6 Brownian Lamination & Homeomorphism Theorem

Exercise 6.1 (Brownian Lamination)**.** *Let* $\mathbf{e} : [0,1] \longrightarrow [0,+\infty)$ *be a normalized Brownian excursion. Recall that we proved that the local minima of* $\mathbf{e}$ *are almost surely distinct. We associate a random closed subset of the closed unit disk* $\overline{\mathbb{D}}$ *to* $\mathbf{e}$ *by the following device. Recall the definition of the pseudo-distance associated to* $\mathbf{e}$*: for* $a, b \in [0,1]$ *we put*

$$d_{\mathbf{e}}(a,b) = \mathbf{e}(a) + \mathbf{e}(b) - 2\inf\big\{\mathbf{e}(u) : u \in [a \wedge b, a \vee b]\big\}.$$

Let $L_{\mathbf{e}}$ *be the union of all segments* $[xy]$ *where* $x = \exp(2i\pi a)$ *and* $y = \exp(2i\pi b)$ *with* $d_{\mathbf{e}}(a,b) = 0$. *A segment* $[xy]$ *with endpoints on* $\mathbb{S}_1$ *is called a* chord*. If two chords* $[xy]$ *and* $[x'y']$ *are such that* $(xy) \cap (x'y') = \varnothing$ *we say that the chords are* non-crossing*. A closed subset of* $\overline{\mathbb{D}}$ *which can be written as a union of non-crossing chords is called a* lamination*.*

1. *Show that* $L_{\mathbf{e}}$ *is a closed subset of* $\overline{\mathbb{D}}$*.*
2. *Show that a.s. if* $a, b, c, d \in [0,1]$ *such that* $d_{\mathbf{e}}(a,b) = d_{\mathbf{e}}(c,d) = 0$ *then*

   $$\textit{either } [e^{2i\pi a}e^{2i\pi b}] = [e^{2i\pi c}e^{2i\pi d}] \quad \textit{or} \quad (e^{2i\pi a}e^{2i\pi b}) \cap (e^{2i\pi c}e^{2i\pi d}) = \varnothing.$$

   Conclude that a.s. $L_{\mathbf{e}}$ *is lamination.*
3. *Show that a.s. the connected components of* $\overline{\mathbb{D}} \backslash L_{\mathbf{e}}$ *are open triangles with vertices on* $\mathbb{S}_1$*.*
4. *Show that a.s.* $L_{\mathbf{e}}$ *is maximal for the inclusion relation among laminations.*

We define a relation on $\overline{\mathbb{D}}$ *using* $\mathbf{e}$*: if* $x, y \in \overline{\mathbb{D}}$*, we put* $x \sim_{\mathbf{e}} y$ *if* x *and* y *belong to a chord* $[e^{2i\pi a}e^{2i\pi b}]$ *with* $d_{\mathbf{e}}(a,b) = 0$ *or if* x *and* y *belong to the closure of some open triangle of* $\overline{\mathbb{D}} \backslash L_{\mathbf{e}}$*.*

5. *Prove that* $\sim_{\mathbf{e}}$ *is a closed equivalence relation and that the quotient space* $\overline{\mathbb{D}}/\sim_{\mathbf{e}}$ *is homeomorphic to the* $\mathbb{R}$*-tree* $T_{\mathbf{e}}$ *coded by* $\mathbf{e}$*.*

   Reminder on quotient topology: *If* X *is topological space and* $\sim$ *an equivalence relation on* X*, we endow* $X/\sim$ *with the finest topology for which the canonical projection* $\pi : X \to X/\sim$ *is continuous. Equivalently, a set* $A \subset X/\sim$ *is open if and only if* $\pi^{-1}(A)$ *is open. We say that* $\sim$ *is closed if the set* $\{(x,y) \in X \times X : x \sim y\}$*, is closed. We admit (or we prove) that if* X *is a compact metric space and* $\sim$ *is closed then* $X/\sim$ *is an Hausdorff space and then compact.*
6. *Show that the local minima of* $\mathbf{e}$ *are dense in* $[0,1]$ *and deduce that* $L_{\mathbf{e}}$ *has an empty interior.*

Exercise 6.2 (Homeomorphism Theorem)**.** *We now consider, together with* $\mathbf{e}$*, the Head of the Brownian snake* Z *driven by* $\mathbf{e}$*. We can do exactly the same procedure for* Z *(in particular we admit that the local minima of* Z *are distinct). Thus* Z *furnishes an equivalence relation* $\sim_Z$ *on* $\overline{\mathbb{D}}$ *is a similar manner as to* $\mathbf{e}$*. We consider* $\mathbb{S}_2$ *the standard Euclidean sphere of radius* 1 *in* $\mathbb{R}^3$ *and put*

$$H_+ = \big\{(x_1,x_2,x_3) \in \mathbb{R}^3, x_1^2 + x_2^2 + x_3^2 = 1 \textit{ and } x_3 \geqslant 0\big\},$$

for the closed North hemisphere of $\mathbb{S}_2$ *and similarly* H_- *denotes the closed South hemisphere. The stereographic projections from the North and South poles enable us to identify* H_+ *and* H_- *with* $\overline{\mathbb{D}}$*. We will associate the function* $\mathbf{e}$ *(resp.* Z*) to the North (resp. South) part of the ball, hence we can define* $\sim_{\mathbf{e}}$ *on* H_+ *and* $\sim_Z$ *on* H_-*.*

1. *Check that* $H_+/\sim_{\mathbf{e}}$ *is still homeomorphic to* $T_{\mathbf{e}}$*.*

We put a relation on $x, y \in \mathbb{S}_2$ by $x \sim y$ if and only if $x, y \in H_+$ and $x \sim_{\mathbf{e}} y$ or $x, y \in H_-$ and $x \sim_Z y$. We admit the following fact about the process $(\mathbf{e}_t, Z_t)_{t\in[0,1]}$. Almost surely, for every $s \in]0, 1[$ such that for some $\varepsilon > 0$ if we have

$$\mathbf{e}_s = \min_{r\in[s-\varepsilon,s]} \mathbf{e}_r \quad \text{or} \quad \mathbf{e}_s = \min_{r\in[s,s+\varepsilon]} \mathbf{e}_r$$

then

$$Z_s > \min_{r\in[s-\delta,s]} Z_r, \text{ for every } 0 < \delta < s \quad \text{and} \quad Z_s > \min_{r\in[s,s+\delta]} Z_r, \text{ for every } 0 < \delta < 1-s.$$

2. *Prove that a.s. $\sim$ is a closed equivalence relation.*

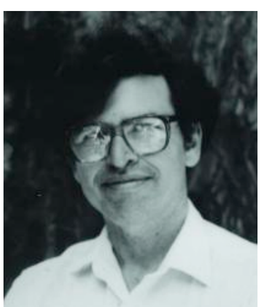
Theorem 6.1 (Moore (1925))**.** *Let $\sim$ be a closed equivalence relation on the two dimensional sphere $\mathbb{S}_2$. Assume that every equivalence class of $\sim$ is a compact path-connected subset of the sphere whose complement is connected. The quotient space $\mathbb{S}_2/\sim$ is homeomorphic to $\mathbb{S}_2$.*

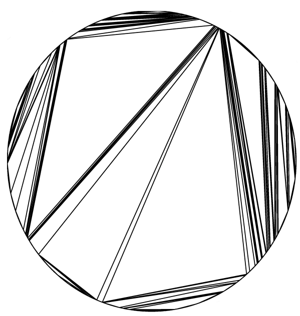
3. *Give an example of a closed equivalence relation $\simeq$ such that the quotient $\mathbb{S}_2/\simeq$ is not homeomorphic to $\mathbb{S}_2$.*

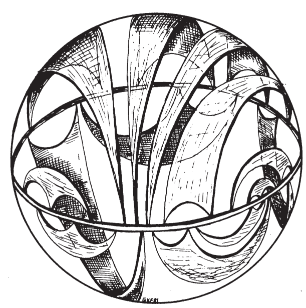
4. *Prove that in our setting $\sim$ a.s. verifies all hypotheses of Moore's Theorem and deduce that almost surely $\mathbb{S}_2/\sim$ is homeomorphic to $\mathbb{S}_2$.*

5. *Does anybody see a link with scaling limits of random planar quadrangulations ?*

Exercise 6.3. *Who is this charming gentleman / What does represent these nice pictures ?*







The third picture is taken from Thurston.

References

[Ald94a] D. Aldous. Recursive self-similarity for random trees, random triangulations and brownian excursion. *Ann. Probab.*, 22(2):527–545, 1994.

[LGP08] Jean-François Le Gall and Frédéric Paulin. Scaling limits of bipartite planar maps are homeomorphic to the 2-sphere. *Geom. Funct. Anal.*, 18(3):893–918, 2008.

7 Introduction to the three-point function

The mutli-pointed bijection. Let (Q, s_1, s_2, s_3) be a planar quadrangulation with 3 distinct distinguished vertices, we call these vertices the *sources* of the quadrangulation. We also impose that the graph distance in Q between any of the sources is larger than 2 (that is no sources are neighbors) and that the sources are not aligned (none of the sources lies on a geodesic path between the two others). The object (Q, s_1, s_2, s_3) is then called *a triply pointed quadrangulation*. Together with the sources we are also given integer *delays*, τ_1, τ_2 and τ_3. We impose that the delays satisfy

$$(\star)\left\{\begin{array}{l} |\tau_i - \tau_j| < \mathrm{d_{gr}}(s_i, s_j),\ \ i, j \in \{1, 2, 3\} \\ \tau_i - \tau_j + \mathrm{d_{gr}}(s_i, s_j) \text{ is even },\ \ i, j \in \{1, 2, 3\}. \end{array}\right.$$

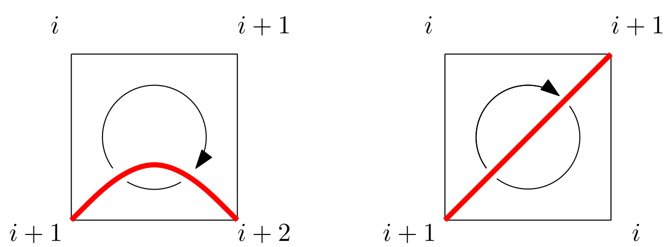
1. Show that we can always find delays satisfying $(\star)$.

We associate a labeling ℓ to the vertices of the map Q by putting

$$\ell(v) = \min_{i \in \{1,2,3\}} \big(\tau_i + \mathrm{d_{gr}}(v, s_i)\big).$$

2. Show that the label of any source is equal to its delays and that two neighboring vertices in the map Q have labels that differ by 1 or -1.

Hence the faces of the quadrangulation Q can be decomposed into two subsets: The faces such that the labels of the vertices are $(i, i+1, i+2, i+1)$ or those satisfying $(i, i+1, i, i+1)$. We add a "red" edge in each face following the rule given by the figure below.



3. Apply this construction to quadrangulation of Fig. 1. (taken from Bouttier and Guitter).

4. Verify that you are left with a labeled map with 3 faces such that for any $i \in \{1, 2, 3\}$

$$\min_{v \text{ incident to face } i} \ell(v) = \tau_i + 1.$$

   Verify also that the labels can vary by $-1, 0$ or 1 along an edge, we say that the map is *well-labeled*. Like in the classical Schaeffer construction, check that if a vertex v is incident to a face i for $i \in \{1, 2, 3\}$ then

$$\mathrm{d_{gr}}(v, s_i) = \ell(v) - \tau_i.$$

   Given the well-labeled map with three faces, how can you reconstruct the original quadrangulation? (No proof)

Three-point function.

5. Can you give the pairwise distances between the sources in the quadrangulation associated to the well-labeled planar map of Fig. 2 without reconstructing the whole quadrangulation?

Let (Q, s_1, s_2, s_3) be a triply-pointed quadrangulation. Now we choose particular values for the delays associated to the sources of a triply-pointed quadrangulation. We let τ_1, τ_2 and τ_3 such that

$$(\clubsuit)\begin{cases}\tau_1 + \tau_2 = -\,\mathrm{d_{gr}}(s_1, s_2),\\ \tau_1 + \tau_3 = -\,\mathrm{d_{gr}}(s_1, s_3),\\ \tau_2 + \tau_3 = -\,\mathrm{d_{gr}}(s_2, s_3).\end{cases}$$

6. Show that these labels satisfy $(\star)$.

7. Show that the map with three faces obtained with Miermont's construction is such that any two faces have a non-empty boundary and satisfy

$$(\heartsuit)\begin{cases}\min\big\{\ell(v) : v \text{ incident to face } 1\big\} = 1 + \tau_1, & \min\big\{\ell(v) : v \text{ incident to faces } 1 \text{ and } 2\big\} = 0\\ \min\big\{\ell(v) : v \text{ incident to face } 2\big\} = 1 + \tau_2, & \min\big\{\ell(v) : v \text{ incident to faces } 1 \text{ and } 3\big\} = 0\\ \min\big\{\ell(v) : v \text{ incident to face } 3\big\} = 1 + \tau_3, & \min\big\{\ell(v) : v \text{ incident to faces } 2 \text{ and } 3\big\} = 0.\end{cases}$$

Theorem 7.1. *The construction presented above is a bijection between on the one hand triply-pointed quadrangulations with n faces such that the sources s_1, s_2 and s_3 are not aligned and on the other hand well-labeled planar maps with 3 faces with n edges satisfying $(\heartsuit)$ for the delays related to the distances between the sources by $(\clubsuit)$.*

8. Let (Q_n, s_1, s_2, s_3) a triply-pointed quadrangulation with n faces. How *would you* show that the triplet
$$n^{-1/4}\big(\,\mathrm{d_{gr}}(s_1, s_2), \mathrm{d_{gr}}(s_1, s_3), \mathrm{d_{gr}}(s_2, s_3)\big),$$
converge in distribution as $n \to \infty$?

Exercise 7.1. *Who are these charming gentlemen ?*

References

[BG08] J. Bouttier and E. Guitter. The three-point function of planar quadrangulations. *J. Stat. Mech. Theory Exp.*, (7):P07020, 39, 2008.

[Mie09] Grégory Miermont. Tessellations of random maps of arbitrary genus. *Ann. Sci. Éc. Norm. Supér. (4)*, 42(5):725–781, 2009.

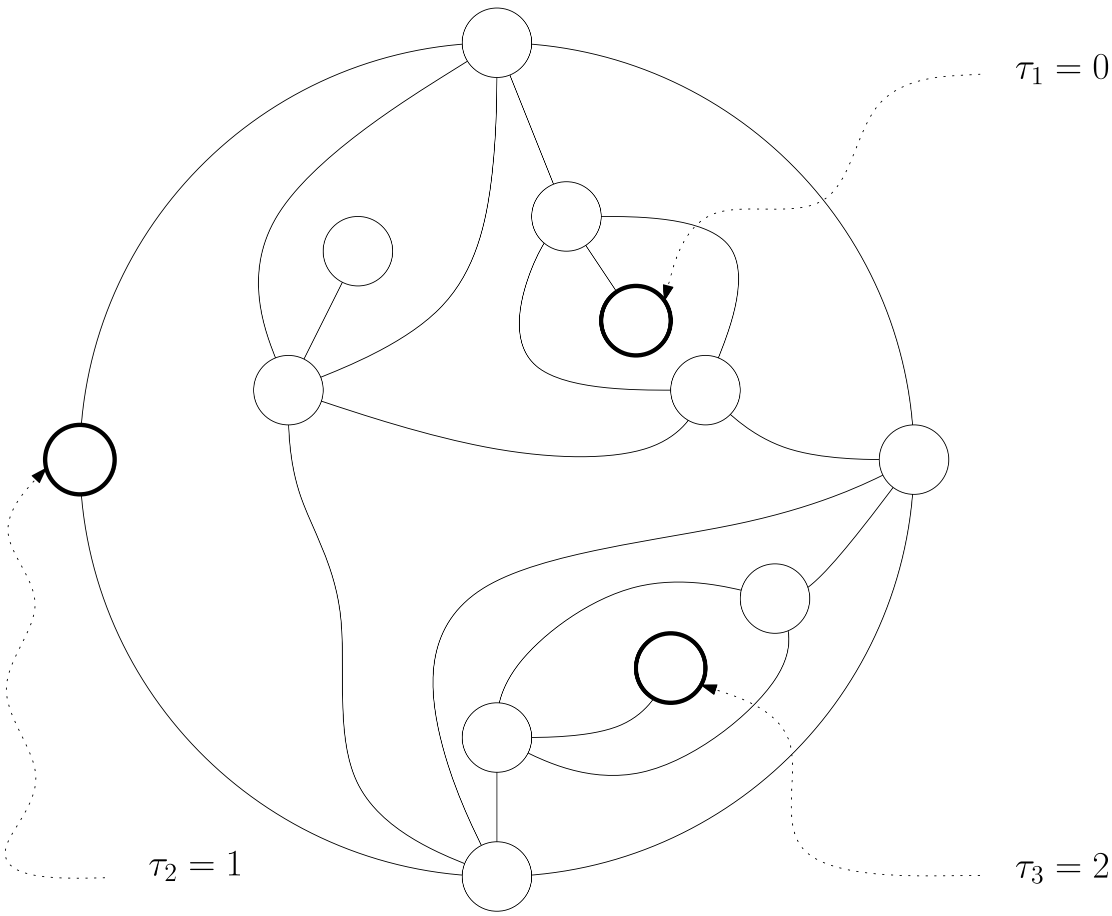


Fig. 1.

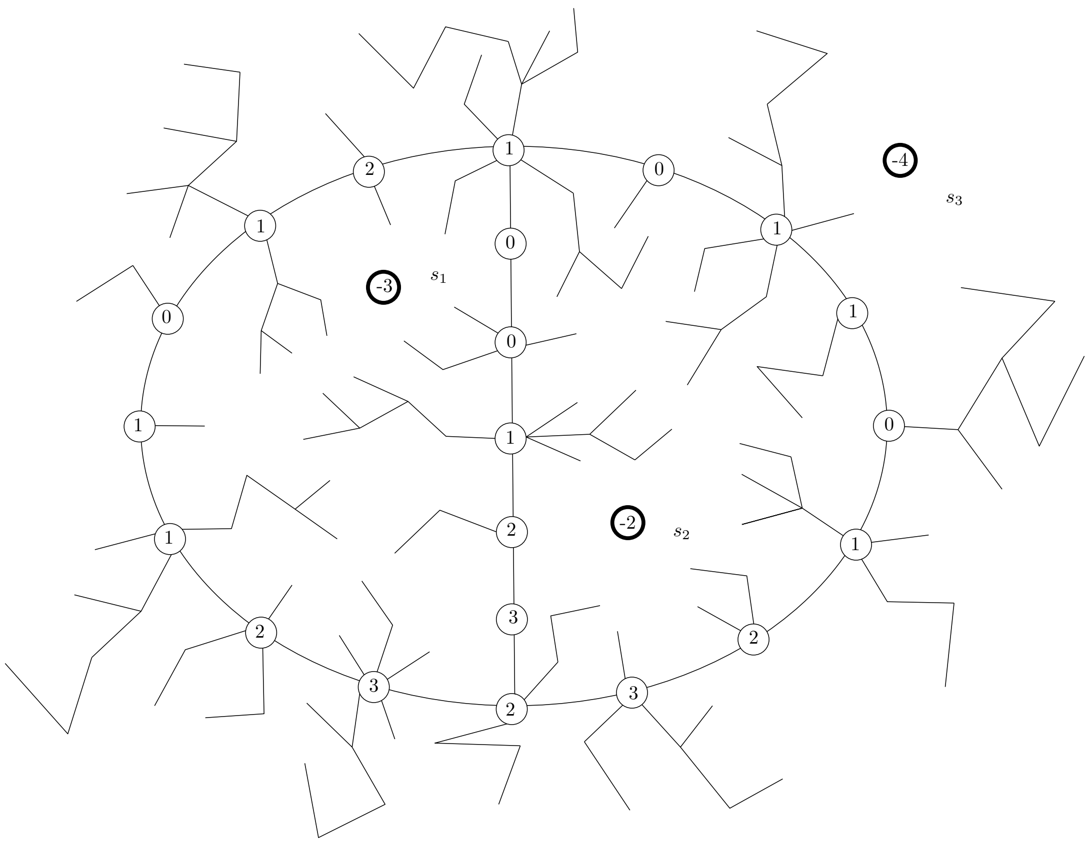


Fig. 2.