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Uniform p-adic cell decomposition and local zeta functions

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1. Introduction

The purpose of this paper is to give a cell decomposition for p-adic fields, uniform in p. This generalizes a cell decomposition for fixed p, proved by Denef [7], [9]. We also give some applications of our cell decomposition. A first implication is a uniform quantifier elimination for p-adic fields. Belair [2], Delon [6] and Weispfenning [16] obtained quantifier elimination in other languages, but the language we use seems more practical for the evaluation of p-adic integrals. As a second application, we reprove results of Denef [10] on the dependence on p of the Igusa local zeta function. In this context we also obtain new results on p-adic integrals over sets definable in a language with cross section.

Let $\mathbb{Q}_p$ be the field of p-adic numbers (p a prime) and $\mathbb{Z}_p$ the ring of p-adic integers. Denote the p-adic valuation on $\mathbb{Q}_p$ by $|\cdot|_p$. Let $h(x)$ be a polynomial in m variables $x=(x_1,\dots,x_m)$ with coefficients in $\mathbb{Z}$, the ring of rational integers. For each prime p, we can define the Igusa local zeta function of h by

$$Z(s,p)=\int_{\mathbb{Z}_p^m}|h(x)|_p^s\,|dx|_p \quad \text{for} \quad s\in\mathbb{R},\ s>0$$

where $|dx|_p$ is the Haar measure on $\mathbb{Q}_p^m$ such that the measure of $\mathbb{Z}_p^m$ is 1.

Igusa [12] proved, using resolution of singularities, that for all p, $Z(s,p)$ is a rational function of p^{-s}. Denef [7] obtained the same result by cell decomposition, a technique which allows you to partition the integration space $\mathbb{Z}_p^m$ into so-called cells, in such a way that on each cell you can compute the integral using a kind of "separation of variables".

We can write $Z(s,p)=R_p(T)/S_p(T)$, where $R_p(T)$, $S_p(T)$ are relatively prime polynomials in $T=p^{-s}$. We define the degree of $Z(s,p)$ as

$$\deg Z(s,p)=\deg R_p(T)-\deg S_p(T).$$

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Denef [10] studied the behaviour of $Z(s, p)$ when the prime p varies. Using resolution of singularities, he proved the following:

(1) The degree of the numerator $R_p(T)$ and of the denominator $S_p(T)$ of $Z(s, p)$ is bounded when $p \to \infty$.

In order to obtain another proof of this result, we tried to find a generalization of cell decomposition suitable for the problem, this means a cell decomposition which is uniform in the prime p.

Denef suggested to modify the first order language he used (i.e. the language of Macintyre [13]) in two ways:

(i) In order to make explicit the dependence on the prime p, which is reflected in the residue field, use a many sorted language which contains variables over the residue field of $\mathbb{Q}_p$.

(ii) Include a symbol for an $\overline{\text{ac}}$-map (see definition 2. 2) in the language instead of Macintyre's n-th power predicates. (This $\overline{\text{ac}}$-map already appeared in [9] but only as an expedient.)

The definition of the new language $\mathscr{L}$, together with a modified definition of cell, appears in Section 2. In Section 3 we prove cell decomposition for a class of valued fields of equicharacteristic zero. Applying this to the ultraproducts $\prod \mathbb{Q}_p/\mathscr{D}$ for all non-principal ultrafilters $\mathscr{D}$ on the index set of prime numbers, we can obtain results in $\mathbb{Q}_p$, uniformly for almost all p. We mention that Macintyre [14] obtained about the same time a different generalization of Denef's cell decomposition, which allows him to prove also the result (1) on the zeta function.

Denef's original cell decomposition [9] implies Macintyre's theorem [13] on quantifier elimination in $\mathbb{Q}_p$. In Section 4 we prove that our cell decomposition gives quantifier elimination in the language $\mathscr{L}$ for a very general class of henselian valued fields of equicharacteristic zero.

Section 5 contains the proof of result (1), based on cell decomposition.

In [10] Denef proves also that

(2) $$\deg Z(s, p) \leqq 0 \quad \text{for almost all } p.$$

We show in Section 6 that this can also be proved using a slight modification of our cell decomposition method.

In the last section, we investigate a more general zeta function

$$Z_\psi(s, p) = \int_{W_p} |h(x)|_p^s \, |dx|_p \quad \text{for} \quad s \in \mathbb{R},\ s > 0$$

where ψ is a formula in some language for valued fields and W_p is the subset of $\mathbb{Q}_p^m$ defined by ψ. If the considered language contains a symbol π for a cross section, the known methods which use resolution of singularities, collapse, but by Denef [8] it is

known that, for fixed p, $Z_\psi(s, p)$ is a rational function of p^{-s}. In Section 7 we show that in a suitable language for valued fields containing a cross section symbol, the results (1) and (2) generalize for $Z_\psi(s, p)$.

I would like to thank J. Denef for suggesting the problem and for pointing out to me the basic ideas for the solution.

2. Some definitions

2. 1. Notation. K denotes a valued field, with valuation $\operatorname{ord}: K \to \Gamma \cup \{\infty\}$, where Γ is an ordered abelian group. We write

$$R = \{x \in K \mid \operatorname{ord} x \geqq 0\} \quad \text{and} \quad P = \{x \in K \mid \operatorname{ord} x > 0\}$$

for the valuation ring respectively valuation ideal of K.

The set of units of R will be denoted by U, i.e. $U = \{x \in R \mid \operatorname{ord} x = 0\}$. We write $\bar{K}$ for the residue field R/P, where $\operatorname{Res}: R \to \bar{K}$ is the canonical projection.

2. 2. Definition. An *angular component map* modulo P on K is a map

$$\overline{\mathrm{ac}}: K \to \bar{K}: x \mapsto \overline{\mathrm{ac}}\, x$$

such that

(i) $\overline{\mathrm{ac}}\, 0 = 0$,

(ii) the restriction of $\overline{\mathrm{ac}}$ to $K^\times$ is a multiplicative morphism from $K^\times$ to $\bar{K}^\times$,

(iii) the restriction of $\overline{\mathrm{ac}}$ to U equals the restriction of Res to U.

Remark. A cross section on a valued field K is a multiplicative morphism $\pi: \Gamma \to K^\times$ such that $\operatorname{ord} \circ \pi = 1_\Gamma$, eventually extended to $\Gamma \cup \{\infty\}$ by $\pi(\infty) = 0$.

On a field K which has a cross section π, we can define an angular component map modulo P $\overline{\mathrm{ac}}$ by putting

$$\overline{\mathrm{ac}}\, x = \operatorname{Res}(x \cdot \pi(-\operatorname{ord} x)) \quad \text{for} \quad x \neq 0.$$

In this paper we will use a first order language with three sorts of variables, namely variables for the elements of the valued field, variables for elements of the residue field and variables for elements of the value group. Our language will contain symbols for the standard field operations in the valued field and in the residue field, and symbols for the usual operations in the value group. We also have a function symbol for the valuation map from the field to the value group, and another function symbol for an angular component map modulo P from the field to the residue field. More precisely, we have

2. 3. Definition. The first order language

$$\mathscr{L} = (\mathbb{L}_K, \mathbb{L}_{\bar{K}}, \mathbb{L}_\Gamma, \mathrm{ord}, \overline{\mathrm{ac}})$$

is a 3-sorted language consisting of

(i) the language $\mathbb{L}_K = \{+, -, ., 0, 1\}$ of fields as field sort or K-sort,

(ii) the language $\mathbb{L}_{\bar{K}} = \{+, -, ., 0, 1\}$ of fields as residue field sort or $\bar{K}$-sort,

(iii) the language $\mathbb{L}_\Gamma$, which is an extension of the language $\{+, 0, \infty, \leqq\}$ of ordered abelian groups with an element ∞ on top, as the value group sort or Γ-sort,

(iv) a function symbol ord from the K-sort to the Γ-sort, which stands for the valuation,

(v) a function symbol $\overline{\mathrm{ac}}$ from the K-sort to the $\bar{K}$-sort, which stands for an angular component map modulo P.

We denote (tuples of) K-sort variables by $x, y, u, t, \ldots$, (tuples of) $\bar{K}$-sort variables by $\xi, \varrho, \zeta, \ldots$, and (tuples of) Γ-sort variables by $k, l, \ldots$.

For more information on many sorted first order logic, see [11], pp. 277—281.

As an example of the language $\mathscr{L}$, we will use in Section 5 the language

$$\mathscr{L}_{\mathrm{PR}} = (\mathbb{L}_K, \mathbb{L}_{\bar{K}}, \mathbb{L}_{\mathrm{PR}\infty}, \mathrm{ord}, \overline{\mathrm{ac}})$$

where $\mathbb{L}_{\mathrm{PR}\infty} = \mathbb{L}_{\mathrm{PR}} \cup \{\infty\}$ and $\mathbb{L}_{\mathrm{PR}}$ is the Presburger language

$$\mathbb{L}_{\mathrm{PR}} = \{+, 0, 1, \leqq\} \cup \{\equiv_n \mid n \in \mathbb{N}, n > 1\}.$$

Notice that, if we interpret $\equiv_n$ as "congruent modulo n" in $\mathbb{Z}$, then $\mathbb{Q}_p$ is a structure for the language $\mathscr{L}_{\mathrm{PR}}$.

2. 4. Notation. In the sequel, K will denote a valued field which satisfies the following conditions:

(i) $\mathrm{char}\, K = 0$,

(ii) $\mathrm{char}\, \bar{K} = 0$,

(iii) K is henselian,

(iv) K has an angular component map modulo P,

(v) $(K, \bar{K}, \Gamma \cup \{\infty\}, \mathrm{ord}, \overline{\mathrm{ac}})$ is a structure for the language $\mathscr{L}$, where the function symbol $\overline{\mathrm{ac}}$ is interpreted as the angular component map modulo P on K, and the interpretations for the other symbols in $\mathscr{L}$ are the standard ones.

The following two definitions are due to Cohen [4].

2. 5. Definition. A formula ψ in $\mathscr{L}$ is called *simple* if ψ doesn't contain any K-quantifiers. A formula φ in $\mathscr{L}$ is called *Γ-simple* if φ doesn't contain any Γ- nor K-quantifiers.

A subset D of K^m or of $K^m \times \bar{K}^n$ is called *simple* (*Γ-simple*) if D is definable by a simple (Γ-simple) formula.

Remark. For given m, n, the simple (Γ-simple) sets form a boolean algebra (i.e. closed under union, intersection and complement).

2. 6. Definition. A function $h: K^m \times \bar{K}^n \to K: (x, \xi) \mapsto h(x, \xi)$ is *strongly definable* (*Γ-strongly definable*) if, for each simple (Γ-simple) formula $\varphi(t, y, \varrho, k)$, there exists a simple (Γ-simple) formula $\psi(x, \xi, y, \varrho, k)$ such that

$$\varphi(h(x, \xi), y, \varrho, k) \leftrightarrow \psi(x, \xi, y, \varrho, k).$$

In the same way, one can define strongly definable (Γ-strongly definable) functions from K^m to K.

2. 7. Remarks. (i) It is clear that the graph of a strongly definable (Γ-strongly definable) function is a simple (Γ-simple) set, so each strongly definable (Γ-strongly definable) function is definable in the language $\mathscr{L}$ in the usual sense.

(ii) For given m, n, one can check that the strongly definable functions form a commutative ring with 1 and the nowhere vanishing functions in this ring are units. The Γ-strongly definable functions form also a commutative ring. The composition of strongly definable (Γ-strongly definable) functions is again strongly definable (Γ-strongly definable). Polynomials with integer coefficients are strongly definable and Γ-strongly definable.

2. 8. Lemma. *Let $h(x, \xi)$ be a function from $K^m \times \bar{K}^n$ to K. Assume that for each positive integer r, and for each polynomial $g(y_1, \ldots, y_r, t)$ in the K-variables $y_1, \ldots, y_r, t$, with integer coefficients, the expressions*

$$\overline{\mathrm{ac}}\, g(y_1, \ldots, y_r, h(x, \xi)) = \varrho \text{ and } \operatorname{ord} g(y_1, \ldots, y_r, h(x, \xi)) = l$$

are equivalent to simple formulas $\psi_1(x, \xi, y_1, \ldots, y_r, \varrho)$ resp. $\psi_2(x, \xi, y_1, \ldots, y_r, l)$. Then h is a strongly definable function.

Proof. The lemma follows from the way in which simple formulas are constructed, and more precisely from the fact that quantifiers over the residue field and over the value group are allowed in simple formulas. ∎

Notice that this proof doesn't work for Γ-strongly definable since in Γ-simple formulas quantifiers over the value group are not allowed.

2. 9. Definition. Let $x = (x_1, \ldots, x_m)$ be K-variables, $\xi = (\xi_1, \ldots, \xi_n)$ $\bar{K}$-variables. Let C be a simple subset of $K^m \times \bar{K}^n$. Let $b_1(x, \xi)$, $b_2(x, \xi)$, $c(x, \xi)$ be strongly definable functions from C to K, λ a positive integer and let $\diamond_1$ resp. $\diamond_2$ be $<$, $\leqq$ or no condition. Set for each $\xi \in \bar{K}^n$,

$$A(\xi) = \{(x, t) \in K^m \times K \mid (x, \xi) \in C, \operatorname{ord} b_1(x, \xi) \diamond_1 \lambda \cdot \operatorname{ord}(t - c(x, \xi)) \diamond_2 \operatorname{ord} b_2(x, \xi), \overline{\mathrm{ac}}(t - c(x, \xi)) = \xi_1\}.$$

Suppose further that for all $\xi, \xi' \in \bar{K}^n$ with $\xi \neq \xi'$ we have $A(\xi) \cap A(\xi') = \emptyset$, then

$$A = \bigcup_{\xi \in \bar{K}^n} A(\xi)$$

is called a *cell in* $K^m \times K$ *with parameters* $(\xi_1, \ldots, \xi_n)$ *and center* $c(x, \xi)$. $A(\xi)$ is called a *fiber* of the cell.

2. 10. Remarks. (i) The representation of a cell in the above way is not unique. More specifically, a cell can have representations with different centers and a different number of parameters. This fact will be explicitly used in the proof of the cell decomposition theorems.

(ii) Analogously, we can define the notion of Γ-cell by replacing in definition 2. 9 the words "simple" by "Γ-simple", and "strongly definable" by "Γ-strongly definable".

3. Cell decomposition

We will now state our two cell decomposition theorems. The proofs proceed along the same lines as Denef's proof of cell decomposition [9], which in turn was based on P. J. Cohen [4]. We notice that the theorems will hold uniformly for all valued fields satisfying the conditions of 2. 4.

3. 1. Cell decomposition theorem I. *Let t be one K-variable and $x = (x_1, \ldots, x_m)$ be m K-variables. Let $f(x, t)$ be a polynomial in t with coefficients in the ring of strongly definable functions in x.*

Then $K^m \times K$ admits a finite partition in cells A, which satisfies:

Each cell $A = \bigcup_{\xi} A(\xi)$ of the partition has parameters $\xi = (\xi_1, \ldots, \xi_n)$ and a center $c(x, \xi)$, such that, if we write

$$f(x, t) = \sum_{i=0}^{d} a_i(x, \xi)\,(t - c(x, \xi))^i,$$

then for all $\xi \in \bar{K}^n$, and for all $(x, t) \in A(\xi)$ we have

$$\operatorname{ord} f(x, t) = \operatorname{ord} a_{i_0}(x, \xi)\,(t - c(x, \xi))^{i_0} = \min_{0 \leq i \leq d} \operatorname{ord} a_i(x, \xi)\,(t - c(x, \xi))^i,$$
$$\overline{\operatorname{ac}} f(x, t) = \xi_{j_0}$$

where $i_0 \in \{0, \ldots, d\}$ and $j_0 \in \{1, \ldots, n\}$ do not depend on (x, ξ, t).

3. 2. Cell decomposition theorem II. *Let $f_1(x, t), \ldots, f_r(x, t)$ be polynomials in t as in theorem* I.

Then $K^m \times K$ admits a finite partition in cells A, which satisfies:

Each cell $A = \bigcup_{\xi} A(\xi)$ of the partition has parameters $\xi = (\xi_1, \ldots, \xi_n)$ and a center $c(x, \xi)$ such that, for all $\xi \in \bar{K}^n$, for all $(x, t) \in A(\xi)$, and for all $i \in \{1, \ldots, r\}$ we have

$$\operatorname{ord} f_i(x, t) = \operatorname{ord} h_i(x, \xi)\,(t - c(x, \xi))^{\nu_i},$$

$$\overline{\mathrm{ac}}\, f_i(x, t) = \xi_{\mu(i)}$$

where the $h_i(x, \xi)$ $(i = 1, \ldots, r)$ *are strongly definable functions, and the* $\nu_i \in \mathbb{N}$ $(i = 1, \ldots, r)$ *and the map* $\mu : \{1, \ldots, r\} \to \{1, \ldots, n\}$ *do not depend on* (x, ξ, t).

3. 3. Remark. Let $\mathscr{D}$ be a non-principal ultrafilter on the index set of the prime numbers. The ultaproduct $\prod \mathbb{Q}_p/\mathscr{D}$ is a valued field which is a structure for the language $\mathscr{L}_{\mathrm{PR}}$ and satisfies the conditions on K in 2. 4. The application of theorem 3. 2 with $\mathscr{L} = \mathscr{L}_{\mathrm{PR}}$ gives cell decomposition on the fields $\prod \mathbb{Q}_p/\mathscr{D}$ uniformly for all non-principal ultrafilters $\mathscr{D}$. By the properties of ultraproducts (see e.g. [3], Ch. 4), a $\mathscr{L}_{\mathrm{PR}}$-sentence which holds in $\prod \mathbb{Q}_p/\mathscr{D}$ for all non-principal ultrafilters $\mathscr{D}$, will hold in $\mathbb{Q}_p$ for almost all p. So, since cells are definable sets, this will yield cell decomposition on $\mathbb{Q}_p$, uniformly for almost all p, in the language $\mathscr{L}_{\mathrm{PR}}$. Using this fact we will be able to obtain in Section 5 results on the zeta function which are uniform in the prime p.

3. 4. Lemma. *We can define (in a non-canonical way) a map* $\pi : \Gamma \to K^\times$ *which satisfies*

(i) $\overline{\mathrm{ac}} \circ \pi$ *is the constant map onto* 1,

(ii) $\operatorname{ord} \circ \pi$ *is the identical map on* Γ.

Proof. For $k \in \Gamma$, choose $x \in K$ such that $\operatorname{ord} x = k$. Because the restriction of $\overline{\mathrm{ac}}$ to U is just Res, we can choose $u \in U$ such that $\overline{\mathrm{ac}}\, u = \overline{\mathrm{ac}}\, x$. Define then $\pi(k) = x/u$. ∎

If the $\overline{\mathrm{ac}}$-map on K is defined using a cross section, then this cross section satisfies the conditions (i) and (ii) on the map π in the above lemma.

The following lemma is a generalization of Hensel's lemma.

3. 5. Lemma. *Let* $f(X) = \sum_{i=0}^{d} a_i (X - c)^i$ *be a polynomial in* $K[X]$. *Let* $\tilde{\alpha} \in K$, $\tilde{\alpha} \neq c$ *such that*

(1) $$\min_{0 \leqq i \leqq d} \operatorname{ord} a_i(\tilde{\alpha} - c)^i = \operatorname{ord} a_{i_0}(\tilde{\alpha} - c)^{i_0} \quad \textit{with} \quad i_0 \geqq 1,$$

(2) $$\operatorname{ord} f(\tilde{\alpha}) > \operatorname{ord} a_{i_0}(\tilde{\alpha} - c)^{i_0},$$

(3) $$\operatorname{ord} f'(\tilde{\alpha}) = \operatorname{ord} a_{i_0}(\tilde{\alpha} - c)^{i_0 - 1}.$$

Then there exists a unique $\alpha \in K$ *such that*

(4) $$f(\alpha) = 0,$$

(5) $$\operatorname{ord}(\alpha - c) = \operatorname{ord}(\tilde{\alpha} - c),$$

(6) $$\overline{\mathrm{ac}}(\alpha - c) = \overline{\mathrm{ac}}(\tilde{\alpha} - c).$$

Proof. We may suppose that $c=0$ since the general case can be reduced to this by a translation. The idea is now to transform f, using a homothetic transformation, into a polynomial to which Hensel's lemma applies. Let $\operatorname{ord}\tilde{\alpha}=l$, and fix a map π as in lemma 3. 4. We put $Y=X/\pi(l)$, $g(Y)=f(X)/a_{i_0}(\pi(l))^{i_0}$ and we define $\tilde{\beta}=\tilde{\alpha}/\pi(l)$. Using conditions (1), (2), (3) in the formulation of the lemma, and lemma 3. 4., one easily checks that $g(Y)$ is a polynomial with coefficients in R, such that $\operatorname{ord}\tilde{\beta}=0$, $\operatorname{ord} g(\tilde{\beta})>0$ and $\operatorname{ord} g'(\tilde{\beta})=0$. By Hensel's lemma there is a unique root β of $g(Y)$ in R such that $\operatorname{Res}\beta=\operatorname{Res}\tilde{\beta}$. So $\operatorname{ord}\tilde{\beta}=0$ implies that $\operatorname{ord}\beta=0$, and $\overline{\mathrm{ac}}\,\beta=\operatorname{Res}\beta=\operatorname{Res}\tilde{\beta}=\overline{\mathrm{ac}}\,\tilde{\beta}=\overline{\mathrm{ac}}\,\tilde{\alpha}$ because the restriction of $\overline{\mathrm{ac}}$ to U is the projection Res. One sees then that $\alpha=\pi(l)\cdot\beta$ satisfies the conditions of the lemma. ∎

3. 6. *Proof of cell decomposition theorem* I. The proof is by induction on the degree of the polynomial $f(x, t)$. In the induction step we use a lemma, lemma 3. 7., which is proved using the induction hypothesis.

Let $f(x, t)$ be of degree d, and suppose theorem I holds for polynomials of degree $\leqq d-1$. The derivative of f with respect to t, denoted by $f'(x, t)$, has degree $d-1$, so applying the induction hypothesis to f', we get a finite partition of $K^m\times K$ in cells $A=\bigcup_{\xi} A(\xi)$ with parameters $\xi=(\xi_1,\dots,\xi_n)$ and center $c(x, \xi)$ such that, for $(x, t)\in A(\xi)$

$$\operatorname{ord} f'(x, t)=\min_{1\leqq i\leqq d}\operatorname{ord}(i\cdot a_i(x, \xi)\,(t-c(x, \xi))^{i-1}). \tag{1}$$

We will further partition A in cells on which theorem I holds.

Suppose the cell A has fiber

$$\begin{aligned} A(\xi)=\{(x, t)\in K^m\times K \mid (x, \xi)\in C,\ &\operatorname{ord} b_1(x, \xi)\diamond_1 \\ &\lambda\cdot\operatorname{ord}(t-c(x, \xi))\diamond_2\operatorname{ord} b_2(x, \xi),\ \overline{\mathrm{ac}}(t-c(x, \xi))=\xi_1\}. \end{aligned} \tag{2}$$

By partitioning the simple set C we may suppose that $a_i(x, \xi)$ is either identically zero or else never zero on C.

Let $I=\{i\in\{0,\dots,d\}\mid \forall(x, \xi)\in C: a_i(x, \xi)\neq 0\}$. We consider now three cases.

First case: $I=\emptyset$. In this case, one has

$$\operatorname{ord} f(x, t)=\infty=\min_{0\leqq i\leqq d}\operatorname{ord}(a_i(x, \xi)\,(t-c(x, \xi))^i)$$

and $\overline{\mathrm{ac}}\,f(x, t)=0$. If we represent A with one parameter ϱ more, in the following way

$$\begin{aligned} A&=\bigcup_{\xi}\bigcup_{\varrho}\{(x, t)\in A(\xi)\mid \varrho=0\} \\ &=\bigcup_{\xi,\varrho}\{(x, t)\mid (x, \xi)\in C,\ \varrho=0,\ \operatorname{ord} b_i(x, \xi)\diamond_i\lambda\cdot\operatorname{ord}(t-c(x, \xi))\ (i=1, 2), \\ &\qquad\qquad \overline{\mathrm{ac}}(t-c(x, \xi))=\xi_1\} \end{aligned}$$

then, on A, $\overline{\mathrm{ac}}\,f(x, t)=\varrho$. So theorem I holds for f on the cell A.

Second case: $I=\{i_0\}$ is a singleton. On $A(\xi)$ we have that

$$f(x,t)=a_{i_0}(x,\xi)\,(t-c(x,\xi))^{i_0},$$

so $\operatorname{ord} f(x,t)=\min\limits_{0\leq i\leq d} a_i(x,\xi)\,(t-c(x,\xi))^i$ and

$$\overline{\mathrm{ac}}\, f(x,t)=\overline{\mathrm{ac}}\, a_{i_0}(x,\xi)\,\overline{\mathrm{ac}}\,(t-c(x,\xi))^{i_0}=\overline{\mathrm{ac}}\, a_{i_0}(x,\xi)\cdot\xi_1^{i_0}$$

by (2). Again we introduce a new parameter ϱ for the cell A:

$$A=\bigcup_{\xi}\bigcup_{\varrho}\{(x,t)\in A(\xi)\mid \varrho=\overline{\mathrm{ac}}\, a_{i_0}(x,\xi)\cdot\xi_1^{i_0}\}.$$

So, on A, theorem I holds for f.

Third case: $\#I>1$. ($\#I$ stands for the cardinality of the set I.) Let

$$J=\{(i,j)\in I\times I\mid i>j\}.$$

For $\theta=(\diamond_{ij})_{(i,j)\in J}\in\Theta=\{<,>\}^J$, put

$$A_\theta(\xi)=\left\{(x,t)\in A(\xi)\mid \forall(i,j)\in J:(i-j)\cdot\operatorname{ord}(t-c(x,\xi))\diamond_{ij}\operatorname{ord}\frac{a_j(x,\xi)}{a_i(x,\xi)}\right\}.$$

For $L\subset J$, $L\neq\emptyset$, put

$$A_L(\xi)=\left\{(x,t)\in A(\xi)\mid \forall(i,j)\in L:(i-j)\cdot\operatorname{ord}(t-c(x,\xi))=\operatorname{ord}\frac{a_j(x,\xi)}{a_i(x,\xi)},\right.$$
$$\left.\forall(i,j)\in J\backslash L:(i-j)\cdot\operatorname{ord}(t-c(x,\xi))\neq\operatorname{ord}\frac{a_j(x,\xi)}{a_i(x,\xi)}\right\}.$$

We can write then

$$A=\left[\bigcup_{\theta\in\Theta}\left(\bigcup_{\xi}A_\theta(\xi)\right)\right]\cup\left[\bigcup_{L\subset J,L\neq\emptyset}\left(\bigcup_{\xi}A_L(\xi)\right)\right]. \tag{3}$$

This obviously gives a finite partition of A in sets $\bigcup\limits_{\xi}A_\theta(\xi)$ and $\bigcup\limits_{\xi}A_L(\xi)$.

First we will show that we can partition each of the sets $\bigcup\limits_{\xi}A_\theta(\xi)$ in a finite number of cells on which theorem I holds for f. Fix a $\theta\in\Theta$. The terms

$$\operatorname{ord} a_i(x,\xi)\,(t-c(x,\xi))^i$$

are strictly ordered on $\bigcup\limits_{\xi}A_\theta(\xi)$ (the ordering depends only on θ). Suppose the smallest term has index i_0.

On $\bigcup_\xi A_\theta(\xi)$ we have that

$$\operatorname{ord} f(x, t) = \min_{0 \leqq i \leqq d} \operatorname{ord} a_i(x, \xi)\,(t - c(x, \xi))^i = \operatorname{ord} a_{i_0}(x, \xi)\,(t - c(x, \xi))^{i_0}.$$

One can check that $\bigcup_\xi A_\theta(\xi)$ is a finite union of cells $B = \bigcup_\xi B(\xi)$ with same parameters ξ and center $c(x, \xi)$, and such that $B(\xi) \subset A_\theta(\xi)$. On such a cell B we have (for $t \neq c(x, \xi)$)

$$\frac{f(x, t)}{a_{i_0}(x, \xi)\,(t - c(x, \xi))^{i_0}} = 1 + \sum_{i \in I \setminus \{i_0\}} \frac{a_i(x, \xi)}{a_{i_0}(x, \xi)} (t - c(x, \xi))^{i - i_0}. \tag{4}$$

The terms in the summation on the right hand side of (4) have strictly positive valuation, so

$$\operatorname{ord}\left(\frac{f(x, t)}{a_{i_0}(x, \xi)\,(t - c(x, \xi))^{i_0}} - 1\right) > 0$$

and

$$\overline{\mathrm{ac}}\, f(x, t) = \overline{\mathrm{ac}}\, a_{i_0}(x, \xi)\,(t - c(x, \xi))^{i_0}.$$

On B we have $\overline{\mathrm{ac}}\,(t - c(x, \xi)) = \xi_1$ because of (2). Hence $\overline{\mathrm{ac}}\, f(x, t) = \overline{\mathrm{ac}}\, a_{i_0}(x, \xi) \cdot \xi_1^{i_0}$. As in the first case, we can represent the cell B with an additional parameter ϱ:

$$B = \bigcup_\xi B(\xi) = \bigcup_\xi \bigcup_\varrho \{(x, t) \in B(\xi) \mid \overline{\mathrm{ac}}\, a_{i_0}(x, \xi) \cdot \xi_1^{i_0} = \varrho\}$$

which shows that theorem I holds for f on B.

Next we consider the remaining part of the partition in (3), namely the sets $\bigcup_\xi A_L(\xi)$ for $L \subset J$, $L \neq \emptyset$. Fix such an L. One can verify that $\bigcup_\xi A_L(\xi)$ is a cell of the form

$$B = \bigcup_\xi \{(x, t) \in K^m \times K \mid (x, \xi) \in \tilde{C},\ \lambda \cdot \operatorname{ord}(t - c(x, \xi)) = \operatorname{ord} \theta(x, \xi),\ \overline{\mathrm{ac}}\,(t - c(x, \xi)) = \xi_1\}$$

where $\theta(x, \xi)$ is a nowhere vanishing, strongly definable function on the simple set $\tilde{C}$.

Let now $\Delta = (\diamond_{ij})_{(i,j) \in J \setminus L} \in \tilde{\Theta} = \{<, >\}^{(J \setminus L)}$ and put

$$\begin{aligned} D_\Delta = \{(x, \xi) \in \tilde{C} \mid \forall (i, j) \in J \setminus L\colon \lambda \cdot \operatorname{ord} a_i(x, \xi) + i \cdot \operatorname{ord} \theta(x, \xi) \diamond_{ij} \\ \lambda \cdot \operatorname{ord} a_j(x, \xi) + j \cdot \operatorname{ord} \theta(x, \xi)\}. \end{aligned} \tag{5}$$

These D_Δ are simple sets, and they form a partition of $\tilde{C}$.

If we write B_Δ for the set

$$\bigcup_{\xi} \{(x, t) \in K^m \times K \mid (x, \xi) \in D_\Delta,\ \lambda \cdot \operatorname{ord}(t - c(x, \xi)) = \operatorname{ord} \theta(x, \xi),\ \overline{\operatorname{ac}}(t - c(x, \xi)) = \xi_1\}$$

then $B = \bigcup_{\Delta \in \tilde{\Theta}} B_\Delta$ gives a partition of B in cells.

We now fix a $\Delta \in \tilde{\Theta}$, and write B for B_Δ and D for D_Δ. By (5), we may suppose that on B the terms $\lambda \cdot \operatorname{ord} a_i(x, \xi) + i \cdot \operatorname{ord} \theta(x, \xi)$ are totally ordered (the ordering only depends on the choosen Δ). So there exists an $i_0 \in I$ such that

$$\lambda \cdot \operatorname{ord} a_{i_0}(x, \xi) + i_0 \cdot \operatorname{ord} \theta(x, \xi) = \min_{i \in I} (\lambda \cdot \operatorname{ord} a_i(x, \xi) + i \cdot \operatorname{ord} \theta(x, \xi)). \tag{6}$$

We can take $i_0 \geqq 1$, except when

$$\lambda \cdot \operatorname{ord} a_0(x, \xi) < \min_{i \neq 0} (\lambda \cdot \operatorname{ord} a_i(x, \xi) + i \cdot \operatorname{ord} \theta(x, \xi)).$$

In this case we have $\operatorname{ord} f(x, t) = \operatorname{ord} a_0(x, \xi) = \min_i \operatorname{ord}(a_i(x, \xi)(t - c(x, \xi))^i)$. Also

$$\operatorname{ord}\left(\frac{f(x, t)}{a_0(x, \xi)} - 1\right) > 0$$

and $\overline{\operatorname{ac}} f(x, t) = \overline{\operatorname{ac}} a_0(x, \xi)$. Writing the cell with a new parameter $\varrho = \overline{\operatorname{ac}} a_0(x, \xi)$, shows in the same way as before that theorem I holds for f on B.

So suppose further $i_0 \geqq 1$. By (6) we have that

$$\operatorname{ord} a_{i_0}(x, \xi)(t - c(x, \xi))^{i_0} = \min_{0 \leqq i \leqq d} \operatorname{ord} a_i(x, \xi)(t - c(x, \xi))^i. \tag{7}$$

Now each fiber $B(\xi)$ of B is splitted up in two (disjoint) parts

$$B_1(\xi) = \{(x, t) \in B(\xi) \mid \operatorname{ord} f(x, t) = \operatorname{ord} a_{i_0}(x, \xi)(t - c(x, \xi))^{i_0}\} \tag{8}$$

and

$$B_2(\xi) = \{(x, t) \in B(\xi) \mid \operatorname{ord} f(x, t) > \operatorname{ord} a_{i_0}(x, \xi)(t - c(x, \xi))^{i_0}\}. \tag{9}$$

Put $B_1 = \bigcup_{\xi} B_1(\xi)$ and $B_2 = \bigcup_{\xi} B_2(\xi)$, then $B = B_1 \cup B_2$.

Choose now a map π as in lemma 3. 4. On $B_1(\xi)$ we have

$$\begin{aligned}\overline{\operatorname{ac}} f(x, t) &= \operatorname{Res} \frac{f(x, t)}{\pi(\operatorname{ord} a_{i_0}(x, \xi)(t - c(x, \xi))^{i_0})} \\ &= \sum_{i \in I} \operatorname{Res} \frac{a_i(x, \xi)(t - c(x, \xi))^i}{\pi(\operatorname{ord} a_{i_0}(x, \xi)(t - c(x, \xi))^{i_0})}.\end{aligned}$$

Set $\tilde{I}=\{i\in I \mid \operatorname{ord} a_i(x,\xi)\,(t-c(x,\xi))^i=\operatorname{ord} a_{i_0}(x,\xi)\,(t-c(x,\xi))^{i_0}\}$ then

$$\begin{aligned}\overline{\mathrm{ac}}\, f(x,t)&=\sum_{i\in\tilde{I}}\overline{\mathrm{ac}}\, a_i(x,\xi)\cdot\overline{\mathrm{ac}}\,(t-c(x,\xi))^i\\&=\sum_{i\in\tilde{I}}\overline{\mathrm{ac}}\, a_i(x,\xi)\cdot\xi_1^i.\end{aligned}$$

B_1 is a cell, since the condition $\operatorname{ord} f(x,t)=\operatorname{ord} a_{i_0}(x,\xi)\,(t-c(x,\xi))^{i_0}$ in (8) is equivalent to

$$\operatorname{Res}\frac{f(x,t)}{\pi(\operatorname{ord} a_{i_0}(x,\xi)\,(t-c(x,\xi))^{i_0})}\neq 0$$

or by the preceding computation, to

$$\sum_{i\in\tilde{I}}\overline{\mathrm{ac}}\, a_i(x,\xi)\cdot\xi_1^i\neq 0$$

and this is a simple condition in (x,ξ). Furthermore we can write B_1 as

$$B_1=\bigcup_{\xi}\bigcup_{\varrho}\{(x,t)\in B_1(\xi)\mid\sum_{i\in\tilde{I}}\overline{\mathrm{ac}}\, a_i(x,\xi)\cdot\xi_1^i=\varrho\}.$$

Since $\overline{\mathrm{ac}}\, f(x,t)=\sum_{i\in\tilde{I}}\overline{\mathrm{ac}}\, a_i(x,\xi)\cdot\xi_1^i$, combining (7) and (8), proves that theorem I holds for f on B_1.

Now we only have to take care of B_2. In the same way as with B_1, one can show that B_2 is a cell which can be written as

$$B_2=\bigcup_{\xi}\{(x,t)\mid(x,\xi)\in\tilde{D},\ \lambda\cdot\operatorname{ord}(t-c(x,\xi))=\operatorname{ord}\theta(x,\xi),\ \overline{\mathrm{ac}}\,(t-c(x,\xi))=\xi_1\}$$

where

$$\tilde{D}=\{(x,\xi)\in D\mid\sum_{i\in\tilde{I}}\overline{\mathrm{ac}}\, a_i(x,\xi)\cdot\xi_1^i=0\quad\text{and}\quad(\exists l)\,(\lambda\cdot l=\operatorname{ord}\theta(x,\xi))\}.$$

(Adding the condition $(\exists l)\,(\lambda\cdot l=\operatorname{ord}\theta(x,\xi))$ doesn't affect B_2 but ensures, since Γ is an ordered abelian group, that for $(x,\xi)\in\tilde{D}$, $\frac{1}{\lambda}\operatorname{ord}\theta(x,\xi)$ is well defined.)

To prove that theorem I holds for f on B_2, we'll have to find a new center for this cell. This is done using Hensel's lemma (or more precisely the generalization of it in lemma 3. 5).

On B_2 we have by (7) and (9)

$$\operatorname{ord} f(x,t)>\min_i\operatorname{ord} a_i(x,\xi)\,(t-c(x,\xi))^i=\operatorname{ord} a_{i_0}(x,\xi)\,(t-c(x,\xi))^{i_0}\tag{10}$$

with $i_0\geqq 1$. Since the characteristic of $\bar{K}$ is zero, $\operatorname{ord} i=0$ for $1\leqq i\leqq d$, so (1) and (7) imply that

$$\operatorname{ord} f'(x,t)=\operatorname{ord}[a_{i_0}(x,\xi)\,(t-c(x,\xi))^{i_0}]-\operatorname{ord}(t-c(x,\xi)).\tag{11}$$

Let now $(x, \xi) \in \tilde{D}$. Since $(x, c(x, \xi))$ doesn't belong to $B_2(\xi)$, we may suppose that $\xi_1 \neq 0$. Choose $u \in K^\times$ such that $\operatorname{ord} u = 0$ and $\overline{\operatorname{ac}}\, u = \xi_1$. Put

$$\tilde{d}(x, \xi) = \pi(\tfrac{1}{\lambda} \cdot \operatorname{ord} \theta(x, \xi)) \cdot u + c(x, \xi).$$

One verifies that $(x, \tilde{d}(x, \xi))$ belongs to $B_2(\xi)$. Hence by (7), (10) and (11)

$$\min_i \operatorname{ord} a_i(x, \xi)\,(\tilde{d}(x, \xi) - c(x, \xi))^i = \operatorname{ord} a_{i_0}(x, \xi)\,(\tilde{d}(x, \xi) - c(x, \xi))^{i_0},$$

$$\operatorname{ord} f(x, \tilde{d}(x, \xi)) > \operatorname{ord} a_{i_0}(x, \xi)\,(\tilde{d}(x, \xi) - c(x, \xi))^{i_0},$$

$$\operatorname{ord} f'(x, \tilde{d}(x, \xi)) = \operatorname{ord} a_{i_0}(x, \xi)\,(\tilde{d}(x, \xi) - c(x, \xi))^{i_0 - 1}.$$

$\tilde{d}(x, \xi)$ satisfies the conditions of lemma 3. 5, so there exists a unique $d(x, \xi)$ such that

$$f(x, d(x, \xi)) = 0, \tag{12}$$

$$\lambda \cdot \operatorname{ord}(d(x, \xi) - c(x, \xi)) = \operatorname{ord} \theta(x, \xi), \tag{13}$$

$$\overline{\operatorname{ac}}(d(x, \xi) - c(x, \xi)) = \xi_1. \tag{14}$$

By the uniqueness property in lemma 3. 5 this $d(x, \xi)$ is uniquely determined, independent of the choice of u and π. This shows that the function $d(x, \xi)$ on $\tilde{D}$ is well defined. By lemma 3. 7 (which is proved using the induction hypothesis) $d(x, \xi)$ is a strongly definable function on $\tilde{D}$.

We can now rewrite B_2 as a cell with center $d(x, \xi)$. Put

$$B_2' = \bigcup_\xi \{(x, t) \mid (x, \xi) \in \tilde{D},\ \lambda \cdot \operatorname{ord}(t - d(x, \xi)) > \operatorname{ord} \theta(x, \xi)\}.$$

Then $B_2 = B_2'$.

Indeed, $B_2 \subset B_2'$: We have

$$\begin{aligned}
\operatorname{Res}\left(\frac{t - c(x, \xi)}{\pi(\frac{1}{\lambda} \cdot \operatorname{ord} \theta(x, \xi))}\right) &= \overline{\operatorname{ac}}(t - c(x, \xi)) \\
&= \xi_1 \\
&= \overline{\operatorname{ac}}(d(x, \xi) - c(x, \xi)) \\
&= \operatorname{Res}\left(\frac{d(x, \xi) - c(x, \xi)}{\pi(\frac{1}{\lambda} \cdot \operatorname{ord} \theta(x, \xi))}\right)
\end{aligned}$$

so

$$\lambda \cdot \operatorname{ord}(t - d(x, \xi)) = \operatorname{ord} \theta(x, \xi) + \lambda \cdot \operatorname{ord}\left(\frac{t - c(x, \xi)}{\pi(\frac{1}{\lambda} \cdot \operatorname{ord} \theta(x, \xi))} - \frac{d(x, \xi) - c(x, \xi)}{\pi(\frac{1}{\lambda} \cdot \operatorname{ord} \theta(x, \xi))}\right) > \operatorname{ord} \theta(x, \xi).$$

Conversely, $B_2' \subset B_2$: By (13) and the definition of B_2'

$$\lambda \cdot \operatorname{ord}(t - c(x, \xi)) = \lambda \cdot \operatorname{ord}[(t - d(x, \xi)) - (c(x, \xi) - d(x, \xi))] = \operatorname{ord}\theta(x, \xi).$$

Also

$$\overline{\mathrm{ac}}(t - c(x, \xi)) = \operatorname{Res}\left(\frac{t - c(x, \xi)}{\pi(\frac{1}{\lambda} \cdot \operatorname{ord}\theta(x, \xi))}\right) = \operatorname{Res}\left(\frac{d(x, \xi) - c(x, \xi)}{\pi(\frac{1}{\lambda} \cdot \operatorname{ord}\theta(x, \xi))}\right) = \overline{\mathrm{ac}}(d(x, \xi) - c(x, \xi)),$$

since

$$\operatorname{ord}\left(\frac{t - c(x, \xi)}{\pi(\frac{1}{\lambda} \cdot \operatorname{ord}\theta(x, \xi))} - \frac{d(x, \xi) - c(x, \xi)}{\pi(\frac{1}{\lambda} \cdot \operatorname{ord}\theta(x, \xi))}\right) = \operatorname{ord}\left(\frac{t - d(x, \xi)}{\pi(\frac{1}{\lambda} \cdot \operatorname{ord}\theta(x, \xi))}\right) > 0.$$

This proves $B_2 = B_2'$.

We are now left to prove that the conditions of theorem I are satisfied on B_2. Using (12), Taylor expansion gives

$$f(x, t) = f'(x, d(x, \xi))\,(t - d(x, \xi)) + \sum_{j=2}^{d} \frac{f^{(j)}(x, d(x, \xi))}{j!}\,(t - d(x, \xi))^j \tag{15}$$

where $f^{(j)}$ stands for the j-th derivative of f with respect to t. We have for $j \in \{2, \ldots, d\}$

$$\begin{aligned}
\operatorname{ord}\frac{f^{(j)}(x, d(x, \xi))}{j!} &= \operatorname{ord}\left(\sum_{i=j}^{d} \binom{i}{j} a_i(x, \xi)\,(d(x, \xi) - c(x, \xi))^{i-j}\right) \\
&\geqq \min_{j \leqq i \leqq d} [\operatorname{ord} a_i(x, \xi)\,(d(x, \xi) - c(x, \xi))^i] - \operatorname{ord}(d(x, \xi) - c(x, \xi))^j \\
&\geqq \operatorname{ord}(a_{i_0}(x, \xi)\,(d(x, \xi) - c(x, \xi))^{i_0}) - \frac{j}{\lambda}\operatorname{ord}\theta(x, \xi) \\
&= \operatorname{ord} f'(x, d(x, \xi)) - \frac{j-1}{\lambda}\operatorname{ord}\theta(x, \xi)
\end{aligned}$$

by (7), (11) and (13).

The definition of B_2' implies that

$$\operatorname{ord}\left(\frac{f^{(j)}(x, d(x, \xi))}{j!}\,(t - d(x, \xi))^j\right) > \operatorname{ord} f'(x, d(x, \xi))\,(t - d(x, \xi)).$$

Hence on B_2 we have

$$\begin{aligned}
\operatorname{ord} f(x, t) &= \operatorname{ord} f'(x, d(x, \xi))\,(t - d(x, \xi)) \\
&= \min_{1 \leqq j \leqq d} \operatorname{ord}\left(\frac{f^{(j)}(x, d(x, \xi))}{j!}\,(t - d(x, \xi))^j\right).
\end{aligned} \tag{17}$$

Equations (15) and (16) also imply that

$$\operatorname{ord}\left(\frac{f(x,t)}{f'(x,d(x,\xi))\,(t-d(x,\xi))}-1\right)>0.$$

So

$$\overline{\mathrm{ac}}\,f(x,t)=\overline{\mathrm{ac}}\,f'(x,d(x,\xi))\cdot\overline{\mathrm{ac}}\,(t-d(x,\xi)). \tag{18}$$

We write B_2 with two additional parameters ϱ and ζ,

$$B_2=\bigcup_{\xi}\bigcup_{\varrho}\bigcup_{\zeta}\{(x,t)\mid(x,\xi)\in\tilde{D},\ \overline{\mathrm{ac}}\,f'(x,d(x,\xi))\cdot\varrho=\zeta,$$
$$\lambda\cdot\operatorname{ord}(t-d(x,\xi))>\operatorname{ord}\theta(x,\xi),\ \overline{\mathrm{ac}}\,(t-d(x,\xi))=\varrho\}.$$

So on B_2, (18) becomes

$$\overline{\mathrm{ac}}\,f(x,t)=\zeta. \tag{19}$$

Since $f'(x,d(x,\xi))$ and $\dfrac{f^{(j)}(x,d(x,\xi))}{j!}$ $(j=2,\dots,d)$ are the coefficients of f written as a polynomial in $t-d(x,\xi)$, (17) and (19) show that theorem I is satisfied for f on the cell B_2. This proves the theorem. ∎

3. 7. Lemma. *Let* $f(x,t)=\sum_{i=0}^{d}a_i(x,\xi)\,(t-c(x,\xi))^i$ *be as in theorem* I. *Suppose theorem* I *holds for polynomials of degree* $\leqq d-1$. *Let* A *be a cell with fiber*

$$A(\xi)=\{(x,t)\in K^m\times K\mid(x,\xi)\in\tilde{D},\ \lambda\cdot\operatorname{ord}(t-c(x,\xi))=\operatorname{ord}\theta(x,\xi),\ \overline{\mathrm{ac}}\,(t-c(x,\xi))=\xi_1\}$$

on which,

(1) $\min_i \operatorname{ord} a_i(x,\xi)\,(t-c(x,\xi))^i=\operatorname{ord} a_{i_0}(x,\xi)\,(t-c(x,\xi))^{i_0}$ *with* $i_0\geqq 1$,

(2) $$\operatorname{ord} f'(x,t)=\operatorname{ord} a_{i_0}(x,\xi)\,(t-c(x,\xi))^{i_0-1}.$$

Suppose $d(x,\xi)$ *is a function from* $\tilde{D}$ *to* K *such that*

(3) $$f(x,d(x,\xi))=0,$$

(4) $$\lambda\cdot\operatorname{ord}(d(x,\xi)-c(x,\xi))=\operatorname{ord}\theta(x,\xi),$$

(5) $$\overline{\mathrm{ac}}\,(d(x,\xi)-c(x,\xi))=\xi_1.$$

Then the function d *is strongly definable.*

Proof. By lemma 2.7, it suffices to prove that for each polynomial $p(y, t)$ with integer coefficients in the K-variables $y=(y_1, \dots, y_r)$ and t, the expressions

$$\operatorname{ord} p(y, d(x, \xi)) = l', \tag{6}$$

$$\overline{\mathrm{ac}}\, p(y, d(x, \xi)) = \zeta' \tag{7}$$

are equivalent to simple formulas.

By (3) and after euclidean division of $p(y, t)$ by $f(x, t)$, we may replace $p(y, t)$ by $p_1(y, x, t)$, a polynomial in t of degree $<d$ whose coefficients are in the ring of strongly definable functions in (y, x). Now we can apply theorem I to $p_1(y, x, t)$. Hence (6) is equivalent to a finite disjunction of expressions of the form

$$\begin{aligned}(\exists \varrho_1) \cdots (\exists \varrho_s)\, [&\operatorname{ord} h(y, x, \varrho) + \nu \cdot \operatorname{ord}(d(x, \xi) - e(y, x, \varrho)) = l' \wedge (y, x, \varrho) \in C \\ &\wedge \operatorname{ord} b_1(y, x, \varrho) \diamond_1 \tilde{\lambda} \cdot \operatorname{ord}(d(x, \xi) - e(y, x, \varrho)) \diamond_2 \operatorname{ord} b_2(y, x, \varrho) \\ &\wedge \overline{\mathrm{ac}}(d(x, \xi) - e(y, x, \varrho)) = \varrho_1]\end{aligned}$$

where $\varrho = (\varrho_1, \dots, \varrho_s)$ are $\bar{K}$-variables, ν and $\tilde{\lambda}$ are positive integers, $\diamond_1$, $\diamond_2$ are $<$, $\leqq$ or no condition, C is a simple set and h, e, b_1, b_2 are strongly definable functions.

This is equivalent to

$$\begin{aligned}(\exists l)\, (\exists \varrho_1) \cdots (\exists \varrho_s)\, [&\operatorname{ord}(d(x, \xi) - e(y, x, \varrho)) = l \wedge \overline{\mathrm{ac}}(d(x, \xi) - e(y, x, \varrho)) = \varrho_1 \\ &\wedge \operatorname{ord} h(y, x, \varrho) + \nu \cdot l = l' \wedge (y, x, \varrho) \in C \wedge \operatorname{ord} b_1(y, x, \varrho) \diamond_1 \tilde{\lambda} \cdot l \diamond_2 \operatorname{ord} b_2(y, x, \varrho)].\end{aligned}$$

This implies that (6) will be equivalent to a simple formula, if we know that the expressions

$$\operatorname{ord}(d(x, \xi) - e(y, x, \varrho)) = l, \tag{8}$$

$$\overline{\mathrm{ac}}(d(x, \xi) - e(y, x, \varrho)) = \zeta \tag{9}$$

are simple. Analogously one obtains that expression (7) is simple if (8) and (9) are simple.

So we are left to prove that (8) and (9) are indeed simple expressions. To get this, we make a disjunction over four disjoint cases which are defined by simple formulas. We show that in each of these cases (8) and (9) are equivalent to simple formulas.

For simplicity we will write d, c, e, θ instead of $d(x, \xi)$, $c(x, \xi)$, $e(y, x, \varrho)$, $\theta(x, \xi)$.

(i) $\lambda \cdot \operatorname{ord}(e-c) < \operatorname{ord} \theta$: We have $d - e = -(e-c)\left(1 - \dfrac{d-c}{e-c}\right)$, so by (4) the ord resp. $\overline{\mathrm{ac}}$ of $(d-e)$ equals the ord resp. $\overline{\mathrm{ac}}$ of $-(e-c)$.

(ii) $\lambda \cdot \operatorname{ord}(e-c) > \operatorname{ord}\theta$: Because $d-e=(d-c)\left(1-\dfrac{e-c}{d-c}\right)$ we have by (4) and (5) that (8) is equivalent to $\operatorname{ord}\theta = \lambda \cdot l$ and (9) is equivalent to $\xi_1 = \zeta$.

(iii) $\lambda \cdot \operatorname{ord}(e-c) = \operatorname{ord}\theta$: ($\alpha$) $\overline{\mathrm{ac}}(e-c) \neq \xi_1$: Fix again a map π as in lemma 3.4. Now we have from (4) and (5) that $\operatorname{Res}((d-c)/\pi(\frac{1}{\lambda}\operatorname{ord}\theta)) \neq \operatorname{Res}((e-c)/\pi(\frac{1}{\lambda}\operatorname{ord}\theta))$. Hence

$$\operatorname{ord}(d-e) = \operatorname{ord}\left((d-c)/\pi\left(\frac{1}{\lambda}\operatorname{ord}\theta\right) - (e-c)/\pi\left(\frac{1}{\lambda}\operatorname{ord}\theta\right)\right) + \frac{1}{\lambda}\operatorname{ord}\theta = \frac{1}{\lambda}\operatorname{ord}\theta$$

and

$$\begin{aligned}\overline{\mathrm{ac}}(d-e) &= \operatorname{Res}\left((d-e)/\pi\left(\frac{1}{\lambda}\operatorname{ord}\theta\right)\right)\\ &= \operatorname{Res}\left((d-c)/\pi\left(\frac{1}{\lambda}\operatorname{ord}\theta\right)\right) - \operatorname{Res}\left((e-c)/\pi\left(\frac{1}{\lambda}\operatorname{ord}\theta\right)\right)\\ &= \overline{\mathrm{ac}}(d-c) - \overline{\mathrm{ac}}(e-c)\\ &= \xi_1 - \overline{\mathrm{ac}}(e-c).\end{aligned}$$

(β) $\overline{\mathrm{ac}}(e-c) = \xi_1$: In this case $(x, e(x, \xi))$ belongs to the fiber $A(\xi)$ of the cell A. So for $t = e(x, \xi)$ conditions (1) and (2) are satisfied. By a computation as in case (iii) (α) we find that

$$\operatorname{ord}(d-e) > \frac{1}{\lambda}\operatorname{ord}\theta. \tag{10}$$

Using (1) one easily derives that

$$\operatorname{ord} f^{(j)}(x, e) \geqq \operatorname{ord} a_{i_0}(x, \xi) + \frac{i_0 - j}{\lambda}\operatorname{ord}\theta \tag{11}$$

where $f^{(j)}$ stands for the j-th derivative of f with respect to t.

Taylor expansion gives

$$0 = f(x, d) = f(x, e) + f'(x, e)(d-e) + \frac{f^{(2)}(x, e)}{2!}(d-e)^2 + \cdots.$$

Hence

$$-\frac{f(x, e)}{f'(x, e)} = (d-e)\left[1 + \sum_{j \geqq 2} \frac{f^{(j)}(x, e)}{j!\, f'(x, e)}(d-e)^{j-1}\right]. \tag{12}$$

From (2), (10) and (11) follows that

$$\operatorname{ord}\left(\frac{f^{(j)}(x, e)}{j!\, f'(x, e)}(d-e)^{j-1}\right) > 0.$$

So (12) implies that

$$\operatorname{ord}(d-e)=\operatorname{ord}\left(\frac{f(x,e)}{f'(x,e)}\right),$$

and

$$\overline{\mathrm{ac}}(d-e)=\overline{\mathrm{ac}}\left(-\frac{f(x,e)}{f'(x,e)}\right). \quad \blacksquare$$

3.8. *Proof of cell decomposition theorem* II. Let $P(A, s)$ stand for the following statement:

A is the intersection of s cells, with parameters $\xi=(\xi_1,\dots,\xi_n)$ and centers, say $c_1,\dots,c_s$. Denote by $A(\xi)$ the intersection of the fibers of the cells of which A is the intersection. For all ξ, for all $(x,t)\in A(\xi)$, and for $i=1,\dots,r$ we have

$$\operatorname{ord} f_i(x,t)=\operatorname{ord}(h_i(x,\xi)\,(t-c_{\eta(i)})^{v_i}),$$
$$\overline{\mathrm{ac}}\, f_i(x,t)=\xi_{\mu(i)}$$

where the $h_i(x,\xi)$ are strongly definable functions, and the non-negative integers v_i, the map $\mu:\{1,\dots,r\}\to\{1,\dots,n\}$, and the map $\eta:\{1,\dots,r\}\to\{1,\dots,s\}$ do not depend on (x,ξ,t).

The application of cell decomposition I to each of the polynomials $f_1,\dots,f_r$ gives us a finite partition of $K^m\times K$ in subsets A such that $P(A,r)$ holds. Theorem II will hold if we find a finite partition of $K^m\times K$ in subsets B such that $P(B,1)$ holds.

We will now show that if we have a set A and an integer s, $1<s\leqq r$, which satisfy $P(A,s)$, then we can partition A in a finite number of sets B such that $P(B,s-1)$ holds. By repeating this process we obtain theorem II.

So suppose A satisfies $P(A,s)$ with $s>1$. Consider two different centers $c_1(x,\xi)$ and $c_2(x,\xi)$ which appear in A. After a partition of A into

$$\bigcup_{\xi}\{(x,t)\in A(\xi)\mid c_1(x,\xi)=c_2(x,\xi)\}$$

and its complement in A, we may suppose that $c_1(x,\xi)\neq c_2(x,\xi)$ for $(x,t)\in A(\xi)$. Now A will be broken up in four disjoint parts $B_1,\dots,B_4$. For simplicity we will write c_1, c_2 instead of $c_1(x,\xi)$, $c_2(x,\xi)$.

Case (i): $\operatorname{ord}(t-c_1)>\operatorname{ord}(c_2-c_1)$. In this case we have

$$t-c_2=(t-c_1)-(c_2-c_1)=-(c_2-c_1)\left(1-\frac{t-c_1}{c_2-c_1}\right).$$

Since $\operatorname{ord}\dfrac{t-c_1}{c_2-c_1}>0$, we have that

$$\operatorname{ord}(t-c_2)=\operatorname{ord}(c_2-c_1)\quad\text{and}\quad\overline{\mathrm{ac}}(t-c_2)=\overline{\mathrm{ac}}(-(c_2-c_1)).$$

If we put

$$B_1 = \bigcup_{\xi} \{(x, t) \in A(\xi) \mid \operatorname{ord}(t - c_1(x, \xi)) > \operatorname{ord}(c_2(x, \xi) - c_1(x, \xi))\},$$

then we can eliminate the center c_2, so $P(B_1, s-1)$ holds.

Case (ii): $\operatorname{ord}(t - c_1) < \operatorname{ord}(c_2 - c_1)$. As in case (i) we can show that

$$B_2 = \bigcup_{\xi} \{(x, t) \in A(\xi) \mid \operatorname{ord}(t - c_1(x, \xi)) < \operatorname{ord}(c_2(x, \xi) - c_1(x, \xi))\}$$

satisfies $P(B_2, s-1)$.

Case (iii): $\operatorname{ord}(t - c_2) > \operatorname{ord}(c_2 - c_1)$. In this case we can eliminate the center c_1.

Case (iv): $\operatorname{ord}(t - c_1) = \operatorname{ord}(c_2 - c_1) = \operatorname{ord}(t - c_2)$. The remaining part of A is

$$B_4 = \bigcup_{\xi} \{(x, t) \in A(\xi) \mid \operatorname{ord}(t - c_1(x, \xi)) = \operatorname{ord}(c_2(x, \xi) - c_1(x, \xi)) = \operatorname{ord}(t - c_2(x, \xi))\}.$$

By putting an additional parameter γ over the residue field in it, we can write B_4 as

$$B_4 = \bigcup_{\xi} \bigcup_{\gamma} \{(x, t) \in A(\xi) \mid \gamma \neq 0, \gamma \neq 1, \operatorname{ord}(t - c_1) = \operatorname{ord}(c_2 - c_1), \overline{\mathrm{ac}}(t - c_1) = \gamma \cdot \overline{\mathrm{ac}}(c_2 - c_1)\}.$$

Indeed, setting $\overline{\mathrm{ac}}\left(\dfrac{t - c_1}{c_2 - c_1}\right) = \gamma$, we have that

$$\operatorname{Res}\left(\frac{t - c_2}{c_2 - c_1}\right) = \operatorname{Res}\left(\frac{t - c_1}{c_2 - c_1}\right) - 1 = \overline{\mathrm{ac}}\left(\frac{t - c_1}{c_2 - c_1}\right) - 1 = \gamma - 1.$$

We can now eliminate the center c_2 since

$$\operatorname{ord}(t - c_2) = \operatorname{ord}(c_2 - c_1) \quad \text{and} \quad \overline{\mathrm{ac}}(t - c_2) = (\gamma - 1) \cdot \overline{\mathrm{ac}}(c_2 - c_1),$$

and $P(B_4, s-1)$ follows.

This proves theorem II. ∎

4. Quantifier elimination

4. 1. Theorem. *Let K be a valued field which is a structure for the language $\mathscr{L}$, and which satisfies the conditions in* 2. 4. *Then K has elimination of K-quantifiers in the language $\mathscr{L}$.*

Proof. We have to show that in K, every $\mathscr{L}$-formula is equivalent to an $\mathscr{L}$-formula without quantifiers over the field sort. For this it suffices to eliminate the K-quantifier $(\exists t)$ from a formula of the form

$$(1) \qquad (\exists t)\, \psi(t, x, \zeta, k)$$

where ψ is a formula in $\mathscr{L}$ without K-quantifiers, t one K-variable, $x=(x_1, \dots, x_m)$ K-variables, ζ and k tuples of resp. $\bar{K}$- and Γ-variables. In ψ atomic formulas of the form $h(x, t)=0$ (where h is a polynomial in (x, t) with integer coefficients) can be replaced by $\overline{\mathrm{ac}}\, h(x, t)=0$. Hence we may suppose that t appears in ψ in the $\bar{K}$-terms $\overline{\mathrm{ac}}\, f_1(x, t), \dots, \overline{\mathrm{ac}}\, f_r(x, t)$ and the Γ-terms $\operatorname{ord} g_1(x, t), \dots, \operatorname{ord} g_s(x, t)$ ($f_1, \dots, f_r$, $g_1, \dots, g_s$ are polynomials in (x, t) with integer coefficients).

Let φ be the formula obtained by replacing in ψ, $\overline{\mathrm{ac}}\, f_i(x, t)$ by a $\bar{K}$-variable ϱ_i $(i=1, \dots, r)$ and $\operatorname{ord} g_j(x, t)$ by a Γ-variable l_j $(j=1, \dots, s)$. Then (1) is equivalent to

$$(\exists t)(\exists \varrho_1)\cdots(\exists \varrho_r)(\exists l_1)\cdots(\exists l_s)\Bigg[\varphi(x, \zeta, k, \varrho_1, \dots, \varrho_r, l_1, \dots, l_s) \wedge\left(\bigwedge_{i=1}^{r} \overline{\mathrm{ac}}\, f_i(x, t)=\varrho_i\right)\wedge\left(\bigwedge_{j=1}^{s} \operatorname{ord} g_j(x, t)=l_j\right)\Bigg].$$

Since φ doesn't contain the variable t, we can push through the quantifier $(\exists t)$ and it suffices to eliminate $(\exists t)$ from

$$(2)\qquad (\exists t)\left[\left(\bigwedge_{i=1}^{r} \overline{\mathrm{ac}}\, f_i(x, t)=\varrho_i\right)\wedge\left(\bigwedge_{j=1}^{s} \operatorname{ord} g_j(x, t)=l_j\right)\right].$$

The application of cell decomposition theorem II (theorem 3. 2) to the polynomials $f_1, \dots, f_r$, $g_1, \dots, g_s$ provides us with a partition of $K^m \times K$ in a finite number of cells $A=\bigcup_{\xi} A(\xi)$ with parameters $\xi=(\xi_1, \dots, \xi_n)$ and center $c(x, \xi)$ such that for $(x, t) \in A(\xi)$ we have

$$\begin{aligned} &\overline{\mathrm{ac}}\, f_i(x, t)=\xi_{\mu(i)} && (i=1, \dots, r),\\ &\operatorname{ord} g_j(x, t)=\operatorname{ord}(h_j(x, \xi)(t-c(x, \xi))^{v_j}) && (j=1, \dots, s)\end{aligned}$$

where μ is a map $\{1, \dots, r\} \to \{1, \dots, n\}$, $v_j \in \mathbb{N}$ and the h_j are strongly definable functions. Hence we may suppose that (2) is equivalent to

$$(3)\qquad (\exists t)(\exists \xi_1)\cdots(\exists \xi_n)\Bigg[(x, t) \in A(\xi) \wedge\left(\bigwedge_{i=1}^{r} \xi_{\mu(i)}=\varrho_i\right) \wedge\left(\bigwedge_{j=1}^{s} \operatorname{ord} h_j(x, \xi)+v_j \operatorname{ord}(t-c(x, \xi))=l_j\right)\Bigg].$$

By the definition of a cell, the condition $(x, t) \in A(\xi)$ is of the form

$$\theta(x, \xi, \operatorname{ord}(t-c(x, \xi))) \wedge \overline{\mathrm{ac}}(t-c(x, \xi))=\xi_1$$

where θ is a simple formula. So, introducing a new Γ-variable l for $\operatorname{ord}(t-c(x, \xi))$, (3) becomes

$$(\exists t)\,(\exists \xi_1)\cdots(\exists \xi_n)\,(\exists l)\left[\theta(x,\xi,l)\wedge\left(\bigwedge_{i=1}^{r}\xi_{\mu(i)}=\varrho_i\right)\right.$$
$$\left.\wedge\left(\bigwedge_{j=1}^{s}\operatorname{ord}h_j(x,\xi)+\nu_j\cdot l=l_j\right)\wedge\operatorname{ord}(t-c(x,\xi))=l\wedge\overline{\operatorname{ac}}(t-c(x,\xi))=\xi_1\right].$$

Pushing through $(\exists t)$ again, we only have to consider

$$(\exists t)\,[\operatorname{ord}(t-c(x,\xi))=l\wedge\overline{\operatorname{ac}}(t-c(x,\xi))=\xi_1]$$

from which the quantifier can easily be eliminated. ∎

4. 2. Remark. By theorem 4. 1 we have that every set which is definable in $\mathscr{L}$ is a simple set, and a function is strongly definable if and only if its graph is definable in $\mathscr{L}$.

4. 3. Corollary. *Let φ be a formula in $\mathscr{L}_{\mathrm{PR}}$, then there exists an $\mathscr{L}_{\mathrm{PR}}$-formula ψ without K-quantifiers such that, for almost all primes p, $\varphi\leftrightarrow\psi$ on $\mathbb{Q}_p$.*

Proof. The corollary follows from theorem 4. 1 using ultraproducts in the same way as in remark 3. 3. ∎

5. Application to zeta functions

5. 1. Theorem. *Let $h(x)\in\mathbb{Z}[x]$ with $x=(x_1,\dots,x_m)$. Let ψ be a $\mathscr{L}_{\mathrm{PR}}$-formula with free K-variables $x_1,\dots,x_m$. Suppose $W_p=\{x\in\mathbb{Q}_p^m\mid\psi(x)\text{ holds}\}$ is bounded for all p. Then we have that for all p*

$$Z_\psi(s,p)=\int_{W_p}|h(x)|_p^s\,|dx|_p$$

is a rational function in p^{-s}, and the degrees of numerator and denominator of this rational function are bounded independently of p.

The following lemma shows how we can compute integrals on simple sets, using cell decomposition.

5. 2. Lemma. *Let y,ϱ,k be tuples of respectively $K,\bar{K}$ and Γ-variables. Let $x=(x_1,\dots,x_m)$ be K-variables, t one K-variable and $\psi(x,t,y,\varrho,k)$ a simple $\mathscr{L}_{\mathrm{PR}}$-formula. Suppose that for almost all p, and for all values of y,ϱ,k in resp. $\mathbb{Q}_p,\bar{\mathbb{Q}}_p,\mathbb{Z}\cup\{\infty\}$*

$$W_p(y,\varrho,k)=\{(x,t)\in\mathbb{Q}_p^m\times\mathbb{Q}_p\mid\psi(x,t,y,\varrho,k)\text{ holds}\}$$

is bounded.

Consider the (parametrized) integral

$$I_p(y,\varrho,k)=\int_{W_p(y,\varrho,k)}|dx|_p\,|dt|_p.$$

Then there exist simple $\mathscr{L}_{\mathrm{PR}}$-formulas φ_i $(i=1,\dots,s)$ such that, for almost all p,

$$I_p(y,\varrho,k)=\frac{1}{p}\sum_{i=1}^{s}\sum_{\xi^{(i)}\in\bar{\mathbb{Q}}_p^{n_i}}\sum_{l\in\mathbb{Z}}p^{-l}\int_{E_p^{(i)}(\xi^{(i)},l,y,\varrho,k)}|dx|_p$$

where $E_p^{(i)}(\xi^{(i)},l,y,\varrho,k)=\{x\in\mathbb{Q}_p^m \mid \varphi_i(x,\xi^{(i)},l,y,\varrho,k)$ *holds*$\}$.

Proof. In the formula ψ, t only appears in terms of the form $\operatorname{ord} f(y,x,t)$ or $\overline{\mathrm{ac}}\, f(y,x,t)$, where f is a polynomial in (y,x,t) with coefficients in $\mathbb{Z}$. By remark 3.3 we can apply to these polynomials cell decomposition on $\mathbb{Q}_p$, uniformly for almost all p. This gives us a finite partition of the (y,x,t)-space in cells $A_p^{(1)},\dots,A_p^{(s)}$. In the sequel of the proof all statements will hold uniformly for almost all p.

To each cell $A_p^{(i)}$ and value of the parameter y, corresponds a subset of $\mathbb{Q}_p^m\times\mathbb{Q}_p$

$$B_p^{(i)}(y)=\{(x,t)\in\mathbb{Q}_p^m\times\mathbb{Q}_p \mid (y,x,t)\in A_p^{(i)}\}.$$

One has then

$$\int_{W_p(y,\varrho,k)}|dx|_p\,|dt|_p=\sum_{i=1}^{s}\int_{W_p(y,\varrho,k)\cap B_p^{(i)}(y)}|dx|_p\,|dt|_p.$$

Suppose now A_p is one of the cells $A_p^{(1)},\dots,A_p^{(s)}$ and $B_p(y)$ is the corresponding subset of $\mathbb{Q}_p^m\times\mathbb{Q}_p$. We will compute

$$\int_{W_p(y,\varrho,k)\cap B_p(y)}|dx|_p\,|dt|_p. \tag{1}$$

Let $\xi=(\xi_1,\dots,\xi_n)$ be the parameters of A_p. Cell decomposition gave us also that on A_p the polynomials f defined above, satisfy

$$\overline{\mathrm{ac}}\, f(y,x,t)=\xi_\mu,$$
$$\operatorname{ord} f(y,x,t)=\operatorname{ord} h(y,x,\xi)+\nu\cdot\operatorname{ord}(t-c(y,x,\xi)).$$

Hence on A_p, $\psi(x,t,y,\varrho,k)$ is equivalent to a simple formula

$$\theta(x,\xi,\operatorname{ord}(t-c(y,x,\xi)),y,\varrho,k).$$

This, combined with the fact that A_p is a cell, implies that $W_p(y,\varrho,k)\cap B_p(y)$ is of the form

$$\bigcup_{\xi\in\bar{\mathbb{Q}}_p^n}\{(x,t)\in\mathbb{Q}_p^m\times\mathbb{Q}_p \mid (y,x,\xi)\in C,\ \operatorname{ord} a_1(y,x,\xi)\diamond_1\lambda\cdot\operatorname{ord}(t-c(y,x,\xi))\diamond_2\operatorname{ord} a_2(y,x,\xi),$$
$$\overline{\mathrm{ac}}(t-c(y,x,\xi))=\xi_1,\ \theta(x,\xi,\operatorname{ord}(t-c(y,x,\xi)),y,\varrho,k)\text{ holds}\}.$$

We can write this set as

$$\bigcup_{\xi\in\bar{\mathbb{Q}}_p^n}\bigcup_{l\in\mathbb{Z}}\{(x,t)\in\mathbb{Q}_p^m\times\mathbb{Q}_p \mid x\in E_p(y,\xi,\varrho,k,l),\ \operatorname{ord}(t-c(y,x,\xi))=l,\ \overline{\mathrm{ac}}(t-c(y,x,\xi))=\xi_1\}$$

where $E_p(y,\xi,\varrho,k,l)$ is the simple set

$$\{x\in\mathbb{Q}_p^m \mid (y,x,\xi)\in C,\ \operatorname{ord}a_1(y,x,\xi)\diamond_1\lambda\cdot l\diamond_2\operatorname{ord}a_2(y,x,\xi),\ \theta(x,\xi,l,y,\varrho,k)\text{ holds}\}.$$

So (1) equals

$$\sum_{\xi\in\bar{\mathbb{Q}}_p^n}\sum_{l\in\mathbb{Z}}\int_{E_p(\xi,l,y,\varrho,k)}\Big(\int_{\substack{\operatorname{ord}(t-c(y,x,\xi))=l\\ \overline{\mathrm{ac}}(t-c(y,x,\xi))=\xi_1}}|dt|_p\Big)\,|dx|_p.$$

Using the substitution $t-c(y,x,\xi)=p^l\cdot u$, we find for the inner integral

$$\int_{\substack{\operatorname{ord}(t-c(y,x,\xi))=l\\ \overline{\mathrm{ac}}(t-c(y,x,\xi))=\xi_1}}|dt|_p=p^{-l}\int_{\substack{\operatorname{ord}u=0\\ \overline{\mathrm{ac}}u=\xi_1}}|du|_p=p^{-l-1}\quad\text{if}\quad\xi_1\neq 0.$$

This proves the lemma. ∎

5. 3. Lemma. *Let φ be a formula in $\mathscr{L}$ which contains only $\bar{K}$- and Γ-variables. Then φ is equivalent to an $\mathscr{L}$-formula of the form*

$$\bigvee_{j=1}^{s}(\chi_j\wedge\theta_j)$$

where χ_j $(j=1,\dots,s)$ is an $\mathbb{L}_{\bar{K}}$-formula and θ_j $(j=1,\dots,s)$ is an $\mathbb{L}_\Gamma$-formula.

Proof. Since φ doesn't contain K-variables, we may suppose that φ is a formula in $\mathbb{L}_{\bar{K}}\cup\mathbb{L}_\Gamma$. Indeed, any K-sort term in φ must be of the form $z\cdot 1$ where $z\in\mathbb{Z}$. The characteristic of the residue field is zero, so if $z\neq 0$ we can replace $\operatorname{ord}(z\cdot 1)$ (resp. $\overline{\mathrm{ac}}(z\cdot 1)$) by 0 (resp. $z\cdot 1$). For $z=0$ we have $\operatorname{ord}0=\infty$ and $\overline{\mathrm{ac}}\,0=0$.

We proceed by induction on the number r of quantifiers in the formula φ.

If $r=0$, we can bring φ in its disjunctive normal form $\bigvee_{j=1}^{s}\psi_j$, where ψ_j is a conjunction of atoms and negations of atoms. Because there are no symbols in $\mathscr{L}$ relating the $\bar{K}$- and Γ-sort, the atoms in ψ_j will be either atoms in $\mathbb{L}_{\bar{K}}$, or atoms in $\mathbb{L}_\Gamma$. So we can write ψ_j as $\chi_j\wedge\theta_j$ where χ_j is a conjunction of atoms and negations of atoms in $\mathbb{L}_{\bar{K}}$, while θ_j is a conjunction of atoms and negations of atoms in $\mathbb{L}_\Gamma$. This proves the induction base.

Suppose now $r>0$. We can bring φ in its prenex normal form $(Q_1)\cdots(Q_r)\psi$ where ψ is an $\mathscr{L}$-formula without quantifiers and (Q_i) is a quantifier over the $\bar{K}$-sort or the Γ-sort. The induction hypothesis implies that φ is equivalent to a formula

$$(1)\qquad (Q_1)\big(\bigvee_j(\chi_j'\wedge\theta_j')\big)$$

where χ_j' is a formula in $\mathbb{L}_{\bar{K}}$, θ_j' a formula in $\mathbb{L}_\Gamma$. Suppose that (Q_1) is a quantifier $(\exists\xi)$, with ξ a $\bar{K}$-variable. We have that φ is equivalent to

$$\bigvee_j \left(((\exists\xi)\chi_j') \wedge \theta_j'\right).$$

Indeed, in (1) the existential quantifier goes through the disjunction and since ξ doesn't appear in θ_j', we can move the quantifier to the first term of the conjunction.

If (Q_1) is of the form $(\forall\xi)$, we can rewrite $\bigvee_j (\chi_j' \wedge \theta_j')$ in conjunctive normal form. The remaining cases: (Q_1) is $(\exists k)$ or $(\forall k)$ with k a Γ-variable, are treated in the same way. This concludes the induction step and proves the lemma. ∎

5. 4. Lemma. *Let $\varphi(l_1, \dots, l_m)$ be a quantifier free formula in $\mathbb{L}_{\mathrm{PR}\infty}$. Then there exists a quantifier free formula $\tilde{\varphi}(l_1, \dots, l_m)$ in $\mathbb{L}_{\mathrm{PR}}$ such that*

$$\varphi(l_1, \dots, l_m) \leftrightarrow \tilde{\varphi}(l_1, \dots, l_m) \quad \textit{for all} \quad (l_1, \dots, l_m) \in \mathbb{Z}^m.$$

Proof. One can check that, for $l_1 \neq \infty, \dots, l_m \neq \infty$, the atomic formulas in φ which contain the symbol ∞, are either identically true or identically false. ∎

5. 5. Lemma. *$\mathbb{Z} \cup \{\infty\}$ has elimination of quantifiers in the language $\mathbb{L}_{\mathrm{PR}\infty}$.*

Proof. It suffices to eliminate the quantifier from a formula of the form

$$(\exists k)\, \varphi(k, l_1, \dots, l_m)$$

where φ is a quantifier free formula in $\mathbb{L}_{\mathrm{PR}\infty}$.

We make a disjunction over the cases $l_i = \infty$ or $l_i \neq \infty$ $(i = 1, \dots, m)$. We also split up into the cases $k = \infty$ or $k \neq \infty$. If $l_i = \infty$ (resp. $k = \infty$), we can eliminate l_i (resp. k) from φ by replacing the occurrences of l_i (resp. k) in φ by ∞. So we only have to consider formulas of the form

$$(\exists k)\, (\varphi(k, l_1, \dots, l_m) \wedge k \neq \infty \wedge l_1 \neq \infty \wedge \dots \wedge l_m \neq \infty).$$

By lemma 5. 4 we may suppose that φ is a formula in $\mathbb{L}_{\mathrm{PR}}$. The lemma follows now from the well known fact that $\mathbb{Z}$ has quantifier elimination in the Presburger language $\mathbb{L}_{\mathrm{PR}}$ (see e.g. [11], p. 188, Thm. 32E). ∎

5. 6. Lemma. *Let $\psi(l_1, \dots, l_m, n)$ be a formula in $\mathcal{L}_{\mathrm{PR}\infty}$, which contains only Γ-variables. Set $E = \{(l_1, \dots, l_m, n) \in \mathbb{Z}^{m+1} \mid \psi(l_1, \dots, l_m, n) \text{ holds}\}$ and suppose that, for almost all p,*

$$J(s) = \sum_{(l_1, \dots, l_m, n) \in E} p^{-ns - l_1 - \dots - l_m}$$

is convergent for $s \in S$, with S an open subset of $\mathbb{R}$. Then there exist polynomials $Q, R \in \mathbb{Z}[X, Y]$ such that, for almost all p, and for $s \in S$

$$J(s) = \frac{Q(p, p^{-s})}{R(p, p^{-s})}.$$

Proof. By lemma 5. 4 and lemma 5. 5 we can take ψ to be a quantifier free formula in $\mathbb{L}_{\mathrm{PR}}$. Put ψ in its disjunctive normal form $\bigvee_i \theta_i$. We may suppose that the disjunction is over disjoint cases, so $J(s)$ will be a finite sum of summations of the form

$$\tilde{J}(s) = \sum_{\theta(l_1, \ldots, l_m, n)} p^{-ns - l_1 - \cdots - l_m}$$

where θ is a conjunction of atomic formulas or their negation. Let d be the product of all integers j such that the symbol $\equiv_j$ appears in the formula θ. We sum over the residue classes of $l_1, \ldots, l_m, n$ modulo d. Let $D \subset \mathbb{N}^{m+1}$ be a complete set of residue classes of $(l_1, \ldots, l_m, n) \bmod d$. Put for $(c_1, \ldots, c_m, c) \in D$

$$E_{(c_1, \ldots, c_m, c)} = \{(l_1, \ldots, l_m, n) \in \mathbb{Z}^{m+1} \mid l_j \equiv c_j \bmod d \, (j = 1, \ldots, m), \\ n \equiv c \bmod d, \ \theta(l_1, \ldots, l_m, n) \text{ holds}\}.$$

On such a set $E_{(c_1, \ldots, c_m, c)}$ the congruences in θ are identically true or identically false. So either $E_{(c_1, \ldots, c_m, c)}$ is empty, or we may suppose that θ is a system of linear inequalities. Set $D' = \{(c_1, \ldots, c_m, c) \in D \mid E_{(c_1, \ldots, c_m, c)} \neq \emptyset\}$. Then

$$\tilde{J}(s) = \sum_{(c_1, \ldots, c_m, c) \in D'} \ \sum_{(l_1, \ldots, l_m, n) \in E_{(c_1, \ldots, c_m, c)}} p^{-ns - l_1 - \cdots - l_m}.$$

Put now $l_j = c_j + d \cdot l'_j$, $n = c + d \cdot n'$ and

$$L = \{(l'_1, \ldots, l'_m, n') \in \mathbb{Z}^{m+1} \mid \theta(c_1 + d l'_1, \ldots, c_m + d l'_m, c + d n') \text{ holds}\}.$$

We can write

$$\tilde{J}(s) = \sum_{(c_1, \ldots, c_m, c) \in D'} p^{-cs - c_1 - \cdots - c_m} \sum_{(l'_1, \ldots, l'_m, n') \in L} p^{-dn's - dl'_1 - \cdots - dl'_m}.$$

Since θ is a system of linear inequalities, L is defined by a system of linear inequalities in $l'_1, \ldots, l'_m, n'$. By [7], lemma 7. 5, there exist polynomials $Q, R \in \mathbb{Z}[X, Y]$ such that

$$\sum_{(l'_1, \ldots, l'_m, n') \in L} p^{-dn's - dl'_1 - \cdots - dl'_m} = \frac{Q(p, p^{-s})}{R(p, p^{-s})}.$$

This proves the lemma. ∎

5. 7. *Proof of theorem* 5. 1. Using Denef's lemma 2. 1 in [9], we see that for each p, the set W_p is definable in $\mathbb{Q}_p^m$ in the language of Macintyre [13]. By [7], theorem 7. 4, it is sufficient to prove the theorem for almost all p. So all the expressions in the sequel of the proof will hold for almost all p.

By theorem 4. 1 we may suppose that $\psi(x)$ is a simple formula. One has

$$Z_\psi(s, p) = \int_{W_p} |h(x)|_p^s \, |dx|_p$$
$$= \sum_{n \in \mathbb{Z}} p^{-ns} \int_{\substack{\psi(x) \\ \operatorname{ord} h(x) = n}} |dx|_p .$$

By repeated application of lemma 5. 2 we find that $\int_{\substack{\psi(x) \\ \operatorname{ord} h(x) = n}} |dx|_p$ is a finite sum of expressions of the form

$$\frac{1}{p^m} \sum_{\xi \in \bar{\mathcal{O}}_p^r} \sum_{\substack{l_1, \ldots, l_m \in \mathbb{Z} \\ \varphi(\xi, l_1, \ldots, l_m, n)}} p^{-l_1 - \cdots - l_m}.$$

The application of lemma 5. 3 with $\mathscr{L} = \mathscr{L}_{\mathrm{PR}}$, shows that φ is equivalent to

$$\bigvee_{k=1}^{d} (\chi_k(\xi) \wedge \theta_k(l_1, \ldots, l_m, n))$$

where χ_k is an $\mathbb{L}_{\bar{K}}$-formula and θ_k is an $\mathbb{L}_{\mathrm{PR}\infty}$-formula. We may suppose that the disjunction is over disjoint cases, so $Z_\psi(s, p)$ is a finite sum of expressions of the form

$$\frac{1}{p^m} \sum_{k=1}^{d} \left[\left(\sum_{\substack{\xi \in \bar{\mathcal{O}}_p^r \\ \chi_k(\xi)}} 1 \right) \cdot \left(\sum_{\substack{l_1, \ldots, l_m, n \in \mathbb{Z} \\ \theta_k(l_1, \ldots, l_m, n)}} p^{-ns - l_1 - \cdots - l_m} \right) \right].$$

The theorem follows now from lemma 5. 6. ∎

6. Some refinements

In this section we will give a more precise result for the degree of the zeta function as a rational function of p^{-s}. For this goal, another specialization of the language $\mathscr{L}$, namely $\mathscr{L}_{\mathrm{tame}}$, will be introduced.

We also need a version of the cell decomposition theorems with Γ-strongly definable functions. This requires some additional technical conditions on the field K, which are stated in 6. 1 below.

6. 1.

Condition 1. *Let y, ϱ, k be tuples of resp. K-, $\bar{K}$- and Γ-variables. Let l_1, l_2 be Γ-variables. Suppose $\psi(l_1, y, \varrho, k)$ is a Γ-simple formula, then there exists a Γ-simple formula φ such that*

$$\psi(l_1, y, \varrho, k) \wedge l_2 \neq \infty \leftrightarrow \varphi(l_1 + l_2, l_2, y, \varrho, k) \text{ in } K.$$

This means essentially that if you substitute $\tilde{l}_1 - \tilde{l}_2$ in a Γ-simple formula, you get again a Γ-simple formula. Since no quantifiers over the value group are allowed in Γ-simple formulas, we need this condition to prove that a nowhere vanishing Γ-strongly definable function is a unit in the ring of Γ-strongly definable functions. This fact will be used implicitly in the proofs of theorems 6. 2 and 6. 3.

Condition 2. *Let λ be a positive integer, l a Γ-variable, y, ϱ, k tuples of resp. K-, $\bar{K}$- and Γ-variables. Suppose $\psi(l, y, \varrho, k)$ is a Γ-simple formula. Then there exists a Γ-simple formula φ such that*

$$\psi(l, y, \varrho, k) \leftrightarrow \varphi(\lambda \cdot l, y, \varrho, k) \text{ in } K.$$

Or formulated in another way: if $\tilde{l}$ is a λ-multiple in Γ, then substituting $\frac{1}{\lambda}\tilde{l}$ in a Γ-simple formula gives a Γ-simple formula in $\tilde{l}$. This condition will be used in the proof of lemma 6. 5.

For our results on the degree of the zeta function, we will need another special case of the language $\mathscr{L}$. The language $\mathscr{L}_{\text{tame}}$ is defined as

$$\mathscr{L}_{\text{tame}} = (\mathbb{L}_K, \mathbb{L}_{\bar{K}}, \mathbb{L}_{\text{tame}\,\infty}, \text{ord}, \overline{\text{ac}})$$

where $\mathbb{L}_{\text{tame}\,\infty} = \mathbb{L}_{\text{tame}} \cup \{\infty\}$ and $\mathbb{L}_{\text{tame}} = \{+, 0, \leqq\} \cup \bigcup_{n \in \mathbb{N} \setminus \{0,1\}} \{\equiv_n c \mid c \in \{1, \ldots, n\}\}$.

The symbol $\equiv_n c$ stands for "congruent with c modulo n". Notice that, for arbitrary p, $\mathbb{Q}_p$ is a structure for $\mathscr{L}_{\text{tame}}$, and that $\mathbb{Q}_p$ satisfies, uniformly for all p, the two conditions above in this language.

6. 2. Cell decomposition theorem I′. *Let K be a structure for the language $\mathscr{L}$ which satisfies the conditions 2. 4 and 6. 1. Let t be one K-variable and $x = (x_1, \ldots, x_m)$ be m K-variables. Let $f(x, t)$ be a polynomial in t with coefficients in the ring of Γ-strongly definable functions in x.*

Then $K^m \times K$ admits a finite partition in Γ-cells A, which satisfies:

Each Γ-cell $A = \bigcup_\xi A(\xi)$ of the partition has parameters $\xi = (\xi_1, \ldots, \xi_n)$ and a center $c(x, \xi)$, such that, if we write

$$f(x, t) = \sum_{i=0}^{d} a_i(x, \xi)\,(t - c(x, \xi))^i,$$

then for all $\xi \in \bar{K}^n$, and for all $(x, t) \in A(\xi)$ we have

$$\text{ord}\, f(x, t) = \text{ord}\, a_{i_0}(x, \xi)\,(t - c(x, \xi))^{i_0} = \min_{0 \leqq i \leqq d} \text{ord}\, a_i(x, \xi)\,(t - c(x, \xi))^i,$$

$$\overline{\text{ac}}\, f(x, t) = \xi_{j_0}$$

where $i_0 \in \{0, \ldots, d\}$ and $j_0 \in \{1, \ldots, n\}$ do not depend on (x, ξ, t).

6. 3. Cell decomposition theorem II′. *Let $\mathscr{L}$ and K be as in theorem* I′. *Let $f_1(x, t), \ldots, f_r(x, t)$ be polynomials in t as in theorem* I′.

Then $K^m \times K$ admits a finite partition in Γ-cells A, which satisfies:

Each Γ-cell $A = \bigcup_{\xi} A(\xi)$ of the partition has parameters $\xi = (\xi_1, \ldots, \xi_n)$ and a center $c(x, \xi)$ such that, for all $\xi \in \bar{K}^n$, for all $(x, t) \in A(\xi)$, and for all $i \in \{1, \ldots, r\}$ we have

$$\operatorname{ord} f_i(x, t) = \operatorname{ord} h_i(x, \xi)\,(t - c(x, \xi))^{v_i},$$
$$\overline{\mathrm{ac}}\, f_i(x, t) = \xi_{\mu(i)}$$

where the $h_i(x, \xi)$ $(i = 1, \ldots, r)$ are Γ-strongly definable functions, and the $v_i \in \mathbb{N}$ $(i = 1, \ldots, r)$ and the map $\mu : \{1, \ldots, r\} \to \{1, \ldots, n\}$ do not depend on (x, ξ, t).

6. 4. *Proof of theorems* 6. 2 *and* 6. 3. We prove cell decomposition theorems I′ and II′ by induction on the degree of the polynomials involved. Cell decomposition theorem I′ (resp. II′) for polynomials $f(x, t)$ (resp. $f_i(x, t)$) of degree $\leqq d$ will be denoted by I'_d (resp. II'_d). The proof proceeds according to the following scheme:

(i) $\mathrm{II}'_{d-1} \Rightarrow$ lemma 6. 5,

(ii) (I'_{d-1} and lemma 6. 5) $\Rightarrow \mathrm{I}'_d$,

(iii) $\mathrm{I}'_d \Rightarrow \mathrm{II}'_d$.

Lemma 6. 5 is an analogon of lemma 3. 7 obtained by replacing the notions "cell" and "strongly definable" by "Γ-cell" resp. "Γ-strongly definable".

The proof of part (ii) and (iii) is completely the same as in 3. 6 resp. 3. 8 and will not be repeated here. The proof of the lemma is written down in 6. 5. Notice that to prove lemma 6. 5, we need II'_{d-1} (i.e. several polynomials), though for lemma 3. 7 only the induction hypothesis of theorem I (one polynomial) was used. The reason for this is that lemma 2. 7 doesn't seem to hold anymore for Γ-strongly definable functions. ∎

6. 5. Lemma (Assuming II'_{d-1}). *Let $f(x, t) = \sum_{i=0}^{d} a_i(x, \xi)\,(t - c(x, \xi))^i$ be as in theorem* I′. *Let A be a Γ-cell with fiber*

$$A(\xi) = \{(x, t) \in K^m \times K \mid (x, \xi) \in D,\ \lambda \cdot \operatorname{ord}(t - c(x, \xi)) = \operatorname{ord} \theta(x, \xi),\ \overline{\mathrm{ac}}(t - c(x, \xi)) = \xi_1\}$$

on which,

(1) $\min_i \operatorname{ord} a_i(x, \xi)\,(t - c(x, \xi))^i = \operatorname{ord} a_{i_0}(x, \xi)\,(t - c(x, \xi))^{i_0}$ *with* $i_0 \geqq 1$,

(2) $$\operatorname{ord} f'(x, t) = \operatorname{ord} a_{i_0}(x, \xi)\,(t - c(x, \xi))^{i_0 - 1}.$$

Suppose $d(x, \xi)$ is a function from D to K such that

$$f(x, d(x, \xi)) = 0, \tag{3}$$

$$\lambda \cdot \operatorname{ord}(d(x, \xi) - c(x, \xi)) = \operatorname{ord}\theta(x, \xi), \tag{4}$$

$$\overline{\mathrm{ac}}(d(x, \xi) - c(x, \xi)) = \xi_1. \tag{5}$$

Then the function d is Γ-strongly definable.

Proof. Let $\theta(t, y, \varrho, k)$ be a Γ-simple formula, where t is one K-variable, y, ϱ, k tuples of resp. K-, $\bar{K}$- and Γ-variables. We have to show that $\theta(d(x, \xi), y, \varrho, k)$ is equivalent to a Γ-simple formula in (x, ξ, y, ϱ, k). Suppose t appears in θ in $\bar{K}$-terms $\overline{\mathrm{ac}}\,\tilde{f}_1(y, t), \ldots, \overline{\mathrm{ac}}\,\tilde{f}_r(y, t)$ and Γ-terms $\operatorname{ord}\tilde{g}_1(y, t), \ldots, \operatorname{ord}\tilde{g}_{r'}(y, t)$, where $\tilde{f}_i$, $\tilde{g}_j$ are polynomials in (y, t) with integer coefficients. By (3) and after euclidean division of $\tilde{f}_1, \ldots, \tilde{f}_r$, $\tilde{g}_1, \ldots, \tilde{g}_{r'}$ by $f(x, t)$, we may replace these polynomials in θ by $f_1(y, x, t), \ldots, f_r(y, x, t)$, $g_1(y, x, t), \ldots, g_{r'}(y, x, t)$, polynomials in t of degree $< d$, with coefficients in the ring of Γ-strongly definable functions in (y, x). To these last $r + r'$ polynomials we can apply II'_{d-1}. This provides us with a partition of the (y, x, t)-space in a finite number of Γ-cells $B = \bigcup_{\zeta} B(\zeta)$ with parameters $\zeta = (\zeta_1, \ldots, \zeta_s)$ and center $e(y, x, \zeta)$ such that on $B(\zeta)$

$$\begin{aligned} &\overline{\mathrm{ac}}\, f_i(y, x, t) = \zeta_{\mu(i)} && (i = 1, \ldots, r), \\ &\operatorname{ord} g_j(y, x, t) = \operatorname{ord} h_j(y, x, \zeta)\,(t - e(y, x, \zeta))^{\nu_j} && (j = 1, \ldots, r') \end{aligned}$$

where $\mu : \{1, \ldots, r\} \to \{1, \ldots, s\}$, $\nu_j \in \mathbb{N}$, and the h_j are Γ-strongly definable functions. So $\theta(d(x, \xi), y, \varrho, k)$ is equivalent to a finite disjunction of expressions of the form

$$(\exists \zeta_1) \cdots (\exists \zeta_s)\, [\theta(d(x, \xi), y, \varrho, k) \wedge (y, x, d(x, \xi)) \in B(\zeta)]. \tag{6}$$

Since the h_j are Γ-strongly definable functions, substituting in θ, $\zeta_{\mu(i)}$ for $\overline{\mathrm{ac}}\, f_i(y, x, t)$ and $\operatorname{ord} h_j(y, x, \zeta) + \nu_j \operatorname{ord}(t - e(y, x, \zeta))$ for $\operatorname{ord} g_j(y, x, t)$, gives a Γ-simple formula, say $\varphi(y, x, \operatorname{ord}(t - e(y, x, \zeta)), \zeta, \varrho, k)$. Combining this with the definition of a Γ-cell, we find (6) to be equivalent to

$$\begin{aligned} (\exists \zeta_1) \cdots (\exists \zeta_s)\, [&\varphi(y, x, \operatorname{ord}(d(x, \xi) - e(y, x, \zeta)), \zeta, \varrho, k) \wedge (y, x, \zeta) \in C \\ &\wedge \operatorname{ord} b_1(y, x, \zeta) \diamond_1 \tilde{\lambda} \operatorname{ord}(d(x, \xi) - e(y, x, \zeta)) \diamond_2 \operatorname{ord} b_2(y, x, \zeta) \\ &\wedge \overline{\mathrm{ac}}(d(x, \xi) - e(y, x, \zeta)) = \zeta_1]. \end{aligned}$$

As in the proof of lemma 3. 7 we make a disjunction over four disjoint cases. For simplicity we will write d, c, e, θ instead of $d(x, \xi)$, $c(x, \xi)$, $e(y, x, \zeta)$, $\theta(x, \xi)$.

(i) $\lambda \cdot \operatorname{ord}(e - c) < \operatorname{ord}\theta$: In this case, we can substitute in (7) $\operatorname{ord}(e - c)$ for $\operatorname{ord}(d - e)$ and $\overline{\mathrm{ac}}(-(e - c))$ for $\overline{\mathrm{ac}}(d - e)$, thus obtaining a Γ-simple formula without $d(x, \xi)$.

(ii) $\lambda \cdot \operatorname{ord}(e-c) > \operatorname{ord}\theta$: Then $\operatorname{ord}(d-e) = \operatorname{ord}(d-c) = \frac{1}{\lambda}\operatorname{ord}\theta$ and

$$\overline{\mathrm{ac}}(d-e) = \overline{\mathrm{ac}}(d-c) = \xi_1.$$

So we can replace in (7) $\overline{\mathrm{ac}}(d-e)$ by ξ_1 and $\operatorname{ord}(d-e)$ by $\frac{1}{\lambda}\operatorname{ord}\theta$. By condition 2 in 6. 1 this gives a Γ-simple formula.

(iii) $\lambda \cdot \operatorname{ord}(e-c) = \operatorname{ord}\theta$: (α) $\overline{\mathrm{ac}}(e-c) \neq \xi_1$: Here we have $\operatorname{ord}(d-e) = \frac{1}{\lambda}\operatorname{ord}\theta$ and $\overline{\mathrm{ac}}(d-e) = \xi_1 - \overline{\mathrm{ac}}(e-c)$. So again we need condition 2 in 6. 1 to get a Γ-simple formula after replacing $\operatorname{ord}(d-e)$ and $\overline{\mathrm{ac}}(d-e)$.

(β) $\overline{\mathrm{ac}}(e-c) = \xi_1$: In this case, using Taylor expansion one finds

$$\operatorname{ord}(d-e) = \operatorname{ord}\left(\frac{f(x,e)}{f'(x,e)}\right) \quad \text{and} \quad \overline{\mathrm{ac}}(d-e) = \overline{\mathrm{ac}}\left(-\frac{f(x,e)}{f'(x,e)}\right). \qquad \blacksquare$$

6. 6. Theorem. *Let $h(x) \in \mathbb{Z}[x]$ with $x = (x_1, \ldots, x_m)$. Let ψ be a Γ-simple $\mathscr{L}_{\mathrm{tame}}$-formula with free K-variables $x_1, \ldots, x_m$. Suppose $W_p = \{x \in \mathbb{Q}_p^m \mid \psi(x) \text{ holds}\}$ is bounded for all p. Then we have that for all p*

$$Z_\psi(s, p) = \int_{W_p} |h(x)|_p^s \, |dx|_p$$

is a rational function in p^{-s}, and for almost all p

$$\deg Z_\psi(s, p) \leqq 0.$$

6. 7. Lemma. *Let y, ϱ, k be tuples of respectively $K, \bar{K}$ and Γ-variables. Let $x = (x_1, \ldots, x_m)$ be K-variables, t one K-variable and $\psi(x, t, y, \varrho, k)$ a Γ-simple $\mathscr{L}_{\mathrm{tame}}$-formula. Suppose that for almost all p, and for all values of y, ϱ, k in resp. $\mathbb{Q}_p$, $\bar{\mathbb{Q}}_p$, $\mathbb{Z} \cup \{\infty\}$*

$$W_p(y, \varrho, k) = \{(x, t) \in \mathbb{Q}_p^m \times \mathbb{Q}_p \mid \psi(x, t, y, \varrho, k) \text{ holds}\}$$

is bounded.

Consider the (parametrized) integral

$$I_p(y, \varrho, k) = \int_{W_p(y, \varrho, k)} |dx|_p \, |dt|_p.$$

Then there exist Γ-simple $\mathscr{L}_{\mathrm{tame}}$-formulas φ_i $(i = 1, \ldots, s)$ such that, for almost all p,

$$I_p(y, \varrho, k) = \frac{1}{p} \sum_{i=1}^{s} \sum_{\xi^{(i)} \in \bar{\mathbb{Q}}_p^{n_i}} \sum_{l \in \mathbb{Z}} p^{-l} \int_{E_p^{(i)}(\xi^{(i)}, l, y, \varrho, k)} |dx|_p$$

where $E_p^{(i)}(\xi^{(i)}, l, y, \varrho, k) = \{x \in \mathbb{Q}_p^m \mid \varphi_i(x, \xi^{(i)}, l, y, \varrho, k) \text{ holds}\}$.

Proof. The proof is completely analogous to the proof of lemma 5. 2, using cell decomposition II′ (theorem 6. 3) with $\mathscr{L} = \mathscr{L}_{\text{tame}}$, instead of cell decomposition II (theorem 3. 2). Notice that the ultraproduct $\prod \mathbb{Q}_p/\mathscr{D}$ satisfies the conditions in 6. 1 in the language $\mathscr{L}_{\text{tame}}$. ∎

6. 8. *Proof of theorem* 6. 6. The rationality of $Z_\psi(s, p)$ follows from theorem 5. 1 and the fact that ψ can be considered as a formula in $\mathscr{L}_{\text{PR}}$. So we are left to prove that $\deg Z_\psi(s, p) \leqq 0$ for almost all p. All the expressions in the sequel of the proof will hold for almost all p.

The polynomial h is bounded on the set W_p, so there exists an $M_p \in \mathbb{N}$ such that

$$|h(x)|_p \leqq p^{M_p} \quad \text{for} \quad x \in W_p.$$

One has

$$Z_\psi(s, p) = \sum_{n=-M_p}^{\infty} p^{-ns} \int_{\substack{\psi(x) \\ \operatorname{ord} h(x) = n}} |dx|_p.$$

Repeated application of lemma 6. 7 shows that $\int_{\substack{\psi(x) \\ \operatorname{ord} h(x) = n}} |dx|_p$ is a finite sum of expressions of the form

$$\frac{1}{p^m} \sum_{\xi \in \bar{\mathbb{Q}}_p^r} \sum_{\substack{l_1, \ldots, l_m \in \mathbb{Z} \\ \varphi(\xi, l_1, \ldots, l_m, n)}} p^{-l_1 - \cdots - l_m}.$$

If we apply lemma 5. 3 with $\mathscr{L} = \mathscr{L}_{\text{tame}}$ to the formula φ, we find that $Z_\psi(s, p)$ is a finite sum of expressions

$$\frac{1}{p^m} \left(\sum_{\substack{\xi \in \bar{\mathbb{Q}}_p^r \\ \chi_k(\xi)}} 1 \right) \cdot \left(\sum_{n=-M_p}^{\infty} \sum_{\substack{l_1, \ldots, l_m \in \mathbb{Z} \\ \theta(l_1, \ldots, l_m, n)}} p^{-ns - l_1 - \cdots - l_m} \right),$$

where χ is an $\mathbb{L}_{\bar{K}}$-formula and θ is a quantifier free formula in $\mathbb{L}_{\text{tame}\,\infty}$. To obtain the theorem we only have to prove that

$$\sum_{n=-M_p}^{\infty} \sum_{\substack{l_1, \ldots, l_m \in \mathbb{Z} \\ \theta(l_1, \ldots, l_m, n)}} p^{-ns - l_1 - \cdots - l_m} \tag{1}$$

is a rational function in p^{-s} of degree $\leqq 0$.

Formula (1) can be written as the sum of

$$\sum_{n=0}^{M_p} p^{ns}\left(\sum_{\substack{l_1,\dots,l_m\in\mathbb{Z}\\ \theta(l_1,\dots,l_m,n)}} p^{-l_1-\cdots-l_m}\right) \tag{2}$$

and

$$\sum_{n=1}^{\infty}\sum_{\substack{l_1,\dots,l_m\in\mathbb{Z}\\ \theta(l_1,\dots,l_m,n)}} p^{-ns-l_1-\cdots-l_m}. \tag{3}$$

Expression (2) is obviously of degree $\leqq 0$ in p^{-s}. For (3) we will prove the assertion using a technique of Denef [10].

Splitting up into the cases $l_i>0$ and $l_i\leqq 0$ and using the substitution $l_i\to -l_i+1$ if $l_i\leqq 0$, we find that (3) is a finite sum of expression of the form

$$p^{-\alpha}\sum_{\substack{(l_1,\dots,l_m,n)\in\mathbb{N}_0^{m+1}\\ \theta(l_1,\dots,l_m,n)}} p^{-ns-\varepsilon_1 l_1-\cdots-\varepsilon_m l_m}, \tag{4}$$

where $\mathbb{N}_0=\mathbb{N}\setminus\{0\}$, $\alpha\in\mathbb{N}_0$, $\varepsilon_i\in\{-1,1\}$. From the definition of $\mathscr{L}_{\text{tame}}$ we see that we may suppose that

$$\{(l_1,\dots,l_m,n)\in\mathbb{N}_0^{m+1}\mid\theta(l_1,\dots,l_m,n)\text{ holds}\}=\Lambda\cap\Delta,$$

where Λ resp. Δ is the set of all $(l_1,\dots,l_m,n)\in\mathbb{N}_0^{m+1}$ satisfying a system of homogeneous $\mathbb{Z}$-linear inequalities resp. congruences. So Λ is the intersection of $\mathbb{N}_0^{m+1}$ and a rational convex polyhedral cone in $\mathbb{R}_+^{m+1}$. From [5], pp. 123—124, follows that such a cone is a finite disjoint union of $\{0\}$ and sets of the form

$$\tilde{\Lambda}=\{\beta_1 v_1+\cdots+\beta_e v_e\mid\beta_j\in\mathbb{R},\ \beta_j>0\ \text{ for } j=1,\dots,e\},$$

where $v_1,\dots,v_e\in\mathbb{N}_0^{m+1}$ can be made part of a $\mathbb{Z}$-module basis for $\mathbb{Z}^{m+1}$. Hence we may suppose that

$$\Lambda=\mathbb{N}_0^{m+1}\cap\tilde{\Lambda}=\{k_1 v_1+\cdots+k_e v_e\mid k_j\in\mathbb{N}_0\ \text{ for } j=1,\dots,e\}.$$

Then (4) can be written as

$$p^{-\alpha}\sum_{(k_1,\dots,k_e)\in\Delta'} p^{-\sum_{j=1}^{e}k_j(A_j s+B_j)} \tag{5}$$

where $A_j\in\mathbb{N}$, $B_j\in\mathbb{Z}$ and Δ' is the set of all $(k_1,\dots,k_e)\in\mathbb{N}_0^e$ satisfying a system of congruences.

Let d be the product of all integers j such that a symbol $\equiv_j c$ appears in the definition of Δ'. Partitioning over all possible residue classes modulo d of $k_1,\dots,k_e$, we may suppose that the system of congruences has the form

$$k_1\equiv v_1 \bmod d,\dots,k_e\equiv v_e \bmod d,$$

with v_j an integer such that $0 < v_j \leqq d$. Hence (5) can be written as

$$p^{-\alpha} \prod_{j=1}^{e} \left(\sum_{\substack{k_j \in \mathbb{N}_0 \\ k_j \equiv v_j \bmod d}} p^{-k_j(A_j s + B_j)} \right),$$

which equals

$$p^{-\alpha} \prod_{j=1}^{e} \frac{p^{-v_j(A_j s + B_j)}}{1 - p^{-d(A_j s + B_j)}}.$$

This proves the theorem. ∎

7. Results with cross section

We introduce the languages $\mathscr{L}_{\mathrm{PR}}(\pi)$ and $\mathscr{L}_{\mathrm{tame}}(\pi)$ obtained by adding a function symbol π from the Γ-sort to the K-sort, to the languages $\mathscr{L}_{\mathrm{PR}}$ resp. $\mathscr{L}_{\mathrm{tame}}$. The symbol π will be interpreted as a cross section on K. We recall that a cross section is a morphism of groups $\pi : \Gamma \to K^{\times}$ which satisfies $\mathrm{ord} \circ \pi = 1_{\Gamma}$. We assume further that $\overline{\mathrm{ac}} \circ \pi$ is the constant map on 1. This agrees with the definition of an $\overline{\mathrm{ac}}$-map using a cross section (see the remark after definition 2. 2).

In these enlarged languages we obtain results similar to the ones in the languages without a cross section. These results extend work for fixed prime p by Denef [8], and have not yet been obtained by the method of resolution of singularities.

7. 1. Theorem. *Let $h(x) \in \mathbb{Z}[x]$ with $x = (x_1, \ldots, x_m)$. Let ψ be a formula without K-quantifiers in $\mathscr{L}_{\mathrm{PR}}(\pi)$, with free K-variables $x_1, \ldots, x_m$. Suppose*

$$W_p = \{x \in \mathbb{Q}_p^m \mid \psi(x) \text{ holds}\}$$

is bounded for all p. Then we have that for all p

$$Z_{\psi}(s, p) = \int_{W_p} |h(x)|_p^s \, |dx|_p$$

is a rational function in p^{-s}, and the degrees of numerator and denominator of this rational function are bounded independently of p.

7. 2. Lemma. *Suppose K is a structure for $\mathscr{L}$, which satisfies the conditions 2. 4 and which has a cross section π such that $\overline{\mathrm{ac}} \circ \pi$ is the constant map onto 1. Let $h(y_1, \ldots, y_r)$ be a polynomial in r variables with integer coefficients. Let $l, k_1, \ldots, k_r$ be Γ-variables, ϱ a $\bar{K}$-variable.*

Then the formulas

$$\text{(1)} \qquad \operatorname{ord} h(\pi(k_1), \dots, \pi(k_r)) = l$$

and

$$\text{(2)} \qquad \overline{\operatorname{ac}}\, h(\pi(k_1), \dots, \pi(k_r)) = \varrho$$

are definable without quantifiers in the language $\mathscr{L}$.

Proof. The proof is the same as in lemma 3. 1 of Cohen [4], and proceeds by induction on the number of terms in h.

Let $h(y_1, \dots, y_r) = \sum_\lambda a_\lambda y_1^{\lambda_1} \cdots y_r^{\lambda_r}$ with $a_\lambda \in \mathbb{Z}$ and $\lambda = (\lambda_1, \dots, \lambda_r)$ a multi-index. We make a partition over a finite number of cases, such that in each case the valuations of the terms of h are totally ordered. This can be done using conditions of the form

$$\operatorname{ord}(a_\lambda \pi(k_1)^{\lambda_1} \cdots \pi(k_r)^{\lambda_r}) \diamond \operatorname{ord}(a_\mu \pi(k_1)^{\mu_1} \cdots \pi(k_r)^{\mu_r})$$

where $\diamond$ is $<$ or $=$. These conditions are definable in $\mathscr{L}$ since $\operatorname{ord} \pi(k_i) = k_i$.

In each case we look at the terms of least valuation. If there is only one such term, say with index μ, we have

$$\operatorname{ord} h(\pi(k_1), \dots, \pi(k_r)) = \operatorname{ord}(a_\mu \pi(k_1)^{\mu_1} \cdots \pi(k_r)^{\mu_r}).$$

So (1) is equivalent to $\mu_1 k_1 + \cdots + \mu_r k_r = l$. Furthermore

$$\overline{\operatorname{ac}}\, h(\pi(k_1), \dots, \pi(k_r)) = \overline{\operatorname{ac}}\, a_\mu$$

which proves (2) to be equivalent to $\overline{\operatorname{ac}}\, a_\mu = \varrho$.

Assume now there are (at least) two terms of least valuation, $a_\lambda \pi(k_1)^{\lambda_1} \cdots \pi(k_r)^{\lambda_r}$ and $a_\mu \pi(k_1)^{\mu_1} \cdots \pi(k_r)^{\mu_r}$ with $\lambda \neq \mu$. Since $\operatorname{ord} a_\lambda = \operatorname{ord} a_\mu = 0$ ($\operatorname{char} \bar{K} = 0$), we have that $\pi(k_1)^{\lambda_1} \cdots \pi(k_r)^{\lambda_r} = \pi(k_1)^{\mu_1} \cdots \pi(k_r)^{\mu_r}$. The sum of these two terms can be replaced in h by $(a_\lambda + a_\mu)\, \pi(k_1)^{\lambda_1} \cdots \pi(k_r)^{\lambda_r}$, thus obtaining a polynomial with one less term. Now the induction hypothesis applies. ∎

7. 3. *Proof of theorem* 7. 1. One can check that ψ can be expressed in the language of Ax and Kochen [1], so by theorem 4. 3 in Denef [8] it suffices to prove the theorem for almost all p. All the expressions in the sequel of the proof will hold for almost all p.

We replace every occurrence of π in the formula ψ i.e. $(\dots \pi(\text{a } \Gamma\text{-term}) \dots)$ by introducing a new Γ-variable k in the following way

$$(\exists k)\, [(\dots \pi(k) \dots) \wedge k = (\text{that } \Gamma\text{-term})].$$

So we can find a simple formula in $\mathscr{L}_{\mathrm{PR}}$ (without cross section!) $\varphi(x, k_1, \dots, k_r, y_1, \dots, y_r)$ (where $y_1, \dots, y_r$ are K-variables) such that

$$\psi(x) \leftrightarrow (\exists k_1) \dots (\exists k_r)\, \varphi(x, k_1, \dots, k_r, \pi(k_1), \dots, \pi(k_r)).$$

These k_i's are unique, so writing k for $(k_1, \dots, k_r)$ and $\pi(k)$ for $(\pi(k_1), \dots, \pi(k_r))$, we have that $Z_\psi(s, p)$ is a finite sum of expressions of the form

$$\sum_{k \in \mathbb{Z}^r} \int_{\varphi(x, k, \pi(k))} |h(x)|_p^s \, |dx|_p.$$

The preceding integral can be written as

$$\sum_{n \in \mathbb{Z}} p^{-ns} \int_{\substack{\varphi(x, k, \pi(k)) \\ \operatorname{ord} h(x) = n}} |dx|_p.$$

To this last integral we apply repeatedly lemma 5. 2 with $(\pi(k_1), \dots, \pi(k_r))$ as values for the parameters $(y_1, \dots, y_r)$. In this way we see that $Z_\psi(s, p)$ is a finite sum of expressions of the form

$$\frac{1}{p^m} \sum_{\xi} \sum_{\substack{n, k, l \\ \theta(\xi, n, k, l, \pi(k))}} p^{-ns - l_1 - \dots - l_m}$$

where θ is a simple $\mathscr{L}_{\mathrm{PR}}$-formula. Lemma 7. 2 implies that $\theta(\xi, n, k, l, \pi(k))$ is equivalent to a simple $\mathscr{L}_{\mathrm{PR}}$-formula $\chi(\xi, n, k, l)$ for almost all p (again using the ultraproduct construction). Since $\chi(\xi, n, k, l)$ doesn't contain the cross section symbol, we can prove as in 5. 7 that the degree of numerator and denominator of $Z_\psi(s, p)$ is bounded with respect to p. ∎

7. 4. Theorem. *Let $h(x) \in \mathbb{Z}[x]$, with $x = (x_1, \dots, x_m)$. Let ψ be a formula in $\mathscr{L}_{\mathrm{tame}}(\pi)$ without K- or Γ-quantifiers, and with free K-variables $x_1, \dots, x_m$. Suppose*

$$W_p = \{x \in \mathbb{Q}_p^m \mid \psi(x) \text{ holds}\}$$

is bounded for all p. Then we have that for all p

$$Z_\psi(s, p) = \int_{W_p} |h(x)|_p^s \, |dx|_p$$

is a rational function in p^{-s}, and for almost all p

$$\deg Z_\psi(s, p) \leqq 0.$$

Proof. The proof is a combination of the proofs of theorem 6. 6 and theorem 7. 1, and is left to the reader. ∎

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