2 Metric Geometry

Exercise 2.1. *Give an example of two (non-compact !) metric spaces X and Y such that $d_{GH}(X,Y)=0$ but $X \neq Y$.*

Exercise 2.2 (Topology)**.** *Suppose that $(X_n)_{n\geqslant 0}$ is a sequence of compact metric spaces such that X_n is homeomorphic to X_0 for every $n \geqslant 0$. Show that $X_n \to X$ in the sense of d_{GH} does not imply X homeomorphic to X_0.*

Exercise 2.3 (Gromov's Compactness Theorem)**.** *The goal of this exercise is to characterize pre-compactness for Gromov-Hausdorff distance. A collection $\mathfrak{X}$ of compact metric spaces is* totally bounded *if*

(i) There exists $C>0$ such that for every $X \in \mathfrak{X}$, the diameter of X is bounded above by C,

(ii) For every $\varepsilon>0$, there exists $N(\varepsilon) \in \mathbb{Z}_+$ such that every $X \in \mathfrak{X}$ admits an ε-net containing no more than $N(\varepsilon)$ points.

1. *Show that every pre-compact collection is totally bounded.*

Now, let $(X_n)_{n\geqslant 1}$ be a totally bounded sequence of compact metric spaces and denote $N(\varepsilon)$ the minimal number of balls of radius ε needed to cover any space X_i. For every n, denote the distance in the space X_n by d_n and let $(x^{(n)}_{k,j})_{k,j\geqslant 1}$ be a sequence of points of X_n such that for every $k \geqslant 1$ the points

$$x^{(n)}_{k,1}, ..., x^{(n)}_{k,N(1/k)}, \text{ form a } 1/k\text{-net in } X_n.$$

2. *Show that we can extract a subsequence $(n_i)_{i\geqslant 1}$ such that for every $(k,j),(k',j') \in \mathbb{Z}_+^2$*

$$\lim_{n\to\infty} d_n\big(x^{(n)}_{k,j}, x^{(n)}_{k',j'}\big) \text{ exists.}$$

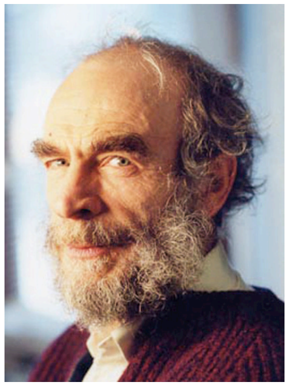
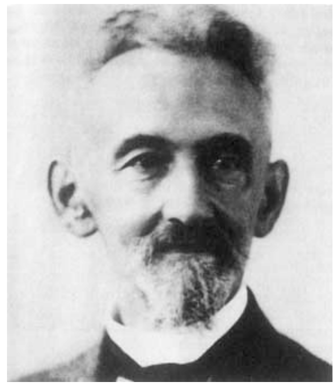
To simplify notation, we suppose that there is no need to take a subsequence. Consider the abstract space $X = \{(k,j)\}_{k,j\geqslant 1}$ endowed with $\Delta((k,j),(k',j')) = \lim d_n\big(x^{(n)}_{k,j}, x^{(n)}_{k',j'}\big)$.

3. *Show that Δ is a pseudo-metric on X and that the completion $\overline{X}$ of X with respect to Δ is compact.*

4. *Prove that $X_n \to \overline{X}$ in the Gromov-Hausdorff sense.*

Exercise 2.4. ** Prove that a sequence of length spaces homeomorphic to the two-dimensional sphere $\mathbb{S}_2$ cannot converge to the standard two-dimensional closed ball $\mathbb{B}_2$.*

Exercise 2.5. *Who are these charming gentlemen ?*

References

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