Yet Another Proof of the Strong Law of Large Numbers

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Abstract. We give a short proof of the strong law of large numbers based on duality for random walks.

Let $X_1, X_2, \dots$ be independent identically distributed real random variables with a finite expectation $\mathbb{E}[X]$ and let $S_n = X_1 + \cdots + X_n$ for $n \geq 0$ be the corresponding random walk. Allegedly one of the most important results in probability theory, Kolmogorov's strong law of large numbers, states that

$$(LLN) \qquad \frac{S_n}{n} \xrightarrow[n\to\infty]{} \mathbb{E}[X] \qquad \text{almost surely (i.e., with probability 1).}$$

There are many proofs of this result, the most standard approaches being either, by the truncation method (see [4, p. 238] or [3]), by Doob's reverse martingale convergence theorem [2], or directly via $0/1$ laws and Birkhoff's ergodic theorem (especially using Garsia's short proof, see [5, Theorem 1.64] or [1]). In this note we propose a new proof based on duality for random walks. First notice that the (LLN) is a consequence of the following result:

Theorem 1. *Let $X_1, X_2, \dots$ be independent identically distributed real random variables with a finite and positive expectation $\mathbb{E}[X] > 0$. Then $\inf_{n\geq 0}(X_1 + \cdots + X_n)$ is finite almost surely.*

Indeed, the theorem implies that for every $\varepsilon > 0$, almost surely $(X_1 - \mathbb{E}[X] + \varepsilon) + \cdots + (X_n - \mathbb{E}[X] + \varepsilon)$ is bounded from below by some (random) constant and, symmetrically, that $(X_1 - \mathbb{E}[X] - \varepsilon) + \cdots + (X_n - \mathbb{E}[X] - \varepsilon)$ is bounded from above by another (random) constant. It follows that with probability one we have

$$\mathbb{E}[X] - \varepsilon \leq \liminf_{n\to\infty} \frac{S_n}{n} \leq \limsup_{n\to\infty} \frac{S_n}{n} \leq \mathbb{E}[X] + \varepsilon.$$

Taking the countable intersection of the above events over rational $\varepsilon > 0$ we deduce the (LLN).

Proof of Theorem 1. Step 1. Bounding the increments from above. Choose $C > 0$ large enough so that by dominated convergence $\mathbb{E}[X\mathbf{1}_{X<C}] > 0$ and put

$$\tilde{X}_i = X_i \mathbf{1}_{X_i<C}, \qquad \text{for } i \geq 0.$$

We will show that the random walk $\tilde{S}_n = \tilde{X}_1 + \cdots + \tilde{X}_n$ is almost surely bounded from below which is sufficient to prove the lemma since $\tilde{S}_i \leq S_i$ for all $i \geq 0$.

doi.org/10.1080/00029890.2022.2115820
MSC: 60-00

Step 2. Duality. For every $n \geq 0$, the independent and identically distributed increments $(\tilde{X}_1, \ldots, \tilde{X}_n)$ have the same law as $(\tilde{X}_n, \ldots, \tilde{X}_1)$. This translates onto the walk $\tilde{S}$ as the so-called *duality* identity [**4**, Chapter XII. 2, p. 394] which states that space and time reversal leaves the distribution of the first n steps invariant:

$$(0 = \tilde{S}_0, \tilde{S}_1, \ldots, \tilde{S}_n)=(\tilde{S}_n - \tilde{S}_n, \tilde{S}_n - \tilde{S}_{n-1}, \ldots, \tilde{S}_n - \tilde{S}_1, \tilde{S}_n - \tilde{S}_0) \quad \text{in law.}$$

Let us apply this identity on a special case, see [**4**, p. 395]. Let $T = \inf\{i \geq 0 : \tilde{S}_i > 0\}$ be the first hitting time of the positive axis by the walk and recall that a time $n \geq 0$ is a weak descending record time if, by definition, $\tilde{S}_n = \min_{0 \leq k \leq n} \tilde{S}_k$. By applying the above equality in law we deduce (see Figure 1) that

$$\text{for all } n \geq 0, \quad \mathbb{P}(T > n) = \mathbb{P}(n \text{ is a weak descending record time}).$$

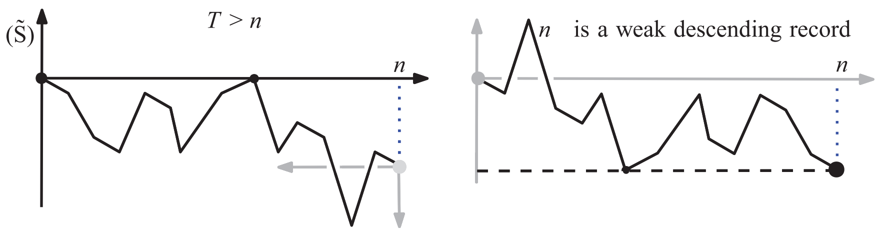

Figure 1. Time and space reversal shows that $\mathbb{P}(T > n) = \mathbb{P}(n \text{ is a descending record time})$.

Summing over $n \geq 0$, we get that $\mathbb{E}[T] = \mathbb{E}[\#\text{ weak descending record times}]$ and the proof is complete if we prove that $\mathbb{E}[T] < \infty$ since this implies that almost surely there are finitely many weak descending records for $\tilde{S}$, hence the walk is bounded from below almost surely.

Step 3. Optional sampling theorem. To prove $\mathbb{E}[T] < \infty$, consider the standard martingale

$$M_n = \tilde{S}_n - \mathbb{E}[X\mathbf{1}_{X<C}]n, \quad \text{for } n \geq 0$$

(for the filtration generated by the X_i's) and apply the optional sampling theorem (see e.g., [**4**, p. 213]) to the bounded stopping time $n \wedge T$ to deduce that

$$0 = \mathbb{E}[M_{n\wedge T}], \quad \text{or, in other words,} \quad \mathbb{E}[X\mathbf{1}_{X<C}]\mathbb{E}[n \wedge T] = \mathbb{E}[\tilde{S}_{n\wedge T}].$$

By definition $\tilde{S}_i \leq 0$ for all $i < T$ and if T is finite we can write $\tilde{S}_T = \tilde{S}_{T-1} + \tilde{X}_T \leq 0 + \tilde{X}_T \leq C$. In all cases $\tilde{S}_{n\wedge T} \leq C$ and so the right-hand side of the last display is also bounded above by C. Letting $n \to \infty$, by monotone convergence we deduce that T has finite expectation. ■

ACKNOWLEDGMENT. We thank Yuval Peres and the probability team at Orsay for helpful feedback.

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