CLIFFORD MODULES

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INTRODUCTION

THIS PAPER developed in part from an earlier version by the last two authors. It is presented here, in its revised form, by the first two authors in memory of their friend and collaborator ARNOLD SHAPIRO.

The purpose of the paper is to undertake a detailed investigation of the role of Clifford algebras and spinors in the KO-theory of real vector bundles. On the one hand the use of Clifford algebras throws considerable light on the periodicity theorem for the stable orthogonal group. On the other hand the use of spinors seems essential in some of the finer points of the KO-theory which centre round the Thom isomorphism. As far as possible we have endeavoured to make this paper self-contained, assuming only a knowledge of the basic facts of K- and KO-theory, such as can be found in [3]. In particular we develop the theory of Clifford algebras from scratch. The paper is divided into three parts.

Part I is entirely algebraic and is the study of Clifford algebras. This contains nothing essentially new, though we formulate the results in a novel way. Moreover the treatment given in §§ 1–3 differs slightly from the standard approach: our Clifford group (Definition (3.1)) is defined via a 'twisted' adjoint representation. This twisting, which is a natural consequence of our emphasis on the *grading*, leads, we believe, to a simplification of the algebra. On the group level our definitions give rise in a natural way to a group† Pin(k) which double covers $O(k)$ and whose connected component Spin(k) double covers $SO(k)$. This group is very convenient for the topological considerations of §§ 13 and 14. In § 4 we determine the structure of the Clifford algebras and express the results in Table 1. The basic algebraic periodicity (8 in the real case, 2 in the complex case) appears at this stage. In § 5 we study Clifford modules, i.e. representations of the Clifford algebras. We introduce certain groups A_k, defined in terms of Grothendieck groups of Clifford modules, and tabulate the results in Table 2. In § 6, using tensor products, we turn $A_* = \sum_{k \geqslant 0} A_k$ into a graded ring and determine its structure. These groups A_k are an algebraic counterpart of the homotopy groups of the stable orthogonal group, as will be shown in Part III.

Part II, which is independent of Part I, is concerned essentially with the 'difference bundle' construction in K-theory. We give a new and more complete treatment of this topic

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† This joke is due to J-P. Serre.

(see [4] and [7] for earlier versions) which includes a Grothendieck-type definition of the relative groups $K(X, Y)$ (Proposition (9.1)) and a product formula for difference bundles (Propositions (10.3) and (10.4)).

In Part III we combine the algebra of Part I with the topology of Part II. We define in § 11 a basic homomorphism

$$\alpha_P : A_k \to \widetilde{KO}(X^V)$$

where P is a principal Spin(k)-bundle over X, $V = P \times_{\text{Spin}(k)} R^k$, and X^V is the Thom complex of V. One of our main results is a product formula for α_P (Proposition (11.3)). Applying this in the case when X is a point gives rise to a ring homomorphism

$$\alpha : A_* \to \sum_{k \geqslant 0} KO^{-k}(\text{point}).$$

Using the periodicity theorem for the stable orthogonal group, as refined in [6], we then verify that α is an isomorphism (Theorem (11.5)). It is this theorem which shows the significance of Clifford algebras in K-theory and it strongly suggests that one should look for a *proof* of the periodicity theorem using Clifford algebras. Since this paper was written a proof on these lines has in fact been found by R. Wood†. It is to be hoped that Theorem (11.5) can be given a more natural and less computational proof.

Using α_P for general X gives us the Thom isomorphism (Theorem (12.3)) in a very precise form. Moreover the product formula for α_P asserts that the 'fundamental class' is multiplicative—just as in ordinary cohomology theory. Developing such a Thom isomorphism with all the good properties was one of our main aims. The treatment we have given is, we claim, more elementary, as well as more complete, than earlier versions which involved heavy use of characteristic classes.

In [7] another approach to the Thom isomorphism is given which has certain advantages over that given here. On the other hand the multiplicative property of the fundamental class does not come out of the method in [7]. To be able to use the advantages of both methods it is therefore necessary to identify the fundamental classes given in the two cases. This is done in §§ 13 and 14.

Finally in § 15 we discuss some other geometrical interpretations of Clifford modules. These throw considerable light on the vector-field problem for spheres.

Although the main interest in this paper lies in the KO-theory, most of what we do applies equally well in the complex case. It is one of the features of the Clifford module approach that the real and complex cases can be treated simultaneously.

<div align="center">PART I</div>

§1. Notation

Let k be a commutative field and let Q be a quadratic form on the k-module E. Let $T(E) = \sum_{i=0}^{\infty} T^i E = k \oplus E \oplus E \otimes E \oplus \ldots$ be the tensor algebra over E, and let $I(Q)$ be the two-sided ideal generated by the elements $x \otimes x - Q(x) \cdot 1$ in $T(E)$. The quotient algebra

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† See also the proof given in: J. MILNOR: Morse Theory, *Ann. Math. Stud.* **51**, (1963).

$T(E)/I(Q)$ is called the Clifford algebra of Q and is denoted by $C(Q)$. We also define $i_Q : E \to C(Q)$ to be the canonical map given by the composition $E \to T(E) \to C(Q)$. Then the following propositions relative to $C(Q)$ are not difficult to verify:

(1.1) $i_Q : E \to C(Q)$ is an injection.

(1.2) Let $\phi : E \to A$ be a linear map of E into a k-algebra with unit A, such that for all $x \in E$, the identity $\phi(x)^2 = Q(x)1$ is valid. Then there exists a unique homomorphism $\bar{\phi} : C(Q) \to B$, such that $\bar{\phi} \cdot i_Q = \phi$. (We refer to $\bar{\phi}$ as the 'extension' of ϕ.)

(1.3) $C(Q)$ is the universal algebra with respect to maps of the type described in (1.2).

(1.4) Let $F^q T(E) = \sum_{i \leqslant q} T^i E$ be the filtered structure in $T(E)$. This filtering induces a filtering in $C(E)$, whose associated graded algebra is isomorphic to the exterior algebra ΛE, on E. Thus $\dim_k C(Q) = 2^{\dim E}$, and if $\{e_i\}$ $(i = 1, ..., n)$ is a base for $i_Q(E)$, then 1 together with the products $e_{i_1} \cdot e_{i_2} \dots e_{i_k}$, $i_1 < i_2 < \dots < i_k$, form a base for $C(Q)$.

(1.5) Let $C^0(Q)$ be the image of $\sum_{i=0}^{i=\infty} T^{2i}(E)$ in $C(Q)$ and set $C^1(Q)$ equal to the image of $\sum_0^\infty T^{2i+1}(E)$ in $C(Q)$. Then this decomposition defines $C(Q)$ as a Z_2-graded algebra. That is:

(a) $C(Q) = \sum_{i=0,1} C^i(Q)$;

(b) If $x_i \in C^i(Q)$, $y_j \in C^j(Q)$, then
$$x_i y_j \in C^k(Q), \qquad k \equiv i + j \bmod 2.$$

That the graded structure of $C(Q)$ should not be disregarded is maybe best brought out by the following:

PROPOSITION (1.6). *Suppose that $E = E_1 \oplus E_2$ is an orthogonal decomposition of E relative to Q, and let Q_i denote the restriction of Q to E_i. Then there is an isomorphism*
$$\psi : C(Q) \cong C(Q_1) \underset{k}{\hat{\otimes}} C(Q_2)$$
of the graded tensor-product of $C(Q_1)$ and $C(Q_2)$ with $C(Q)$.

Recall first, that the graded tensor product of two graded algebras $A = \sum_{\alpha=0,1} A^\alpha$, $B = \sum_{\alpha=0,1} B^\alpha$, is by definition the algebra whose underlying vector space is $\sum_{\alpha,\beta=0,1} A^\alpha \otimes B^\beta$, with multiplication defined by:
$$(u \otimes x_i) \cdot (y_j \otimes v) = (-1)^{ij} u y_j \otimes x_i v, \quad x_i \in C^i(Q), \ y_j \in C^j(Q).$$
This graded tensor product is denoted by $A \hat{\otimes} B$; and is again a graded algebra:
$$(A \hat{\otimes} B)^k = \sum A^i \otimes B^j \quad (i + j \equiv k(2)).$$

Proof of the proposition. Define $\psi : E \to C(Q_1) \underset{k}{\hat{\otimes}} C(Q_2)$ by the formula, $\psi(e) = e_1 \otimes 1 + 1 \otimes e_2$, where e_1 and e_2 are the orthogonal projections of e on E_1 and E_2. Then
$$\psi(e)^2 = (e_1 \otimes 1 + 1 \otimes e_2)^2 = \{Q_1(e_1) + Q_2(e_2)\}(1 \otimes 1) = Q(e)(1 \otimes 1).$$
Hence ψ extends to an algebra homomorphism $\psi : C(Q) \to C(Q_1) \hat{\otimes} C(Q_2)$, by (1.2). Checking the behavior of ψ on basis elements now shows that ψ is a bijection. Note that the graded structure entered through the formula $(e_1 \otimes 1 + 1 \otimes e_2)^2 = e_1^2 \otimes 1 + 1 \otimes e_2^2$ which is valid as $e_i \in C^1(Q_i)$.

The algebra $C(Q)$ also inherits a canonical antiautomorphism from the tensor algebra $T(E)$. Namely if $x = x_1 \otimes x_2 \ldots \otimes x_k \in T^k(E)$, then the map $x \to x^t$, given by

$$x_1 \otimes x_2 \otimes \ldots \otimes x_k \to x_k \otimes \ldots \otimes x_2 \otimes x_1$$

clearly defines an antiautomorphism of $T(E)$, which preserves $I(Q)$ because $\{x \otimes x - Q(x) \cdot 1\}^t = x \otimes x - Q(x) \cdot 1$. Hence this operation induces a well defined antiautomorphism on $C(Q)$ which we also denote by $x \to x^t$ and refer to as the transpose. The transpose is the identity map on $i_Q(E) \subset C(Q)$.

The following two operations on $C(Q)$ will also be useful:

DEFINITION (1.7). *The canonical automorphism of $C(Q)$ is defined as the 'extension' of the map $\alpha : E \to C(Q)$, given by $\alpha(x) = -i_Q(x)$. (It is clear that $\{\alpha(x)\}^2 = Q(x)1$ and so α is well-defined by (1.1)). We denote this automorphism by α.*

DEFINITION (1.8). *Let $x \to \bar{x}$ be defined by the formula $x \to \alpha(x^t)$. This 'bar operation' is then an antiautomorphism of $C(Q)$.*

Note. (1) The identity $\alpha(x^t) = \{\alpha(x)\}^t$ holds as both are antiautomorphisms which extend the map $E \to C(Q)$ given by $x \to -i_Q(x)$;

(2) The grading on $C(Q)$ may be defined in terms of α: $C^i(Q) = \{x \in C(Q) | \alpha(x) = (-1)^i x\}$, $i = 0, 1$.

§2.   The algebras C_k

We are interested in the algebras $C(Q_k)$, where Q_k is a negative definite form on k-space over the real numbers. Quite specifically, we let $\mathbf{R}^k$ denote the space of k-tuples of real numbers, and define $Q_k(x_1, \ldots, x_k) = -\sum x_i^2$. Then we define C_k as the algebra $C(Q_k)$ and identify $\mathbf{R}^k$ with $i_{Q_k}\mathbf{R}^k \subset C_k$ and $\mathbf{R}$ with $\mathbf{R} \cdot 1 \subset C_k$. For $k = 0$, $C_k = \mathbf{R}$.

PROPOSITION (2.1). *The algebra C_1 is isomorphic to $\mathbf{C}$ (the complex numbers) considered as an algebra over $\mathbf{R}$. Further*

$$C_k \cong C_1 \hat{\otimes} C_1 \hat{\otimes} \ldots \hat{\otimes} C_1 \qquad (k \text{ factors}).$$

Clearly C_1 is generated by 1 and e_1, where 1 denotes the real number 1 in $\mathbf{R}^1$. Hence $e_1^2 = -1$. The formula $C_k \cong C_1 \hat{\otimes} \ldots \hat{\otimes} C_1$ now follows from repeated application of Proposition (1.6).

We will denote the k-tuple, $(0, \ldots, 1, \ldots, 0)$ with 1 in the ith position by e_i. The e_i, $i \leqslant k$ then form a base of $\mathbf{R}^k \subset C_k$.

COROLLARY (2.2). *The e_i, $i = 1, \ldots, k$, generate C_k multiplicatively and satisfy the relations*

$$(2.3) \qquad\qquad e_j^2 = -1, \qquad e_i e_j + e_j e_i = 0; \qquad i \neq j.$$

C_k may be identified with the universal algebra generated over $\mathbf{R}$ by a unit, 1, and the symbols e_i, $i = 1, \ldots, k$, subject to the relations (2.3).

§3.   The groups, Γ_k, Pin(k), and Spin (k)

Let C_k^* denote the multiplicative group of invertible elements in C_k.

DEFINITION (3.1). *The Clifford group Γ_k is the subgroup of those elements $x \in C_k^*$ for which $y \in \mathbf{R}^k$ implies $\alpha(x)yx^{-1} \in \mathbf{R}^k$.*

It is clear enough that Γ_k is a subgroup of C_k, because α is an automorphism. We also write $\alpha(x)\mathbf{R}^k x^{-1} \subset \mathbf{R}^k$ for the condition defining Γ_k. As α and the transpose map $\mathbf{R}^k$ into itself, it is then also evident that we have:

PROPOSITION (3.2). *The maps $x \to \alpha(x)$, $x \to x^t$ preserve Γ_k, and respectively induce an automorphism and an antiautomorphism of Γ_k. Hence $x \to \bar{x}$ is also an antiautomorphism of Γ_k.*

The group Γ_k comes to us with a ready-made homomorphism $\rho : \Gamma_k \to \mathrm{Aut}(\mathbf{R}^k)$. By definition $\rho(x)$, for $x \in \Gamma_k$, is the linear map $\mathbf{R}^k \to \mathbf{R}^k$ given by $\rho(x) \cdot y = \alpha(x)yx^{-1}$. We refer to ρ as the *twisted adjoint representation* of Γ_k on $\mathbf{R}^k$. This representation ρ turns out to be nearly faithful.

PROPOSITION (3.3). *The kernel of $\rho : \Gamma_k \to \mathrm{Aut}(\mathbf{R}^k)$ is precisely $\mathbf{R}^*$, the multiplicative group of nonzero multiples of $1 \in C_k$.*

Proof. Suppose $x \in \mathrm{Ker}(\rho)$. This implies

$$(3.4) \qquad\qquad \alpha(x)y = yx \qquad\qquad \text{for all } y \in \mathbf{R}^k.$$

Write $x = x^0 + x^1$, $x^i \in C_k^i$. Then (3.4) becomes

$$(3.5) \qquad\qquad x^0 y = yx^0$$

$$(3.6) \qquad\qquad x^1 y = -yx^1.$$

Let $e_1, \ldots, e_k$ be our orthonormal base for $\mathbf{R}^k$, and write $x^0 = a^0 + e_1 b^1$ in terms of this basis. Here $a^0 \in C_k^0$ does not involve e_1 and $b^1 \in C_k^1$ does not involve e_1. By setting $y = e_1$ in (3.5) we get $a^0 + e_1 b^1 = e_1 a^0 e_1^{-1} + e_1^2 b^1 e_1^{-1} = a_0 - e_1 b^1$. Hence $b^1 = 0$. That is, the expansion of x^0 does not involve e_1. Applying the same argument with the other basis elements we see that x^0 does not involve any of them. Hence x^0 is a multiple of 1. Next we write x^1 in the same form: $x^1 = a^1 + e_1 b^0$ and set $y = e_1$. We then obtain $a^1 + e_1 b^0 = -\{e_1 a^1 e_1^{-1} + e_1^2 b^0 e_1^{-1}\} = a^1 - e_1 b^0$. We again conclude that x^1 does not involve the e_i. Hence x^1 is a multiple of 1. On the other hand $x^1 \in C_k^1$ whence $x^1 = 0$. This proves that $x = x_0 \in \mathbf{R}$ and as x is invertible $x \in \mathbf{R}^*$. Q.E.D.

Consider now the function $N : C_k \to C_k$ defined by

$$(3.7) \qquad\qquad N(x) = x \cdot \bar{x}.$$

If $x \in \mathbf{R}^k$, then $N(x) = x(-x) = -x^2 = -Q_k(x)$. Thus $N(x)$ is the square of the length in $\mathbf{R}^k$ relative to the positive definite form $-Q_k$.

PROPOSITION (3.8). *If $x \in \Gamma_k$ then $N(x) \in \mathbf{R}^*$.*

Proof. We show that $N(x)$ is in the kernel of ρ. Let then $x \in \Gamma_k$, whence for every $y \in \mathbf{R}^k$ we have

$$\alpha(x)yx^{-1} = y', \qquad y' = \rho(x)v \in \mathbf{R}^k.$$

Applying the transpose we obtain: (as $y^t = y$)

$$(x^t)^{-1} y \alpha(x)^t = \alpha(x)yx^{-1}$$

whence $y\alpha(x^t)x = x^t\alpha(x)y$. This implies that $\alpha(x^t)x$ is in the kernel of ρ, and hence in $\mathbf{R}^*$ by (3.3). It follows that $x^t\alpha(x) \in \mathbf{R}^*$, whence $N(x^t) \in \mathbf{R}^*$. However $x \to x^t$ is an antiauto-morphism of Γ_k, by (3.2). Hence $N(\Gamma_k) \subset \mathbf{R}^*$.

PROPOSITION (3.9). $N : \Gamma_k \to \mathbf{R}^*$ *is a homomorphism. Moreover* $N(\alpha x) = N(x)$.

Proof. $N(xy) = xy\,\bar{y}\bar{x} = x\,N(y)\bar{x} = N(x)\cdot N(y)$, $N(\alpha(x)) = \alpha(x)x^t = \alpha N(x) = N(x)$.

PROPOSITION (3.10). $\rho(\Gamma_k)$ *is contained in the group of isometries of* $\mathbf{R}^k$.

Proof. Using (3.9) and the fact that $\mathbf{R}^k - \{0\} \subset \Gamma_k$ we have

$$N(\rho(x)\cdot y) = N(\alpha(x)y\,x^{-1}) = N(\alpha(x))N(y)N(x^{-1}) = N(y).$$

Q.E.D.

THEOREM (3.11). *Let* Pin(k) *be the kernel of* $N : \Gamma_k \to \mathbf{R}^*$, $k \geqslant 1$, *and let* $O(k)$ *denote the group of isometries of* $\mathbf{R}^k$. *Then* $\rho|$Pin(k) *is a surjection of* Pin(k) *onto* $O(k)$ *with kernel* Z_2, *generated by* $-1 \in \Gamma_k$. *We thus have the exact sequence*

$$1 \to Z_2 \to \text{Pin}(k) \xrightarrow{\rho} O(k) \to 1.$$

Proof. We show first that ρ is onto. For this purpose consider $e_1 \in \mathbf{R}^k$. We have $N(e_1) = -e_1e_1 = +1$, and

$$\alpha(e_1)e_i e_1^{-1} = \begin{cases} -e_i & \text{if} \quad i = 1 \\ e_i & \text{if} \quad i \neq 1. \end{cases}$$

Thus $e_1 \in$ Pin(k), and $\rho(e_1)$ is the reflection in the hyperplane perpendicular to e_1. Applying the same argument to any orthonormal base $\{e_i\}$ in $\mathbf{R}^k$, we see that the unit sphere

$$\{x \in \mathbf{R}^k | N(x) = 1\}$$

is in Pin(k) whence all the orthogonal reflections in hyperplanes of $\mathbf{R}^k$ are in $\rho\{$Pin$(k)\}$. But these are well known to generate $O(k)$. Thus ρ maps Pin(k) onto $O(k)$. Consider next the kernel of this map, which clearly consists of the intersection Ker $\rho \cap \{N(x) = 1\}$. Thus the kernel of $\rho|$Pin(k) consists of the multiples $\lambda \cdot 1$, with $N(\lambda 1) = 1$. Thus $\lambda^2 = +1$ which implies $\lambda = \pm 1$.

DEFINITION (3.12). *For* $k \geqslant 1$ *let* Spin(k) *be the subgroup of* Pin(k) *which maps onto* $SO(k)$ *under* ρ.

The groups Pin(k) and Spin(k) are double coverings of $O(K)$ and $SO(k)$ respectively. As such they inherit the Lie-structure of the latter groups. One may also show that these groups are closed subgroups of C_k^* and get at their Lie structure in this way.

PROPOSITION (3.13). *Let* Pin$(k)^i =$ Pin$(k) \cap C_k^i$. *Then* Pin$(k) = \cup_{i=0,1}$ Pin$(k)^i$, *and* Spin$(k) =$ Pin$(k)^0$.

Proof. Let $x \in$ Pin(k). Then $\rho(x)$ is equal to the composition of a certain number of reflections in hyperplanes: $\rho(x) = R_1 \circ \ldots \circ R_n$. We may choose elements $x_i \in \mathbf{R}^k$, such that $\rho(x_i) = R_i$. Hence, by (3.11), $x = \pm x_1 x_2 \ldots x_n$ and is therefore either in C_k^0 or in C_k^1. Finally x is in Spin(k) if and only if the number n in the above decomposition of $\rho(x)$ is even, i.e. if and only if $x \in$ Pin$(k)^0$.

PROPOSITION (3.14). *When* $k \geqslant 2$, *the restriction of* ρ *to* Spin(k) *is the nontrivial double covering of* $SO(k)$.

Proof. It is sufficient to show that $+1$, -1, the kernel of $\rho|\text{Spin}(k)$, can be connected by an arc in $\text{Spin}(k)$. Such an arc is given by:

$$\lambda : t \to \cos t + \sin t \cdot e_1 e_2 \qquad 0 \leqslant t \leqslant \pi.$$

COROLLARY (3.15). *When $k \geqslant 2$, $\text{Spin}(k)$ is connected and, when $k \geqslant 3$, simply-connected.*

This is clear from the fact that $SO(k)$ is connected for $k \geqslant 2$, and that $\pi_1\{SO(k)\} = Z_2$ if $k \geqslant 3$.

We note finally that $\text{Spin}(1) = Z_2$, while $\text{Pin}(1) = Z_4$.

All the preceding discussion can be extended to the complex case. We define α, t on $C_k \otimes_{\mathbf{R}} \mathbf{C}$ by

$$\alpha(x \otimes z) = \alpha(x) \otimes z$$

$$(x \otimes z)^t = x^t \otimes \bar{z}$$

and we take the bar operation and N to be defined in terms of α, t as before.

DEFINITION (3.16). Γ_k^c *is the subgroup of invertible elements $x \in C_k \otimes_{\mathbf{R}} \mathbf{C}$ for which $y \in \mathbf{R}^k$ implies $\alpha(x)yx^{-1} \in \mathbf{R}^k$.*

Propositions (3.2)–(3.10) go through with $\mathbf{R}^*$ replaced by $\mathbf{C}^*$ and (3.11) becomes:

THEOREM (3.17). *Let $\text{Pin}^c(k)$ be the kernel of $N : \Gamma_k^c \to \mathbf{C}^*$, $k \geqslant 1$, then we have an exact sequence:*

$$(3.18) \qquad 1 \to U(1) \to \text{Pin}^c(k) \to O(k) \to 1$$

where $U(1)$ is the subgroup consisting of elements $1 \otimes z \in C_k \otimes_{\mathbf{R}} \mathbf{C}$ with $|z| = 1$.

COROLLARY (3.19). *We have a natural isomorphism*

$$\text{Pin}(k) \times_{Z_2} U(1) \to \text{Pin}^c(k),$$

where Z_2 acts on $\text{Pin}(k)$ and $U(1)$ as $\{\pm 1\}$.

Proof. The inclusions $\text{Pin}(k) \subset C_k$, $U(1) \subset \mathbf{C}$ induce an inclusion

$$\text{Pin}(k) \times_{Z_2} U(1) \to C_k \otimes_{\mathbf{R}} \mathbf{C},$$

and it follows from the definitions that this factors through a homomorphism:

$$\psi : \text{Pin}(k) \times_{Z_2} U(1) \to \text{Pin}^c(k).$$

Now we have an obvious exact sequence

$$(3.20) \qquad 0 \to U(1) \to \text{Pin}(k) \times_{Z_2} U(1) \to \text{Pin}(k)/_{Z_2} \to 1$$

and ψ induces a homomorphism of (3.20) into (3.18). The 5-lemma and (3.11) now complete the proof.

We define $\text{Spin}^c(k)$ as the inverse image of $SO(k)$ in the homomorphism

$$\text{Pin}^c(k) \to O(k).$$

Then from (3.19) we have

$$\text{Spin}^c(k) \cong \text{Spin}(k) \times_{Z_2} U(1).$$

The groups $\text{Spin}^c(k)$ are particularly relevant to an understanding of the relationship

between spinors and complex structure, as we proceed to explain. The natural homomorphism

$$j : U(k) \to SO(2k)$$

does not lift to Spin(2k), as one easily verifies. However the homomorphism

$$l : U(k) \to SO(2k) \times U(1)$$

defined by

$$l(T) = j(T) \times \det T$$

does lift to $\mathrm{Spin}^c(2k)$. This follows at once from elementary topological considerations and the fact that

$$\det : U(k) \to U(1)$$

induces an isomorphism of fundamental groups.

Explicitly the lifted map

$$\tilde{l} : U(k) \to \mathrm{Spin}^c(2k)$$

is given as follows. Let $T \in U(k)$ be expressed, relative to an orthonormal base $f_1, \ldots, f_k$ of $\mathbf{C}^k$, by the diagonal matrix

$$\begin{pmatrix} \exp it_1 & & & \\ & \exp it_2 & & \\ & & \ddots & \\ & & & \exp it_k \end{pmatrix}$$

Let $e_1, \ldots, e_{2k}$ be the corresponding base of $\mathbf{R}^{2k}$, so that

$$e_{2j-1} = f_j \qquad e_{2j} = if_j$$

Then

$$\tilde{l}(T) = \prod_{j=1}^{k} \left(\cos t_j/2 + \sin t_j/2 \cdot e_{2j-1}e_{2j} \right) \times \exp\left(\frac{i\Sigma t_j}{2}\right).$$

§4.   Determination of the algebras C_k

In the following we will write $\mathbf{R}$, $\mathbf{C}$, and $\mathbf{H}$ respectively for the real, complex and quarternion number-fields. If F is any one of these fields, $F(n)$ will be the full $n \times n$ matrix algebra over F. The following are well known identities among these:

$$(4.1) \quad \begin{cases} F(n) \cong \mathbf{R}(n) \otimes_{\mathbf{R}} F, \ \mathbf{R}(n) \otimes_{\mathbf{R}} \mathbf{R}(m) \cong \mathbf{R}(nm) \\ \mathbf{C} \otimes_{\mathbf{R}} \mathbf{C} \cong \mathbf{C} \oplus \mathbf{C} \\ \mathbf{H} \otimes_{\mathbf{R}} \mathbf{C} \cong \mathbf{C}(2) \\ \mathbf{H} \otimes_{\mathbf{R}} \mathbf{H} \cong \mathbf{R}(4). \end{cases}$$

To compute the algebras C_k one now proceeds as follows: Let C'_k be the universal $\mathbf{R}$-algebra generated by a unit and the symbols e'_i ($i = 1, \ldots, k$) subject to the relations $(e'_i)^2 = +1$; $e'_i e'_j + e'_j e'_i = 0$, $i \neq j$. Thus C'_k may be identified with $C(-Q_k)$.

PROPOSITION (4.2). *There exist isomorphisms:*

$$(4.3) \quad \begin{aligned} C_k \otimes_{\mathbf{R}} C'_2 &\cong C'_{k+2} \\ C'_k \otimes_{\mathbf{R}} C_2 &\cong C_{k+2}. \end{aligned}$$

Proof. Denote by R'^k the space spanned by the e'_i in C'_k.

Consider the linear map $\psi : R'^{k+2} \to C_k \otimes C'_2$ defined by

$$\psi(e'_i) = \begin{cases} e_{i-2} \otimes e'_1 e'_2 & 2 \leqslant i \leqslant k \\ 1 \otimes e'_i & 1 \leqslant i \leqslant 2. \end{cases}$$

Then it is easily seen that ψ satisfies the universal property (1.1) for C'_k and hence extends to an algebra homomorphism $\psi : C'_{k+2} \to C_k \otimes C'_2$. As the map takes basis elements into basis elements and the spaces in question have equal dimension, it follows that ψ is a bijection. If we now replace the dashed symbols by the undashed ones and apply the same argument we obtain the second isomorphism.

Now it is clear that

$$C_1 \cong \mathbf{C}, \qquad C'_1 \cong \mathbf{R} \oplus \mathbf{R}$$
$$C_2 \cong \mathbf{H}, \qquad C'_2 \cong \mathbf{R}(2).$$

Hence (4.1) and repeated application of (4.3) yields the following table:

<div align="center">TABLE 1</div>

k	C_k	C'_k	$C_k \otimes_{\mathbf{R}} \mathbf{C} = C'_k \otimes_{\mathbf{R}} \mathbf{C}$
1	$\mathbf{C}$	$\mathbf{R} \oplus \mathbf{R}$	$\mathbf{C} \oplus \mathbf{C}$
2	$\mathbf{H}$	$\mathbf{R}(2)$	$\mathbf{C}(2)$
3	$\mathbf{H} \oplus \mathbf{H}$	$\mathbf{C}(2)$	$\mathbf{C}(2) \oplus \mathbf{C}(2)$
4	$\mathbf{H}(2)$	$\mathbf{H}(2)$	$\mathbf{C}(4)$
5	$\mathbf{C}(4)$	$\mathbf{H}(2) \oplus \mathbf{H}(2)$	$\mathbf{C}(4) \oplus \mathbf{C}(4)$
6	$\mathbf{R}(8)$	$\mathbf{H}(4)$	$\mathbf{C}(8)$
7	$\mathbf{R}(8) \oplus \mathbf{R}(8)$	$\mathbf{C}(8)$	$\mathbf{C}(8) \oplus \mathbf{C}(8)$
8	$\mathbf{R}(16)$	$\mathbf{R}(16)$	$\mathbf{C}(16)$

Note that (4.2) implies $C_4 \cong C'_4$; $C_{k+4} \cong C_k \otimes C_4$; $C_{k+8} \cong C_k \otimes C_8$; further $C_8 \cong \mathbf{R}(16)$, whence if $C_k \cong F(m)$ then, $C_{k+8} \cong F(16m)$. Thus both columns are in a quite definite sense of period 8. If we move up eight steps, the field is left unaltered, while the dimension is multiplied by 16. Note also the considerably simpler behavior of the complexifications of these algebras, which of course can be interpreted as the Clifford algebra of Q_k over the complex-numbers. Over the complex field, the period is 2.

§5. Clifford modules

We will now describe the set of **R**- and **C**- modules for the algebras C_k. We write $M(C_k)$ for the free abelian group generated by the irreducible Z_2-graded C_k-modules, and $N(C_k^0)$ for the corresponding group generated by the (ungraded) C_k^0-modules. The corresponding objects for the complex algebras $C_k \otimes_{\mathbf{R}} \mathbf{C}$ are denoted by $M^c(C_k)$ and $N^c(C_k^0)$.

PROPOSITION (5.1). *Let* $R : M \mapsto M^0$ *be the functor which assigns to a graded* C_k-*module* $M = M^0 \oplus M^1$ *the* C_k^0-*module* M^0. *Then* R *induces isomorphisms*

(5.2) $$M(C_k) \cong N(C_k^0).$$

Proof. If M^0 is a C_k^0-module, let

$$S(M^0) = C_k \otimes_{C_k^0} M^0.$$

The left action of C_k on C_k then defines $S(M^0)$ as a graded C_k-module. We now assert that $S \circ R$ and $R \circ S$ are naturally isomorphic to the identity. In the first case the isomorphism is induced by the 'module-map' $C_k \otimes M^0 \to M$, while in the second case the map $M^0 \to 1 \otimes M^0$ induces the isomorphism.

We of course also have the corresponding formula:

(5.3) $M^c(C_k) \cong N^c(C_k^0)$.

PROPOSITION (5.4). *Let $\phi : \mathbf{R}^k \to C_{k+1}^0$ be defined by $\phi(e_i) = e_i e_{k+1}$, $i = 1, ..., k$. Then ϕ extends to yield an isomorphism $C_k \cong C_{k+1}^0$.*

Proof. $\phi(e_i)^2 = e_i e_{k+1} e_i e_{k+1} = -1$. Hence ϕ extends. As it maps distinct basis elements onto distinct basis elements the extension is an isomorphism.

In view of these two propositions and Table 1, we may now write down the group $M(C_k)$ etc., explicitly. This is done in Table 2, where we also tabulate the following quantities:

Let $i : C_k \to C_{k+1}$ be the inclusion which extends the inclusion $\mathbf{R}^k \to \mathbf{R}^{k+1}$, let $i^* : M(C_{k+1}) \to M(C_k)$ be the induced homomorphism, and set $A_k = $ cokernel of i^*. Similarly define A_k^c as $M^c(C_k)/i^*\{M^c(C_{k+1})\}$ and finally define $a_k[a_k^c]$ as the $\mathbf{R}[\mathbf{C}]$-dimension of M^0 when M is an irreducible graded module for $C_k[C_k \otimes_\mathbf{R} \mathbf{C}]$.

TABLE 2

k	C_k	$M(C_k)$	A_k	a_k	$M^c(C_k)$	A_k^c	a_k^c
1	$\mathbf{C}(1)$	$\mathbf{Z}$	$\mathbf{Z}_2$	1	$\mathbf{Z}$	0	1
2	$\mathbf{H}(1)$	$\mathbf{Z}$	$\mathbf{Z}_2$	2	$\mathbf{Z} \oplus \mathbf{Z}$	$\mathbf{Z}$	1
3	$\mathbf{H}(1) \oplus \mathbf{H}(1)$	$\mathbf{Z}$	0	4	$\mathbf{Z}$	0	2
4	$\mathbf{H}(2)$	$\mathbf{Z} \oplus \mathbf{Z}$	$\mathbf{Z}$	4	$\mathbf{Z} \oplus \mathbf{Z}$	$\mathbf{Z}$	2
5	$\mathbf{C}(4)$	$\mathbf{Z}$	0	8	$\mathbf{Z}$	0	4
6	$\mathbf{R}(8)$	$\mathbf{Z}$	0	8	$\mathbf{Z} \oplus \mathbf{Z}$	$\mathbf{Z}$	4
7	$\mathbf{R}(8) \oplus \mathbf{R}(8)$	$\mathbf{Z}$	0	8	$\mathbf{Z}$	0	8
8	$\mathbf{R}(16)$	$\mathbf{Z} \oplus \mathbf{Z}$	$\mathbf{Z}$	8	$\mathbf{Z} \oplus \mathbf{Z}$	$\mathbf{Z}$	8

$$M_{k+8} \cong M_k, \qquad A_{k+8} \cong A_k, \qquad a_{k+8} = 16a_k$$
$$M_{k+2}^c \cong M_k^c, \qquad A_{k+2}^c \cong A_k^c, \qquad a_{k+2}^c = 2a_k^c.$$

Most of the entries in Table 2 follow directly from Table 1, because the algebras $F(n)$ are simple and hence have only one class of irreducible modules, the one given by the action of $F(n)$ on the n-tuples of elements in F. The only entries which still need clarification are therefore A_{4n} and A_{2n}^c.

Before explaining these entries observe that if $M = M^0 \oplus M^1$, then $M^* = M^1 \oplus M^0$. i.e. the module obtained from M by merely interchanging labels, is again a graded module. This operation therefore induces an involution on $M(C_k)$ and $M^c(C_k)$ which we again denote by $*$.

PROPOSITION (5.5). *Let x and y be the classes of the two distinct irreducible graded modules in $M(C_{4n})$. Then*

$$(5.6) \qquad\qquad x^* = y, \qquad y^* = x.$$

COROLLARY (5.7). $A_{4n} \cong \mathbf{Z}$.

Indeed if z generates $M(C_{4n+1})$, then $z^* = z$ as there is only one irreducible graded module for C_{4n+1}. Hence as $(i^*z)^* = i^*(z^*)$ we see that $i^*z = x + y$, by a dimension count.

To prove (5.5) we require the following lemma which is quite straight-forward and will be left to the reader.

LEMMA (5.8). *Let $y \in \mathbf{R}^k$, $y \neq 0$ and denote by $A(y)$ the inner automorphism of C_k induced by y. Thus $A(y) \cdot w = ywy^{-1}$. We also write $A(y)$ for the induced automorphism on $M(C_k)$. Similarly $A^0(y)$ denotes the restriction of $A(y)$ to C_k^0, as well as the induced automorphism on $N(C_k^0)$. Then we have*

$$(5.9) \qquad \begin{aligned} A(y) \cdot x &= x^* & x \in M(C_k) \\ A^0(y) \cdot R(x) &= R(x^*), \\ A^0(e_k)\phi(w) &= \phi\{\alpha(w)\}. \end{aligned}$$

Here $R : M(C_k) \mapsto N(C_k^0)$ is the functor introduced earlier, and $\phi : C_{k-1} \to C_k$, the map introduced in (5.4), while α is the canonical automorphism of C_k.

It now follows from these isomorphisms, that $*$ on $M(C_{4n})$ corresponds to the action of α on the ungraded modules of C_{4n-1}. Now the centre of C_{4n-1} is spanned by 1 and $w = e_1 e_2 \ldots e_{4n-1}$. Further $w^2 = +1$. Hence the projections of C_{4n-1} on the two ideals which make up C_{4n-1} are $(1 + w)/2$ and $(1 - w)/2$. Hence α interchanges these, and therefore clearly interchanges the two irreducible C_{4n-1} modules.

Finally, the evaluation $A_{2n}^c \cong \mathbf{Z}$ proceeds in an entirely analogous fashion.

Actually in the complex case there is a relation with Grassmann algebras which we shall now describe. Give $\mathbf{C}^k$ the standard Hermitian metric. Then the complex Grassmann algebra

$$\Lambda(\mathbf{C}^k) = \sum_{j=0}^{k} \Lambda^j(\mathbf{C}^k)$$

inherits a natural metric. In terms of an orthonormal basis $f_1, \ldots, f_k$ of $\mathbf{C}^k$ the elements

$$f_{i_1} \wedge f_{i_2} \wedge \ldots \wedge f_{i_k} \qquad\qquad i_1 < i_2 < \ldots < i_k$$

form an orthonormal basis of $\Lambda(\mathbf{C}^k)$. For each $v \in \mathbf{C}^k$ let d_v denote the (vector space) endomorphism of $\Lambda(\mathbf{C}^k)$ given by the exterior product:

$$d_v(w) = v \wedge w,$$

and let δ_v denote its adjoint with respect to the metric. We now define a pairing

$$(5.10) \qquad\qquad \mathbf{C}^k \otimes_{\mathbf{R}} \Lambda(\mathbf{C}^k) \to \Lambda(\mathbf{C}^k)$$

by

$$v \otimes w \to d_v(w) - \delta_v(w).$$

One verifies that

$$(d_v - \delta_v)^2 w = -\|v\|^2 w$$

so that (5.10) makes $\Lambda(\mathbf{C}^k)$ into a complex module for the Clifford algebra C_{2k} (identifying $\mathbf{C}^k$ with $\mathbf{R}^{2k}$ as usual) i.e. into a module for $C_{2k} \otimes_{\mathbf{R}} \mathbf{C}$. Moreover $\Lambda(\mathbf{C}^k)$ has a natural Z_2-grading

$$\Lambda^0 = \sum \Lambda^{2r}$$
$$\Lambda^1 = \sum \Lambda^{2r+1}$$

compatible with (5.10). A dimension count then shows that $\Lambda(\mathbf{C}^k)$ must be one of the two irreducible Z_2-graded modules for $C_{2k} \otimes_{\mathbf{R}} \mathbf{C}$. Now if $u = iv$ we see that

$$(d_v - \delta_v)(d_u - \delta_u)(1) = -i\|v\|^2(1) \qquad\qquad 1 \in \Lambda^0(\mathbf{C}^k).$$

Hence $\Lambda(\mathbf{C}^k)$ is a $(-i)^k$-module, and so we get

PROPOSITION (5.11). $\Lambda(\mathbf{C}^k)$ *is a graded* $C_{2k} \otimes_{\mathbf{R}} \mathbf{C}$-module *defining the class*

$$(-1)^k (\mu^c)^k \in A_k^c.$$

Remark. Using the explicit formula for $\tilde{\iota}: U(k) \to \mathrm{Spin}^c(2k)$ given in §3 it is easy to verify the commutativity of the following diagram

$$
\begin{array}{ccc}
U(k) & \xrightarrow{\tilde{\iota}} & \mathrm{Spin}^c(2k) \\
\downarrow{\scriptstyle i} & & \downarrow{\scriptstyle \sigma} \\
\mathrm{End}(\mathbf{C}^k) & \xrightarrow{\Lambda} & \mathrm{End}(\Lambda(\mathbf{C}^k))
\end{array}
$$

Here Λ is the functorial homomorphism, i is the inclusion and σ is the homomorphism induced by the action of $C_{2k} \otimes_{\mathbf{R}} \mathbf{C}$ on $\Lambda(\mathbf{C}^k)$ defined above.

§6. The multiplicative properties of the Clifford modules

If M and N are graded C_k and C_l modules, respectively, then their graded tensor product $M \hat{\otimes} N$ is in a natural way a graded module over $C_k \hat{\otimes} C_l$. By definition $(M \hat{\otimes} N)^0 = M^0 \otimes N^0 \oplus M^1 \otimes N^1$ and $(M \hat{\otimes} N)^1 = M^0 \otimes N^1 \oplus M^1 \otimes N^0$, the action of $C_k \hat{\otimes} C_l$ on $M \hat{\oplus} N$ being given by:

$$(6.1) \qquad (x \otimes y) \cdot (m \otimes n) = (-1)^{qi}(x \cdot m) \otimes (y \cdot n), \qquad y \in C_l^q, \qquad m \in M^i (q, i = 0, 1).$$

We also have the isomorphism $\phi_{k,l}: C_{k+l} \to C_k \hat{\otimes} C_l$ defined by the linear extension of the map

$$\phi_{k,l}(e_i) = \begin{cases} e_i \otimes 1 & 1 \leqslant i \leqslant k \\ 1 \otimes e_{k+i} & k < i \leqslant k + l. \end{cases}$$

The operation $(M, N) \mapsto M \hat{\otimes} N \mapsto \phi_{k,l}^*(M \hat{\otimes} N)$ is easily seen to give rise to a pairing

$$M(C_k) \otimes_{\mathbf{Z}} M(C_l) \to M(C_{k+l})$$

and thus induces a $\mathbf{Z}$-graded ring structure on the direct sum $M_* = \sum_0^\infty M(C_k)$. We denote this product by $(u, v) \to u \cdot v$. It is clearly associative.

PROPOSITION (6.2). *The following formulae are valid for* $u \in M(C_k)$, $v \in M(C_l)$

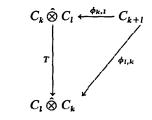$$(6.3) \qquad\qquad (u \cdot v)^* = u \cdot v^*$$

$$(6.4) \qquad\qquad u \cdot v = \begin{cases} v \cdot u & \text{if } kl \text{ is even} \\ (v \cdot u)^* & \text{if } kl \text{ is odd.} \end{cases}$$

(6.5)   If $i^* : M(C_k) \to M(C_{k-1})$ is the restriction homomorphism, as defined in §5, then

$$u \cdot i^* v = i^*(u \cdot v) \qquad k \geqslant 1.$$

The formulae (6.3) and (6.5) follow immediately from the definitions.

Proof of (6.4). We have the diagram:

$$C_k \hat{\otimes} C_l \xleftarrow{\ \phi_{k,l}\ } C_{k+l}$$

where T is the isomorphism $x \otimes y \to (-1)^{pq} y \otimes x$, $x \in C_k^p$, $y \in C_l^q$. Now the composition $\phi_{l,k}^{-1} \circ T \circ \phi_{k,l} : C_{k+l} \to C_{k+l}$ is an automorphism σ of C_{k+l}, which clearly is the linear extension of the map which permutes the first k elements of the basis $\{e_i\}$ with the last l elements

$$\sigma(e_i) = \begin{cases} e_{i+l} & 1 \leqslant i \leqslant k \\ e_{i-k} & k < i \leqslant k+l \end{cases}$$

Thus σ is the composition of inner automorphisms by elements in $\mathbf{R}^k - \{0\}$. It follows therefore from (5.9) that the effect of σ on $M(C_k)$ is equal to the effect of the operation (*) applied kl times. If we combine this with the fact that $T^*(N \hat{\otimes} M) \cong M \hat{\otimes} N$, whence

$$\phi_{k,l}^*(N \hat{\otimes} M) \cong \sigma^* \circ \phi_{l,k}^* \cdot (M \hat{\otimes} N),$$

we obtain the desired formula.

COROLLARY (6.6). *Let* $\lambda \in M(C_8)$ *be the class of an irreducible module of* C_8. *Then multiplication by* λ *induces an isomorphism:* $M(C_k) \cong M(C_{k+8})$.

Proof. This follows from our table of the a_k, in all cases except when $k = 4n$. In that case let x, y be the generators corresponding to the two irreducible graded modules of C_k. Then we know that $x^* = y$. Now $\lambda \cdot x \in M(C_{k+8})$ is the class of one of the irreducible graded modules of C_{k+8} by a dimension count. Hence by (6.4) $\lambda \cdot y = \lambda(x^*) = (\lambda x)^*$ corresponds to the other generator.

COROLLARY (6.7). *The image of* $i^* : M_* \to M_*$ *is an ideal, and hence the quotient ring* $A_* = \sum_0^\infty A_k$ *inherits a ring structure from* M_*.

This follows from (6.5). The element λ above projects into a class—again called λ—in A_8, and we clearly have:

PROPOSITION (6.8) *Multiplication by* λ *induces an isomorphism* $A_k \cong A_{k+8}$, $k \geqslant 0$.

The complete ring-structure of A_* is given by:

THEOREM (6.9). A_* *is the anticommutative graded ring generated by a unit* $1 \in A_0$, *and by elements* $\xi \in A_1$, $\mu \in A_4$, $\lambda \in A_8$ *with relations:* $2\xi = 0$, $\xi^3 = 0$, $\mu^2 = 4\lambda$.

Proof. As $A_1 \cong Z_2$, it is clear that $2\xi = 0$. From the fact that $a_1 = 1$, and $a_2 = 2$, we conclude that ξ_1^2 generates A_2. There remains the computation of μ^2. To settle this case we

introduce a notion which will be of use later in any case. Let $k = 4n$, and let $\omega = e_1 \ldots e_{4n}$. Then as we have already remarked, the centre of C_k^0 is generated by 1 and ω, whence, as $\omega^2 = +1$, the projection of C_k^0 on its two ideals is given by $(1 \mp \omega)/2$. It follows that if M is an irreducible graded C_k-module, then ω acts on M^0 as the scalar $\varepsilon = \pm 1$. In general we call a graded module for C_k an ε-module, $(\varepsilon = \pm 1)$ if ω acts as ε on M^0. Now because $e_i \omega = -\omega e_i$, it follows immediately that if M is an ε-module, then M^* is a $(-\varepsilon)$-module, i.e., ω acts as $-\varepsilon$ on M^1, and finally, that if M is an ε-module and M' an ε'-module for C_k then $M \hat{\otimes} M'$ is an $\varepsilon\varepsilon'$-module for C_{2k}.

With this understood, let μ be the class of an irreducible C_4-module M in A_4. Then M is of type ε. Hence $M \hat{\otimes} M$ is of type $\varepsilon^2 = +1$ in C_8. Now if $\lambda \in A_8$ is chosen as the class of the irreducible $(+1)$-module W of C_8 it follows that $M \hat{\otimes} M \cong 4W$ by a dimension count, and so finally that $\mu^2 = 4\lambda$.

The corresponding propositions for the complex modules are clearly also valid. Thus we may define M_*^c and A_*^c, and now already the generator μ^c corresponding to an irreducible $C_2 \otimes_{\mathbf{R}} \mathbf{C}$-module yields periodicity. In fact the following is checked readily.

THEOREM (6.10). *The ring A_*^c is isomorphic to the polynomial ring $\mathbf{Z}[\mu^c]$.*

We consider again the element $\omega = e_1 \ldots e_k \in C_k$. For $k = 2l$ we have $\omega^2 = (-1)^l$. Hence if M is an irreducible complex graded C_k-module then ω acts on M^0 as the complex scalar $\varepsilon = \pm i^l$. We call a complex graded C_k-module an ε-module if ω acts as ε on M^0. Let $\mu_l^c \in M^c(C_{2l})$ denote the generator given by an irreducible i^l-module. Then $\mu_l^c = (\mu^c)^l$ where $\mu_1^c = \mu^c$.

Comparing our conventions in the real and complex cases we see that if M is a real ε-module for C_{4n} then $M \otimes_{\mathbf{R}} \mathbf{C}$ is a complex $(-1)^n \varepsilon$-module for C_{4n}. Now we choose $\mu \in A_4$ to be the class of an irreducible (-1)-module. Then in the homomorphism $A_* \to A_*^c$ given by complexification $\mu \to 2(\mu^c)^2$. From (6.9) and (6.10) we then deduce

(6.11) $$\lambda \to (\mu^c)^4$$

under complexification.

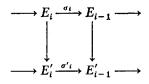<div align="center">PART II</div>

§7. Sequences of bundles

In this and succeeding sections we shall show how one can give a Grothendieck-type definition for the relative groups $K(X, Y)$. This will apply equally to real or complex vector bundles and we will just refer to vector bundles. For simplicity we shall work in the category of finite CW-complexes (and pairs of complexes).

For $Y \subset X$ we shall consider the set $\mathscr{C}_n(X, Y)$ of sequences

$$E = (0 \to E_n \xrightarrow{\sigma_n} E_{n-1} \xrightarrow{\sigma_{n-1}} \ldots \to E_1 \xrightarrow{\sigma_1} E_0 \to 0)$$

where the E_i are vector bundles on X, the σ_i are homomorphisms defined on Y and the sequence is exact on Y. An isomorphism $E \to E'$ in $\mathscr{C}_n$ will mean a diagram

$$\longrightarrow E_i \xrightarrow{\sigma_i} E_{i-1} \longrightarrow$$

$$\longrightarrow E'_i \xrightarrow{\sigma'_i} E'_{i-1} \longrightarrow$$

in which the vertical arrows are isomorphisms on X and the squares commute on Y.

An *elementary* sequence in $\mathscr{C}_n$ is one in which

$$E_i = E_{i-1}, \qquad \sigma_i = 1 \qquad \text{for some } i$$
$$E_j = 0 \qquad\qquad\qquad \text{for } j \neq i, i-1.$$

The direct sum $E \oplus F$ of two sequences is defined in the obvious way. We consider now the following equivalence relation:

DEFINITION (7.1). $E \sim F \Leftrightarrow$ *there exist elementary sequences* P^i, $Q^j \in \mathscr{C}_n$ *so that*

$$E \oplus P^1 \oplus \ldots \oplus P^r \cong F \oplus Q^1 \oplus \ldots \oplus Q^s.$$

In other words this is the equivalence relation generated by isomorphism and addition of elementary sequences. The set of equivalence classes will be denoted by $L_n(X, Y)$. *The operation* $\oplus$ *induces on* L_n *an abelian semi-group structure. If* $Y = \varnothing$ *we write* $L_n(X) = L_n(X, \varnothing)$.

If $E \in \mathscr{C}_n$ then we can consider the sequence in $\mathscr{C}_{n+1}$ obtained from E by just defining $E_{n+1} = 0$. In this way we get inclusions

$$\mathscr{C}_1 \to \mathscr{C}_2 \to \ldots \to \mathscr{C}_n \to$$

and we put $\mathscr{C} = \mathscr{C}_\infty = \lim \mathscr{C}_n$. These induce homomorphisms

$$L_1 \to L_2 \to \ldots \to L_n \to$$

and it is clear that

$$L = L_\infty = \lim_{\longrightarrow} L_n$$

is obtained from $\mathscr{C}$ by an equivalence relation as above applied now to sequences of finite but unbounded length.

LEMMA (7.2). *Let E, F be vector bundles on X and $f: E \to F$ a monomorphism on Y. Then if* $\dim F > \dim E + \dim X$, f *can be extended to a monomorphism on X and any two such extensions are homotopic rel. Y.*

Proof. Consider the fibre bundle $\operatorname{Mon}(E, F)$ on X whose fibre at $x \in X$ is the space of all monomorphisms $E_x \to F_x$. This fibre is homeomorphic to $GL(n)/GL(n-m)$ where $n = \dim F$, $m = \dim E$, and so it is $(n-m-1)$-connected. Hence cross-sections can be extended and are all homotopic if

$$\dim X \leqslant n - m - 1 = \dim F - \dim E - 1.$$

But a cross-section of $\operatorname{Mon}(E, F)$ is just a global monomorphism $E \to F$.

LEMMA (7.3). $L_n(X, Y) \to L_{n+1}(X, Y)$ *is an isomorphism for* $n \geqslant 1$.
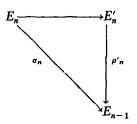
Proof. Let $\bar{\mathscr{C}}_{n+1}$ denote the subset of $\mathscr{C}_{n+1}$ consisting of sequences E such that

$$\dim E_n > \dim E_{n+1} + \dim X. \tag{1}$$

If $n \geqslant 1$ then given any $E \in \mathscr{C}_{n+1}$ we can add an elementary sequence to it so that it will satisfy (1). Hence $\tilde{\mathscr{C}}_{n+1} \to L_{n+1}$ is surjective. Now let $E \in \tilde{\mathscr{C}}_{n+1}$, then by (7.2) σ_{n+1} can be extended to a monomorphism σ'_{n+1} on the whole of X. Put $E'_n = \mathrm{Coker}\, \sigma'_{n+1}$, let P denote the elementary sequence with $P_{n+1} = P_n = E_{n+1}$, and let

$$E' = (0 \to E'_n \xrightarrow{\rho'_n} E_{n-1} \xrightarrow{\sigma_{n-1}} E_{n-2} \to \ldots \xrightarrow{\sigma_1} E_0 \to 0),$$

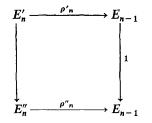where ρ'_n is defined by the commutative diagram on Y:

A splitting of the exact sequence on X

$$0 \to E_{n+1} \xrightarrow{\sigma'_{n+1}} E_n \to E'_n \to 0$$

then defines an isomorphism in $\mathscr{C}_{n+1}$

$$P \oplus E' \cong E.$$

If σ''_{n+1} is another extension of σ_{n+1} leading to a sequence E'', then by (7.2) $E'_n \cong E''_n$ and this isomorphism can be taken to extend the given one on Y, i.e., the diagram

commutes on Y. Hence $E' \cong E''$ in $\mathscr{C}_n$ and so we have a well-defined map $E \mapsto E'$ from the isomorphism classes in $\tilde{\mathscr{C}}_{n+1}$ to the isomorphism classes in $\mathscr{C}_n$. Moreover, if

$$Q = (0 \to Q_{n+1} \to Q_n \to 0), \qquad R = (0 \to R_i \to R_{i-1} \to 0) \qquad (i \leqslant n)$$

are elementary sequences, then

$$(E \oplus Q)' \cong E', \qquad (E \oplus R)' \cong E' \oplus R.$$

Hence the class of E' in L_n depends only on the class of E in L_{n+1}. Since $\tilde{\mathscr{C}}_{n+1} \to L_{n+1}$ is surjective it follows that $E \to E'$ induces a map $L_{n+1} \to L_n$. From its construction it is immediate that its composition in either direction with $L_n \to L_{n+1}$ is the identity, and this completes the proof.

From (7.3) we deduce, by induction on n, and then passing to the limit:

PROPOSITION (7.4). *The homomorphisms $L_1(X, Y) \to L_n(X, Y)$ are isomorphisms for* $1 \leqslant n \leqslant \infty$.

§8. Euler characteristics

DEFINITION (8.1) *An Euler characteristic for $\mathscr{C}_n$ is a natural homomorphism* (i.e. *a natural transformation of functors*)

$$\chi : L_n(X, Y) \to K(X, Y)$$

which for $Y = \varnothing$ *is given by*

$$\chi(E) = \sum_{i=0}^{n} (-1)^i E_i.$$

Remark. It is clear that, if $Y = \varnothing$, $E \mapsto \sum (-1)^i E_i$ gives a well-defined map $L_n(X) \to K(X)$.

LEMMA (8.2). *Let χ be an Euler characteristic for $\mathscr{C}_1$ then*

$$\chi : L_1(X) \to K(X)$$

is an isomorphism.

Proof. χ is an epimorphism by definition of $K(X)$. Suppose $\chi(E) = 0$, then $E_1 \oplus F \cong E_0 \oplus F$ for some F (in fact F can be taken trivial). Hence if

$$P : 0 \to F \to F \to 0$$

is the elementary sequence defined by F, $E \oplus P$ is isomorphic to the elementary sequence defined by $E_1 \oplus F$. Hence $E \sim 0$ in $\mathscr{C}_1(X)$ and so $E = 0$ in $L_1(X)$. To conclude we need the following elementary lemma:

LEMMA (8.3). *Let A be a semi-group with an identity element 1, B a group, $\phi : A \to B$ an epimorphism with $\phi^{-1}(1) = 1$. Then ϕ is an isomorphism.*

Proof. It is sufficient to prove that A is a group, i.e., has inverses. Let $a \in A$, then from the hypotheses there exists $a' \in A$ so that

$$\phi(a') = \phi(a)^{-1}.$$

Hence

$$\phi(a \cdot a') = \phi(a) \cdot \phi(a') = 1,$$

and so $aa' = 1$ as required.

LEMMA (8.4). *Let χ be an Euler characteristic for $\mathscr{C}_1$, and let Y be a point. Then*

$$\chi : L_1(X, Y) \to K(X, Y)$$

is an isomorphism.

Proof. Consider the diagram

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & L_1(X, Y) & \xrightarrow{\alpha} & L_1(X) & \xrightarrow{\beta} & L_1(Y) \\
& & \Big\downarrow{\scriptstyle\chi} & & \Big\downarrow{\scriptstyle\chi} & & \Big\downarrow{\scriptstyle\chi} \\
0 & \longrightarrow & K(X, Y) & \longrightarrow & K(X) & \longrightarrow & K(Y).
\end{array}
$$

By (8.2) and (8.3) and the exactness of the bottom line it will be sufficient to show the

exactness of the top line. Now $\beta\alpha = 0$ obviously and so we have to show

(i) $\qquad\qquad \alpha^{-1}(0) = 0$;

(ii) if $\qquad \beta(E) = 0 \qquad$ then $\qquad E \in \operatorname{Im} \alpha$.

We consider (ii) first. Since Y is a point, and $\chi : L_1(Y) \cong K(Y)$, $\beta(E) = 0$ is equivalent to

$$\dim E_1 | Y = \dim E_0 | Y.$$

But then we can certainly find an isomorphism

$$\sigma : E_1 | Y \longrightarrow E_0 | Y,$$

showing that $E \in \operatorname{Im}(\alpha)$. Finally we consider (i). Thus let

$$E = (0 \longrightarrow E_1 \overset{\sigma}{\longrightarrow} E_0 \longrightarrow 0)$$

be an element of $\mathscr{C}_1(X, Y)$ and suppose $\alpha(E) = 0$ in $L_1(X)$. Then $\chi\alpha(E) = 0$ in $K(X)$, and hence, if we suppose $\dim E_i > \dim X$ (as we may), there is an isomorphism

$$\tau : E_1 \longrightarrow E_0$$

on the whole of X. Then $\sigma\tau^{-1} \in \operatorname{Aut}(E_0 | Y)$. Since Y is a point this automorphism is homotopic to the identity† and hence can be extended to an element $\rho \in \operatorname{Aut}(E_0)$. Then $\rho\tau : E_1 \to E_0$ is an isomorphism extending σ. This shows that E represents 0 in $L_1(X, Y)$ as required.

LEMMA (8.5). *Let χ be an Euler characteristic for $\mathscr{C}_1$, then χ is an equivalence of functors $L_1 \to K$.*

Proof. Consider, for any pair (X, Y), the commutative diagram

$$
\begin{array}{ccc}
L_1(X/Y, Y/Y) & \overset{\chi}{\longrightarrow} & K(X/Y, Y/Y) \\
\Big\downarrow{\phi} & & \Big\downarrow{\psi} \\
L_1(X, Y) & \overset{\chi}{\longrightarrow} & K(X, Y).
\end{array}
$$

Since ψ is an isomorphism (by definition) and χ on the top line is an isomorphism by (8.4) it will be sufficient (by (8.3)) to prove that ϕ is an epimorphism. Now any element ξ of $L_1(X, Y)$ can be represented by a sequence

$$E = (0 \longrightarrow E_1 \overset{\sigma}{\longrightarrow} E_0 \longrightarrow 0)$$

where E_0 is a product bundle. But then we can define a 'collapsed bundle' $E_1' = E_1/\sigma$ over X/Y and a collapsed sequence $E' \in \mathscr{C}_1(X/Y, Y/Y)$ defining an element $\xi' \in L_1(X/Y, Y/Y)$. Then $\xi = \phi(\xi')$ and so ϕ is an epimorphism.
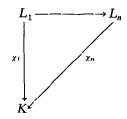
LEMMA (8.6). *Let χ, χ' be two Euler characteristics for $\mathscr{C}_1$. Then $\chi = \chi'$.*

Proof. Let $T = \chi'\chi^{-1}$ (which is well-defined by (8.5)). This is a natural automorphism of $K(X, Y)$ which is the identity when $Y = \varnothing$. Replacing X by X/Y and considering the exact sequence for $(X/Y, Y/Y)$ we deduce that $T = 1$, i.e., that $\chi' = \chi$.

---

† This argument needs modification in the real case since $GL(n, R)$ is not connected: we replace E_i by $E_i \oplus 1$ and σ, τ by $\sigma \oplus 1$, $\tau \oplus (-1)$.

From (8.6) and (7.4) we deduce

LEMMA (8.7). *There is a bijective correspondence* $(\chi_1 \longleftrightarrow \chi_n)$ *between Euler characteristics for* $\mathscr{C}_1$ *and* $\mathscr{C}_n$ *such that the diagram*

commutes.

These lemmas show that there is at most one Euler characteristic. In the next section we shall prove that it exists by giving a direct construction.

§9. The difference bundle

Given a pair (X, Y) define $X_i = X \times \{i\}$ $i = 0, 1$, $A = X_0 \cup_Y X_1$ (obtained by identifying $y \times \{0\}$ and $y \times \{1\}$ for all $y \in Y$). Then we have retractions

$$\pi_i : A \to X_i$$

so that we get split exact sequences:

$$0 \longrightarrow K(A, X_i) \xrightarrow{\rho^*_i} K(A) \underset{j^*_i}{\overset{\pi^*_i}{\rightleftarrows}} K(X_i) \longrightarrow 0$$

Also, if we regard the index $i \in Z_2$, the natural map $X \to X_i$ gives an inclusion

$$\phi_i : (X, Y) \to (A, X_{i+1}),$$

which induces an isomorphism

$$\phi^*_i : K(A, X_{i+1}) \to K(X, Y).$$

Now let $E \in \mathscr{C}_1(X, Y)$,

$$E = (0 \to E_1 \xrightarrow{\sigma} E_0 \to 0),$$

and construct the vector bundle F on A by putting E_i on X_i and identifying on Y by σ. It is clear that the isomorphism class of F depends only on the isomorphism class of E in $\mathscr{C}_1(X, Y)$. Let $F_i = \pi^*_i(E_i)$. Then $F|X_i \cong F_i$ and so $F - F_i \in \text{Ker } j^*_i$. We define an element $d(E) \in K(X, Y)$ by

$$\rho^*_1(\phi^*_0)^{-1} d(E) = F - F_1.$$

It is clear that d is additive:

$$d(E \oplus E') = d(E) + d(E').$$

Also if E is elementary $F \cong F_1$ so that $d(E) = 0$. Hence d induces a homomorphism

$$d : L_1(X, Y) \to K(X, Y)$$

which is clearly natural. Moreover if $Y = \varnothing$, $A = X_0 + X_1$, $F = E_0 \times \{0\} + E_1 \times \{1\}$ (disjoint sum), $F_i = E_i \times \{0\} + E_i \times \{1\}$ and so

$$d(E) = E_0 - E_1.$$

Thus d is an Euler Characteristic in the sense of §8. The existence of this d together with the lemmas of §8 lead to the following proposition:

PROPOSITION (9.1). *For any integer n with $1 \leqslant n \leqslant \infty$ there exists a unique natural homomorphism*

$$\chi : L_n(X, Y) \to K(X, Y)$$

which, for $Y = \varnothing$, is given by

$$\chi(E) = \sum_{i=0}^{n} (-1)^i E_i.$$

Moreover χ is an isomorphism.

The unique χ given by (9.1) will be referred to as the *Euler characteristic*. From (8.6) we see that we may effectively identify the χ for different n.

Two elements E, $F \in \mathscr{C}_n(X, Y)$ are called *homotopic* if they are isomorphic to the restrictions to $X \times \{0\}$ and $X \times \{1\}$ of an element in $\mathscr{C}_n(X \times I, Y \times I)$.

PROPOSITION (9.2). *Homotopic elements in $\mathscr{C}_n(X, Y)$ define the same elements in $L_n(X, Y)$.*

Proof. This follows at once from (9.1) and the homotopy invariance of $K(X, Y)$.

Proposition (9.1) shows that we could take $L_n(X, Y)$ (for any $n \geqslant 1$) as a *definition* of $K(X, Y)$. This would be a Grothendieck-type definition.

We shall now give a method for constructing the inverse of $j : L_1(X, Y) \to L_n(X, Y)$. If $E \in \mathscr{C}_n(X, Y)$, then by introducing metrics we can define the adjoint sequence E^* with maps $\sigma_i^* : E_{i-1} \to E_i$. Consider the sequence

$$F = (0 \to F_1 \xrightarrow{\tau} F_0 \to 0)$$

where $F_0 = \bigoplus_i E_{2i}$, $F_1 = \bigoplus_i E_{2i+1}$ and

$$\tau(e_1, e_3, e_5, \dots) = (\sigma_1 e_1, \sigma_2^* e_2 + \sigma_3 e_3, \sigma_4^* e_3 + \sigma_5 e_5, \dots).$$

Since, on Y, we have the decomposition

$$E_{2i} = \sigma_{2i+1}(E_{2i+1}) \oplus \sigma_{2i}^*(E_{2i-1})$$

it follows that $F \in \mathscr{C}_1(X, Y)$. If $E \in \mathscr{C}_1$ then $E = F$. Since two choices of metric in E are homotopic it follows by (9.2) that F will be a representative for $j^{-1}(E)$.

§10. Products

In this section we shall consider complexes of vector bundles, i.e., sequences

$$0 \longrightarrow E_n \xrightarrow{\sigma_n} E_{n-1} \xrightarrow{\sigma_{n-1}} \dots \longrightarrow E_0 \longrightarrow 0$$

in which $\sigma_{i-1}\sigma_i = 0$ for all i.

LEMMA (10.1). *Let $E_0, \dots, E_n$ be vector bundles on X,*

$$0 \longrightarrow E_n \xrightarrow{\sigma_n} E_{n-1} \longrightarrow \dots \longrightarrow E_0 \longrightarrow 0$$

a complex on Y. Then the σ_i can be extended so that this becomes a complex on X.

Proof. By induction on the cells of $X - Y$ it is sufficient to consider the case when X is obtained from Y by attaching one cell. Thus let

$$X = Y \cup_f e^k$$

where $f \colon S^{k-1} \to Y$ is the attaching map. If B^k denotes the unit ball in $\mathbf{R}^k$, with boundary S^{k-1}, then X is the quotient of $Y + B^k$ by an identification map π induced by f. The bundle $\pi^* E_i$ is then the disjoint sum of $E_i | Y$ and a trivial bundle $B^k \times V_i$. The homomorphism $\sigma_i \colon E_i \to E_{i-1}$ on Y lifts to give a homomorphism $\tau_i \colon S^{k-1} \times V_i \to S^{k-1} \times V_{i-1}$, i.e. a map $S^{k-1} \to \mathrm{Hom}(V_i, V_{i-1})$. Extend each τ_i to B^k by defining

$$\tau_i(u) = \|u\| \sigma_i(u) \qquad\qquad u \in B^k.$$

This induces an extension of the σ_i to X preserving the relations $\sigma_{i-1}\,\sigma_i = 0$, as required.

We now introduce the set $\mathscr{D}_n(X, Y)$ of complexes of length n on X acyclic (i.e. exact) on Y. Two such complexes are *homotopic* if they are isomorphic to the restrictions to $X \times \{0\}$ and $X \times \{1\}$ of an element in $\mathscr{D}_n(X \times I, Y \times I)$. By restricting the homomorphisms to Y we get a natural map

$$\Phi \colon \mathscr{D}_n(X, Y) \to \mathscr{C}_n(X, Y).$$

LEMMA (10.2). $\Phi \colon \mathscr{D}_n \to \mathscr{C}_n$ *induced a bijective map of homotopy classes.*

Proof. Applying (10.1) we see that Φ itself is surjective. Next, applying (10.1) to the pair

$$(X \times I, X \times \{0\} \cup X \times \{1\} \cup Y \times I)$$

we see that

$$\Phi(E) \text{ homotopic to } \Phi(F) \Rightarrow E \text{ homotopic to } F$$

which completes the proof.

If $E \in \mathscr{D}_n(X, Y)$, $F \in \mathscr{D}_m(X', Y')$ then $E \otimes F$ is a complex on $X \times X'$ acyclic on $X \times Y \cup Y \times X'$ so that

$$E \otimes F \in \mathscr{D}_{n+m}(X \times X', X \times Y' \cup Y \times X').$$

This product is additive and compatible with homotopies. Hence it induces a bilinear product on the homotopy classes. From (10.2) and (9.2) it follows that it induces a natural product

$$L_n(X, Y) \otimes L_m(X', Y') \to L_{n+m}(X \times X', X \times Y' \cup Y \times X').$$

PROPOSITION (10.3). *The tensor product of complexes induces a natural product*

$$L_n(X, Y) \otimes L_m(X', Y') \to L_{n+m}(X \times X', X \times Y' \cup Y \times X')$$

and

$$\chi(ab) = \chi(a)\chi(b) \tag{1}$$

where χ is the Euler characteristic.

Proof. The formula (1) is certainly true when $Y = Y' = \varnothing$. On the other hand there is a unique natural extension of the product $K(X) \otimes K(X') \to K(X \times X')$ to the relative case (cf. [3]). Hence, by (9.1), formula (1) is also true in the general case.

Remark. This result is essentially due to Douady (*Séminaire Bourbaki* (1961) No. 223).

PROPOSITION (10.4). *Let*

$$E = (0 \longrightarrow E_1 \xrightarrow{\sigma} E_0 \longrightarrow 0) \in \mathscr{D}_1(X, Y)$$

$$E' = (0 \longrightarrow E_1' \xrightarrow{\sigma'} E_0' \longrightarrow 0) \in \mathscr{D}_1(X', Y')$$

and choose metrics in all the bundles. Let

$$F = (0 \longrightarrow F_1 \xrightarrow{\tau} F_0 \longrightarrow 0) \in \mathscr{D}_1(X \times X', X \times Y' \cup Y \times X')$$

be defined by

$$F_1 = E_0 \otimes E_1' \oplus E_1 \otimes E_0'$$

$$F_0 = E_0 \otimes E_0' \oplus E_1 \otimes E_1'$$

$$\tau = \begin{pmatrix} 1 \otimes \sigma', & \sigma \otimes 1 \\ \sigma^* \otimes 1, & -1 \otimes \sigma'^* \end{pmatrix}$$

where σ^*, σ'^* denote the adjoints of σ, σ'. Then

$$\chi(F) = \chi(E) \cdot \chi(E').$$

Proof. By (10.3) $\chi(E) \cdot \chi(E') = \chi(E \otimes E')$. Now the construction of §9 for the inverse of $j_2 : L_1 \to L_2$ turns $E \otimes E'$ into F and so $\chi(E \otimes E') = \chi(F)$.

PART III

§11. Clifford bundles

In this section and the next we shall consider the Thom complex of a vector bundle. If V is a (real) Euclidean vector bundle over X (i.e. the fibres have a positive definite inner product) we denote by X^V the one-point compactification of V and refer to it as the Thom complex of V. It inherits a natural structure of CW-complex (with base point) from that of X. An alternative description which is also useful is the following. Let $B(V)$, $S(V)$ denote the unit ball and unit sphere bundles of V, then X^V may be identified with $B(V)/S(V)$. A technical point which arises here is that $(B(V), S(V))$ is not obviously a CW-pair. However the following remarks show that there is no real loss of generality in assuming that $(B(V), S(V))$ is a CW-pair.

1. If X is a differentiable manifold then $(B(V), S(V))$ is a manifold with boundary and hence triangulable.

2. Every vector bundle over a finite complex is induced by a map of the base space into a differentiable manifold (namely a Grassmannian).

There are of course more satisfactory ways of dealing with this point but a lengthy discussion would be out of place in this context.

With our assumption therefore we have the isomorphism

$$\tilde{K}(X^V) \cong K(B(V), S(V))$$

where $\tilde{K}$ denotes K modulo the base point.

Since each fibre V_x of V is a vector space with a positive definite quadratic form Q_x, we can form the *Clifford bundle* $C(V)$ of V. This will be a bundle of algebras whose fibre at

x is the Clifford algebra $C(-Q_x)$. Contained in $C(V)$ are bundles of groups, Pin(V) and Spin(V). All these bundles are associated to the principal $O(k)$-bundle of V by the natural action of $O(k)$ on C_k, Pin(k), Spin(k).

By a graded Clifford module of V we shall mean a Z_2-graded vector bundle E (real or complex) over X which is a graded $C(V)$-module. In other words $E = E^0 \oplus E^1$ and we have vector bundle homomorphisms

$$V \otimes_{\mathbf{R}} E^0 \to E^1, \qquad V \otimes_{\mathbf{R}} E^1 \to E^0$$

(denoted simply by $v \otimes e \to v(e)$) such that

$$v(v(e)) = -\|v\|^2 e \tag{1}$$

For notational convenience we shall consider real modules only. The complex case is entirely parallel.

Let $E = E^0 \oplus E^1$ be a graded $C(V)$-module. Then E^0 is a Spin(V)-module and by integration over the fibres of Spin(V) we can give E^0 a metric invariant under Spin(V). This can then be extended to a metric on E invariant under Pin($\dot{V}$) and such that E^0 and E^1 are orthogonal complements. If now $v \in V_x$ and $v \neq 0$ then $v/\|v\| \in \mathrm{Pin}(V_x)$. Hence we deduce, for all $v \in V_x$ and $e \in E_x$,

$$\|ve\| = \|v\| \cdot \|e\|.$$

This, together with (1), implies that the adjoint of

$$v : E^0_x \to E^1_x \qquad \text{is} \qquad -v : E^1_x \to E^0_x.$$

Let $\pi : B(V) \to X$ be the projection map and let

$$\sigma(E) : \pi^* E^1 \to \pi^* E^0$$

be given by multiplication by $-v$, i.e.

$$\sigma(E)_v(e) = -ve.$$

Then

$$0 \longrightarrow \pi^* E^1 \xrightarrow{\sigma(E)} \pi^* E^0 \longrightarrow 0 \tag{2}$$

is an element of $\mathscr{D}_1(B(V), S(V))$ and hence defines an element $\chi_V(E)$ of $KO(B(V), S(V))$, or equivalently an element of $\widetilde{KO}(X^V)$. If the $C(V)$-module structure of E extends to a $C(V \oplus 1)$-module structure (1 denoting the trivial line-bundle) then the isomorphism $\sigma(E)$ extends from $S(V)$ to $S^+(V \oplus 1)$ the 'upper hemisphere' of $S(V \oplus 1)$. Since the pairs $(B(V), S(V))$ and $(S^+(V \oplus 1), S(V))$ are clearly equivalent it follows that $\chi_V(E)$ will, in this case, be zero.

Following §5, which is the special case $X = $ point, we now define $M(V)$ as the Grothendieck group of graded $C(V)$-modules, and we let $A(V)$ denote the cokernel of the natural homomorphism

$$M(V \oplus 1) \to M(V).$$

Then the construction described above gives rise to a homomorphism

$$\chi_V : A(V) \to \widetilde{KO}(X^V).$$

This homomorphism is of fundamental importance in the theory, and our next step is to discuss its multiplicative properties.

Let V, W be Euclidean vector bundles over X, Y respectively. Then we have a natural homeomorphism

$$X^V \divideontimes Y^W \approx X \times Y^{V \oplus W}$$

which induces a homomorphism (or 'cup-product')

$$\widetilde{KO}(X^V) \otimes \widetilde{KO}(Y^W) \to \widetilde{KO}(X \times Y^{V \oplus W}).$$

If $a \in \widetilde{KO}(X^V)$, $b \in \widetilde{KO}(Y^W)$ the image of $a \otimes b$ will simply be written as ab.

PROPOSITION (11.1). *The following diagram commutes*

$$
\begin{array}{ccc}
A(V) \otimes A(W) & \xrightarrow{\;\mu\;} & A(V \oplus W) \\
\downarrow{\scriptstyle \chi_V \otimes \chi_W} & & \downarrow{\scriptstyle \chi_{V \oplus W}} \\
\widetilde{KO}(X^V) \otimes \widetilde{KO}(Y^W) & \longrightarrow & \widetilde{KO}(X \times Y^{V \oplus W})
\end{array}
$$

where μ is induced by the graded tensor product of Clifford modules. Thus

$$\chi_{V \oplus W}(E \,\hat{\otimes}\, F) = \chi_V(E)\chi_W(F).$$

Proof. Let E, F be graded $C(V)$- and $C(W)$-modules and let them both be given invariant metrics as above. Applying Proposition (10.2) it follows that

$$\chi_V(E) \cdot \chi_W(F) \in KO(B(V) \times B(W), B(V) \times S(W) \cup S(V) \times B(W))$$

is equal to $\chi(G)$ where

$$G \in \mathscr{D}_1(B(V) \times B(W), B(V) \times S(W) \cup S(V) \times B(W))$$

is defined by

$$G_1 = \pi^*(E^0 \otimes F^1 \oplus E^1 \otimes F^0)$$
$$G_0 = \pi^*(E^0 \otimes F^0 \oplus E^1 \otimes F^1)$$

and $\tau : G_1 \to G_0$ is given by

$$\tau = \begin{pmatrix} 1 \otimes \sigma(F), & \sigma(E) \otimes 1 \\ -\sigma(E) \otimes 1, & 1 \otimes \sigma(F) \end{pmatrix}$$

(since $\sigma(E)^* = -\sigma(E)$, $\sigma(F)^* = -\sigma(F)$). Thus, at a point $v \oplus w \in V \oplus W$, τ is given by the matrix

$$\tau_{v \oplus w} = \begin{pmatrix} 1 \otimes -w, & -v \otimes 1 \\ v \otimes 1, & 1 \otimes -w \end{pmatrix} = -\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 1 \otimes w, & v \otimes 1 \\ v \otimes 1, & -1 \otimes w \end{pmatrix}$$

where v, w denote module multiplication by v, w. Hence

$$\tau = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\sigma(E \,\hat{\otimes}\, F) \qquad\qquad (3)$$

On the other hand let $B'(V \oplus W)$ denote the ball of radius 2 and let

$$S'(V \oplus W) = \overline{B'(V \oplus W) - B(V \oplus W)},$$

so that the inclusions

$$i : B(V \oplus W), S(V \oplus W) \to B'(V \oplus W), S'(V \oplus W)$$

$$j : B(V) \times B(W), B(V) \times S(W) \cup S(V) \times B(W) \to B'(V \oplus W), S'(V \oplus W)$$

are both homotopy equivalences. Let

$$H \in \mathscr{D}_1(B'(V \oplus W), S'(V \oplus W))$$

be defined by $\sigma(E \hat{\otimes} F)$. Then $i^*(H)$ defines the element $\chi_{V \oplus W}(E \hat{\otimes} F)$, while (3) shows that $j^*(H)$ and G define the same element of $KO(B(V) \times B(W), B(V) \times S(W) \cup S(V) \times B(W))$. Hence we have

$$\chi_V(E) \cdot \chi_W(F) = \chi_{V \oplus W}(E \otimes F)$$

as required.

Suppose now that P is a principal Spin(k)-bundle over X, $V = P \times_{\text{Spin }(k)} \mathbf{R}^k$ the associated vector bundle. If M is a graded C_k-module then $E = P \times_{\text{Spin }(k)} M$ will be a graded $C(V)$-module. In this way we obtain a homomorphism of groups

$$\beta_P : A_k \to A(V).$$

Similarly in the complex case we obtain

$$\beta_P^c : A_k^c \to A^c(V).$$

PROPOSITION (11.2). *Let* P, P' *be* Spin (k), Spin (l) *bundles over* X, X' *and let* $V = P \times_{\text{Spin }(k)} \mathbf{R}^k$, $V' = P' \times_{\text{Spin }(l)} \mathbf{R}^l$. *Let* P'' *be the* Spin($k + l$)-*bundle over* $X \times X'$ *induced from* $P \times P'$ *by the standard homomorphism*

$$\text{Spin } (k) \times \text{Spin } (l) \to \text{Spin}(k + l).$$

Then if $a \in A_k$, $b \in A_l$, *we have*

$$\beta_{P''}(ab) = \beta_P(a)\beta_{P'}(b).$$

A similar formula holds for β_P^c.

The verification of this result is straightforward and is left to the reader.

Let $\alpha_P : A_k \to \widetilde{KO}(X^V)$ be defined by $\alpha_P = \chi_V \beta_P$.

Then from Propositions (11.1) and (11.2) we deduce

PROPOSITION (11.3). *With the notation of* (11.2) *we have*

$$\alpha_{P''}(ab) = \alpha_P(a)\alpha_{P'}(b),$$

and a similar formula for α_P^c.

If we apply all the preceding discussion to the case when X is a point (and P denotes the trivial Spin(k)-bundle) we get maps

$$\alpha : A_k \to \widetilde{KO}(S^k) \qquad \text{in the real case}$$
$$\alpha^c : A_k^c \to \widetilde{K}(S^k) \qquad \text{in the complex case.}$$

Proposition (11.3) then yields the following corollary, as a special case:

COROLLARY (11.4). *The maps*

$$\alpha : A_* \to \sum_{k \geq 0} KO^{-k}(\text{point})$$

$$\alpha^c : A_*^c \to \sum_{k \geq 0} K^{-k}(\text{point})$$

are ring homomorphisms.

Now the rings A_* and A_*^c were explicitly determined in §6 (Theorems (6.9) and (6.10)). On the other hand the additive structure of $B_* = \sum KO^{-k}(\text{point})$ and $B_*^c = \sum K^{-k}(\text{point})$ was determined in [5], while their multiplicative structure was (essentially) given in [6]. These results may be summarized as follows:

(i) B_*^c is the polynomial ring generated by an element $x \in B_2^c$ corresponding to the reduced Hopf bundle on $P_1(\mathbf{C}) = S^2$;

(ii) B_* contains a polynomial ring $\mathbf{Z}[y]$ with $y \in B_8$, and $y \to x^4$ under the complexification map $B_* \to B_*^c$;

(iii) As a module over $\mathbf{Z}[y]$, B_* is freely generated by elements $1, a, b, z$ where $a \in B_1$, $b \in B_2$, $z \in B_4$, subject to the relations $2a = 0$, $2b = 0$.

If we use Stiefel-Whitney classes then a simple calculation shows that

$$w_2(a^2) \neq 0$$

where we regard $a^2 \in \tilde{K}(S^2)$. Thus we must have $a^2 = b$.

Consider now the ring homomorphism

$$\alpha^c : A_*^c \to B_*^c.$$

It is immediate from the definition of α^c that $\alpha^c(\mu^c)$ gives the reduced Hopf bundle on S^2. Hence from (6.10) we deduce that α^c is an isomorphism.

Consider next the ring homomorphism

$$\alpha : A_* \to B_*.$$

Because of the commutative diagram

$$
\begin{array}{ccc}
A_* & \xrightarrow{\;\alpha\;} & B_* \\
\downarrow & & \downarrow \\
A_*^c & \xrightarrow{\;\alpha^c\;} & B_*^c
\end{array}
$$

the results on α^c together with (6.11) and (ii) above imply that

$$\alpha(\lambda) = y.$$

Similarly using (6.9) and (iii) above we get

$$\alpha(\mu) = z.$$

It remains to consider $\alpha(\xi)$ and $\alpha(\xi^2)$. But as in the complex case it is immediate that $\alpha(\xi)$ is the reduced Hopf bundle on $P_1(\mathbf{R}) = S^1$. Since a is the unique non-zero element of B_1 we must therefore have

$$\alpha(\xi) = a.$$

Using (6.9) and (ii), (iii) above it follows that α is an isomorphism. Thus we have established:

THEOREM (11.5). *The maps*

$$\alpha : A_* \to \sum_{k \geqslant 0} KO^{-k}(\text{point})$$

and

$$\alpha^c : A_*^c \to \sum_{k \geqslant 0} K^{-k}(\text{point})$$

are ring isomorphisms.

As remarked in the introduction this theorem shows clearly the intimate relation between Clifford algebras and the periodicity theorems. It is to be hoped that a less computational proof of (11.5) will eventually be found and that the theorem will then appear as the foundation stone of K-theory.

We shall conclude this section by taking up again the relation between Clifford and Grassmann algebras mentioned in §3. Let V be a complex vector bundle over X, $\Lambda(V)$ its Grassmann bundle, i.e. the bundle whose fibre at $x \in X$ is the Grassmann algebra $\Lambda(V_x)$. Let $\pi : V \to X$ be the projection and consider the complex

$$\Lambda_V \quad : \longrightarrow \pi^*(\Lambda^r(V)) \overset{d}{\longrightarrow} \pi^*(\Lambda^{r+1}(V)) \longrightarrow$$

where d is given by the exterior product:

$$d_v(w) = v \wedge w \qquad\qquad v \in V_x, \; w \in \Lambda(V_x).$$

This is acyclic outside the zero-section and hence defines an element

$$\chi(\Lambda_V) \in \tilde{K}(X^V)$$

On the other hand, if we give V a Hermitian metric, and use the homomorphism

$$\bar{l} : U(k) \to \mathrm{Spin}^c(2k) \qquad\qquad k = \dim_C V$$

we obtain a principal $\mathrm{Spin}^c(2k)$-bundle P over X, and hence a homomorphism

$$\alpha_P^c : A_{2k}^c \to \tilde{K}(X^V).$$

The relation between α_P^c and $\chi(\Lambda_V)$ is then given by:

PROPOSITION (11.6). $\chi(\Lambda_V) = \alpha_P^c((\mu^c)^k)$.

Proof. Applying the construction at the end of §9 for the inverse of

$$j_k : L_1 \to L_k$$

to the complex Λ_V, we obtain a sequence

$$E = (0 \longrightarrow E_1 \overset{\sigma}{\longrightarrow} E_0 \longrightarrow 0)$$

where

$$E_0 = \pi^*\Lambda^k \oplus \pi^*\Lambda^{k-2} \oplus \ldots$$
$$E_1 = \pi^*\Lambda^{k-1} \oplus \pi^*\Lambda^{k-3} \oplus \ldots$$
$$\sigma_v = d_v + \delta_v.$$

In fact we could equally well have taken

$$\sigma_v = d_v - \delta_v$$

in §5. In view of (5.10), (5.11) and the final remark of §5 this shows that

$$\chi(\Lambda_V) = \alpha_P^c((\mu^c)^k)$$

as required.

Remark. The multiplicative property of Grassmann algebras:

$$\Lambda(V \oplus W) \cong \Lambda(V) \otimes \Lambda(W)$$

can be used directly to establish a product formula for $\chi(\Lambda_V)$. This corresponds of course to (11.3).

§12. The Thom isomorphism

We begin with some brief remarks on the Thom isomorphism for general cohomology theories.

Let F be a generalized cohomology theory with products. Thus $F^{\#}(X) = \sum F^q(X)$ is a graded anti-commutative ring with identity and $F^{\#}(X, Y)$ is a graded $F^{\#}(X)$-module. Moreover the product must be compatible with the coboundary in the sense that

$$\delta(ab) = \delta(a) \cdot b + (-1)^{\alpha} a \delta b$$

where $\alpha = \deg a$ and a, b belong to suitable F-groups.

In $\tilde{F}^n(S^m)$ we have a canonical element σ^n which corresponds to the identity element $1 = \sigma^0 \in F^0(\text{point}) = \tilde{F}^0(S^0)$ under suspension. $\tilde{F}^{\#}(S^n)$ is then a free module over $F^{\#}(\text{point})$ generated by σ^n.

Suppose now that V is a real vector bundle of dimension n over X. We choose a metric in V and introduce the pair $(B(V), S(V))$ (or the Thom complex X^V). For each point $P \in X$ we consider the inclusion

$$i_P : P^V \to X^V$$

and the induced homomorphism

$$i_P^* : \tilde{F}^n(X^V) \to \tilde{F}^n(P^V).$$

Suppose now that V is *oriented*, then for each $P \in X$ we have a well-defined suspension isomorphism

$$S_P : F^0(P) \to \tilde{F}^n(P^V).$$

We let $\sigma_P^n = S_P(1)$. We shall say that V is *F-orientable* if there exists an element $\mu_V \in \tilde{F}^n(X^V)$ such that, for all $P \in X$,

$$i_P^*(\mu_V) = \sigma_P^n.$$

A definite choice of such a μ_V will be called an *F-orientation* of V. Then we have the following general Thom isomorphism theorem:

THEOREM (12.1). *Let V be an F-oriented bundle over X with orientation class μ_V. Then $\tilde{F}^{\#}(X^V)$ is a free $F^{\#}(X)$-module with generator μ_V.*

Proof. Multiplication by μ_V defines a homomorphism of the F-spectral sequence of X into the $\tilde{F}$-spectral sequence of X^V which is an isomorphism on E_2 (the Thom isomorphism for cohomology) and hence on E_∞. Hence

$$a \to \mu_V a$$

gives an isomorphism $F^{\#}(X) \to \tilde{F}^{\#}(X^V)$ as stated.†

Applying (12.1) to the special theories K, KO we obtain††:

THEOREM (12.2). *Let V be an oriented real vector bundle of dimension n over X. Then*

(i) *if $n \equiv 0 \mod 2$ and there is an element $\mu_V \in \tilde{K}(X^V)$ whose restriction to $\tilde{K}(P^V)$ for each $P \in X$ is the generator, then $\tilde{K}^*(X^V)$ is a free $K^*(X)$-module generated by μ_V;*

---

† One can also use the Mayer–Victoris sequence instead of the spectral sequence.
†† We use K^*, KO^* to denote the sum of K^q, KO^q over the period (2, or 8) in distinction with K^* which is the sum over all integers.

(ii)   *if $n \equiv 0 \bmod 8$ and there is an element $\mu_V \in \widetilde{KO}(X^V)$ whose restriction to each $\widetilde{KO}(P^V)$ for each $P \in X$ is the generator, then $\widetilde{KO}^*(X^V)$ is a free $\widetilde{KO}^*(X)$-module generated by μ_V.*

Remark. Since K^0 (point) $\cong KO^0$(point) $\cong \mathbf{Z}$ these groups are generated by the identity element of the ring. This element and its suspensions are what we mean by *the generator*.

Suppose now that V has a Spin-structure, i.e., that we are given a principal Spin(n)-bundle P and an isomorphism

$$V \cong P \times_{\mathrm{Spin}(n)} \mathbf{R}^n.$$

Then from §11 we have a homomorphism

$$\alpha_P : A_n \to \widetilde{KO}(X^V).$$

Similarly if V has a Spin^c-structure, i.e. we are given a principal Spin^c(n)-bundle P and an isomorphism

$$V \cong P \times_{\mathrm{Spin}^c(n)} \mathbf{R}^n$$

then we get a homomorphism

$$\alpha_P^c : A_n^c \to \widetilde{K}(X^V).$$

In the real case assume $n = 8k$ and in the complex case $n = 2k$, and put

$$\mu_V = \alpha_P(\lambda^k)$$

$$\mu_V^c = \alpha_P^c((\mu^c)^k).$$

Then by the naturality of α_P, α_P^c and Theorem (11.1) we see that μ_V, μ_V^c define KO and K orientations of V and hence (12.2) gives:

THEOREM (12.3). (i) *Let P be a* Spin($8k$)-*bundle $V = P \times_{\mathrm{Spin}(8k)} \mathbf{R}^{8k}$. Then $\widetilde{KO}^*(X^V)$ is a free $KO^*(X)$-module generated by μ_V;* (ii) *Let P be a* Spin^c($2k$)-*bundle, $V = P \times_{\mathrm{Spin}^c(2k)} \mathbf{R}^{2k}$. Then $\widetilde{K}^*(X^V)$ is a free $K^*(X)$-module generated by μ_V^c.*

Remark. It is easy to see, by considering the first differentials in the spectral sequence, that the existence of a Spin (Spin^c)-structure is necessary for $KO(K)$-orientability. Theorem (12.3) shows that these conditions are also sufficient.

(12.3) together with (11.3) shows that, for Spin bundles, we have a Thom isomorphism for KO and K with all the good formal properties. It is then easy to show that for Spin-manifolds one can define a functorial homomorphism

$$f_! : KO^*(Y) \to KO^*(X) \qquad \text{for maps} \qquad f : Y \to X,$$

and similarly for Spin^c-manifolds in K-theory. This improves the results of [2].

§13. The sphere

The purpose of these next sections is to identify the generator of $\widetilde{KO}(X^V)$ (for a V with Spinor structure and dim $\equiv 0 \bmod 8$) given in §12 with that given in [7]. Essentially we have to study the sphere as a homogeneous space of the spinor group. This actually leads to simpler formulae (Proposition (13.2)) for the characteristic map of the tangent bundle than one gets from using the orthogonal group.

We recall first the existence of an isomorphism $\phi : C_k \to C^0_{k+1}$ (Proposition (5.2)) and we note that, on C^0_k, ϕ coincides with the standard inclusion $C_k \to C_{k+1}$. We introduce the following notation: $K = \mathrm{Spin}(k+1)$, $H = \phi(\mathrm{Pin}(k)) = H^0 + H^1$, $H^0 = \phi(\mathrm{Spin}(k))$ (where $+$ here denotes *disjoint* sums of the two components).

$$S^k = \text{unit sphere in } \mathbf{R}^{k+1}$$

$$S_+ = S^k \cap \{x_{k+1} \geqslant 0\}, \qquad S_- = S^k \cap \{x_{k+1} \leqslant 0\}$$

$$S^{k-1} = S^+ \cap S^-.$$

We consider S^k as the orbit space of e_{k+1} for the group K operating on $\mathbf{R}^{k+1}$ by the representation ρ. Thus $K/H^0 = S^k$ and we have the principal H^0-bundle

$$K \xrightarrow{\ \pi\ } K/H^0.$$

Let $K_+ = \pi^{-1}(S_+)$, $K_- = \pi^{-1}(S_-)$. We shall give explicit trivializations of K_+ and K_-, and the identification will then give the 'characteristic map' of the sphere.

We parametrize S_+ by use of 'polar co-ordinates':

$$(x, t) = \mathrm{Cos}\, t . e_{k+1} + \mathrm{Sin}\, t . x \qquad x \in S_{k-1}, \qquad\qquad 0 \leqslant t \leqslant \pi/2.$$

Now define a map $\beta_+ : S_+ \times H^0 \to K_+$ by

$$\beta_+(x, t, h^0) = (-\mathrm{Cos}\, t/2 + \mathrm{Sin}\, t/2 . x\, e_{k+1}) h^0.$$

Since

$$\rho((-\mathrm{Cos}\, t/2 + \mathrm{Sin}\, t/2 . x e_{k+1}) h^0) e_{k+1}$$

$$= (-\mathrm{Cos}\, t/2 + \mathrm{Sin}\, t/2 . x e_{k+1}) e_{k+1} (-\mathrm{Cos}\, t/2 + \mathrm{Sin}\, t/2 . x e_{k+1})^{-1}$$

$$= (-\mathrm{Cos}\, t/2 + \mathrm{Sin}\, t/2 . x e_{k+1})^2 e_{k+1}$$

$$= \mathrm{Cos}\, t . e_{k+1} + \mathrm{Sin}\, t . x = (x, t),$$

it follows that β_+ is an H^0-bundle isomorphism.

Similarly we parametrize S_- by

$$(x, t) = -\mathrm{Cos}\, t . e_{k+1} + \mathrm{Sin}\, t . x \qquad x \in S_{k-1}, \qquad\qquad 0 \leqslant t \leqslant \pi/2.$$

Note that for points of S_{k-1} the two parametrizations agree (putting $t = \pi/2$). Now define a map $\beta_- : S_- \times H^1 \to K_-$ by

$$\beta_-(x, t, h^1) = (\mathrm{Cos}\, t/2 + \mathrm{Sin}\, t/2 . x e_{k+1}) h^1.$$

Since

$$\rho((\mathrm{Cos}\, t/2 + \mathrm{Sin}\, t/2 . x e_{k+1}) h^1) e_{k+1}$$

$$= (\mathrm{Cos}\, t/2 + \mathrm{Sin}\, t/2 . x e_{k+1})(-e_{k+1})(\mathrm{Cos}\, t/2 + \mathrm{Sin}\, t/2 . x e_{k+1})^{-1}$$

$$= -(\mathrm{Cos}\, t/2 + \mathrm{Sin}\, t/2 . x e_{k+1})^2 e_{k+1} = -\mathrm{Cos}\, t . e_{k+1} + \mathrm{Sin}\, t . x,$$

it follows that β_- is an H^0-bundle isomorphism.

Putting $t = \pi/2$ above we get

$$\beta_+(x, \pi/2, h^0) = (-\mathrm{Cos}\, \pi/4 + \mathrm{Sin}\, \pi/4 . x e_{k+1}) h^c$$

$$\beta_-(x, \pi/2, h^1) = (\mathrm{Cos}\, \pi/4 + \mathrm{Sin}\, \pi/4 . x e_{k+1}) h^1.$$

These are the same point of $K_+ \cap K_-$ if

$$h^1 = -(\mathrm{Cos}\ \pi/4 - \mathrm{Sin}\ \pi/4 . xe_{k+1})^2 h^0$$
$$= xe_{k+1}h^0.$$

Thus we have a commutative diagram

$$
\begin{array}{ccc}
S_{k-1} \times H^0 & \xrightarrow{\ \beta_+\ } & K_+ \cap K_- \\
\downarrow{\scriptstyle \delta} & & \downarrow{\scriptstyle 1} \\
S_{k-1} \times H^1 & \xrightarrow{\ \beta_-\ } & K_+ \cap K_-
\end{array}
$$

where

$$\delta(x, h^0) = (x, xe_{k+1}h^0). \tag{1}$$

LEMMA (13.1). *If we regard H^0 as (left) operating on both factors of $S_+ \times H^0$ and $S_- \times H^1$, then β_+ and β_- are compatible with left operation.*

Proof (i) $\beta_+ g(x, t, h^0) = \beta_+(g(x), t, gh^0)$

$$= (-\mathrm{Cos}\ t/2 + \mathrm{Sin}\ t/2 . gxg^{-1}e_{k+1})gh^0$$
$$= g\beta_+(x, t, h^0)$$

where $g \in H^0$ and $g(x) = \rho_{k+1}(g) . x = gxg^{-1}$.

(ii) $\beta_- g(x, t, h^1) = \beta_-(\mathrm{Cos}\ t/2 + \mathrm{Sin}\ t/2 . gxg^{-1}e_{k+1})gh^1$

$$= g\beta_-(x, t, h^1).$$

Since $\phi(x) = xe_{k+1}$ for $x \in \mathbf{R}^k$ formula (1) above can be rewritten

$$\delta(x, g) = (x, xg) \qquad x \in \mathbf{R}^k, g \in \mathrm{Spin}(k).$$

Summarizing our results therefore we get:

PROPOSITION (13.2). *The principal $\mathrm{Spin}(k)$-bundle $\mathrm{Spin}(k+1) \to S^k$ is isomorphic to the bundle obtained from the two bundles*

$$S_+ \times \mathrm{Pin}^0(k) \to S_+$$
$$S_- \times \mathrm{Pin}^1(k) \to S_-$$

by the identification

$$(x, g) \leftrightarrow (x, xg) \qquad for\ x \in S^{k-1}, g \in \mathrm{Pin}^0(k).$$

Moreover this isomorphism is compatible with left multiplication by $\mathrm{Spin}(k)$.

Here $\mathrm{Pin}^0(k) = \mathrm{Spin}(k)$ and $\mathrm{Pin}^1(k)$ are the two components of $\mathrm{Pin}(k)$.

§14. Spinor bundles

Let P^0 be a principal $\mathrm{Spin}(k)$-bundle over X and put

$$P^1 = P^0 \times_{\mathrm{Spin}(k)} \mathrm{Pin}^1(k), \qquad Q = P^0 \times_{\mathrm{Spin}(k)} \mathrm{Spin}(k+1)$$
$$T^k = P^0 \times_{\mathrm{Spin}(k)} S^k = T_+ \cup T_-, \text{ where}$$
$$T_+ = P^0 \times_{\mathrm{Spin}(k)} S_+, \qquad T_- = P^0 \times_{\mathrm{Spin}(k)} S_-$$
$$\pi_+ : T_+ \to X, \qquad \pi_- : T_- \to X \text{ the projections.}$$

Consider now the two commutative diagrams

$$
\begin{array}{ccc}
P^0 \times_{\mathrm{Spin}(k)}(S_+ \times \mathrm{Pin}^0(k)) & \xrightarrow{\ \lambda^0\ } & P^0 \\
\downarrow & & \downarrow \\
T_+ & \xrightarrow{\ \pi_+\ } & X
\end{array}
$$

$$
\begin{array}{ccc}
P^0 \times_{\mathrm{Spin}(k)}(S_- \times \mathrm{Pin}^1(k)) & \xrightarrow{\ \lambda^1\ } & P^1 \\
\downarrow & & \downarrow \\
T_- & \xrightarrow{\ \pi_-\ } & X.
\end{array}
$$

where $\lambda^i(p, s, g) = pg$, $p \in P^0$, $s \in S_\pm$, $g \in \mathrm{Pin}^i(k)$, $i = 0, 1$.

These allow us to identify the two Spin(k) bundles occurring in the first column with $\pi_+^*(P^0)$ and $\pi_-^*(P^1)$ respectively. Now because of the left compatibility in (13.2) we immediately get

PROPOSITION (14.1). *The principal* Spin(k)-*bundle* $Q \to T^k$ *is isomorphic to the bundle obtained from the two bundles*

$$
\pi_+^*(P^0) \longrightarrow T_+, \qquad \pi_-^*(P^1) \longrightarrow T_-
$$

by the identification

$$
(p, s, g) \longleftrightarrow (p, s, sg)
$$

for $s \in S^{k-1}$, $g \in \mathrm{Spin}(k)$ *and* $p \in P^0$.

Now suppose that $M = M^0 \oplus M^1$ is a graded C_k-module. Then we have a natural isomorphism

$$
M^1 \cong \mathrm{Pin}^1(k) \times_{\mathrm{Spin}(k)} M^0.
$$

Hence

$$
\begin{aligned}
P^1 \times_{\mathrm{Spin}(k)} M^0 &= P^0 \times_{\mathrm{Spin}(k)} \mathrm{Pin}^1(k) \times_{\mathrm{Spin}(k)} M^0 \\
&\cong P^0 \times_{\mathrm{Spin}(k)} M^1.
\end{aligned}
$$

From (14.1) and this isomorphism we obtain:

PROPOSITION (14.2). *The vector bundle* $Q \times_{\mathrm{Spin}(k)} M^0$ *over* T^k *is isomorphic to the bundle obtained from the two bundles*

$$
\pi_+^*(P^0 \times_{\mathrm{Spin}(k)} M^0) \to T_+, \qquad \pi_-^*(P^0 \times_{\mathrm{Spin}(k)} M^1) \to T_-
$$

by the identification

$$
(p, s, m) \longleftrightarrow (p, s, sm) \qquad for \qquad p \in P^0,\ s \in S^{k-1},\ m \in M^0.
$$

Note. Here we have identified $\pi_+^*(P^0)$ with $P^0 \times S_+$, and $\pi_+^*(P^0 \times_{\mathrm{Spin}(k)} M^0)$ with $\pi_+^*(P^0) \times_{\mathrm{Spin}(k)} M^0$ etc.

Let us consider now the construction of §11 which assigned to any graded C_k-module M and any Spin(k)-bundle P^0 an element $\alpha_{P^0}(M) \in KO(B(V), S(V))$ where $V = P^0 \times_{\mathrm{Spin}(k)} \mathbf{R}^k$. This construction depended on the 'difference bundle' of §9. In our present case the spaces

A, X_0, X_1 of §9 can be effectively replaced by T^k, T_+, T_- and we see from (14.2) (and the fact that $s^2 = -1$ for $s \in S_{k-1}$) that the bundle F of §9 is isomorphic to the bundle $Q \times {}_{\text{Spin}(k)} M^0$. Now from the split exact sequence of the pair (T^k, T_-) and the isomorphisms

$$KO(T^k, T_-) \cong KO(T_+, T^{k-1}) \cong KO(B(V), S(V))$$

we obtain a natural projection

$$KO(T^k) \to KO(B(V), S(V)).$$

Then what we have shown may be stated as follows:

THEOREM (14.3). *Let* P^0 *be a principal* Spin(k)-*bundle*, M *a graded* C_k-*module*, $Q = P^0 \times {}_{\text{Spin}(k)} \text{Spin}(k+1)$, $V = P^0 \times {}_{\text{Spin}(k)} \mathbf{R}^k$, $T^k = Q/\text{Spin}(k)$, $E^0 = Q \times {}_{\text{Spin}(k)} M^0$, $p : KO(T^k) \to KO(B(V))$, $S(V)$ *the natural projection, then*

$$\alpha_{P^0}(M) = p(E^0).$$

If $k \equiv 0 \bmod 8$ and M is an irreducible $(+1)$-module then $p(E^0)$ is the element of $KO(B(V), S(V))$ used in [7] as the fundamental class. Thus (14.3) implies that this class coincides with our class μ_V. For some purposes, such as the behaviour under our definition of μ_V is more convenient. For others, such as computing the effect of representations, the definition in [7] is better. (14.3) enables us to switch from one to the other.

The proof of (14.3) carries over without change to the complex case, Spin being replaced by Spin^c throughout.

§15. Geometric interpretation of Clifford modules

Consider the data of §11. Thus V is a vector-bundle over X, $C(V)$ the corresponding Clifford bundle, and E a graded real Clifford module for V. The construction of χ_V in that section then depended on a particular geometric interpretation of the pairing

(15.1) $$V \otimes E^1 \to E^0$$

induced by the $C(V)$-structure on E. More precisely we passed from (15.1) to the family of maps

(15.2) $$S(V_x) \times E_x^1 \to E_x^0 \qquad x \in X,$$

which describe a definite isomorphism along $S(V)$, of E^0 and E^1 lifted to $B(V)$, and so by the difference construction a definite element $\chi_V(E) \in KO(B(V), S(V))$.

There are two other geometric interpretations of (15.2) which we will discuss here briefly. The first one leads to a rather uniform description of the bundles on stunted projective spaces, while the second one explains the relation between Clifford modules and the vector field problem.

A. *The generalized* χ_V.

Let V be a Euclidean (real) vector bundle over X, $S(V)$ its unit sphere bundle. The group Z_2 then acts on $S(V)$ by the antipodal map, and we denote the projective bundle $S(V)/Z_2$ by $P(V)$. The projection $P(V) \to X$ will be denoted by π, and $\xi(V)$ shall stand for the line bundle induced over $P(V)$ by the nontrivial representation of Z_2 on $\mathbf{R}^1$:

$$\xi(V) = S(V) \times {}_{Z_2} \mathbf{R}^1$$

Consider now the data at the beginning of this section, in particular the induced family of maps:

$$S(V_x) \times E_x^1 \to E_x^0 \qquad x \in X.$$

We can clearly divide by Z_2 on the left due to the bilinearity of the inducing map. Thus we obtain maps

(15.3) $$S(V_x) \times_{Z_2} E_x^1 \to E_x^0 \qquad x \in X,$$

which may be interpreted directly as an explicit isomorphism

$$\phi(V, E) : \xi(V) \otimes \pi^*(E^1) \to \pi^*(E^0).$$

We now let $W \subset V$ be a sub-bundle, and consider a graded $C(W)$-module E. The bundles $\xi(V) \otimes \pi^*E^1$ and π^*E^0 then become explicitly isomorphic along $P(W) \subset P(V)$ by means of $\phi(W, E)$, and so determine a well-defined difference element $\chi(V, W)E \in KO(P(V), P(W))$.

The linear extension of this construction now leads to a homomorphism,

(15.4) $$\chi(V, W) : M(W) \to KO(P(V), P(W)),$$

and an analogous homomorphism

$$\chi^c(V, W) : M^c(W) \to K(P(V), P(W))$$

in the complex case. (15.4) is the desired generalization of the χ_W in §11. Before justifying this assertion, we remark that $\chi(V, W)$ clearly vanishes on those $C(W)$-modules which are restrictions of $C(V)$-modules. Hence if we set $A(V, W)$ equal to the cokernel of the restriction map $M(V) \xrightarrow{i^*} M(W)$, then $\chi(V, W)$ induces a homomorphism

(15.5) $$A(V, W) \to KO(P(V), P(W)).$$

To see that the operation $\chi(V, W)$ indeed generalizes our earlier χ, one may proceed as follows: Let $V = W \oplus 1$, and let $f : B(W) \to P(V)$ be the fibre map which sends $w \in W_x$, into the line spanned by $(w, (1 - \|w\|^2))$ in $P(V)$. Thus f induces an isomorphism of $B(W)/S(W)$ with $P(V)/P(W)$. Now one just checks that the following diagram is commutative:

(15.6)
$$\begin{array}{ccc}
M(W) & \xrightarrow{\chi(V,W)} & KO\{P(V), P(W)\} \\
\Big\Updownarrow & & f^* \Big\Updownarrow \\
M(W) & \xrightarrow{\chi_W} & KO(B(W), S(W)).
\end{array}$$

It would be possible to extend a considerable portion of our work on χ_W to $\chi(W, V)$, but this does not seem justified by any application at present. However we wish to draw attention to the following property of $\chi(V, W)$.

PROPOSITION (15.7). *Let X be a point. Then the sequence*

(15.8) $$M(V) \xrightarrow{i^*} M(W) \xrightarrow{\chi(V,W)} KO(P(V)P(W)) \to 0$$

is exact. A similar result holds in the complex case.

In other words, over a point, the relation $A(V, W) \cong KO(P(V)/P(W))$ holds. As we gave a complete survey of the groups M_k and their inclusions in §5, this proposition gives the desired uniform description of the KO (and K) of a stunted real projective space. For example, taking dim $V = k$, dim $W = 1$, we obtain

$$\widetilde{KO}((P_{k+1}) \cong KO(P_{k-1}, P_0)) \cong Z_{a_k},$$

where a_k is the kth. Radon–Hurwitz number.

We know of no really satisfactory proof of proposition (15.7), primarily because we know of no good algebraic description of the higher KO^i of these spaces. On the other hand it is easy to show that $A(V, W) \to KO(P(V), P(W))$ is onto. For this purpose consider the diagram associated with a triple of vector-spaces $W \subset V' \subset V$

$$
\begin{array}{ccccc}
0 & & & & 0 \\
\uparrow & & & & \uparrow \\
KO(P(V'), P(W)) & \leftarrow & KO(P(V), P(W)) & \leftarrow & KO(P(V), P(V')) \\
\uparrow & & \uparrow & & \uparrow \\
0 \leftarrow A(V', W) & \leftarrow & A(V, W) & \leftarrow & A(V, V')
\end{array}
$$

(15.9)

whose horizontal rows are exact; the upper one by the exact sequence of a triple, the lower one by the definition of the A-groups. We know, by (15.6), that $\chi(V, W)$ is a bijection if $\dim V - \dim W \leqslant 1$. Hence, arguing by induction on $\dim V - \dim W$ we may assume that the vertical homomorphisms of (15.9) are also exact. But then the middle homomorphism must be onto, proving the assertion for the next higher value of $\dim W - \dim V$.

The proof of proposition (15.7) may now be completed either by obtaining a lower bound for the groups in question from the spectral sequence of KO-theory, or by a detailed analysis of the sequence (15.9), which unfortunately involves several special cases. In view of the fact that a computation of $KO(P(k)/P(l))$ is now already in the literature [1] we will not pursue this argument further here.

B. *Relation with the vector-field problem*

We again consider the pairing

$$V \times E^0 \to E^1$$

of §11, but now focus our attention on the induced maps:

(15.10) $$V_x \times_{Z_2} S(E_x^0) \to E_x^1 \qquad x \in X.$$

Note that this is only relevant if E is a *real* module.

The geometric interpretation of (15.10) is clear: if $\pi : P(E^0) \to X$ is the projective bundle of E^0 over X, and ξ is the canonical line bundle over $P(E^0)$, then (15.10) describes a definite injection:

(15.11) $$\omega(V, E) : \pi^* V \otimes \xi \to \pi^* E^1.$$

It is possible to give (15.11) a more geometric setting if $S(V)$ admits a section, s. One may then use $w(V, E)$ to 'trivialize' a certain part of the 'tangent bundle along the fibres' of $P(E^0)$. Recall first that this bundle, which we will denote by $\mathscr{T}_F(E^0)$, is described in the following manner. The bundle $\xi = \xi(E^0)$ is canonically embedded in $\pi^*(E^0)$, whence $\pi^*(E^0)/\xi$ is well defined. Then we have

(15.12) $$\mathscr{T}_F(E^0) = (\pi^*(E^0)/\xi) \otimes \xi.$$

With this understood, let V' be the quotient of V by the line bundle determined by s:

$$0 \to 1 \xrightarrow{s} V \to V' \to 0$$

and let $s_* : E^0 \to E^1$ be the isomorphism induced by multiplication by $s(x)$ in E_x^0. It is then quite easy to check that the homomorphism $s_*^{-1} \cdot \omega(V, E) : \pi^* V \otimes \xi \to \pi^* E^0$ induces an injection

$$\pi^* V' \otimes \xi \to \pi^* E^0 / \xi.$$

Tensoring this homomorphism with ξ, we obtain the desired injection:

(15.13) $$\omega(s, V, E) : \pi^* V' \to \mathcal{T}_F(E^0).$$

Let us now again restrict the whole situation to a point. Then if dim $V = k$, dim $E^0 = m$, V' will be a trivial bundle of dimension $k - 1$, and $\mathcal{T}_F(E^0)$ will be the tangent-bundle of projective $(m - 1)$-space P_{m-1}.

Applying the results of §5 we conclude that the following proposition is valid:

PROPOSITION (15.14). *Let $m = \lambda a_k$ where a_k is the kth. Radon–Hurwitz number. Then the tangent bundle of P_{m-1} (and hence of S_{m-1}) contains a $(k - 1)$-dimensional trivial bundle.*

The work of Adams [1], gives the converse of this proposition: if the tangent bundle of S_{m-1} contains a trivial $(n - 1)$-bundle, then $m = \lambda a_n$.

We remark in closing that on the other hand the generalized vector-field question is still open. This question is: let ξ be the line bundle over P_n, then what is the maximum dimension of a trivial bundle in $m\xi$, $m \geqslant n$. Thus the vector field problem solves this question for $m = n$. The general solution would, by virtue of the work of M. Hirsch, give the most economical immersions of P_n in Euclidean space.

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