6 Brownian Lamination & Homeomorphism Theorem

Exercise 6.1 (Brownian Lamination). *Let $\mathbf{e} : [0,1] \longrightarrow [0,+\infty)$ be a normalized Brownian excursion. Recall that we proved that the local minima of $\mathbf{e}$ are almost surely distinct. We associate a random closed subset of the closed unit disk $\overline{\mathbb{D}}$ to $\mathbf{e}$ by the following device. Recall the definition of the pseudo-distance associated to $\mathbf{e}$: for $a, b \in [0,1]$ we put*

$$d_{\mathbf{e}}(a,b) = \mathbf{e}(a) + \mathbf{e}(b) - 2 \inf \big\{ \mathbf{e}(u) : u \in [a \wedge b, a \vee b] \big\}.$$

Let $L_{\mathbf{e}}$ be the union of all segments $[xy]$ where $x = \exp(2i\pi a)$ and $y = \exp(2i\pi b)$ with $d_{\mathbf{e}}(a,b) = 0$. A segment $[xy]$ with endpoints on $\mathbb{S}_1$ is called a chord. *If two chords $[xy]$ and $[x'y']$ are such that $(xy) \cap (x'y') = \varnothing$ we say that the chords are* non-crossing. *A closed subset of $\overline{\mathbb{D}}$ which can be written as a union of non-crossing chords is called a* lamination.

1. *Show that $L_{\mathbf{e}}$ is a closed subset of $\overline{\mathbb{D}}$.*
2. *Show that a.s. if $a, b, c, d \in [0,1]$ such that $d_{\mathbf{e}}(a,b) = d_{\mathbf{e}}(c,d) = 0$ then*

   $$\text{either } [e^{2i\pi a} e^{2i\pi b}] = [e^{2i\pi c} e^{2i\pi d}] \quad \text{or} \quad (e^{2i\pi a} e^{2i\pi b}) \cap (e^{2i\pi c} e^{2i\pi d}) = \varnothing.$$

   Conclude that a.s. $L_{\mathbf{e}}$ is lamination.
3. *Show that a.s. the connected components of $\overline{\mathbb{D}} \backslash L_{\mathbf{e}}$ are open triangles with vertices on $\mathbb{S}_1$.*
4. *Show that a.s. $L_{\mathbf{e}}$ is maximal for the inclusion relation among laminations.*

We define a relation on $\overline{\mathbb{D}}$ using $\mathbf{e}$: if $x, y \in \overline{\mathbb{D}}$, we put $x \sim_{\mathbf{e}} y$ if x and y belong to a chord $[e^{2i\pi a} e^{2i\pi b}]$ with $d_{\mathbf{e}}(a,b) = 0$ or if x and y belong to the closure of some open triangle of $\overline{\mathbb{D}} \backslash L_{\mathbf{e}}$.

5. *Prove that $\sim_{\mathbf{e}}$ is a closed equivalence relation and that the quotient space $\overline{\mathbb{D}} / \sim_{\mathbf{e}}$ is homeomorphic to the $\mathbb{R}$-tree $T_{\mathbf{e}}$ coded by $\mathbf{e}$.*

   Reminder on quotient topology: *If X is topological space and $\sim$ an equivalence relation on X, we endow $X/\sim$ with the finest topology for which the canonical projection $\pi : X \to X/\sim$ is continuous. Equivalently, a set $A \subset X/\sim$ is open if and only if $\pi^{-1}(A)$ is open. We say that $\sim$ is closed if the set $\{(x,y) \in X \times X : x \sim y\}$, is closed. We admit (or we prove) that if X is a compact metric space and $\sim$ is closed then $X/\sim$ is an Hausdorff space and then compact.*

6. *Show that the local minima of $\mathbf{e}$ are dense in $[0,1]$ and deduce that $L_{\mathbf{e}}$ has an empty interior.*

Exercise 6.2 (Homeomorphism Theorem). *We now consider, together with $\mathbf{e}$, the Head of the Brownian snake Z driven by $\mathbf{e}$. We can do exactly the same procedure for Z (in particular we admit that the local minima of Z are distinct). Thus Z furnishes an equivalence relation $\sim_Z$ on $\overline{\mathbb{D}}$ is a similar manner as to $\mathbf{e}$. We consider $\mathbb{S}_2$ the standard Euclidean sphere of radius 1 in $\mathbb{R}^3$ and put*

$$H_+ = \big\{ (x_1, x_2, x_3) \in \mathbb{R}^3, x_1^2 + x_2^2 + x_3^2 = 1 \text{ and } x_3 \geqslant 0 \big\},$$

for the closed North hemisphere of $\mathbb{S}_2$ and similarly H_- denotes the closed South hemisphere. The stereographic projections from the North and South poles enable us to identify H_+ and H_- with $\overline{\mathbb{D}}$. We will associate the function $\mathbf{e}$ (resp. Z) to the North (resp. South) part of the ball, hence we can define $\sim_{\mathbf{e}}$ on H_+ and $\sim_Z$ on H_-.

1. *Check that $H_+/\sim_{\mathbf{e}}$ is still homeomorphic to $T_{\mathbf{e}}$.*

We put a relation on $x, y \in \mathbb{S}_2$ by $x \sim y$ if and only if $x, y \in H_+$ and $x \sim_{\mathbf{e}} y$ or $x, y \in H_-$ and $x \sim_Z y$. We admit the following fact about the process $(\mathbf{e}_t, Z_t)_{t\in[0,1]}$. Almost surely, for every $s \in]0,1[$ such that for some $\varepsilon > 0$ if we have

$$\mathbf{e}_s = \min_{r\in[s-\varepsilon,s]} \mathbf{e}_r \quad \text{or} \quad \mathbf{e}_s = \min_{r\in[s,s+\varepsilon]} \mathbf{e}_r$$

then

$$Z_s > \min_{r\in[s-\delta,s]} Z_r, \text{ for every } 0<\delta<s \quad \text{and} \quad Z_s > \min_{r\in[s,s+\delta]} Z_r, \text{ for every } 0<\delta<1-s.$$

2. *Prove that a.s. $\sim$ is a closed equivalence relation.*

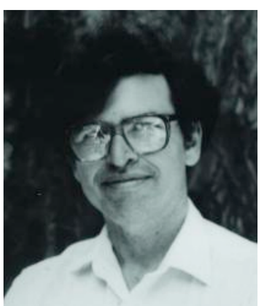
Theorem 6.1 (Moore (1925))**.** *Let $\sim$ be a closed equivalence relation on the two dimensional sphere $\mathbb{S}_2$. Assume that every equivalence class of $\sim$ is a compact path-connected subset of the sphere whose complement is connected. The quotient space $\mathbb{S}_2/\sim$ is homeomorphic to $\mathbb{S}_2$.*

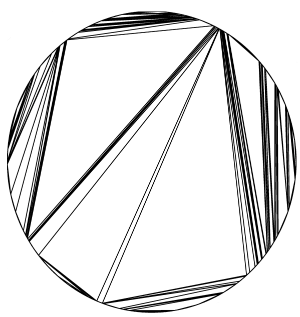
3. *Give an example of a closed equivalence relation $\simeq$ such that the quotient $\mathbb{S}_2/\simeq$ is not homeomorphic to $\mathbb{S}_2$.*

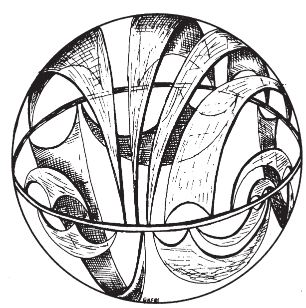
4. *Prove that in our setting $\sim$ a.s. verifies all hypotheses of Moore's Theorem and deduce that almost surely $\mathbb{S}_2/\sim$ is homeomorphic to $\mathbb{S}_2$.*

5. *Does anybody see a link with scaling limits of random planar quadrangulations ?*

Exercise 6.3. *Who is this charming gentleman / What does represent these nice pictures ?*







The third picture is taken from Thurston.

References

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