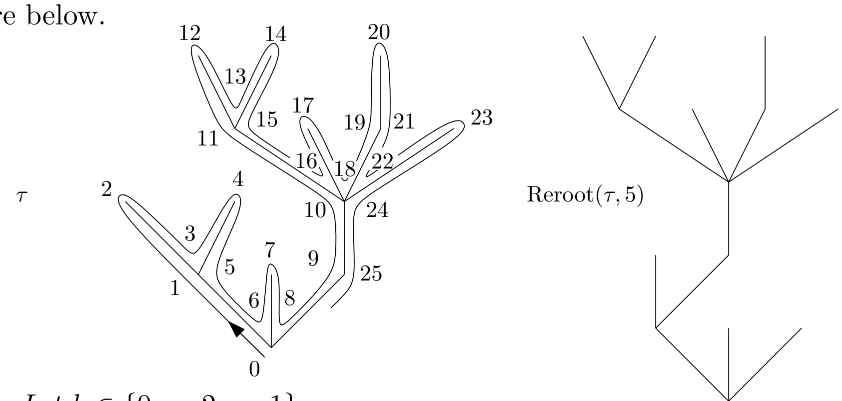
4 Properties of the CRT

Let $\mathbf{A}_n$ be the set of all rooted oriented trees with n edges. If $\tau \in \mathbf{A}_n$ and $k \in \{0, ..., 2n-1\}$ we define $\mathrm{Reroot}(\tau, k)$ the tree τ re-rooted at the k-th corner in the counter clockwise contour of τ. See figure below.



Exercise 4.1. *Let $k \in \{0, ..., 2n-1\}$.*

1. *Show that* $\mathrm{Reroot}(\cdot, k) : \mathbf{A}_n \to \mathbf{A}_n$ *is a bijection. Deduce that if τ_n is uniform over $\mathbf{A}_n$ then* $\mathrm{Reroot}(\tau_n, k)$ *is also uniform over $\mathbf{A}_n$.*

Let $\tau \in \mathbf{A}_n$ and denote its contour function by $(C_n(t), t \in [0, 2n])$. We also write for $x \in [0,1]$, $\mathbf{C}_n(x) = C_n(2nx)$. For every $r \geqslant$ let $\overline{r} = r - \lfloor r \rfloor$ be the fractional part of r. If $g : [0,1] \longrightarrow [0, +\infty[$ such that $g(0) = g(1) = 0$ and $s, t \in [0,1]$ we recall the notation

$$m_g(s,t) = \inf \big\{ g(u), u \in [s \wedge t, s \vee t] \big\}.$$

2. *Show that the contour function $(\mathfrak{C}_n(x), x \in [0, 2n])$ of the rooted oriented tree* $\mathrm{Reroot}(\tau, k)$ *is given for every $t \in [0,1]$ by*

$$\mathfrak{C}_n(t) = \mathbf{C}_n\left(\frac{k}{2n}\right) + \mathbf{C}_n\left(\overline{\frac{t+k}{2n}}\right) - 2m_{\mathbf{C}_n}\left(\frac{k}{2n}, \overline{\frac{k+t}{2n}}\right).$$

3. *Deduce from the previous considerations that if $(\mathbf{e}(t))_{t \in [0,1]}$ is a normalized Brownian excursion and $x \in [0,1]$ then the process $(\mathfrak{e}(t))_{t \in [0,1]}$ defined by*

$$\mathfrak{e}(t) = \mathbf{e}(x) + \mathbf{e}(\overline{x+t}) - 2m_{\mathbf{e}}(x, \overline{x+t}),$$

has the same distribution as $(\mathbf{e}(t))_{t \in [0,1]}$. What does it imply for the Brownian Continuum Random Tree ?

Let $\mathcal{T}$ be a real tree. For $x \in \mathcal{T}$, the *multiplicity* of x is the number of connected components of $\mathcal{T} \backslash \{x\}$. A *leaf* is a point of multiplicity one. In the following $(\mathbf{e}(t))_{t \in [0,1]}$ denotes a normalized Brownian excursion and $\mathcal{T}_{\mathbf{e}}$ its associated $\mathbb{R}$-tree, in particular we denote $p_{\mathbf{e}} : [0,1] \longrightarrow \mathcal{T}_{\mathbf{e}}$ the canonical projection.

Exercise 4.2.

1. *Show that almost surely $p_{\mathbf{e}}(0)$ is a leaf of $\mathcal{T}_{\mathbf{e}}$. Deduce that for every $x \in [0,1]$, $p_{\mathbf{e}}(x)$ is almost surely a leaf of $\mathcal{T}_{\mathbf{e}}$.*

2. *Show that a.s. the local minima of a Brownian motion are pairwise distinct.*

3. *Deduce that almost surely $\mathcal{T}_{\mathbf{e}}$ has only countable points of multiplicity 3 but no point of multiplicity strictly larger than 3.*

Exercise 4.3. *Who are these charming gentlemen ?*


References

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[MM06] Jean-François Marckert and Abdelkader Mokkadem. Limit of normalized quadrangulations: the Brownian map. *Ann. Probab.*, 34(6):2144–2202, 2006.