7 Second approach to the Homeomorphism Theorem

Let $(\mathbf{e}, Z)$ be a normalized Brownian excursion and the Head of the Brownian snake driven by $\mathbf{e}$. We associate a pseudo-distance on $[0,1]$ to $\mathbf{e}$ by setting for $a, b \in [0,1]$

$$\mathrm{d}_{\mathbf{e}}(a,b) = \mathbf{e}(a) + \mathbf{e}(b) - 2\inf\big\{\mathbf{e}(u) : u \in [a \wedge b, a \vee b]\big\}.$$

Let $L_{\mathbf{e}}$ be the union of segments $[xy]$ where $x = \exp(2i\pi a)$ and $y = \exp(2i\pi b)$ with $\mathrm{d}_{\mathbf{e}}(a,b) = 0$. Finally, we define an equivalence relation on $\overline{\mathbb{D}}$ using $\mathbf{e}$: if $x, y \in \overline{\mathbb{D}}$, we put $x \sim_{\mathbf{e}} y$ if x and y belongs to an chord $[e^{2i\pi a} e^{2i\pi b}]$ with $\mathrm{d}_{\mathbf{e}}(a,b) = 0$ or if x and y belong to the closure of some open triangle of $\overline{\mathbb{D}} \backslash L_{\mathbf{e}}$. We proved that the quotient set $\overline{\mathbb{D}}/\sim_{\mathbf{e}}$ is homeomorphic to the $\mathbb{R}$-tree $T_{\mathbf{e}}$ coded by the function $\mathbf{e}$.
We can do exactly the same procedure for Z (in particular we admit that the local minima of Z are distinct). Thus Z furnishes an equivalence relation $\sim_Z$ on $\overline{\mathbb{D}}$ is a similar manner as to $\mathbf{e}$. We consider $\mathbb{S}_2$ the standard Euclidean sphere of radius 1 in $\mathbb{R}^3$ and put

$$H_+ = \big\{(x_1, x_2, x_3) \in \mathbb{R}^3, x_1^2 + x_2^2 + x_3^2 = 1 \text{ and } x_3 \geqslant 0\big\},$$

for the closed North hemisphere of $\mathbb{S}_2$ and similarly H_- denotes the closed South hemisphere. The stereographic projections from the North and South poles enable us to identify H_+ and H_- with $\overline{\mathbb{D}}$. We will associate the function $\mathbf{e}$ (resp. Z) to the North (resp. South) part of the ball, hence we can define $\sim_{\mathbf{e}}$ on H_+ and $\sim_Z$ on H_-.

1. Check that $H_+/\sim_{\mathbf{e}}$ is still homeomorphic to $T_{\mathbf{e}}$.

We put a relation on $x, y \in \mathbb{S}_2$ by $x \sim y$ if and only if $x, y \in H_+$ and $x \sim_{\mathbf{e}} y$ or $x, y \in H_-$ and $x \sim_Z y$. We admit the following fact about the process $(\mathbf{e}_t, Z_t)_{t \in [0,1]}$. Almost surely, for every $s \in]0,1[$ such that for some $\varepsilon > 0$ if we have

$$\mathbf{e}_s = \min_{r \in [s-\varepsilon, s]} \mathbf{e}_r \quad \text{or} \quad \mathbf{e}_s = \min_{r \in [s, s+\varepsilon]} \mathbf{e}_r$$

then

$$Z_s > \min_{r \in [s-\delta, s]} Z_r, \text{ for every } 0 < \delta < s \quad \text{and} \quad Z_s > \min_{r \in [s, s+\delta]} Z_r, \text{ for every } 0 < \delta < 1-s.$$

2. Prove that a.s. $\sim$ is a closed equivalence relation.

Theorem 7.1 (Moore (1925))**.** *Let $\sim$ be a closed equivalence relation on the two dimensional sphere $\mathbb{S}_2$. Assume that every equivalence class of $\sim$ is a compact path-connected subset of the sphere whose complement is connected. The quotient space $\mathbb{S}_2/\sim$ is homeomorphic to $\mathbb{S}_2$.*

3. Give an example of a closed equivalence relation $\simeq$ such that the quotient $\mathbb{S}_2/\simeq$ is not homeomorphic to $\mathbb{S}_2$.

4. Prove that in our setting $\sim$ a.s. verifies all hypotheses of Moore's Theorem and deduce that almost surely $\mathbb{S}_2/\sim$ is homeomorphic to $\mathbb{S}_2$.

5. Does anybody see a link with scaling limits of random planar quadrangulations ?

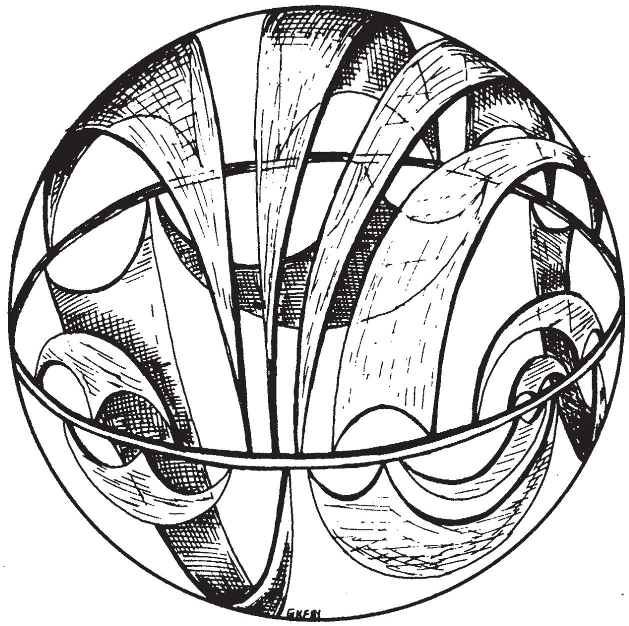

FIGURE 10. A pattern of identifications of a circle, here represented as the equator, whose quotient space is topologically a sphere. This defines, topologically, a sphere-filling curve.

(Taken from Thurston)

References

[LGP08] Jean-François Le Gall and Frédéric Paulin. Scaling limits of bipartite planar maps are homeomorphic to the 2-sphere. *Geom. Funct. Anal.*, 18(3):893–918, 2008.