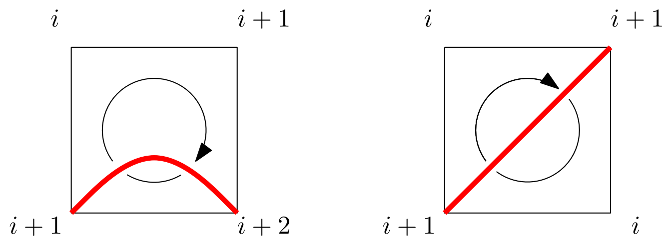
5 Schaeffer's Bijection

Exercise 5.1. *The goal of this exercise is to introduce the reverse construction of the Schaeffer bijection. In the following Q is a finite quadrangulation with n faces with a distinguished oriented edge $\overrightarrow{e}$ with endpoints e_- and e_+. We denote the graph distance in Q by $\mathrm{d_{gr}}$.*

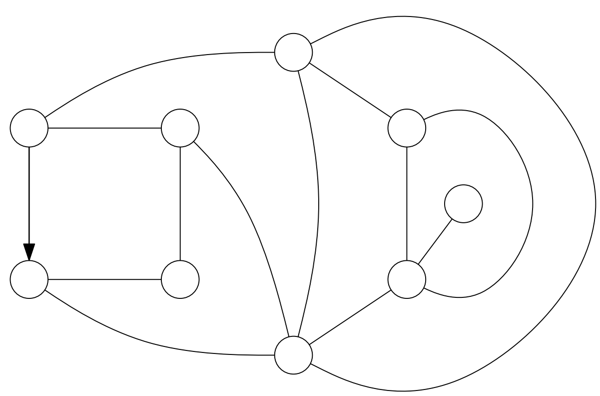
1. *Show that Q is bipartite: if u and v are two neighboring vertices then*

$$\left| \mathrm{d_{gr}}(e_-, u) - \mathrm{d_{gr}}(e_-, v) \right| = 1.$$

Hence the faces of the quadrangulation Q can be decomposed into two subsets: The faces f such that the distance to e_- of the vertices of f are $(i, i+1, i+2, i+1)$ or $(i, i+1, i, i+1)$. We add a "red" edge in each face following the rule given by the figure below.



2. *Apply this construction to this quadrangulation.*[1]



We denote $\mathfrak{T}$ the graph formed by the red edges and the vertices of Q they span. We aim at proving that $\mathfrak{T}$ is a tree that spans $V \backslash \{e_-\}$.

3. *Show that e_- is not a vertex of $\mathfrak{T}$.*

4. *Using Euler's formula, show that it suffices to prove that $\mathfrak{T}$ has no cycle.*

5. *Suppose that we can form a simple cycle $\mathcal{C}$ with the red edges. Pick a vertex u of this cycle whose distance to e_- is minimal among the vertices of the cycle. Show that we can find two vertices v and v' which are separated by $\mathcal{C}$ and such that*

$$\mathrm{d_{gr}}(e_-, v) = \mathrm{d_{gr}}(e_-, v') = \mathrm{d_{gr}}(e_-, u) - 1.$$

6. *Find a contradiction.*

[1] For the connoisseurs this is Le Gall's map.

Exercise 5.2 (Re-rooting). *Let $\mathcal{Q}_n^*$ be the set of all rooted quadrangulations with n faces and an extra oriented edge $\vec{e}$. If $(q, \vec{e}) \in \mathcal{Q}_n^*$ we put $\mathfrak{R}(q, \vec{e})$ for the map q but re-rooted at $\vec{e}$.*

1. *Show that the image of the uniform distribution on $\mathcal{Q}_n^*$ by $\mathfrak{R}$ is the uniform distribution over the set of rooted quadrangulations with n faces.*

Let Q_n be a uniform rooted quadrangulation with n faces. Conditionally on Q_n, let $(X_n)_{n \geqslant 0}$ be a simple random walk on Q_n starting from the end point of the root-edge of Q_n and denote $(\vec{E}_1, \vec{E}_2, ...)$ the oriented edges traversed by the walk.

2. *Prove that for every $k \geqslant 1$, $\mathfrak{R}(Q_n, \vec{E}_k) = Q_n$, in distribution.*

Exercise 5.3. *Who are these charming gentlemen ?*

References

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[CV81] Robert Cori and Bernard Vauquelin. Planar maps are well labeled trees. *Canad. J. Math.*, 33(5):1023–1042, 1981.

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