3 Stick-Breaking Construction of the CRT

Exercise 3.1 (Aldous-Broder Algorithm and stick breaking construction of the CRT.). *Given a finite connected graph $G = (V, E)$, how can we sample a uniform spanning tree of G ? In general it is hard to list all the spanning trees of a given graph. However, there exist stochastic algorithms that sample from this set without knowing it. The algorithm we describe here is due to Aldous & Broder.*
Recipe: Let $G = (V, E)$ be a connected graph and $v_0 \in V$ a vertex. We perform a simple random walk on G starting from v_0. For any $v \in G$, we denote by T_v the hitting time of v by the walk and by E_{T_v} the edge traversed by the walk just before it hits v.

1. *Show that the set of edges $\left\{E_{T_v}, v \in V\backslash\{v_0\}\right\}$, defines a spanning tree of G.*

Theorem 3.1. *This (unrooted) spanning tree is uniform over the set of all spanning trees of G.*

Now, we will focus on the case of $G = \mathbb{K}_n$ the complete graph over n vertices [1]. Denote by $(X_k)_{k\geqslant 0}$ the simple random walk on $\mathbb{K}_n$ (the dependence in n is implicit). That is $(X_k)_{k\geqslant 0}$ is a sequence of independent variables uniformly distributed over the vertices of $\mathbb{K}_n$. We introduce the first time the walk hits its past and the corresponding vertex

$$T_1^{(n)} = \inf\left\{k \geqslant 1 : X_k \in \{X_0, ..., X_{k-1}\}\right\} \text{ and } P_1^{(n)} = X_{T_1^{(n)}}.$$

We define $T_2^{(n)} = \inf\{k > T_1^{(n)} : X_k \in \{X_0, ..., X_{k-1}\}\}$, $P_2^{(n)} = X_{T_2^{(n)}}$ ... by induction.
We also recall "the birthday paradox": Assume that a year has n days and that people are born equally likely each day of the year. Then among $\sqrt{n}$ people chosen at random two of them are born the same day with a big probability.

2. *What is the rough order of $T_1^{(n)}$ as $n \to \infty$ (don't look below !).*
3. *Show that $n^{-1/2}T_1^{(n)}$ converges in distribution as $n \to \infty$ and identify the limit law.*
4. *More generally, show that for any $k \in \mathbb{Z}_+$,*

$$n^{-1}\left(\frac{(T_1^{(n)})^2}{2}, \frac{(T_2^{(n)})^2}{2}, ..., \frac{(T_k^{(n)})^2}{2}\right),$$

   converges in distribution towards the first k points of a standard Poisson point process on $\mathbb{R}_+$ with intensity 1.
5. *Show that for any $k \in \mathbb{Z}_+$, conditionally on $(T_1^{(n)}, ..., T_k^{(n)}, X_0, X_1, ..., X_{T_k-1})$ the point $P_k^{(n)}$ is uniformly distributed over $\{X_0, ..., X_{T_k^{(n)}-1}\}$.*
6. *Describe the continuous limit of the construction of the spanning tree over $\mathbb{K}_n$.*

Exercise 3.2. ** Prove Theorem 3.1. in the case of the complete graph.*

[1] We have seen (tutorial 1) that the number of spanning trees of $\mathbb{K}_n$ is n^{n-2}

Exercise 3.3. *Who are these charming gentlemen ?*

References

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