Einstein deformations of hyperbolic metrics

Olivier Biquard

Introduction

The simplest nontrivial examples of Einstein metrics are certainly rank one symmetric spaces. We are interested in those of negative curvature, that is the hyperbolic spaces $\mathbb{K}\mathbf{H}^m$ ($m \geqslant 2$), where $\mathbb{K}$ is the field of real numbers ($\mathbb{R}$), complex numbers ($\mathbb{C}$), quaternions ($\mathbb{H}$) or the algebra of octonions ($\mathbb{O}$); in the last case we have only the Cayley hyperbolic plane $\mathbb{O}\mathbf{H}^2$. They are the noncompact duals of the projective spaces $\mathbb{K}\mathbf{P}^m$. We normalize the metric so that the maximum of the sectional curvature is -1. We denote by d the real dimension of $\mathbb{K}$ (so $d = 1$, 2, 4 or 8) and by $n = md$ the real dimension of $\mathbb{K}\mathbf{H}^m$.

The boundary sphere $\mathbb{S}^{n-1}$ of a hyperbolic space carries a rich geometric structure, namely a conformal Carnot-Carathéodory metric. Let see this first in the real and complex examples.

The real hyperbolic space (with constant sectional curvature -1) is the unit ball $\mathbb{B}^n$ in $\mathbb{R}^n$, with the metric

$$g = 4\frac{\mathrm{euc}}{(1-\rho^2)^2}\ ,$$

where euc is the flat metric on $\mathbb{R}^n$ and ρ the radius. The metric g induces a metric on the boundary $\mathbb{S}^{n-1}$

$$\gamma = \lim_{\rho\to 1}(1-\rho^2)^2 g_{\mathbb{S}_\rho}\ ; \tag{1}$$

the function $(1-\rho^2)$ is a *defining function* for the boundary, and the metric γ depends on the choice of the defining function only up to a conformal factor, so that the conformal class $[\gamma]$ is well defined. We shall say (following LeBrun's terminology) that $[\gamma]$ is the *conformal infinity* of g.

The complex hyperbolic space (with constant holomorphic sectional curvature -4) is the unit ball of $\mathbb{C}^m$ with the Bergman metric

$$g = \frac{\mathrm{euc}}{1-\rho^2} + \frac{\rho^2(d\rho^2 + (Id\rho)^2)}{(1-\rho^2)^2}\ .$$

Now equation (1) would lead to a very degenerate tensor on the boundary, so we consider instead

$$\gamma = \lim_{\rho\to 1}(1-\rho^2) g_{\mathbb{S}_\rho}\ ; \tag{2}$$

this metric is infinite, except on the distribution $V = \ker \eta$, where $\eta = Id\rho$ is a connection 1-form for the $\mathbb{S}^1$-bundle $\mathbb{S}^{2m-1} \to \mathbb{C}\mathbf{P}^{m-1}$. Such a metric defined on a contact distribution is called a Carnot-Carathéodory metric. Again, only the conformal class $[\gamma]$ is well defined and we extend the previous terminology to call it the *conformal infinity* of g.

These two examples fit in with the following more general picture. Fix a base point $*$ in the hyperbolic space and denote by r the distance to $*$ and by $\mathbb{S}_r$ the radius r sphere around $*$. The metric γ on the boundary sphere $\mathbb{S}$ of the hyperbolic space $\mathbb{K}\mathbf{H}^m$ is defined as

$$\gamma = \lim_{r\to\infty} e^{-2r} g_{\mathbb{S}_r}\,. \tag{3}$$

This metric is infinite except on a distribution V of codimension 1 (complex case), 3 (quaternionic case) or 7 (octonionic case). In the real case it is finite and $V = T\mathbb{S}$. The brackets of vector fields in V generate the whole tangent bundle $T\mathbb{S}$, making γ into a Carnot-Carathéodory metric. Moreover, there is a contact form η with values in $\mathrm{Im}(\mathbb{K}) = \mathbb{R}$, $\mathbb{R}^3$ or $\mathbb{R}^7$, such that the metric is exactly

$$g = dr^2 + \sinh^2(2r)\eta^2 + \sinh^2(r)\gamma\,.$$

In the real case, the η^2 term does not appear. To give a sense to the formula in the three other cases, we have to choose a supplementary subspace to the distribution $V \subset T\mathbb{S}$. This is given here by the fibers of the fibration

$$\begin{array}{ccc} \mathbb{S}^{d-1} & \longrightarrow & \mathbb{S}^{n-1} \\ & & \downarrow \\ & & \mathbb{K}\mathbf{P}^{m-1} \end{array}$$

Of course, all this depends on the choice of the base point $*$, but the conformal class $[\gamma]$ is well defined by (3) and will be called the *conformal infinity* of g.

The symmetric metrics are Einstein, $\mathrm{Ric}^g = -\lambda g$ with

$$\lambda = n-1,\ n+2,\ n+8,\ 36$$

in the real, complex, quaternionic and octonionic cases respectively. In this article, we will explain how all (in a sense to be precised) Einstein deformations of the hyperbolic metric are obtained as solutions of the following problem: given a Carnot-Carathéodory metric γ on the boundary, compatible in some sense with a contact structure, find a metric g in the interior such that

(i) $\mathrm{Ric}^g = -\lambda g$;

(ii) g has $[\gamma]$ as conformal infinity.

This problem has now a long history. In the complex case, one can try to find Kähler-Einstein deformations. The problem is solved by the theorem of Cheng and Yau [CY80]: they prove in particular that any smooth strictly pseudoconvex domain in $\mathbb{C}^m$ admits a unique complete Kähler-Einstein metric, which is asymptotic to the CR-structure of the boundary as in equation (2). High order approximate formal solutions near the boundary had been constructed earlier by Fefferman [Fef76] for this complex Monge-Ampère equation, and the regularity of the solution near the boundary has been determined by Lee and Melrose [LM82].

In the real case, LeBrun [LeB82] has solved, in dimension 4, a local problem near $\mathbb{S}^3$, or more generally near any real-analytic 3-manifold M. Since equation (i) with initial data (ii) is an underdetermined local Cauchy problem, he needs the following additional condition:

(iii) g is self-dual.

This means that the Weyl tensor W^g of the metric is a selfdual 2-form. Using Penrose twistor correspondence, he proves that for any real analytic γ on M, there is a unique solution g of equations (i), (ii) and (iii) in a neighborhood of M, up to diffeomorphisms. Moreover, if t is a defining function for the boundary M, the tensor $t^2 g$ is smooth up to the boundary, so that $[\gamma]$ is the conformal infinity of g in a strong sense. We have given earlier a weaker definition, since, for example in the complex case, such a regularity does not hold for the solutions produced by Cheng and Yau: the asymptotics provided by Lee and Melrose contain logarithmic terms.

Now, condition (iii) is special to dimension 4; in order to get a generalization to any dimension, Fefferman and Graham [FG85] have replaced condition (iii) by a less geometric condition: in a coordinate system such that

$$g = t^{-2}\left(dt^2 + \sum_1^{n-1} g_{ij}(x,t)dx^i dx^j\right),$$

where t is a defining function for the boundary M, they ask

(iii') $g_{ij}(x,t)$ is an even function of t.

This condition is independent of the coordinate system. They prove that, for n even, given a metric γ on M^{n-1}, the equations (i), (ii) and (iii') have a unique formal solution, which converges in a neighborhood of M if γ is real analytic (for n odd, there exist conformal structures $[\gamma]$ for which there is no formal solution). The aim of their study was to construct conformal invariants of $[\gamma]$ from Riemannian invariants of the canonical metric g. It seems that this theorem is a rediscovery of a theorem of Schouten and Haantjes [SH36a, SH36b].

Finally, in the case of quaternionic hyperbolic space, LeBrun [LeB91] has constructed an infinite dimensional family of quaternionic Kähler deformations, using the twistor correspondence for quaternionic Kähler metrics.

Explicit solutions are known only in one case: SU_2-invariant solutions of (i), (ii) and (iii) in $\mathbb{B}^4$. Given any left-invariant metric or Carnot-Carathéodory metric γ on $\mathbb{S}^3$, Hitchin [Hit95], using twistor theory, has found formulas for the solutions in terms of elliptic functions. The case when γ is the metric of a Berger sphere had been established earlier by Pedersen [Ped86].

These explicit examples are very special, since for general boundary data, one can not hope to solve more than equations (i) and (ii) if one wants global solutions. These have been produced, when γ is close to the standard metric on the sphere at infinity, by Graham and Lee [GL91] in the real case, by the author [Biq97] in the three other cases. In the rest of this survey, we explain this solution. In section 1, we define the Carnot-Carathéodory metrics needed at infinity, and then state the main theorem (theorem 3). In section 2, we give an idea of the proof, and in section 3, we ask some questions related to the problem.

1 Asymptotically symmetric metrics

Given a conformal Carnot-Carathéodory metric γ on $\mathbb{S}$, in order to solve problem (i)-(ii), we begin by producing a first order solution. This leads to the definition of special Carnot-Carathéodory metrics on the boundary, and then to the notion of "asymptotically symmetric metrics". We close the section by stating the theorem giving the solution to problem (i)-(ii).

1.1 Carnot-Carathéodory metrics

If γ is defined on a distribution V, choose a supplementary V_2 in $T\mathbb{S}$, and a 1-form η with values in $\mathbb{R}^{d-1}$ and kernel V. We define

$$g = dr^2 + \sinh^2(2r)\eta^2 + \sinh^2(r)\gamma\,; \tag{4}$$

this metric is not smooth at $r = 0$, so we modify it for $r \leqslant 1$ so that it extends smoothly in the interior; we can do so with smooth dependence to the parameter γ. Choose a local basis of vector fields on $\mathbb{S}$ in the following way: $(X_1, \ldots, X_{d-1})$ is a basis of V_2 such that $(\eta(X_i))$ is the standard basis of $\mathbb{R}^{d-1}$, and $(X_d, \ldots, X_n)$ is a γ-orthonormal basis of V. Define now the orthonormal basis $\xi_0 = \partial_r$, $\xi_i = X_i/\sinh(2r)$ for $1 \leqslant i < d$, and $\xi_i = X_i/\sinh(r)$ for $i \geqslant d$. Using the form of the metric, we see easily that

$$\begin{array}{lll} [\xi_0, \xi_i] & = -2\xi_i + O(e^{-r}), & 1 \leqslant i < d, \\ [\xi_0, \xi_i] & = -\xi_i + O(e^{-r}), & i \geqslant d, \\ [\xi_i, \xi_j] & = O(e^{-r}), & i \geqslant 1,\ 1 \leqslant j < d, \\ [\xi_i, \xi_j] & = \sum_1^d b_{ij}^k \xi_k + O(e^{-r}), & i, j \geqslant d, \end{array} \tag{5}$$

where the $b_{ij}^k = -d\eta_k(X_i, X_j)$ are the coefficients of the tensor induced by the bracket $V \otimes V \to V_2$. In particular, we see that at infinity, the bracket structure on an orthonormal basis depends only on this tensor. We shall require that this tensor is the same as in our model, that is the symmetric space.

This motivates the following definition. Let $H = U_{m-1}$, $Sp_{m-1}Sp_1$ or $Spin_7$ in the complex, quaternionic or octonionic cases. As is well known, rank one symmetric spaces are characterized by the fact that their spheres are homogeneous, and the group H is precisely the isotropy group of $\mathbb{S}^{n-1}$ when we represent it as a quotient under isometries of $\mathbb{K}\mathbf{H}^m$, that is as a homogeneous sphere U_m/U_{m-1}, $Sp_mSp_1/Sp_{m-1}Sp_1$ or $Spin_9/Spin_7$.

Definition 1 *A* Carnot-Carathéodory H-metric *on the sphere $\mathbb{S}^{n-1}$ is the data of a Carnot-Carathéodory metric γ on a distribution of codimension $d-1$, such that there exists a 1-form η with values in $\mathrm{Im}(\mathbb{K}) = \mathbb{R}^{d-1}$ and kernel V, satisfying*

- complex case: *the restriction to V of $d\eta$ is a symplectic form, compatible with γ (that is, $d\eta(\cdot,\cdot) = \gamma(I\cdot,\cdot)$ with I an almost complex structure on V);*
- quaternionic case: *the three 2-forms $(d\eta_1, d\eta_2, d\eta_3)$ on V give a quaternionic structure on V, compatible with γ (that is, $d\eta_i(\cdot,\cdot) = \gamma(I_i\cdot,\cdot)$ for almost complex structures I_i which satisfy the commutation relations of the quaternions);*
- octonionic case: *the seven 2-forms $(d\eta_i)_{i=1,\ldots,7}$ give an octonionic structure on V, compatible with γ (that is, $d\eta_i(\cdot,\cdot) = \gamma(I_i\cdot,\cdot)$ for almost complex structures I_i which satisfy the commutation relations of the octonions).*

The meaning of this definition is that, in each case, we get a H-structure on V, with $H \subset SO(\gamma)$, compatible in some sense with the symplectic form $d\eta$. In the real case, the isotropy group is $H = SO(n-1)$, and we need nothing more than a metric γ on $\mathbb{S}^{n-1}$. In the complex case, there are lots of metrics compatible with a given contact form, but any deformation of a contact structure V is diffeomorphic to V: when we study this case, we may fix V and vary the almost complex structure on V. In the quaternionic and octonionic cases, the situation is completely different, because the metric is completely determined by the contact form η, and more precisely by the fundamental 4-form $\sum d\eta_i^2$, whose stabilizer is H (except for $m=2$ in the quaternionic case). Finally, remark that given the H-structure on V, the contact form η with values in $\mathbb{R}^{d-1}$ is unique in the complex case, defined up to the action of the sections of the trivial SO_3-bundle (resp. SO_7-bundle) on $\mathbb{R}^3$ (resp. $\mathbb{R}^7$) in the quaternionic (resp. octonionic) case.

In all cases, the definition means that for a $\mathbb{K}$-basis (e_i) on V, we have

$$d\eta\left(\sum x_i e_i, \sum y_i e_i\right) = -\mathrm{Im} \sum \overline{x_i} y_i \,.$$

For the question of existence of such structures in the quaternionic and octonionic cases, see section 3.5.

1.2 Solution of (i)-(ii)

We can now come back to our metric g defined from a Carnot-Carathéodory H-metric γ, and to formula (5). We see that the coefficients b_{ij}^k are the same for g and for the symmetric metric, and can be chosen constant in an adapted basis.

From this follows easily that the curvature tensor of g at infinity is asymptotic to the curvature tensor of the symmetric metric, that is

$$|R^g - R^{sym}| = O(e^{-r})\,; \tag{6}$$

in particular,

$$|\mathrm{Ric}^g + \lambda g| = O(e^{-r})\,,$$

so we get the promised first order solution.

Because of the form (6) for the curvature, we shall say that g is asymptotically symmetric. We want to define more precisely this notion. For this, we introduce a little analysis: we need the usual Hölder spaces $C^{k,\alpha}$ for the metric g, and the weighted versions

$$C^{k,\alpha}_\delta = \cosh(-\delta r) C^{k,\alpha}\,.$$

This weighted space corresponds to functions decreasing as $e^{-\delta r}$, with their derivatives. Now we suppose that the Carnot-Carathéodory metric has regularity $C^{2,\alpha}$; then it is easy to see that $R^g - R^{sym}$ belongs to C^α_1. Therefore, the following definition is natural.

Definition 2 *We say that a metric h is* asymptotically symmetric *if, for some g constructed as above, one has $h - g \in C^{2,\alpha}_1$. We say that the* conformal infinity *of h is $[\gamma]$.*

Of course, the order $\delta = 1$ is arbitrary, as is the number of derivatives that we require. We are now able to state the theorem giving the solution to the problem (i)-(ii) of the introduction.

Theorem 3 *If a conformal Carnot-Carathéodory H-metric $[\gamma]$ is close enough (in $C^{2,\alpha}$ norm) to the symmetric conformal infinity, then there is an asymptotically symmetric metric h, solution to the problem (i)-(ii). This metric is locally unique, that is if we have another solution h' close enough to h, then there exists a diffeomorphism ϕ, equal to the identity at the boundary, such that $h' = \phi_* h$.*

In the real case, this is the theorem of Graham and Lee [GL91], with a slight technical difference in dimension 4, because they had to ask a regularity $C^{3,\alpha}$ on γ: this reflects the difference between their method and ours.

In the complex case, in dimension $2m-1 \geqslant 5$, the integrability of the CR-structure (sufficiently close to the standard CR-structure) is equivalent to the solution h being Kähler-Einstein. Indeed, if a CR-deformation of $\mathbb{S}^{2m-1}$ is integrable, it can be realized as a strongly pseudoconvex hypersurface in $\mathbb{C}^m$ (see [Tan75, theorem 9.4]); then the theorem of Cheng and Yau furnishes a Kähler-Einstein solution to the problem (i)-(ii). Conversely, if the CR-structure is not integrable, the solution h cannot be Kähler-Einstein.

Again in the complex case, recall that all nearby contact structures are diffeomorphic, so we may suppose that the distribution V is fixed. It follows that all the metrics constructed by the metric in the real or complex cases are mutually bounded; in fact, it will be shown that *these metrics exhaust all the bounded Einstein deformations of the symmetric metric in the real and complex cases* (with sufficient regularity at infinity). On the contrary, *there is no bounded Einstein deformation of the symmetric metric in the quaternionic and octonionic cases.* Finally, the theorem holds around other metrics than the symmetric one, provided that some L^2-obstruction space (see definition 4) vanishes. For all these complements to the theorem, see section 2.3.

2 Sketch of proof of the theorem

Recall that we want to solve the problem (i)-(ii) of the introduction. The symmetric metric is a solution; given a perturbation of the data at the boundary, we have constructed a first order approximate solution g, already satisfying the boundary condition (ii).

2.1 Gauge fixing condition

We now want to solve the equation (i), that is $\mathrm{Ric}^h + \lambda h = 0$, for h close to g. As is well known, this is not an elliptic problem, because the equation is invariant under the action of the group of diffeomorphisms. This reflects also in the fact that the Ricci tensor always satisfies the Bianchi identity $\delta^h \mathrm{Ric}^h = -\frac{1}{2} ds^h$, where s^h is the scalar curvature. However, the method to overcome this difficulty is now well known: one can use harmonic coordinates to break the invariance, or more globally require that the identity map $(X,h) \rightarrow (X,\mathrm{ref})$ is a harmonic map, where ref if a reference metric. In the real case, Graham and Lee [GL91] choose for ref the first approximation g. We choose another condition, much in the spirit of the Coulomb gauge in gauge theory: it is a linear condition, essentially the infinitesimal version of the previous harmonicity condition:

$$B^g(h) := \delta^g h + \frac{1}{2} d\mathrm{tr}\,^g h = 0\,. \tag{7}$$

There are two reasons why this is the correct choice. First we want to prove that this is a "gauge fixing condition" for the action of the diffeomorphism group, that is: given h close to g, prove that there is a unique diffeomorphism ϕ (equal to the identity on the boundary), such that $\phi_* h$ satisfies condition (7). It is sufficient to check this infinitesimally: the diffeomorphism group acts infinitesimally on g by taking the vector field X to the symmetrized covariant derivative $(\delta^g)^* X$, so the problem to solve is

$$B^g\left((\delta^g)^* X\right) = -B^g(h)\,.$$

But one has the formula

$$B^g(\delta^g)^* = \frac{1}{2}((D^g)^* D^g - \mathrm{Ric}^g)\,, \tag{8}$$

where D^g is the covariant derivative; as we consider essentially metrics g with negative curvature, the analysis explained later in section 2.3 proves easily that the operator $B^g(\delta^g)^*$ is an isomorphism.

The second reason why the gauge condition (7) is the right one is that a solution of equation (i) put in this gauge will satisfy

$$\Phi^g(h) := \mathrm{Ric}^h + \lambda h + (\delta^h)^* \left(\delta^g h + \frac{1}{2} d\mathrm{tr}^{\,g} h \right) = 0\,. \tag{9}$$

The linearization of the nonlinear second order differential operator Φ^g at g is now very simple:

$$d_g\Phi^g(\dot h) = \frac{1}{2}(D^g)^* D^g \dot h + \frac{1}{2}(\mathrm{Ric}^g \circ \dot h + \dot h \circ \mathrm{Ric}^g + 2\lambda \dot h) - \overset{\circ}{R^g}\ \dot h\,.$$

Here, $(\overset{\circ}{R^g}\ k)_{X,Y} = \sum k(R^g_{e_i,X}Y, e_i)$. In particular the operator Φ^g becomes elliptic. Conversely, using the Bianchi identity $B^h(\mathrm{Ric}^h) = 0$, a solution h of (9) will satisfy

$$B^h(\delta^h)^* \left(\delta^g h + \frac{1}{2} d\mathrm{tr}^{\,g} h \right) = 0\,.$$

Therefore, using equation (8), we see that h actually satisfies the gauge condition (7) and the initial equation (i).

We deduce from these considerations that the resolution of equation (i) modulo diffeomorphisms near g is completely reduced to problem (9).

2.2 L^2-obstruction

Let us now look at the linearization $d_g\Phi^g$ in the case where g is Einstein. The operator is then reduced to

$$d_g\Phi^g(\dot h) = \frac{1}{2}(D^g)^* D^g \dot h - \overset{\circ}{R^g}\ \dot h\,.$$

The kernel consists of the infinitesimal Einstein deformations of g.

Definition 4 *If g is an Einstein asymptotically symmetric metric, we define the L^2-infinitesimal deformation space, $L^2\mathbf{H}^1(g)$, as the L^2-kernel of $d_g\Phi^g$.*

By a Weitzenböck formula (see [Bes87, lemma 12.71]), it is easy to prove that

$$2\int \langle d_g\Phi^g(\dot h), \dot h\rangle \geqslant (n-2)(-\sup K^g) \int |\dot h|^2\,. \tag{10}$$

Therefore the operator $d_g\Phi^g$ is an isomorphism in L^2 if g has negative curvature, and in particular if g is symmetric. This is the argument used by Koiso [Koi78] to prove that compact quotients of irreducible symmetric spaces of noncompact type and dimension greater than 2 do not admit Einstein deformations.

However, in the noncompact case, L^2-theory for the operator $d_g\Phi^g$ is not enough. Since this is a nonlinear problem, we need to work in Hölder spaces rather than L^2-spaces. More importantly, the L^2-condition gives a strong decay at infinity: functions like $\exp(-\delta r)$ are in L^2 if $\delta > \mathcal{H}$, where the critical exponent $\mathcal{H}$ is easily seen from (4) to be

$$\mathcal{H} = \frac{n-1}{2},\ \frac{n}{2},\ \frac{n}{2}+4,\ 11 \tag{11}$$

in the real, complex, quaternionic and octonionic cases respectively. Therefore, in order to understand bounded deformations of g, we need to understand the behavior of the operator $d_g\Phi^g$ in the weighted Hölder spaces $C^{k,\alpha}_\delta$, for $\delta = 0$.

We have also another problem to solve: if g is now the first order approximate solution to the problem (i)-(ii) that we have constructed before for some $C^{2,\alpha}$ data γ on the boundary, then we have $\Phi^g(g) \in C^\alpha_1$ and we want to find an exact solution h of $\Phi^g(h) = 0$ with $h - g \in C^{2,\alpha}_1$. Basically, if g is a good enough approximate solution, this has a chance to be true if the differential $d_g\Phi^g$ is an isomorphism for the weight $\delta = 1$. The space $L^2\mathbf{H}^1(g)$ then appears as the obstruction for $d_g\Phi^g$ being an isomorphism in L^2. We shall see below that there is no other obstruction.

Before we proceed to the analysis, we need the following lemma on the eigenvalues of the curvature acting on symmetric 2-tensors. The only proof I know is by checking case by case.

Lemma 5 *For the rank 1 symmetric metric g, the highest eigenvalue of $\mathring{R}^g$ is* 4 *(except in the real case, it is* 1*), and the other eigenvalues are negative.*

The value 1 for $\mathbb{R}\mathbf{H}^m$ if due to the choice of the sectional curvature equal to -1, instead of -4 for example for the holomorphic sectional curvature of $\mathbb{C}\mathbf{H}^m$.

2.3 Analysis and resolution

On the hyperbolic space, the analysis of an operator $\Phi = D^*D + P$, where P is some zero order homogeneous selfadjoint operator, is difficult, and we shall not give any proof here, but simply give some intuition on the problem. Suppose for example that we know that Φ is an isomorphism in L^2, that is essentially for the weight $\delta = \mathcal{H}$. The question is: for which range of weights (δ_0, δ_1) does the operator Φ remain an isomorphism?

There is an elementary (but non optimal) approach: using Kato's inequality $|Ds| \geqslant |d|s||$ and the maximum principle, one can prove that the interval (δ_0, δ_1) contains the interval (δ_0', δ_1') for the scalar operator $d^*d + \nu$, where ν is the smallest eigenvalue of P. Now one can see that for functions $f(r)$ depending only on r, one has

$$(d^*d + \nu)f = -\partial_r^2 f - 2\mathcal{H}\partial_r f + \nu f + O(e^{-r})(f, \partial_r f)\,. \tag{12}$$

The operator $-\partial_r^2 - 2\mathcal{H}\partial_r + \nu$ is called the *indicial operator*. It governs the behavior at infinity of the operator $d^*d + \nu$ because differentiating along the other directions always has a weight $\exp(-r)$ or $\exp(-2r)$. It is not difficult to see that the solutions $\exp(-\delta r)$ of the indicial operator give the values of δ_0' and δ_1', that is

$$\begin{aligned} \delta_0' &= \mathcal{H} - \sqrt{\mathcal{H}^2 + \nu}\,, & (13)\\ \delta_1' &= \mathcal{H} + \sqrt{\mathcal{H}^2 + \nu}\,. & (14) \end{aligned}$$

Let us apply this result to $d_g\Phi^g$, using lemma 5. We have $\nu = -8$ (-2 in the real case) so we cannot catch the weight $\delta = 0$ but we can try to catch the weight $\delta = 1$. One can see easily, using (11), that $\delta_0' < 1$ if $n > 4$ in the real case, $n > 9$ in the complex case, and always in the quaternionic and octonionic cases. This is the analysis result used in the real case by Graham and Lee, and the technical restriction in dimension 4 we mentioned after theorem 3 comes from here: in this case, they have to find a higher order approximation before applying this analysis.

Now come back to our operator $\Phi = D^*D + P$. In order to get the optimal values of δ_0 and δ_1, one cannot use Kato's inequality, because this neglects zero order terms in D^*D. Actually, there is an indicial operator as in (12) for D^*D itself, given by the asymptotic behavior of D^*D:

$$-\partial_r^2 - 2\mathcal{H}\partial_r + \mathcal{C}\,,$$

where $\mathcal{C}$ is now some zero order operator. The above discussion remains true when we replace the smallest eigenvalue ν of P by the smallest eigenvalue μ of $\mathcal{C} + P$, and we get

$$\begin{aligned} \delta_0 &= \mathcal{H} - \sqrt{\mathcal{H}^2 + \mu}\,, & (15)\\ \delta_1 &= \mathcal{H} + \sqrt{\mathcal{H}^2 + \mu}\,. & (16) \end{aligned}$$

This is a considerably more difficult result, because one cannot use the maximum principle, which forgets the zero order term $\mathcal{C}$. These operators are probably a matter for the beautiful theory of *edge operators*, see for example [Maz91a] in the real case, [EMM91, Mel92] in the complex case. In our symmetric case, there is an alternative approach using some elementary harmonic analysis [Biq97]. Now apply this theory to the operator $d_g\Phi^g$. A calculation gives $\mu = 0$ in the real and complex cases, and $\mu > 0$ in the quaternionic and octonionic cases. In the last two cases, we deduce that $d_g\Phi^g$ is an isomorphism $C^{2,\alpha} \to C^\alpha$, so Φ^g is a local isomorphism, which means that there is no bounded Einstein deformation. On the contrary, in the real and complex cases, there are lots of bounded Einstein deformations, corresponding to sections on the sphere at infinity of the eigenbundle associated to the eigenvalue $\mu = 0$. This eigenbundle can be made explicit: in the real case, it is $\mathrm{Sym}_0^2 T\mathbb{S}^{n-1}$, so that Einstein infinitesimal deformations are given by conformal deformations of the boundary metric. In the complex case, recall that we have the contact distribution V with a symplectic form and a complex structure I, and the eigenbundle can be seen to be the subspace of $\mathrm{Sym}^2_{\mathbb{R}} V$ consisting of symmetric 2-tensors k on V such that $k(I\cdot, I\cdot) = -k(\cdot,\cdot)$. This is exactly the tangent space to metrics on V which remain compatible with the symplectic form. Thus we see that our theorem 3 gives all bounded Einstein deformations of the symmetric metric, such that the data on the boundary has regularity $C^{2,\alpha}$.

Now pass to the problem of actually producing the Einstein deformations. Denote the symmetric metric by g_0, and g the first approximation to the solution of problem (i)-(ii). In the complex case, one can fix the contact structure and deform only the almost complex structure. In the real and complex cases, the metrics remain mutually bounded, so that the weighted Hölder spaces for g and g_0 remain equal, and the problem is solved by applying the implicit function theorem to the equation $\Phi^g(h) = 0$ at $g = g_0$, using the analysis above for the weight $\delta = 1$. In the quaternionic and octonionic cases, this is not possible, because g and g_0 are no more close, but a more constructive method proves that if γ is close enough to the standard metric on the boundary, so that $h = g$ is a very good approximate solution, then one can deform g into a solution h of $\Phi^g(h) = 0$. This proves theorem 3.

The analysis above for the symmetric space has a counterpart for any asymptotically symmetric metric. Indeed, in the complex case, any contact structure is locally diffeomorphic to the standard contact structure, so that it is locally possible to approximate an asymptotically symmetric metric by a symmetric metric. In the real case, this is of course even simpler. In both cases, using this local approximation, one can graft the isomorphism obtained for the symmetric model to construct a parametrix for an operator $\Phi = D^*D+P$, and prove that Φ is Fredholm for $\delta \in (\delta_0, \delta_1)$, where δ_0 and δ_1 are given by formulas (15)-(16). In particular, if Φ is an isomorphism in L^2, it remains an isomorphism $C^{2,\alpha}_\delta \to C^\alpha_\delta$ for δ in this range. In the quaternionic and octonionic cases, our special contact structures are not locally diffeomorphic, so that such an approximation by the symmetric model seems difficult; at least, one can use the first more elementary method above to prove a similar statement, but with the weight δ restricted to (δ'_0, δ'_1), where δ'_0 and δ'_1 are given by formulas (13)-(14); as we have seen earlier, this is probably not the optimal interval, but it is sufficient for these two cases. The application of these considerations is that theorem 3 remains true around any asymptotically symmetric Einstein metric g, provided that the L^2-obstruction space $L^2\mathbf{H}^1(g)$ vanishes. In view of (10), this is true in particular if g has negative curvature.

3 Open questions

3.1 Regularity

There are two questions on regularity. The first question is: suppose γ is smooth, what can be said on the regularity of the solution g up to the boundary? In the real case, Graham and Lee have constructed a high order approximate formal solution: the resolution stops at the critical weight $\delta_1 = 2\mathcal{H} = n-1$ in the notations of section 2.3; this enables them to prove that, if $n > 4$, the solution satisfies $t^2 g \in C^{n-2,\alpha}(\mathbb{B}^n)$, where t is some defining function of the boundary and the Hölder spaces are taken with respect to the flat metric on the ball. There is no doubt that such a high order approximate formal solution can be constructed in the other cases and that the resolution stops at the weight δ_1. Is it possible to construct an expansion in power series? this expansion should eventually contain logarithmic terms, as does the Lee-Melrose expansion for the Cheng-Yau metric.

The second question on regularity goes in the opposite direction: we have required a regularity $C^{2,\alpha}$ for the boundary metric γ, because the first order approximation can then be merely defined by (4), but it is obvious that this is not optimal. What is the lowest regularity on the Carnot-Carathéodory metric which enables to solve the problem?

3.2 Uniqueness

The Einstein metric with given conformal infinity is locally unique modulo the action of the group of diffeomorphisms, so an important question is to get a more global answer: namely, if g and h are two asymptotically symmetric Einstein metrics, with the same conformal infinity, does there exist a diffeomorphism ϕ, equal to the identity on the sphere at infinity, such that $h = \phi_* g$?

There are partial answers to this question, when g is the real or complex hyperbolic metric; these answers are rigidity results for the scalar curvature rather than for the Ricci tensor, and the idea comes from Witten's spinorial proof of the positive mass conjecture [Wit81]. Namely, in [MO89], Min-Oo has proven that a metric h which is strongly asymptotic to the real hyperbolic metric g and such that $s^h \geqslant s^g$, is equal to g; here strongly asymptotic means that $h - g$ and one derivative are $O(e^{-(n+\varepsilon)r})$ (one can give weaker L^1 and L^2 bounds instead). For other rigidity results based on Min-Oo's theorem, see [Leu93]. In [Her97], Herzlich has proven a similar statement in the case where g is the complex hyperbolic metric and h is a Kähler metric, strongly asymptotic to g, such that $s^h \geqslant s^g$; here, strongly asymptotic means that $h - g$ and one derivative are $O(e^{-(n+2+\varepsilon)r})$, and the same for the difference between the two complex structures (again one can use some integral bounds instead).

3.3 Global existence

Related to the uniqueness statement is the very difficult question to know for which $[\gamma]$ on the boundary one can solve problem (i)-(ii). The problem is completely open, and is probably related to a better understanding of the obstruction space $L^2\mathbf{H}^1(g)$.

3.4 Conformal geometry

If one believes in the uniqueness of the solution (up to diffeomorphism), then it is clear that the conformal geometry of $[\gamma]$ must be related to the Riemannian geometry of g. An interesting step in this direction in the real case is the following result of Lee [Lee95], relating the Yamabe invariant of $[\gamma]$ and the spectrum of g. The essential L^2 spectrum of an asymptotically real hyperbolic metric is $[(n-1)^2/4, \infty)$, without embedded eigenvalues [Maz91b]; moreover the real hyperbolic metric has purely continuous spectrum consisting of this ray. If g is an Einstein asymptotically hyperbolic metric with conformal infinity $[\gamma]$, such that the conformal class $[\gamma]$ on $\mathbb{S}^{n-1}$ contains a metric with positive scalar curvature, then the infimum of the L^2-spectrum of the scalar Laplacian of g is again $(n-1)^2/4$ (that is, there is no discrete eigenvalues below the continuous spectrum, as for the symmetric metric).

3.5 Quaternionic and octonionic cases

In the quaternionic and octonionic cases, even the basis theorem 3 remains mysterious. In the real and complex cases, we know precisely that the Carnot-Carathéodory metrics are given by sections on some bundles. In the quaternionic and octonionic cases, existence of the corresponding Carnot-Carathéodory metrics is already a complicated differential system! it is an important question to understand these metrics. In the quaternionic case, examples are provided by LeBrun's twistorial construction in [LeB91], but the metrics g share the same holonomy $Sp_m Sp_1$ as the hyperbolic metric; are there $[\gamma]$ for which the metric g is not quaternionic Kähler? In the octonionic case, we have no example at all; note that in this case, it is impossible to deform the hyperbolic metric preserving the holonomy $Spin_9$, since any metric with holonomy $Spin_9$ must be locally symmetric.

References

[Bes87] A. L. Besse, *Einstein manifolds*, Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)], vol. 10, Springer-Verlag, Berlin, 1987.

[Biq97] O. Biquard, *Métriques d'Einstein asymptotiquement symétriques*, Preprint, École Polytechnique, 1997.

[CY80] S. Y. Cheng and S. T. Yau, *On the existence of a complete Kähler metric on noncompact complex manifolds and the regularity of Fefferman's equation*, Comm. Pure Appl. Math. **33** (1980), 507–544.

[EMM91] C. L. Epstein, R. B. Melrose, and G. A. Mendoza, *Resolvent of the Laplacian on strictly pseudoconvex domains*, Acta Math. **167** (1991), 1–106.

[Fef76] C. L. Fefferman, *Monge-Ampère equations, the Bergman kernel, and geometry of pseudoconvex domains*, Ann. Math. (2) **103** (1976), 395–416.

[FG85] C. L. Fefferman and C. R. Graham, *Conformal invariants*, Astérisque **hors série** (1985), 95–116, The mathematical heritage of Élie Cartan (Lyon, 1984).

[GL91] C. R. Graham and J. M. Lee, *Einstein metrics with prescribed conformal infinity on the ball*, Adv. Math. **87** (1991), 186–225.

[Her97] M. Herzlich, *Scalar curvature and rigidity of odd-dimensional complex hyperbolic spaces*, Preprint, École Polytechnique (to appear in Math. Ann.), 1997.

[Hit95] N. J. Hitchin, *Twistor spaces, Einstein metrics and isomonodromic deformations*, J. Differential Geom. **42** (1995), 30–112.

[Koi78] N. Koiso, *Nondeformability of Einstein metrics*, Osaka J. Math. **15** (1978), 419–433.

[LeB82] C. R. LeBrun, *$\mathcal{H}$-space with a cosmological constant*, Proc. Roy. Soc. London Ser. A **380** (1982), 171–185.

[LeB91] C. R. LeBrun, *On complete quaternionic-Kähler manifolds*, Duke Math. J. **63** (1991), 723–743.

[Lee95] J. M. Lee, *The spectrum of an asymptotically hyperbolic Einstein manifold*, Comm. Anal. Geom. **3** (1995), 253–271.

[Leu93] M. C. Leung, *Pinching theorem on asymptotically hyperbolic spaces*, Internat. J. Math. **4** (1993), 841–857.

[LM82] J. M. Lee and R. B. Melrose, *Boundary behaviour of the complex Monge-Ampère equation*, Acta Math. **148** (1982), 159–192.

[Maz91a] R. Mazzeo, *Elliptic theory of differential edge operators. I*, Comm. Partial Differential Equations **16** (1991), 1615–1664.

[Maz91b] R. Mazzeo, *Unique continuation at infinity and embedded eigenvalues for asymptotically hyperbolic manifolds*, Amer. J. Math. **113** (1991), 25–45.

[Mel92] R. B. Melrose, *Calculus of conormal distributions on manifolds with corners*, Internat. Math. Res. Notices **3** (1992), 51–61.

[MO89] M. Min-Oo, *Scalar curvature rigidity of asymptotically hyperbolic spin manifolds*, Math. Ann. **285** (1989), 527–539.

[Ped86] H. Pedersen, *Einstein metrics, spinning top motions and monopoles*, Math. Ann. **274** (1986), 35–59.

[SH36a] J. A. Schouten and J. Haantjes, *Beitraege zur allgemeinen (gekruemmten) konformen Differentialgeometrie*, Math. Ann. **112** (1936), 594–629.

[SH36b] J. A. Schouten and J. Haantjes, *Beitraege zur allgemeinen (gekruemmten) konformen Differentialgeometrie. II*, Math. Ann. **113** (1936), 568–583.

[Tan75] N. Tanaka, *A differential geometric study on strongly pseudo-convex manifolds*, Kinokuniya Book-Store Co. Ltd., Tokyo, 1975, Lectures in Mathematics, Department of Mathematics, Kyoto University, No. 9.

[Wit81] E. Witten, *A new proof of the positive energy theorem*, Comm. Math. Phys. **80** (1981), 381–402.

CMAT, École Polytechnique, F-91128 Palaiseau Cedex (FRANCE)
e-mail : `biquard@math.polytechnique.fr`