Appendix to “Irreducibility, Brill–Noether’ loci, and Vojta’s inequality” by Thomas J. Tucker

On a curve $C$ with no $g^1_d$ such that $W_d(C)$ contains finitely many elliptic curves

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The aim of this appendix is to complement a construction from [DF] (and to correct an error in the proof of prop. 5.7 of this article). The setup is the following: let $E$ be a complex elliptic curve and let $S$ be its second symmetric product, with $s : S \to E$ the sum map. To avoid confusion between addition of divisors and addition of points on $E$, we write $(x)$ for the divisor associated with a point $x$ of $E$. We define two divisors on $S$ by setting $F = s^{-1}(o)$ and $H = \{(o) + (x) \mid x \in E\}$. For any divisor $D$ on $S$ and any $x \in E$, we set $D_x = D + s^*((x) - (o))$. We denote $\sim$ numerical equivalence for divisors.

Fix an integer $d \geq 4$ and a point $x_0$ on $E$, and set $L = \mathcal{O}_S((d+1)H - F_{x_0})$. A general curve $C$ in $|L|$ is smooth of genus $(d^2)/2 + 1 (|DF|).

1. On the Kawamata locus of $W_d(C)$

Our aim is to prove that the Kawamata locus of $W_d(C)$, i.e., the union of all (translated) non-zero abelian varieties in $W_d(C)$, is 1-dimensional. We will analyze pencils of low degree on $C$, using as in [DF] the following result of Reider ([R]).

**Theorem (I. Reider).**— Let $L$ be a nef line bundle on a smooth projective surface $S$ and let $C$ be a smooth curve in $|L|$. Let $A$ be a base-point-free $g^1_\delta$ on $C$ such that $\delta < L^2/4$. There exists a divisor $D$ on $S$ such that:
- a) $h^0(S, D) \geq 2$;
- b) $h^0(C, D - A) > 0$;
- c) $C \cdot D < 2\delta$;
- d) $(C - D) \cdot D \leq \delta$.

For any divisor $D$ on $S$, we denote by $\overline{D}$ its restriction to $C$.

**Corollary.** — Let $A$ be a base-point-free $g^1_\delta$ on $C$ with $\delta \leq 2d$ and $\delta \leq 3d - 10$. One of the following possibilities occur:
- a) there exist $x \in E$ and an effective divisor $B$ on $C$ such that $A + B \equiv 2H_x$, and $\delta \geq 2d - 4$;
- b) there exist $x \in E$ and an effective divisor $B$ on $C$ such that $A + B \equiv 3H_x - F$, and $\delta \geq 2d - 4$;
- c) one has $A = 4H - 2F$ and $\delta = 2d - 2$. 
Proof. A case-by-case inspection shows that the divisor $D$ in Reider’s theorem satisfies one of the following: $D \sim 2H$ and $\delta \geq 2d - 4$, $D \sim 3H - F$ and $\delta \geq 2d - 4$, or $D \equiv 4H - 2F$ and $\delta \geq 2d - 2$. In the first (resp. second) (resp. last) case, $|D|$ induces a base-point-free $g^2_{2d}$ (resp. a $g^1_{2d-1}$ with at most 3 distinct base points by [DF], prop. 4.2) (resp. a base-point-free $g^1_{2d-2}$). Therefore, we are in case a) (resp. b)) (resp. c)).

As noted in [DF], the variety $W_d(C)$ contains a translate $E_0$ of $E$, to wit the image of the morphism $\psi : x \mapsto \overline{H}_x$.

Theorem.— Assume $C$ is general in $|L|$ and $d \geq 4$. The Kawamata locus of $W_d(C)$ is 1-dimensional. More precisely,

a) $W_{d-1}(C)$ contains no non-zero abelian varieties;

b) the only non-zero abelian varieties contained in $W_d(C)$ are translates of $E_0$ by torsion points;

c) for $d \geq 9$, the only non-zero abelian variety contained in $W_d(C)$ is $E_0$.

Proof. Let $A$ be a non-zero (translated) abelian variety in $W_\delta(C)$, with $\delta \leq d$. By a theorem of Mori (see proof of prop. 5.4 of [DF]), $JC/s^*JE$ is simple hence $A$ maps to a point in this quotient. It follows that $A$ is a translate of $E$. Take $\delta$ minimal, so that a linear system corresponding to a general point $a$ of $A$ has only one element, which we will denote by $D_a$, and so that the divisors $D_a$ have no common point. Since $E_0$ does not come from a morphism, one shows as in the proof of Lemma 5 of [AH] that $E_0 + A \subset W^2_{d+\delta}(C)$.

Assume $d \geq 9$ and let $y \in E, a \in A$ and $p \in C$; the corollary of Reider’s theorem applied to $\overline{H}_y + D_a - p$ yields that

a) either there exist $x \in E$ and an effective divisor $B$ on $C$ such that, for all $e \in E$, one has $\overline{H}_{y-e} + D_{a+e} - p + B \equiv 2\overline{H}_x$;

b) or there exist $x \in E$ and an effective divisor $B$ on $C$ such that, for all $e \in E$, one has $\overline{H}_{y-e} + D_{a+e} - p + B \equiv 3\overline{H}_x - \overline{F}$;

c) or $\delta = d - 1$ and, for all $e \in E$, one has $\overline{H}_{y-e} + D_{a+e} - p \equiv 4\overline{H} - 2\overline{F}$.

Case c) cannot occur because $p$ is fixed in $|4\overline{H} - 2\overline{F} + p|$ by Riemann–Roch. For the same reason $B' = B - p$ is effective in cases a) and b). In case b), we get $D_{a+e} + B' \equiv 2\overline{H}_{x-y+e} - \overline{F}$. But this is impossible since $H^0(C, 2\overline{H}_{x-a+e} - 2\overline{F}) = 0$. Therefore, we are in case a), and $\overline{H}_{x-y+e} = D_{a+e} + B'$. When $e$ varies, the left-hand side varies; the $\overline{H}_{x-y+e}$’s having no fixed point, we must have $B' = 0$ and $\overline{H}_{x-y+e} = D_{a+e}$. It follows that $A = E_0$.

Assume now $d \geq 4$, and set

$$\Gamma = \{ \alpha \in JC \mid E_0 + \alpha \subset W_d(C) \}.$$ 

We will use theorem 2 of [AH]; one should however be careful: first one needs to add to the hypotheses of this theorem that the embedding $A \subset W_d(C)$ does not come from a morphism (this is true in our case by prop. 5.14 of [DF]). Second, the proof of the theorem given in [AH] is incomplete: the proof of lemma 6 on which it relies is wrong when $\dim(A) > 1$ (this fortunately does not concern us) and the case $r_2 = 3$ and $\dim(A) = 1$ needs a separate treatment (which was provided by Abramovich in a private communication). The
conclusion of the theorem is then \( g(C) \leq \binom{\delta}{2} + 1 \), which implies for example \( \delta = d \). However, we get more from the proof, to wit that if there is equality, then \( r_k = \binom{k+1}{2} - 1 \) for \( 2 \leq k \leq d \). The same reasoning shows in our case that the same holds for the sum of \( k \) generic elements of \( E_0 + \alpha_1, \ldots, E_0 + \alpha_k \), where \( \alpha_1, \ldots, \alpha_k \in \Gamma \). In particular, for \( \alpha_1 \) generic in \( E_0 + \alpha_1 \), for \( \alpha_2 \) generic in \( E_0 + \alpha_2 \) and for \( x \) generic in \( E \), we have

\[
h^0(C, (k - 2)H + D_a + D_a) = \binom{k + 1}{2}
\]

for \( 2 \leq k \leq d \). Since \( K_C = (d - 2)H + H_{\alpha_1} \), Riemann–Roch implies

\[
h^0(C, 2H - D_{\alpha_1}) = 1 \quad \text{and} \quad h^0(C, 3H - D_{\alpha_1} - D_{\alpha_2}) = 1,
\]

i.e., the curves \( E_0 - \alpha_1 \) and \( E_0 - \alpha_1 - \alpha_2 \) are contained in \( W_d(C) \). This proves that \( \Gamma \) is a closed (proper) subgroup of \( JC \) hence is a translate of \( E \) by a finite group. \( \blacksquare \)

2. Erratum for [DF]

The following proposition corrects the part of the proof of prop. 5.7 of [DF] which is incomplete (to wit the case \( d = 4 \)).

**Proposition.**— Assume \( d = 4 \) and \( C \) general in \( |L| \). Then \( C \) has no \( g_1 \).

**Proof.** Let \( A \) be a base-point-free \( g_1 \) on \( C \), with \( \delta \leq 4 \). Following [DF] (5.12), one constructs a rank 2 vector bundle \( T \) on \( S \) that fits into an exact sequence

\[
(*) \quad 0 \to H^0(A)^* \otimes \mathcal{O}_S \to T \to \mathcal{O}_C(C - A) \to 0 .
\]

Note that \( H^2(T) = 0 \) and \( \chi(T) = 10 - \delta \) by Riemann–Roch. If \( h = h^0(T) > 6 \), the kernel of the map \( \wedge^2H^0(T) \to H^0(\wedge^2T) \simeq H^0(S, C) \simeq \mathbb{C}^10 \) meets the \((2h - 3)\)-dimensional set of decomposable vectors off the origin. One proceeds as in [DF] (where the numbers at the top of page 246 are all wrong) to show that there exists a divisor \( D \) on \( S \) that fits into an exact sequence

\[
0 \to \mathcal{O}_S(D) \to T \to \mathcal{I}_Z(C - D) \to 0,
\]

where \( Z \) is a finite subscheme of \( S \); moreover, either \( D \sim 2H \) or \( D \sim 3H - F \). Then, \( h^0(T) \leq h^0(D) + h^0(C - D) = 5 \), which is a contradiction (this remark avoids the lengthy proof in [DF]).

It follows that \( h^0(T) = 6 \), \( h^1(T) = 0 \) and \( \delta = 4 \). Assume first that there is a non-zero morphism \( u : T \to T \otimes \omega_S \). We argue as in [L]; since \( H^0(\omega_S^2) = 0 \), the morphism \( \wedge^2u \) vanishes hence \( u \) drops rank everywhere. Then \( N = (\text{Im } u)^* \) is a line bundle on \( S \) which is a subsheaf of \( T \otimes \omega_S \); there is a morphism \( T \to N \) which is surjective off a finite subset of \( S \). Note that by Riemann–Roch, one has \( h^0(\mathcal{O}_C(C - A)) \geq 5 > 2 = h^1(H^0(A)^* \otimes \mathcal{O}_S)^* \), hence the exact sequence \((*)\) shows that \( T \) is generated by global sections off a finite subset of \( S \), hence so is \( N \). It follows that either \( h^0(S, N) \geq 2 \), or \( N \not\simeq \mathcal{O}_S \); but the latter cannot occur since \( \text{Hom}(T, \mathcal{O}_S) = 0 \). Tensoring \((*)\) by \( \omega_S \otimes N^* \), we see that \( H^0(T \otimes \omega_S \otimes N^*) \neq 0 \) implies

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$H^0(\omega_C \otimes N^* \otimes \mathcal{O}_C(-A)) \neq 0$ and in particular $(5H - F) \cdot (3H - N) \geq 4$. Furthermore, there is an exact sequence

$$0 \to N \to T \otimes \omega_S \to \mathcal{I}_Z(\omega_S^{\otimes 2} \otimes N^*(C)) \to 0,$$

which implies $N \cdot (H + F - N) \leq c_2(T \otimes \omega_S) = 0$. A case-by-case analysis shows that the only possibility is $N \sim 2H$; but then $H^0(\omega_C \otimes N^*(-A)) \neq 0$ implies $A \equiv \mathcal{P}_x$, which is not a pencil.

Hence $\text{Hom}(T, T \otimes \omega_S)$ vanishes, and so does $H^2(\text{End} \ T)$ by duality. Dualizing (\star) yields

$$0 \to T^* \to H^0(A) \otimes \mathcal{O}_S \to \mathcal{O}_C(A) \to 0.$$ 

Tensoring by $T$, we get $H^1(T \otimes A) = 0$. We now follow another construction of [L], where a moduli space $P$ is constructed which parametrizes triples $(C, A, l)$, where $C$ is a smooth curve in $[L]$, $A$ is a base-point-free $g^1_1$ on $C$, and $l$ is a surjective morphism $H \otimes \mathcal{O}_S \to A$ which induces an isomorphism on global sections, two such morphisms being identified if they differ by multiplication by a non-zero scalar. Let $\pi : P \to [L]$ be the forgetful morphism. The tangent space to $P$ at $(C, A, l)$ is identified with the kernel $\tilde{H}^0(T \otimes A)$ of the map $H^0(T \otimes A) \to H^1(\text{End} \ T) \xrightarrow{\text{Tr}} H^1(\mathcal{O}_S)$; the tangent space to $[L]$ at $C$ is identified with the kernel $\tilde{H}^0(C, L)$ of the map $H^0(C, L) \to H^1(\mathcal{O}_S)$.

There is an exact sequence ([L], page 304)

$$\tilde{H}^0(T \otimes A) \xrightarrow{T(\cdot, A, l)\pi} \tilde{H}^0(C, L) \longrightarrow (\text{Ker } \mu)^* \longrightarrow \tilde{H}^1(T \otimes A)$$

where $\mu : H^0(A) \otimes H^0(\omega_C \otimes A^*) \to H^0(\omega_C)$ is the Petri map. By the base-point-free pencil trick, its kernel is isomorphic to $H^0(\omega_C \otimes (A^{\otimes 2})^*)$, which has by Riemann–Roch dimension at least $h^0(A^{\otimes 2}) - 2 > 0$. Since $H^1(T \otimes A)$ vanishes, $T(\cdot, A, l)$ is not surjective, hence neither is $\pi$ by generic smoothness. This shows that there is no $g^1_1$ on a generic $C$ in $[L]$, and finishes the proof of the proposition. 

REFERENCES


