1 Galton-Watson Trees

Exercise 1.1 (Catalan number)**.** *1. Show that there exists a bijection between the set $\mathcal{B}_n$ of rooted, oriented binary trees with $2n$ edges and the set $\mathbf{A}_n$ of rooted, oriented (general) trees with n edges.*

2. The generating function of $\mathcal{B}_n$ is by definition

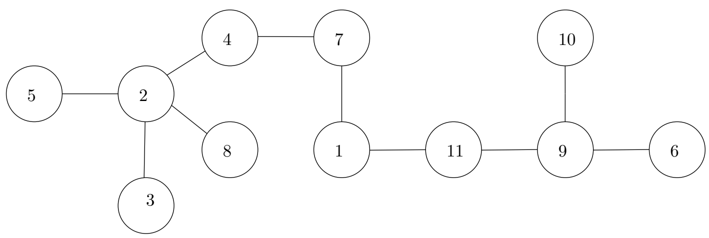
$$B(z) = \sum_{n \geqslant 0} z^{2n} \# \mathcal{B}_n.$$

Show that B satisfy $B(z) = 1 + z^2 B(z)^2$, or equivalently $zB(z) = z(1 + (zB(z))^2)$.

3. Apply Lagrange inversion formula (or solve the equation) to find

$$\# \mathcal{B}_n = \frac{1}{n+1} \binom{2n}{n}.$$

A *Cayley tree* is a labeled tree with n vertices without any orientation nor distinguished point. In other words it is a spanning tree on $\mathbb{K}_n$, the complete graph over n vertices. See figure below.



Exercise 1.2. *Let T be a (rooted, oriented) Galton-Watson tree with Poisson(1) offspring distribution. We denote by $|T|$ the number of edges of T and put*

$$P(z) = \sum_{n \geqslant 0} \mathbb{P}(|T| = n) z^n.$$

1. Show that $P = \mathrm{e}^{-1} \exp(zP)$.

2. Apply Lagrange Inversion Theorem to get

$$\mathbb{P}(|T| = n-1) = \frac{n^{n-1} \mathrm{e}^{-n}}{n!}, \text{ for all } n \geqslant 1.$$

3. Let T_n be a Galton-Watson tree with Poisson(1) offspring distribution conditioned to have n vertices. Then assign the labels $\{1, ..., n\}$ uniformly at random to the vertices of T_n and forget the ordering and the root of T_n. Show that the resulting unordered labeled tree is uniform over the set $\mathcal{C}_n$ of Cayley trees over $\{1, ..., n\}$. Deduce

$$\# \mathcal{C}_n = n^{n-2}.$$

Lagrange Inversion Theorem. Let $\phi(u) = \sum_{k \geqslant 0} \phi_k u^k$ be a power series of $\mathbb{C}[[u]]$ with $\phi_0 \neq 0$. Then, the equation $y = z\phi(y)$ admits a unique solution in $\mathbb{C}[[z]]$ whose coefficients are given by

$$y(z) = \sum_{n=1}^{\infty} y_n z^n, \qquad \text{with } y_n = \frac{1}{n} [u^{n-1}] \phi(u)^n,$$

where $[u^{n-1}]\phi(u)^n$ stands for the coefficient in front of u^{n-1} in $\phi(u)^n$.

Exercise 1.3 (Proof of Lagrange Inversion Theorem)**.** *1. Show that the coefficients of y as a series of z are determined by the coefficients of ϕ and the equation $y = z\phi(y)$ in $\mathbb{C}[[z]]$.*

2. Assume that ϕ is a polynomial function.

   (a) Show that y is analytic around 0.

   (b) Using Cauchy formula, show that

$$n[z^n]y = ny_n = \frac{1}{2i\pi}\int_{\mathcal{C}} \frac{y'(z)dz}{z^n}, \qquad \text{where } \mathcal{C} \text{ is a small anticlockwise contour around } 0.$$

   (c) With a change of variable, prove the Lagrange Inversion Theorem in this case.

3. Deduce the general case.

Exercise 1.4. ** Who are these charming gentlemen ?*

Note that the fourth one already had a cell-phone.

References

[Ald98] David Aldous. Tree-valued Markov chains and Poisson-Galton-Watson distributions. In *Microsurveys in discrete probability (Princeton, NJ, 1997)*, volume 41 of *DIMACS Ser. Discrete Math. Theoret. Comput. Sci.*, pages 1–20. Amer. Math. Soc., Providence, RI, 1998.

[FS09] Philippe Flajolet and Robert Sedgewick. *Analytic combinatorics*. Cambridge University Press, Cambridge, 2009.