $\mu_i > 0$ and $\sigma_i = \pm 1$. Let us prove that $\sigma_i = -1$ for $i \leqslant g$. Indeed, by construction, the orientation generated by $\widetilde{\Gamma}^*$ on the contour c_i coincides with its orientation as a part of the boundary of the surface P_1. This orientation induces the orientation of the line $\ell(C_i)$ indicated in Fig. 11.1. The map F^{-1} sends it into the orientation of the imaginary axis I in the direction in which the values $\operatorname{Im} z$ decrease (see Fig. 11.1). This means exactly that $\sigma_i = -1$.

Thus,

$$\widehat{C}_{g+1}^* \widehat{C}_i^* = -\sigma_{g+1} \begin{pmatrix} \mu_{g+1} & 0 \\ 0 & \mu_{g+1}^{-1} \end{pmatrix} F_i^* \begin{pmatrix} \mu_i & 0 \\ 0 & \mu_i^{-1} \end{pmatrix} (F_i^*)^{-1},$$

and hence

$$\omega_{\Gamma^*}(d_i) = \operatorname{sgn}\big(\operatorname{Tr}(D_i)\big) = \operatorname{sgn}\big(\operatorname{Tr}(\widehat{C}_{g+1}^* \widehat{C}_i^*)\big) = -\sigma_{g+1};$$

therefore, $\omega_{\widetilde{\Gamma}^*}(d_i)$ is the same for all $i \leqslant g$.

Lemma 11.2. *Let ω be a non-singular Arf function on a real algebraic curve (P,τ) of type $(g,k,0)$. Then there is a standard basis*

$$\{c_i, d_i \ (i = 1, \dots, g)\} \in H_1(P, \mathbb{Z}_2)$$

such that $c_i, \dots, c_g$ are pairwise disjoint invariant contours, $\tau(d_i) = d_i + c_{g+1} + \widetilde{c}_i$, where

$$\widetilde{c}_i = \begin{cases} 0 & \text{for } i \leqslant k, \\ c_i & \text{for } i > k, \end{cases}$$

and $\omega(d_i) = \omega(d_j)$ for any $i, j \leqslant g$.

Proof. By Lemma 2.1, there is a real Fuchsian group $\widetilde{\Gamma}$ such that $(P,\tau) = [\widetilde{\Gamma}]$. By Lemma 4.2, there is a lifting $\widetilde{\Gamma}^*$ of $\widetilde{\Gamma}$ such that $\omega_{\widetilde{\Gamma}^*} = \omega$. Therefore, the assertion of Lemma 11.2 follows from Lemma 11.1.

2. Let (P,τ) be a real algebraic curve. Arf functions ω_1 and ω_2 on (P,τ) are said to be *topologically equivalent* if there is a homeomorphism $\varphi\colon P \to P$ such that $\varphi\tau = \tau\varphi$ and the induced automorphism $\varphi\colon H_1(P,\mathbb{Z}_2) \to H_1(P,\mathbb{Z}_2)$ satisfies the condition $\omega_1 = \omega_2\varphi$.

Theorem 11.1 [41]**.** *All singular Arf functions on an arbitrary real curve* (P,τ) *are topologically equivalent .*

Proof. We have $P^\tau = \varnothing$ by Lemma 3.2. Therefore, by Lemma 1.2, there is a set of pairwise disjoint, invariant contours $c_1,\dots,c_{g+1}$ such that $P\setminus\bigcup_{i=1}^{g+1} c_i$ decomposes into two spheres P_1 and P_2 with holes. We join any contour c_i to the contour c_{g+1} by a simple segment $\ell_i \subset P_1$. Let us consider a simple closed contour $d_i = \ell_i\cup\tau\ell_i\cup r_i\cup r_{g+1}$, where $r_j\subset c_j$. Let ω_1 and ω_2 be singular Arf functions on (P,τ). By Lemma 3.3, $\omega_1(c_i)=\omega_2(c_i)=0$. For any i with $\omega_1(d_i)\neq\omega_2(d_i)$ we apply to P the Dehn twist along c_i, that is, cut P along c_i and paste together along the same contour after a rotation of 2π. We can readily see that the homeomorphism φ thus obtained commutes with τ. On the other hand, for such i we have

$$\omega_2(\varphi(d_i)) = \omega_2(d_i+c_i) = \omega_2(d_i)+\omega(c_i)+1 = \omega_1(d_i).$$

Thus, $\omega_2\varphi=\omega_1$.

Theorem 11.2 [41]**.** *Let* (P,τ) *be a real algebraic curve of type* $(g,k,0)$*. Then non-singular Arf functions on* (P,τ) *are topologically equivalent if and only if they have the same topological type* (g,δ,k_α)*.*

Proof. Let ω_1 and ω_2 be non-singular Arf functions on (P,τ) of type (g,δ,k_α). Using Lemma 11.2, we associate with the Arf function ω_m a standard basis $\{c_i^m, d_i^m$ $(i=1,\dots,g)\}$, where the c_i^m are pairwise disjoint invariant contours and $\omega_m(d_i^m)=\omega_m(d_j^m)$ for any $i,j\leqslant g$. After renumbering, we may assume that $c_1^m,\dots,c_{k_0+k_1}^m$ are ovals and that $\omega_m(c_i^m)=0$ for $i\leqslant k_0$ and $\omega_m(c_i^m)=1$ for $i>k_0$. By Theorem 3.2 we have $k_0\equiv g+1 \pmod 2$, and hence

$$\delta(P,\omega_m) = \sum_{j=1}^{g+1}\omega_m(c_j^m)\omega_m(d_j^m) = \sum_{j=k_0+1}^{g}\omega_m(d_j^m) = \omega_m(d_j^m).$$

Thus, $\omega_1(d_j^1)=\delta=\omega_2(d_j^2)$. By Lemma 1.2, the set $c_1^m,\dots,c_g^m$ can be supplemented by a contour c_{g+1}^m to form a complete set of invariant contours. Let

$$P_1^m\cup P_2^m = P\setminus\bigcup_{i=1}^{g+1} c_i^m.$$

We choose c_{g+1}^m so that the homeomorphism $\varphi\colon P_1^1\to P_1^2$ can be extended to a homeomorphism $\varphi\colon P^1\to P^2$ that sends $\{c_i^1,d_i^1\}$ into $\{c_i^2,d_i^2\}$ and commutes with τ. Then $\omega_1=\omega_2\varphi$.

Theorem 11.3 [41]**.** *Let* (P,τ) *be a real algebraic curve of type* $(g,k,1)$*. Then Arf functions on* (P,τ) *are topologically equivalent if and only if they have the same topological type* $(g,\widetilde{\delta},k_\alpha^\gamma)$*.*

Proof. Let ω_1 and ω_2 be Arf functions on (P,τ) of type $(g,\widetilde{\delta},k_\alpha^\gamma)$. The ovals of P^τ decompose P into two connected components P_1 and P_2. By assumption, $\omega_1\big|_{P_1}$ and $\omega_2\big|_{P_1}$ have the same topological type, and hence by [47], § 8, there is a homeomorphism $\varphi_1 \colon P_1 \to P_1$ that sends $\omega_1\big|_{P_1}$ into $\omega_2\big|_{P_1}$. Since the topological types of ω_1 and ω_2 coincide, we can choose φ_1 such that ovals similar with respect to ω_1 pass to ovals similar with respect to ω_2. We now set $\varphi_2 = \tau\varphi_1\tau \colon P_2 \to P_2$. Then $\varphi_1 \cup \varphi_2 \colon P \to P$ commutes with τ and sends ω_1 into ω_2.

3. In the rest of this section a spinor bundle is understood to mean a rank-one spinor bundle.

Theorem 5.1 establishes a one-to-one correspondence between spinor bundles on a real algebraic curve (P,τ) and real Arf functions on this curve. By the *type of a spinor bundle* we mean the type of the corresponding Arf function.

By the *moduli space* of spinor bundles on real algebraic curves we mean the space of pairs $((P,\tau),(e,E))$, where (P,τ) is a real algebraic curve and (e,E) is a spinor bundle on (P,τ). By Theorem 5.1, there are only finitely many spinor bundles on a real curve, and therefore the topology of the moduli space of real curves induces a topology in the moduli space of spinor bundles on real curves.

Theorem 11.4 [36]. *The space of spinor bundles on non-separating real algebraic curves decomposes into the connected components $S_p(g,\delta,k_\alpha)$, where (g,δ,k_α) is an arbitrary topological type of a non-singular Arf function on a non-separating real curve. Each of the components $S(g,\delta,k_\alpha)$ is diffeomorphic to*

$$\mathbb{R}^{3g-3}/\operatorname{Mod}_{g,\delta,k_\alpha}$$

(*where* $\operatorname{Mod}_{g,\delta,k_\alpha}$ *is a discrete group of diffeomorphisms*) *and is a* $\binom{k}{k_0}\cdot 2^{g-1}$*-sheeted covering of* $M_{g,k,0}$*, where* $k = k_0 + k_1$.

Proof. By definition, to any $\psi \in \widetilde{T}_{0,g+1}$ there corresponds a sequential set

$$V = (C_1,\dots,C_{g+1}) \in \operatorname{Aut}(\Lambda)$$

of type $(0,g+1)$ which, together with

$$\widehat{C}_i = \begin{cases} \overline{C}_i & \text{for } i \leqslant k, \\ \widetilde{C}_i & \text{for } i > k, \end{cases}$$

generates a real Fuchsian group $\widetilde{\Gamma} = \Gamma^k_{\psi,g+1}$. On a real curve $(P,\tau) = [\widetilde{\Gamma}]$, we consider a homology basis $\{c_i, d_i\ (i = 1,\dots,g)\} \in H_1(P,\mathbb{Z}_2)$ that corresponds to the shifts $\{C_i, D_i = \widetilde{C}_{g+1}\widehat{C}_i\ (i = 1,\dots,g)\}$. We introduce a non-singular real Arf function $\omega = \omega_\psi$ defined by the conditions $\omega(c_i) = 0$ for $i \leqslant k_0$, $\omega(c_i) = 1$ for $i > k_0$, $\omega(d_i) = 0$ for $i < g$, and $\omega(d_g) = \delta$. By Theorem 3.2 we have $k_0 \equiv g+1 \pmod 2$, which immediately implies that ω is a non-singular real Arf function of type (g,δ,k_α). By Theorem 5.1, a spinor bundle $\Omega(\psi) \in S_p(g,\delta,k_\alpha)$ is associated with this Arf function. The correspondence $\psi \mapsto \Omega(\psi)$ induces a map $\Omega \colon T_{0,g+1} \to S_p(g,\delta,k_\alpha)$.

Let us prove that $\Omega(T_{0,g+1}) = S_p(g,\delta,k_\alpha)$. Indeed, by Theorem 2.2, the map

$$\Psi \colon T_{0,g+1} \xrightarrow{\Omega} S_p(g,\delta,k_\alpha) \xrightarrow{\Phi} M_{g,k,0},$$

where Φ is the natural projection, satisfies the condition

$$\Psi(T_{0,g+1}) = M_{g,k,0}.$$

The fibre of the map Ψ is represented by the group $\mathrm{Mod}_{g,k,0}$ of all autohomeomorphisms of the curve (P,τ), that is, the autohomeomorphisms of P that commute with τ. By Theorem 11.2, this group $\mathrm{Mod}_{g,k,0}$ acts transitively on the set of non-singular real Arf functions of type (g,δ,k_α) and hence, by Theorem 5.1, transitively on the fibres $\Phi^{-1}((P,\tau))$. Thus,

$$\Omega(T_{0,g+1}) = S_p(g,\delta,k) \quad \text{and} \quad S_p(g,\delta,k_\alpha) = T_{0,g+1}/\,\mathrm{Mod}_{g,\delta,k_\alpha},$$

where

$$\mathrm{Mod}_{g,\delta,k_\alpha} \subset \mathrm{Mod}_{g,k,0}\,.$$

By [47], § 4, the space $T_{0,g+1}$ is diffeomorphic to $\mathbb{R}^{3g-3}$. By Theorem 3.2, the index of the subgroup $\mathrm{Mod}_{g,\delta,k_\alpha}$ in $\mathrm{Mod}_{g,k_0+k_1,0}$ is equal to $\binom{k}{k_0}\cdot 2^{g-1}$.

Theorem 11.5 [36]. *The space of spinor bundles on separating real algebraic curves decomposes into connected components $S_p(g,\widetilde{\delta},k_\alpha^\gamma)$, where $(g,\widetilde{\delta},k_\alpha^\gamma)$ is an arbitrary topological type of an Arf function on a separating real algebraic curve. Each of the components $S_p(g,\widetilde{\delta},k_\alpha^\gamma)$ is diffeomorphic to $\mathbb{R}^{3g-3}/\,\mathrm{Mod}_{g,\widetilde{\delta},k_\alpha^\gamma}$ (where $\mathrm{Mod}_{g,\widetilde{\delta},k_\alpha^\gamma}$ is a discrete group of diffeomorphisms) and covers $M_{g,k,1}$ with $\binom{k}{k_0}\cdot\binom{k_0}{k_0^0}\cdot\binom{k_1}{k_1^0}\cdot 2^{\widetilde{g}-2}\cdot(2^{\widetilde{g}}+m)$ sheets, where $m=2^{\widetilde{g}}$ for $k_1>0$, $m=1$ for $\widetilde{\delta}=0$, $m=-1$ for $k_1=0$ and $\widetilde{\delta}=1$, and $k_\alpha=k_\alpha^0+k_\alpha^1$, $k=k_0+k_1$, and $g=2\widetilde{g}+k-1$.*

Proof. By definition, to each $\psi\in\widetilde{T}_{\widetilde{g},k}$ there corresponds a sequential set $V=\{A_i,B_i\ (i=1,\dots,\widetilde{g}), C_i\ (i=1,\dots,k)\}$ of type $(\widetilde{g},k)$ which, together with $\overline{C}_i$ $(i=1,\dots,k)$, generates a real Fuchsian group $\widetilde{\Gamma}=\Gamma_\psi^k$. On a real curve $(P,\tau)=[\widetilde{\Gamma}]$ we consider a homology basis $\{a_i,b_i,a_i',b_i'\ (i=1,\dots,\widetilde{g}),\ c_i,d_i\ (i=1,\dots,k-1)\}\in H_1(P,\mathbb{Z}_2)$ generated by the shifts

$$\{A_i,B_i,\overline{C}_kA_i\overline{C}_k,\overline{C}_kB_i\overline{C}_k\ (i=1,\dots,\widetilde{g}),\ C_i,\overline{C}_k\overline{C}_i\ (i=1,\dots,k-1)\}.$$

To be definite, let $k_1^1>0$ (the other cases can be treated similarly). We consider the real Arf function $\omega=\omega_\psi$ determined by the following conditions: 1) $\omega(a_i)=\omega(b_i)=\omega(a_i')=\omega(b_i')=\varepsilon_i$, where $\varepsilon_i=0$ for $i<\widetilde{g}$ and $\varepsilon_i=\widetilde{\delta}$ for $i=\widetilde{g}$; 2) $\omega(c_i)=0$ for $i\leqslant k_0$ and $\omega(c_i)=1$ for $i>k_0$; 3) $\omega(d_i)=0$ for $i=k_0^0+1,\dots,k_0$ and for $i=k_0+k_1^0+1,\dots,k-1$ and $\omega(d_i)=1$ otherwise. By Theorem 5.1, a spinor bundle $\Omega(\psi)\in S_p(g,\widetilde{\delta},k_\alpha^\gamma)$ corresponds to this Arf function. The rest of the proof coincides almost literally with the corresponding part of the proof of Theorem 11.4.

§ 12. Real algebraic $N=1$ supercurves, and their moduli space

1. We recall some definitions (see [4] and [47], § 11).

Let $L=L(\mathbb{K})$ be the Grassmann algebra with infinitely many generators $1,\ell_1,\ell_2,\dots$ over a field $\mathbb{K}$. Each of the elements $a\in L(\mathbb{K})$ is a finite linear combination of monomials $\ell_{i_1}\wedge\dots\wedge\ell_{i_n}$ with coefficients in $\mathbb{K}$, that is,

$$a = a^{\#} + \sum a_ie_i + \sum_{ij} a_{ij}e_i\wedge e_j + \cdots\,.$$

The correspondence $a\mapsto a^{\#}$ defines an epimorphism $\#\colon L(\mathbb{K})\to\mathbb{K}$.

A monomial $\ell_{i_1} \wedge \dots \wedge \ell_{i_n} \neq 0$ is said to be *even* if n is even and *odd* if n is odd. The constants are also regarded as even monomials. The linear combinations of even (odd) monomials with coefficients in $\mathbb{K}$ form the linear space $L_0(\mathbb{K})$ of even (the linear space $L_1(\mathbb{K})$ of odd) elements of the algebra $L(\mathbb{K})$. The superanalogue of a linear space is the set

$$\mathbb{K}^{(n|m)} = \bigl\{(z_1, \dots, z_n \,|\, \theta_1, \dots, \theta_m) \colon z_i \in L_0(\mathbb{K}),\ \theta_j \in L_1(\mathbb{K})\bigr\}.$$

For the field $\mathbb{K}$ we take the field $\mathbb{C}$ of complex numbers or the field $\mathbb{R}$ of real numbers.

The set

$$\Lambda^{NS} = \bigl\{(z \,|\, \theta_1, \dots, \theta_N) \in \mathbb{C}^{(1|N)} \mid \operatorname{Im} z^{\#} > 0\bigr\}$$

is called the *upper N super half-plane*. In this section we deal with the 1 super half-planes $\Lambda^S = \Lambda^{1S}$. The group $\operatorname{Aut}(\Lambda^S)$ of automorphisms of the super domain Λ^S consists of transformations $A = A[a, b, c, d, \sigma \,|\, \varepsilon, \delta]$ of the form

$$A(z \,|\, \theta) = \left(\frac{az+b}{cz+d} - \frac{(ad-bc)(\varepsilon+\delta z)}{(cz+d)^2}\theta,\ \frac{\sigma\sqrt{ad-bc}}{cz+d}\left(\theta + \varepsilon + \delta z + \frac{1}{2}\varepsilon\delta\theta\right)\right),$$

where $a, b, c, d \in L_0(\mathbb{R})$, $\sigma = \pm 1$, $\varepsilon, \delta \in L_1(\mathbb{R})$, $(ad-bc)^{\#} > 0$, and the symbol $\sqrt{\Delta}$ stands for an element of $L_0(\mathbb{R})$ that is uniquely determined by the properties $(\sqrt{\Delta}\,)^2 = \Delta$ and $(\sqrt{\Delta}\,)^{\#} > 0$.

The correspondence

$$A \mapsto A^{\#}, \quad \text{where} \quad A^{\#}(z) = \frac{a^{\#}z + b^{\#}}{c^{\#}z + d^{\#}},$$

generates an epimorphism

$$\#\colon \operatorname{Aut}(\Lambda^S) \to \operatorname{Aut}(\Lambda).$$

The transformations that are mapped by this epimorphism into hyperbolic transformations are said to be *superhyperbolic*.

With an automorphism $A = A[a, b, c, d, \sigma \,|\, \varepsilon, \delta]$ we associate the matrix

$$\overline{J}(A) = \frac{\sigma}{\sqrt{a^{\#}d^{\#} - c^{\#}d^{\#}}} \begin{pmatrix} a^{\#} & b^{\#} \\ c^{\#} & d^{\#} \end{pmatrix} \in SL(2, \mathbb{R}).$$

A subgroup $\Gamma \subset \operatorname{Aut}(\Lambda^S)$ is said to be *super Fuchsian* if $\Gamma^{\#} = \#(\Gamma)$ is a Fuchsian group and $\#\colon \Gamma \to \Gamma^{\#}$ is an isomorphism. In this section we study (unless otherwise stated) only super Fuchsian groups that consist of superhyperbolic automorphisms of Λ^S.

The quotient set $P = \Lambda^S/\Gamma$ is called an ($N = 1$) *Riemann supersurface* (or a *super Riemann surface*). The correspondence $\overline{J}$ generates a lifting

$$J^* \colon \Gamma^{\#} \to \Gamma^* \subset SL(2, \mathbb{R}).$$

The type of the corresponding Arf function $\omega_\Gamma = \omega_{\Gamma^*}$ on $P^\# = \Lambda/\Gamma^\#$ is called the *type of the supersurface*.

2. We now let $\widetilde{\mathrm{Aut}}(\Lambda^S)$ be the group generated by $\mathrm{Aut}(\Lambda^S)$ together with the involutions

$$\sigma_\pm \colon (z \mid \theta) \mapsto (-\overline{z} \mid \pm\overline{\theta}).$$

If $C \in \mathrm{Aut}(\Lambda^S)$ is a hyperbolic automorphism, then there is an element $g \in \mathrm{Aut}(\Lambda^S)$ such that $g^{-1}Cg(z) = (\lambda z \mid \sqrt{\lambda}\,\theta)$, where $\lambda^\# > 0$. We set

$$\overline{C}^{\pm} = g\sigma_\pm g^{-1}, \qquad \widetilde{C}^{\pm} = \sqrt{C}\,\overline{C}_{\pm} \in \widetilde{\mathrm{Aut}}(\Lambda^S),$$

where $g^{-1}\sqrt{C}\,g(z \mid \theta) = (\sqrt{\lambda}\,z \mid \sqrt[4]{\lambda}\,\theta)$. Let us extend $\#\colon \mathrm{Aut}(\Lambda^S) \to \mathrm{Aut}(\Lambda)$ to a map $\#\colon \widetilde{\mathrm{Aut}}(\Lambda^S) \to \widetilde{\mathrm{Aut}}(\Lambda)$ by setting $\#(\sigma_\pm) = \sigma_\pm^\# \colon z \mapsto -\overline{z}$.

A subgroup $\widetilde{\Gamma} \subset \widetilde{\mathrm{Aut}}(\Lambda^S)$ is said to be a *real super Fuchsian group* if $\widetilde{\Gamma}^\#$ is a real Fuchsian group. To a real super Fuchsian group $\widetilde{\Gamma}$ there correspond the super Fuchsian group $\Gamma = \widetilde{\Gamma} \cap \mathrm{Aut}(\Lambda^S)$, the Riemann supersurface $P = \Lambda^S/\Gamma$, and the *real algebraic supercurve* $[\Gamma] = (P, \tau)$, where $\tau = (\widetilde{\Gamma} \setminus \Gamma)/\Gamma \colon P \to P$ is a superantiholomorphic involution. Corresponding to the supercurve (P, τ) is the real algebraic curve

$$\#(P, \tau) = (P^\#, \tau^\#) = [\widetilde{\Gamma}^\#],$$

called the *substructure of the supercurve* (P, τ). We can readily see that ω_Γ is a real Arf function on $(P^\#, \tau^\#)$. Its topological type is called the *topological type of the real supercurve* (P, τ).

3. Let $t = (\widetilde{g}, \delta, k_\alpha)$ be the topological type of a Riemann supersurface of genus $\widetilde{g}$ with k holes. Denote by M^t the set of all such supersurfaces. By [47], § 12, it is "uniformized" by the space

$$T^t = \widetilde{T}^t / \mathrm{Aut}(\Lambda^S),$$

where $\widetilde{T}^t$ is the space of monomorphisms $\psi\colon \gamma_{\widetilde{g},n} \to \mathrm{Aut}(\Lambda^S)$ (where $n = \widetilde{g} + k$) such that $\psi(v_{\widetilde{g},n})^\#$ is a sequential set of type (g, k) and $\Lambda^S/\psi(\gamma_{\widetilde{g},n}) \in M^t$, and the group $\mathrm{Aut}(\Lambda^S)$ acts by conjugations.

A set Q is said to be *strongly diffeomorphic to* $\mathbb{R}^{(p,q)}$ if there is an embedding $Q \subset \mathbb{R}^{(p|q)}$ such that $Q^\#$ is diffeomorphic to $\mathbb{R}^p$ and $Q = \#^{-1}(\#(Q))$. By [47], § 12, T^t is strongly diffeomorphic to

$$\mathbb{R}^{(p|q)}/\mathbb{Z}_2 = \mathbb{R}^{(6\widetilde{g}+3k-6|4\widetilde{g}+2k-4)}/\mathbb{Z}_2.$$

Moreover,

$$M^t = T^t / \mathrm{Mod}_t,$$

where Mod_t is a discrete group.

Theorem 12.1 ([36], [38])**.** *The moduli space of real algebraic supercurves with non-separating substructure decomposes into connected components of the form* $S(g, \delta, k_\alpha)$, *where* (g, δ, k_α) *is an arbitrary topological type of a non-singular Arf* *function on a non-separating real curve. Each of the components has a representation*

$$S(g,\delta,k_\alpha) = T_{g,\delta,k_\alpha} / \operatorname{Mod}_{g,\delta,k_\alpha},$$

where T_{g,δ,k_α} *is strongly diffeomorphic to* $\mathbb{R}^{(3g-3|2g-2)}/\mathbb{Z}_2$ *and* $\operatorname{Mod}_{g,\delta,k_\alpha}$ *is a discrete group.*

Proof. We set $t=(0,0,k_0,g+1-k_0)$. By definition, to any $\psi \in \widetilde{T}^t$ there corresponds a set

$$V=(C_1,\dots,C_{g+1}) \in \operatorname{Aut}(\Lambda^S)$$

such that $V^\# = (C_1^\#,\dots,C_{g+1}^\#)$ is a sequential set of type $(0,g+1)$. This set, together with

$$\widehat{C}_i = \begin{cases} \overline{C}_i^{\,+} & \text{for } i \leqslant k, \\ \widetilde{C}_i^+ & \text{for } k<i<g, \\ \widetilde{C}_g^+ & \text{for } i=g,\ \delta=0, \\ \widetilde{C}_i^- & \text{for } i=g,\ \delta=1, \end{cases}$$

(where $k=k_0+k_1$) generates a real super Fuchsian group $\widetilde{\Gamma}$. On the real curve $(P^\#,\tau^\#)=[\widetilde{\Gamma}^\#]$, we consider a homology basis

$$(c_i,d_i\ (i=1,\dots,g)) \in H_1(P,\mathbb{Z}_2)$$

that corresponds to the shifts

$$(C_i, D_i = \widetilde{C}_{g+1}\widehat{C}_i\ (i=1,\dots,g)).$$

In this case the Arf function $\omega=\omega_\Gamma$ satisfies the conditions $\omega(c_i)=0$ for $i\leqslant k_0$, $\omega(c_i)=1$ for $i>k_0$, $\omega(d_i)=0$ for $i<g$, and $\omega(d_g)=\delta$. Thus, the correspondence $\psi \mapsto [\widetilde{\Gamma}]$ induces a map

$$\Omega\colon (\widetilde{T}^t) \to S(g,\delta,k_\alpha).$$

Under this map conjugate ψ's are mapped into the same supercurves, and hence a map

$$\Omega\colon T^t \to S(g,\delta,k_\alpha)$$

is well defined.

We prove that $\Omega(T^t)=S(g,\delta,k_\alpha)$. Let

$$(P,\tau) \in S(g,\delta,k_\alpha).$$

It follows from Lemma 1.2 and Theorem 11.2 that there are simple closed contours $\{c_i,d_i\ (i=1,\dots,g)\}$ on $(P^\#,\tau^\#)$ such that: 1) $\tau^\#(c_i)=c_i$ and $(P^\#)^{\tau^\#}=\bigcup_{i=1}^k c_i$; 2) the elements of $H_1(P^\#,\mathbb{Z}_2)$ representing these contours satisfy the conditions $\tau^*(d_i)=-d_i+c+\widehat{c}_i$, where $c=\sum_{i=1}^g c_i$ and

$$\widehat{c}_i = \begin{cases} 0 & \text{for } i\leqslant k, \\ c_i & \text{for } i>k; \end{cases}$$

3) the Arf function $\omega = \omega_\Gamma$ satisfies the conditions

$$\omega(c_i) = \begin{cases} 0 & \text{for } i \leqslant k_0, \\ 1 & \text{for } i > k_0, \end{cases}$$

$\omega(d_i) = 0$ for $i < g$, and $\omega(d_g) = \delta$. The contours $\{c_i\}$ decompose the surface $P^{\#}$ into components $P_1^{\#}$ and $P_2^{\#}$. We set $P_1 = \#^{-1}(P_1^{\#})$. By [47], § 12, we have $P_1 = \Lambda^s/\psi(\gamma_{0,g+1})$, where $\psi \in \widetilde{T}^t$. It follows immediately from our constructions that $\Omega(\psi) = (P, \tau)$, and $\Omega(\psi') = \Omega(\psi)$ if and only if $\psi' = \psi\alpha$, where $\alpha \in \mathrm{Mod}_{g,\delta,k_\alpha}$, and $\mathrm{Mod}_{g,\delta,k_\alpha}$ stands for the group in Theorem 11.5.

Theorem 12.2 ([36], [38]). *The moduli space of real algebraic supercurves with separating substructure decomposes into connected components $S(g,\widetilde{\delta},k_\alpha^\gamma)$ that correspond to arbitrary topological types $t = (g,\widetilde{\delta},k_\alpha^\gamma)$ of Arf functions on separating real curves. Each of the components is of the form*

$$T^t/\,\mathrm{Mod}_{g,\widetilde{\delta},k_\alpha^\gamma},$$

where T^t is strongly diffeomorphic to $\mathbb{R}^{(3g-3|2g-2)}/\mathbb{Z}_2$ and $\mathrm{Mod}_{g,\widetilde{\delta},k_\alpha^\gamma}$ is a discrete group.

Proof. We set $k_0 = k_0^0 + k_0^1$, $k_1 = k_1^0 + k_1^1$, $k = k_0 + k_1$, $\widetilde{g} = \frac{1}{2}(g+1-k)$, and $t = (\widetilde{g},\widetilde{\delta},k_\alpha)$. By definition, to any $\psi \in \widetilde{T}^t$ there corresponds a set $V = (A_i, B_i\ (i=1,\dots,\widetilde{g}),\ C_i\ (i=1,\dots,k)) \subset \mathrm{Aut}(\Lambda^S)$ such that $V^{\#} = \{A_i^{\#}, B_i^{\#}\ (i=1,\dots,\widetilde{g}),\ C_i^{\#}(i=1,\dots,k)\}$ is a sequential set of type $(\widetilde{g},k)$. Together with

$$\widehat{C}_i = \begin{cases} \overline{C}_j^{\,+} & \text{for } i \leqslant k_0^0 \text{ and for } k_0 < i \leqslant k_0 + k_1^0, \\ \overline{C}_i^{\,-} & \text{for } k_0^0 < i \leqslant k_0 \text{ and for } i > k_0 + k_1^0, \end{cases}$$

the set V generates a real super Fuchsian group $\widetilde{\Gamma}$. The correspondence $\psi \mapsto [\widetilde{\Gamma}]$ defines a map $\Omega\colon T^t \to S(g,\widetilde{\delta},k_\alpha^\gamma)$. The rest of the proof repeats the corresponding part of the proof of Theorem 12.1 with obvious modifications.

§ 13. Real algebraic $N=2$ supercurves

1. We recall some definitions of [29] and [47], § 13. By $A[a,b,c,d,\ell\,|\,\varepsilon]$ we denote a map $A\colon \Lambda^{2S} \to \Lambda^{2S}$ of the form

$$A(z\,|\,\theta_1,\theta_2) = \biggl(\frac{az+b+\delta^{11}\theta_1+\delta^{12}\theta_2}{cz+d+\delta^{21}\theta_1+\delta^{22}\theta_2}\;\bigg|\;\frac{\ell^{11}\theta_1+\ell^{12}\theta_2+\varepsilon^{11}z+\varepsilon^{12}}{cz+d+\delta^{21}\theta_1+\delta^{22}\theta_2}, \frac{\ell^{21}\theta_1+\ell^{22}\theta_2+\varepsilon^{21}z+\varepsilon^{22}}{cz+d+\delta^{21}\theta_1+\delta^{22}\theta_2}\biggr),$$

where $a,b,c,d \in L_0(\mathbb{R})$, $\ell \in GL(2,L_0(\mathbb{R}))$, and $\varepsilon^{ij},\delta^{ij} \in L_1(\mathbb{R})$.

According to [29], the automorphism group $\mathrm{Aut}(\Lambda^{2S})$ of the super domain Λ^{2S} consists of $A[a,b,c,d,\ell\,|\,\varepsilon]$, where

$$\begin{pmatrix} -c & a \\ -d & b \end{pmatrix}\begin{pmatrix} \delta^{11} & \delta^{12} \\ \delta^{21} & \delta^{22} \end{pmatrix} = \begin{pmatrix} \varepsilon^{21} & \varepsilon^{11} \\ \varepsilon^{22} & \varepsilon^{12} \end{pmatrix}\begin{pmatrix} \ell^{11} & \ell^{12} \\ \ell^{21} & \ell^{22} \end{pmatrix}$$

and

$$ad - bc - \varepsilon^{11}\varepsilon^{12} - \varepsilon^{21}\varepsilon^{22} = \ell^{11}\ell^{22} + \ell^{21}\ell^{12} + \delta^{11}\delta^{22} + \delta^{12}\delta^{21} = \Delta,$$

where $\Delta^{\#} > 0$, and

$$\ell^{11}\ell^{21} + \delta^{11}\delta^{21} = \ell^{12}\ell^{22} + \delta^{12}\delta^{22} = 0.$$

It can be shown by direct calculation that any automorphism $A[a,b,c,d,\ell \,|\, \varepsilon]$ is of one of the two types

1) (non-twisted) $(\ell^{12})^{\#} = (\ell^{21})^{\#} = 0$, $(\ell^{11}\ell^{22})^{\#} > 0$,
2) (twisted) $(\ell^{11})^{\#} = (\ell^{22})^{\#} = 0$, $(\ell^{12}\ell^{21})^{\#} > 0$.

A non-twisted (twisted) automorphism is uniquely determined by the parameters a, b, c, d, ε^{ij}, ℓ^{11} (by the parameters $a,b,c,d,\varepsilon^{ij},\ell^{12}$, respectively).These parameters can take arbitrary values such that $a,b,c,d,\ell^{ij} \in L_0(\mathbb{R})$, $\varepsilon^{ij} \in L_1(\mathbb{R})$, $(ad-bc)^{\#} > 0$, and $(\ell^{11} + \ell^{12})^{\#} \neq 0$.

The correspondence $A \mapsto A^{\#}$, where

$$A = A[a,b,c,d,\ell \,|\, \varepsilon],$$
$$A^{\#}(z) = \frac{a^{\#}z + b^{\#}}{c^{\#}z + d^{\#}},$$

generates an epimorphism $\#\colon \operatorname{Aut}(\Lambda^{2S}) \to \operatorname{Aut}(\Lambda)$. A transformation that is mapped into a hyperbolic transformation under this epimorphism is said to be *superhyperbolic*.

A subgroup $\Gamma \subset \operatorname{Aut}(\Lambda^{2S})$ is called an $N = 2$ *super Fuchsian group* if $\Gamma^{\#} = \#(\Gamma)$ is a Fuchsian group and $\#\colon \Gamma \to \Gamma^{\#}$ is an isomorphism. Unless otherwise stated, in this section we treat only $N = 2$ super Fuchsian groups that consist of superhyperbolic automorphisms of Λ^{2S}.

With an automorphism $A = A[a,b,c,d,\ell \,|\, \varepsilon]$ we associcate the matrix

$$\overline{J}(A) = \frac{\sigma}{\sqrt{a^{\#}d^{\#} - b^{\#}c^{\#}}} \begin{pmatrix} a^{\#} & b^{\#} \\ c^{\#} & d^{\#} \end{pmatrix} \in SL(2,\mathbb{R}),$$

where $\sigma = \sigma(A) = \operatorname{sgn}(\ell^{11} + \ell^{12} + \ell^{21} + \ell^{22})^{\#}$.

If $\Gamma \in \operatorname{Aut}(\Lambda^{2S})$ is an $N = 2$ super Fuchsian group, then the correspondence $\overline{J}\colon \Gamma \to SL(2,\mathbb{R})$ is a monomorphism, and hence defines a lifting $J^*\colon \Gamma^{\#} \to \overline{J}(\Gamma)$.

Let $\Gamma \subset \operatorname{Aut}(\Lambda^{2S})$ be an $N = 2$ super Fuchsian group. The quotient set Λ^{2S}/Γ is called a *Riemann* $N = 2$ *supersurface* or an $N = 2$ *super Riemann surface*. Two $N = 2$ supersurfaces $P_1 = \Lambda^{2S}/\Gamma_1$ and $P_2 = \Lambda^{2S}/\Gamma_2$ are assumed to be equal if Γ_1 and Γ_2 are conjugate in $\operatorname{Aut}(\Lambda^{2S})$. The projections $\#\colon \Lambda^{2S} \to \Lambda$ and $\#\colon \Gamma \to \Gamma^{\#}$ determine a projection $\#\colon P \to P^{\#} = \Lambda/\Gamma^{\#}$.

By [47], § 7, the lifting J^* defines an Arf function

$$\omega_P^1\colon H_1(P^{\#},\mathbb{Z}_2) \to \mathbb{Z}_2.$$

Let us introduce functions $\Omega_i = \Omega_i(\Gamma)\colon \Gamma \to \mathbb{Z}_2 = \{0,1\}$ $(i = 1,2)$ by setting

$$\Omega_1(A) = \begin{cases} 0 & \text{for } \sum\limits_{i,j\in\{1,2\}} (h^{ij})^{\#} < 0, \\ 1 & \text{otherwise,} \end{cases}$$

and

$$\Omega(A) = \Omega_1(A) + \Omega_2(A) = \begin{cases} 0 & \text{for } h^{12} = h^{21} = 0, \\ 1 & \text{for } h^{11} = h^{22} = 0. \end{cases}$$

We can readily see that Ω_1 induces ω_P^1 and that Ω is a homomorphism inducing a homomorphism $\omega_P^0 \colon H_1(P^\#, \mathbb{Z}_2) \to \mathbb{Z}_2$. By ω_P^2 we denote the Arf function $\omega_P^1 + \omega_P^0$ generated by Ω_2.

An $N = 2$ super Riemann surface P is said to be *non-twisted* if $\omega_P^0 = 0$. By its *topological type* we mean the topological type (g, δ, k_α) of the Arf function $\omega_P^1 = \omega_P^2$. For $\omega_P^0 \neq 0$, the Riemann surface is said to be *twisted*. By its *topological type* we mean the topological type $(g, \delta_1, \delta_2, k_{\alpha\beta})$ of the pair of Arf functions (ω_P^1, ω_P^2), where $\delta_i = \delta(P^\#, \omega_i)$ and $k_{\alpha\beta}$ is the number of holes c_i of the surface $P^\#$ such that $\omega_1(c_i) = \alpha$ and $\omega_2(c_i) = \beta$ [47], § 8.

2. An $N = 2$ superanalogue of the group $\widetilde{\mathrm{Aut}}(\Lambda)$ is the group $\widetilde{\mathrm{Aut}}(\Lambda^{2S})$ generated by $\mathrm{Aut}(\Lambda^{2S})$ together with the map $\sigma \colon (z \,|\, \theta_1, \theta_2) \mapsto (-\overline{z} \,|\, \overline{\theta}_1, \overline{\theta}_2)$. We extend $\# \colon \mathrm{Aut}(\Lambda^{2S}) \to \mathrm{Aut}(\Lambda)$ to a homomorphism $\# \colon \widetilde{\mathrm{Aut}}(\Lambda^{2S}) \to \widetilde{\mathrm{Aut}}(\Lambda)$ by assuming that $\#(\sigma) \colon z \mapsto -\overline{z}$. A subgroup $\widetilde{\Gamma} \subset \widetilde{\mathrm{Aut}}(\Lambda^{2S})$ is called a *real* $N = 2$ *super Fuchsian group* if $\Gamma = \widetilde{\Gamma} \cap \mathrm{Aut}(\Lambda^{2S})$ is an $N = 2$ super Fuchsian group, $\widetilde{\Gamma} \neq \Gamma$, and $\Lambda^\# / \Gamma^\#$ is a compact surface. In this case, the pair $(\Lambda^{2S}/\Gamma, \widetilde{\Gamma}/\Gamma)$ is called a *real algebraic* $N = 2$ *supercurve*.

Real $N = 2$ supercurves $(\Lambda^{2S}/\Gamma_1, \widetilde{\Gamma}_1/\Gamma_1)$ and $(\Lambda^{2S}/\Gamma_2, \widetilde{\Gamma}_2/\Gamma_2)$ are assumed to be equal if there is an $h \in \widetilde{\mathrm{Aut}}(\Lambda^{2S})$ such that $\widetilde{\Gamma}_2 = h \widetilde{\Gamma}_1 h^{-1}$. The projection $\#$ sends a real supercurve $(P, \tau) = (\Lambda/\Gamma, \widetilde{\Gamma}/\Gamma)$ into the real curve $(P^\#, \tau^\#) = (\Lambda^\#/\Gamma^\#, \widetilde{\Gamma}^\#/\Gamma^\#)$.

Let $(P, \tau) = (\Lambda^{2S}/\Gamma, \widetilde{\Gamma}/\Gamma)$ be a real $N = 2$ supercurve and let $C \subset \Gamma$ correspond to an oval or to an invariant contour c (disjoint from the ovals). Replacing Γ by a conjugate group, we may assume that $C(z \,|\, \theta_1, \theta_2) = (\lambda z \,|\, h^1\theta_j, h^2\theta_{3-j})$. In this case the group $\widetilde{\Gamma}$ contains an element S_C of the form $S_C(z \,|\, \theta_1, \theta_2) = (-\rho\overline{z} \,|\, l^1\overline{\theta}_i, l^2\overline{\theta}_{3-i})$, where $\rho^\# > 0$ and $\ell^1\ell^2 = \rho^2$, $\rho = 1$ if c is an oval, and $(S_C)^2 = C$ if c is an invariant contour. We set $\mu(c) = 0$ for $i = 1$ and $\mu(c) = 1$ for $i = 2$.

If $\omega_1 = \omega_2$ (where $\omega_i = \omega_P^i$), then $\mu(c)$ is the same for all ovals and invariant contours c (disjoint with the ovals). This enables us to define the invariant $\mu(P, \tau) = \mu(c)$.

If $\omega_1 \neq \omega_2$, then the kernel of the homomorphism $\Omega \colon \Gamma \to \mathbb{Z}_2$ forms a subgroup Γ_* of index two. On the surface $P_*^\# = \Lambda^\#/\Gamma_*^\#$ the involutions in the set $\{F = S_C \,|\, \mu(c) = \mu\}$ generate the involution $\tau_\mu^\#$ $(\mu \in \mathbb{Z}_2)$. We set $\rho_\mu(P, \tau) = \varepsilon(P_*^\#, \tau_\mu^\#)$.

Let $M(g, \varepsilon)$ be the set of all real algebraic $N = 2$ supercurves (P, τ) such that $g(P^\#) = g$ and $\varepsilon(P^\#, \tau^\#) = \varepsilon \in \mathbb{Z}_2$. The structure of an $N = 2$ supercurve defines two Arf functions $\omega_i = \omega_P^i \colon H_1(P^\#, \mathbb{Z}_2) \to \mathbb{Z}_2$. We set $\chi(P) = 0$ if $\omega_1 = \omega_2$ and $\chi(P) = 1$ if $\omega_1 \neq \omega_2$. The invariant $\chi \in \mathbb{Z}_2$ decomposes $M(g, \varepsilon)$ into the subsets $M(g, \varepsilon, \chi) = \{(P, \tau) \in M(g, \varepsilon) \,|\, \chi(P) = \chi\}$.

By Theorem 3.2, the number of ovals c with the properties $\omega_i(c) = 0$ has the parity of $g + 1$. For $(P, \tau) \in M(g, 0, 0)$, denote by $k_\alpha(P, \tau)$ the number of ovals c such that $\omega_1(c) = \omega_2(c) = \alpha \in \mathbb{Z}_2$. We decompose the set $M(g, 0, 0)$

into the subsets

$$\begin{aligned}&M(g,0,0,k_\alpha,\delta,\mu)\\&\qquad=\bigl\{(P,\tau)\in M(g,0,0)\,|\,k_\alpha(P,\tau)=k_\alpha,\ \delta(\omega_1)=\delta(\omega_2)=\delta,\ \mu(P,\tau)=\mu\bigr\}.\end{aligned}$$

For $(P,\tau)\in M(g,0,1)$ we denote by $k^\mu_{\alpha\beta}(P,\tau)$ the number of ovals $c\subset P^\tau$ such that

$$\omega_1(c)=\alpha,\qquad \omega_2(c)=\beta,\qquad \mu(c)=\mu\in\mathbb{Z}_2.$$

We set

$$\begin{aligned}&M(g,0,1,k^\mu_{\alpha\beta},\delta_i,\rho_i)\\&\qquad=\bigl\{(P,\tau)\in M(g,0,1)\,|\,k^\mu_{\alpha\beta}(P,\tau)=k^\mu_{\alpha\beta},\ \delta(\omega_i)=\delta_i,\ \rho_i(P,\tau)=\rho_i\bigr\}.\end{aligned}$$

By [7] and [35], we have $M(g,0,1,k^\mu_{\alpha\beta},\delta_i,\rho_i)=0$ for $\rho_1=\rho_2=1$ and also for $k^0_{01}+k^0_{10}+k^1_{01}+k^1_{10}>0$ and $\rho_1+\rho_2>0$.

Let $(P^\#,\tau^\#)$ be a real algebraic curve such that $\varepsilon(P^\#,\tau^\#)=1$ and suppose that $\omega\colon H_1(P^\#,\mathbb{Z}_2)\to\mathbb{Z}_2$ is an Arf function with $\omega(\tau^\# a)=\omega(a)$ for all $a\in H_1(P^\#,\mathbb{Z}_2)$. The ovals $c_1,\dots,c_k$ decompose $P^\#$ into components $P^\#_1$ and $P^\#_2$. We set

$$\eta_\omega(P^\#,\tau^\#)=\delta(P^\#_1,\omega'),$$

where ω' is the restriction of ω to $P^\#_1$. In particular, $\eta_\omega(P^\#,\tau^\#)=0$ if there is an oval c such that $\omega(c)=1$.

Let

$$\begin{aligned}&M(g,1,0,k^\gamma_\alpha,\eta,\mu)\\&\qquad=\bigl\{(P,\tau)\in M(g,1,0)\,|\,k^\gamma_\alpha(P^\#,\tau^\#,\omega_1)=k^\gamma_\alpha,\ \eta_{\omega_1}(P,\tau)=\eta,\ \mu(P,\tau)=\mu\bigr\}.\end{aligned}$$

Assume now that $(P,\tau)\in M(g,1,1)$. We denote by $k^{0\mu}_{\alpha\beta}(P,\tau)$ (by $k^{1\mu}_{\alpha\beta}(P,\tau)$) the number of ovals c_i that are similar to c_1 with respect to ω_1 (that are not similar to c_1 with respect to ω_1, respectively) and such that $\omega_1(c_i)=\alpha$, $\omega_2(c_i)=\beta$, and $\mu(c_i)=\mu$. The set of numbers $k^{\gamma\mu}_{\alpha\beta}=k^{\gamma\mu}_{\alpha\beta}(P,\tau)$ is defined up to a permutation $k^{\gamma\mu}_{\alpha\beta}\mapsto k^{1-\gamma,\mu}_{\alpha\beta}$ related to the choice of c_1. We set

$$M(g,1,1,k^{\gamma\mu}_{\alpha\beta},\eta_i)=\bigl\{(P,\tau)\in M(g,1,1)\,|\,k^{\gamma\mu}_{\alpha\beta}(P,\tau)=k^{\gamma\mu}_{\alpha\beta},\ \eta_{\omega_i}(P,\tau)=\eta_i\bigr\}.$$

Thus, we obtain the following theorem.

Theorem 13.1 [46]**.** 1) *The set $M(g,\varepsilon,0)$ of real supercurves (P,τ) of genus g with the property $\omega_1(P)=\omega_2(P)$ decomposes into the subsets*

$$M(g,0,0,k_\alpha,\delta,\mu),\qquad M(g,1,0,k^\gamma_\alpha,\eta,\mu),$$

where

$$\alpha,\gamma,\delta,\eta,\mu\in\mathbb{Z}_2,\quad 0\leqslant k_0+k_1\leqslant g,\quad 1\leqslant\sum_{\alpha\gamma}k^\gamma_\alpha\leqslant g+1,$$

$$\sum_{\alpha\gamma}k^\gamma_\alpha\equiv g+1\pmod 2,\quad k_0\equiv g+1\pmod 2,\quad k^0_0+k^1_0\equiv g+1\pmod 2$$

$$\textit{and}\quad \eta=0\quad\textit{for}\quad k^0_1+k^1_1>0.$$

Among these subsets, only $M(g,1,0,k^\gamma_\alpha,\eta,\mu)$ and $M(g,1,0,k^{1-\gamma}_\alpha,\eta,\mu)$ coincide.

2) *The set* $M(g,\varepsilon,1)$ *of real supercurves* (P,τ) *of genus* g *with the property* $\omega_1(P)\neq\omega_2(P)$ *decomposes into the subsets*

$$M(g,0,1,k^{\mu}_{\alpha\beta},\delta_i,\rho_i),\qquad M(g,1,1,k^{\gamma\mu}_{\alpha\beta},\eta_i),$$

where

$$\alpha,\beta,\gamma,\mu,i,\delta_i,\rho_i,\eta_i\in\mathbb{Z}_2,\quad 0\leqslant\sum_{\alpha\beta\mu}k^{\mu}_{\alpha\beta}\leqslant g,$$
$$1\leqslant\sum_{\alpha\beta\gamma\mu}k^{\gamma\mu}_{\alpha\beta}\leqslant g+1,\quad \sum_{\alpha\beta\gamma\mu}k^{\gamma\mu}_{\alpha\beta}\equiv g+1 \pmod 2,$$
$$\sum_{\mu\beta}k^{\mu}_{0\beta}\equiv\sum_{\mu\alpha}k^{\mu}_{\alpha0}\equiv\sum_{\gamma\mu\beta}k^{\gamma\mu}_{0\beta}\equiv\sum_{\gamma\mu\alpha}k^{\gamma\mu}_{\alpha0}\equiv g+1 \pmod 2,$$
$$\rho_1+\rho_2<2,\quad \rho_1=\rho_2=0\quad \textit{for}\quad k^0_{01}+k^1_{01}+k^0_{10}+k^1_{10}>0,$$
$$\eta_1=0\quad\textit{for}\quad\sum_{\beta\gamma\mu}k^{\gamma\mu}_{1\beta}>0\quad\textit{and}\quad\eta_2=0\quad\textit{for}\quad\sum_{\alpha\gamma\mu}k^{\gamma\mu}_{\alpha1}>0.$$

Among these subsets, only $M(g,1,1,k^{\gamma\mu}_{\alpha\beta},\eta_i)$ *and* $M(g,1,1,k^{1-\gamma,\mu}_{\alpha\beta},\eta_i)$ *coincide.*

§ 14. The moduli space of the real algebraic $N=2$ supercurves

1. Let (P,τ) be real algebraic curves. By a *double Arf function* on (P,τ) we mean a pair (ω,α), where $\omega\colon H_1(P,\mathbb{Z}_2)\to\mathbb{Z}_2$ is an Arf function on (P,τ) and $\alpha\colon H_1(P,\mathbb{Z}_2)\to\mathbb{Z}_2$ is a homomorphism such that $\alpha\tau=\alpha$. Double Arf functions (ω_1,α_1) and (ω_2,α_2) on (P_1,τ_1) and (P_2,τ_2) are said to be *topologically equivalent* if there is a homeomorphism $\varphi\colon P_1\to P_2$ such that $\varphi\tau=\tau\varphi$, $\omega_1=\omega_2\varphi$, and $\alpha_1=\alpha_2\varphi$.

By § 13, a real algebraic $N=2$ supercurve gives rise to a double Arf function $(\omega_P,\alpha_P)=(\omega^1_P,\omega^0_P)$ on $(P^{\#},\tau^{\#})$.

Theorem 14.1 [46]**.** *Real algebraic* $N=2$ *supercurves* P_1 *and* P_2 *give rise to topologically equivalent double Arf functions if and only if the topological types of* P_1 *and* P_2 *coincide or differ from each other by a simultaneous replacement of* μ *by* $1-\mu$, $k^{\mu}_{\alpha\beta}$ *by* $k^{1-\mu}_{\alpha\beta}$, *and* $k^{\gamma\mu}_{\alpha\beta}$ *by* $k^{\gamma,1-\mu}_{\alpha\beta}$.

Proof. All the topological invariants associated with a supercurve P, except for $\mu(c)$ for ovals and invariant contours c, are uniquely determined by a pair of Arf functions (ω^1_P,ω^2_P), and hence are preserved under homeomorphisms $\varphi\colon P^{\#}_1\to P^{\#}_2$ that agree with $\tau^{\#}_i$. Thus, the topological equivalence of the double Arf functions $(\omega^1_{P_i},\omega^0_{P_i})$ implies the conditions on the types of P_1 and P_2 indicated in the theorem.

In the case $\varepsilon(P^{\#}_i,\tau^{\#}_i)=1$ the proof of the converse assertion repeats the proof of Theorem 11.3. Let us prove the converse assertion for curves $(P^{\#}_i,\tau^{\#}_i)$ of type $(g,k,0)$. We consider standard bases (which exist by Theorem 1.1)

$$\{a^i_j,b^i_j\ (i=1,\dots,2r),\ c^i_j,d^i_j\ (i=1,\dots,s)\}\in H_1(P^{\#}_j,\mathbb{Z}_2),$$

where

$$(P_i^{\#})^{\tau_i^{\#}} = \bigcup_{j=1}^{k} c_j^i, \qquad s-k \leqslant 1,$$

$$\tau_i^{\#}(a_j^i) = a_{2r+1-j}^i, \quad \tau_i^{\#}(b_j^i) = -b_{2r+1-j}^i,$$
$$\tau_i^{\#}(c_j^i) = c_j^i, \qquad \tau_i^{\#}(d_j^i) = -d_j^i + c^i;$$

here $c^i = \sum_{j=1}^{s} c_j^i$ and

$$\tau_i^{\#}(d_s^i) = -d_s^i + c^i + c_s^i \quad \text{for} \quad s > k.$$

By Theorem 11.2, these bases can be chosen so that

$$\omega_{P_1}^1(d_j^1) = \omega_{P_2}^1(d_j^2)$$

for all $j \leqslant s$. Under the conditions of Theorem 14.1, we can renumber c_j^i and d_j^i so that

$$\omega_{P_1}^m(c_j^1) = \omega_{P_2}^m(c_j^2), \qquad \omega_{P_1}^m(d_j^1) = \omega_{P_2}^m(d_j^2)$$

for all j and for $m = 1, 2$. Moreover, by [47], Theorem 8.1, under the conditions of Theorem 14.1 we can pass to a basis of the same form so that c_j^i and d_j^i are preserved and

$$\omega_{P_1}^m(a_j^1) = \omega_{P_2}^m(a_j^2) \quad \text{and} \quad \omega_{P_1}^m(b_j^1) = \omega_{P_2}^m(b_j^2)$$

for all j. Then a homeomorphism $\varphi\colon P_1^{\#} \to P_2^{\#}$ that maps one of the bases into another generates a topological equivalence of the double Arf functions.

2. We recall the description of the moduli space of $N = 2$ super Riemann surfaces [47], § 14.

Let $2t$ be the topological type of a Riemann $N = 2$ supersurface of genus $\widetilde{g}$ with k holes. By M we denote the set of all such supersurfaces. It is "uniformized" by the space

$$T^{2t} = \widetilde{T}^{2t} / \operatorname{Aut}(\Lambda^{2S}),$$

where $\widetilde{T}^{2t}$ is the space of monomorphisms $\psi\colon \gamma_{\widetilde{g},n} \to \operatorname{Aut}(\Lambda^{2S})$ (where $n = \widetilde{g} + k$) such that $\psi(v_{\widetilde{g},n})^{\#}$ is a sequential set of type (g,k), $\Lambda^{2S}/\psi(\gamma_{\widetilde{g},n}) \in M^{2t}$, and the group $\operatorname{Aut}(\Lambda^{2S})$ acts by conjugation [47], § 14. According to [47], § 14, T^{2t} is strongly diffeomorphic to

$$\mathbb{R}^{(p|q)}/\mathbb{Z}_2 = \mathbb{R}^{(8\widetilde{g}+4k-b(2t)|8\widetilde{g}+4k-8)}/(\mathbb{Z}_2)^2,$$

where $b(2t) = 8$ for the surface of twisted type and $b(2t) = 7$ otherwise. Moreover,

$$M^{2t} = T^{2t}/\operatorname{Mod}_{2t},$$

where Mod_{2t} is a discrete group.

3. Let us pass now to the description of the moduli space of real algebraic $N = 2$ supercurves.

Theorem 14.2 [46]**.** 1) *The moduli space $M(g,\varepsilon,0)$ of real algebraic $N=2$ supercurves (P,τ) of genus g with $\omega_1(P)=\omega_2(P)$ decomposes into the connected components*

$$M(g,0,0,k_\alpha,\delta,\mu), \qquad M(g,1,k_\alpha^\gamma,\eta,\mu),$$

where

$$\alpha,\gamma,\delta,\eta,\mu\in\mathbb{Z}_2, \quad 0\leqslant k=k_0+k_1\leqslant g, \quad 1\leqslant\sum_{\alpha\gamma}k_\alpha^\gamma\leqslant g+1,$$
$$k=\sum_{\alpha\gamma}k_\alpha^\gamma\equiv g+1 \pmod 2, \quad k_0\equiv g+1 \pmod 2,$$
$$k_0^0+k_0^1\equiv g+1 \pmod 2 \quad \text{and} \quad \eta=0 \quad \text{for} \quad k_1^0+k_1^1>0.$$

Among these components, only $M(g,1,0,k_\alpha^\gamma,\eta,\mu)$ and $M(g,1,0,k_\alpha^{1-\gamma},\eta,\mu)$ coincide. Each of the components $M(\chi)$ is of the form $T(\chi)/\operatorname{Mod}(\chi)$, where $T(\chi)$ is strongly diffeomorphic to $\mathbb{R}^{(4g-3+\mu k\,|\,4g-4)}/(\mathbb{Z}_2)^2$, and $\operatorname{Mod}(\chi)$ is a discrete group.

2) *The moduli space $M(g,\varepsilon,1)$ of real algebraic $N=2$ supercurves (P,τ) of genus g for which $\omega_1(P)\neq\omega_2(P)$ decomposes into connected components of the form*

$$M(g,0,1,k_{\alpha\beta}^\mu,\delta_i,\rho_i), \qquad M(g,1,1,k_{\alpha\beta}^{\gamma\mu},\eta_i),$$

where

$$\alpha,\beta,\gamma,\mu,i,\delta_i,\rho_i,\eta_i\in\mathbb{Z}_2, \quad 0\leqslant\sum_{\alpha\beta\mu}k_{\alpha\beta}^\mu\leqslant g,$$
$$1\leqslant\sum_{\alpha\beta\gamma\mu}k_{\alpha\beta}^{\gamma\mu}\leqslant g+1, \quad \sum_{\alpha\beta\gamma\mu}k_{\alpha\beta}^{\gamma\mu}\equiv g+1 \pmod 2,$$
$$\sum_{\mu\beta}k_{0\beta}^\mu\equiv\sum_{\mu\alpha}k_{\alpha0}^\mu\equiv\sum_{\gamma\mu\beta}k_{0\beta}^{\gamma\mu}\equiv\sum_{\gamma\mu\alpha}k_{\alpha0}^{\gamma\mu}\equiv g+1 \pmod 2,$$
$$\rho_1+\rho_2<2, \quad \rho_1=\rho_2=0 \quad \text{for} \quad k_{01}^0+k_{01}^1+k_{10}^0+k_{10}^1>0,$$
$$\eta_1=0 \quad \text{for} \quad \sum_{\beta\gamma\mu}k_{1\beta}^{\gamma\mu}>0 \quad \text{and} \quad \eta_2=0 \quad \text{for} \quad \sum_{\alpha\gamma\mu}k_{\alpha1}^{\gamma\mu}>0.$$

Among these components, only $M(g,1,1,k_{\alpha\beta}^{\gamma\mu},\eta_i)$ and $M(g,1,1,k_{\alpha\beta}^{1-\gamma,\mu},\eta_i)$ coincide. Each of these components $M(\chi)$ is of the form $T(\chi)/\operatorname{Mod}(\chi)$, where $\operatorname{Mod}(\chi)$ is a discrete group, the space $T(\chi)$ is strongly diffeomorphic to the quotient $\mathbb{R}^{(4g-4+k^1|4g-4)}/(\mathbb{Z}_2)^2$, and k^1 is equal to $\sum_{\alpha\beta\gamma}k_{\alpha\beta}^{\gamma1}$ or $\sum_{\alpha\beta}k_{\alpha\beta}^1$.

Proof. Let $\widetilde{\Gamma}$ be a real $N=2$ super Fuchsian group, let $(P,\tau)=[\widetilde{\Gamma}]$, let c be an oval or an invariant contour of the curve $(P^\#,\tau^\#)$ that does not intersect ovals, and let $C\subset\Gamma=\widetilde{\Gamma}\cap\operatorname{Aut}(\Lambda^{2S})$ be the shift corresponding to it. Replacing $\widetilde{\Gamma}$ by a conjugate group, we may assume that $C(z\,|\,\theta_1,\theta_2)=(\rho z\,|\,l_1\theta_i,l_2\theta_{3-i})$. By $\widehat{C}\subset\widetilde{\Gamma}\setminus\Gamma$ we denote an element such that: 1) $\widehat{C}C\widehat{C}^{-1}=C$; 2) $\widehat{C}^2=1$ if c is an oval; 3) $\widehat{C}^2=C$ if c is an invariant contour. If $\mu(c)=0$ and c is an oval, then

$$\widehat{C}(z\,|\,\theta_1,\theta_2)=(-\overline{z}\,|\,\pm\overline{\theta}_1,\pm\overline{\theta}_2).$$

If $\mu(c) = 0$ and c is an invariant contour, then

$$\widehat{C}(z \,|\, \theta_1, \theta_2) = (-\sqrt{\rho}\,\overline{z} \,|\, \pm\sqrt{|\ell_1|}\,\overline{\theta}_1, \pm\sqrt{|\ell_2|}\,\overline{\theta}_2).$$

If $\mu(c) = 1$ and c is an oval, then

$$\widehat{C}(z \,|\, \theta_1, \theta_2) = (-\overline{z} \,|\, h\overline{\theta}_2, h^{-1}\overline{\theta}_1).$$

If $\mu(c) = 1$ and c is an invariant contour, then

$$\widehat{C}(z \,|\, \theta_1, \theta_2) = (-\sqrt{\rho}\,\overline{z} \,|\, h\overline{\theta}_2, \sqrt{\rho}\,h^{-1}\overline{\theta}_1).$$

The rest of the proof repeats that of Theorems 12.1 and 12.2 with the space T^t replaced by T^{2t} and Theorems 11.2 and 11.3 by Theorem 14.1. The single essential difference arises only when associating a map $\widehat{C}_i$ with a shift C_i belonging to the set

$$\{A_i, B_i \ (i = 1, \dots, \widetilde{g}),\ C_i \ (i = 1, \dots, m)\} = \psi(V_{\widetilde{g},m}), \qquad \psi \in T^{2t}.$$

The above arguments show that if $\mu(c_i) = 0$ for a contour c_i corresponding to C_i, then the map $\widehat{C}_i$ is determined by the shift C_i with the same arbitrariness as in the case $N = 1$ (§ 12). For $\mu(c_i) = 1$ the choice of $\widehat{C}_i$ depends on a single additional arbitrary parameter $h \in L_0(\mathbb{R})$. However, if c_i is not an oval, then the condition $\widehat{C}_i^2 = C_i$ fixes one of the parameters in $L_0(\mathbb{R})$ on which an arbitrary element $C_i \in \mathrm{Aut}_2(\Lambda^{2S})$ depends. It is this that determines the dimension of the superlinear spaces that uniformize the connected components of the moduli space of real $N = 2$ supercurves.

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