HILBERT SQUARES OF K3 SURFACES AND DEBARRE–VOISIN VARIETIES

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Abstract. The Debarre–Voisin hyperkähler fourfolds are built from alternating 3-forms on a 10-dimensional complex vector space, which we call trivectors. They are analogous to the Beauville–Donagi fourfolds associated with cubic fourfolds. In this article, we study several trivectors whose associated Debarre–Voisin variety is degenerate, in the sense that it is either reducible or has excessive dimension. We show that the Debarre–Voisin varieties specialize, along general 1-parameter degenerations to these trivectors, to varieties isomorphic or birationally isomorphic to the Hilbert square of a K3 surface.

1. Introduction

Throughout this article, the notation $U_m$, $V_m$, or $W_m$ means an $m$-dimensional complex vector space. Let $\sigma \in \bigwedge^3 V_10^\vee$ be a nonzero alternating 3-form (which we call a trivector). The Debarre–Voisin variety associated with $\sigma$ is the scheme

$$ K_\sigma := \{ [W_6] \in \text{Gr}(6, V_{10}) \mid \sigma|_{W_6} = 0 \} $$

whose points are the 6-dimensional vector subspaces of $V_{10}$ which are totally isotropic for $\sigma$. If $\epsilon_6$ is the dual of the rank-6 tautological subbundle on $\text{Gr}(6, V_{10})$, the scheme $K_\sigma$ is the zero-locus of the section of the rank-20 vector bundle $\bigwedge^3 \epsilon_6$ that corresponds to $\sigma$.

We also consider the Plücker hyperplane section

$$ X_\sigma := \{ [U_3] \in \text{Gr}(3, V_{10}) \mid \sigma|_{U_3} = 0 \}. $$

It was proved in [DV] that when $\sigma$ is general, $K_\sigma$ is a smooth hyperkähler fourfold which is deformation equivalent to the Hilbert square of a K3 surface (one says that $K_\sigma$ is of K3[2]-type). This result was reproved in [KLSV] by a different argument but both articles use a specialization of $\sigma$ to a general element of the discriminant hypersurface in $\bigwedge^3 V_{10}^\vee$, where $X_\sigma$ acquires a node and $K_\sigma$ becomes singular along a surface.

The situation is very similar to the case of hypersurfaces $X_f \subseteq \mathbb{P}(V_6)$ defined by cubic polynomials $f \in \text{Sym}^3 V_6^\vee$. When $f$ is general, the variety

$$ F_f = \{ [W_2] \in \text{Gr}(2, V_6) \mid f|_{W_2} = 0 \} $$

of lines contained in $X_f$ was shown in [BD] to be a hyperkähler fourfold of K3[2]-type. When the cubic fourfold $X_f$ degenerates to a nodal cubic $X_{f_0}$, the variety $F_f$ specializes to the singular fourfold $F_{f_0}$, which is birationally isomorphic to the symmetric product of a K3 surface (CG).

Beauville and Donagi however do not use the nodal specialization in order to prove that $F_f$ is of K3[2]-type. They show that for $X_f$ in a divisor in the space of cubic fourfolds, the hyperkähler fourfold $F_f$ is actually isomorphic to the Hilbert square of a K3 surface. This divisor (called the Pfaffian divisor) consists of those (smooth) cubic fourfolds that are obtained by intersecting

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the cubic in $\mathbf{P}(\Lambda^3 W_6)$ defined by the Pfaffian equation $\omega^{\wedge 3} = 0$ with a general linear subspace $\mathbf{P}(V_0) \subset \mathbf{P}(\Lambda^3 W_6)$.

Another remarkable divisor, first introduced by Hassett in [H] and studied by Looijenga in [LO] and Laza in [L], is obtained by blowing up, in the parameter space of all cubics, the closed semistable orbit corresponding to chordal cubics $X_{f_0} \subset \mathbf{P}(V_0)$ ([L] Section 4.1.1]). Along a general 1-parameter deformation $(f_t)_{t \in \Delta}$ of $f_0$, the limit of the varieties $F_t$ is (after a base change of degree 2) birationally isomorphic to the Hilbert square of a degree-2 K3 surface; it is also a 4-to-1 cover of a nonreduced component of $F_{f_0}$ (see [vD] Section 3.2).

We find in this article several divisors in blow ups of the moduli space of Debarre–Voisin varieties which have properties analogous to the Hassett divisor. These divisors are obtained in the following way: while general trivectors in $\mathbf{P}(\Lambda^3 V_{10}^\vee)$ have finite stabilizers in $\text{SL}(V_{10})$, we consider instead special trivectors $\sigma_0$ with positive-dimensional stabilizers and we blow up their $\text{SL}(V_{10})$-orbits in $\mathbf{P}(\Lambda^3 V_{10}^\vee)$. The stabilizers along the exceptional divisors of the resulting blown up space for the induced $\text{SL}(V_{10})$-action are generically finite, thus producing divisors in the quotient (this is a Kirwan blow up).

We describe the corresponding Debarre–Voisin varieties $K_{\sigma_0}$. In the simplest cases (Sections 1.1.2 and 1.1.4), they are still smooth but of dimension greater than 4; there is an excess vector bundle $\mathcal{F}$ of rank $\dim(K_{\sigma_0}) - 4$ on $K_{\sigma_0}$ that measures the defect of transversality of the section $\sigma_0$ of $\wedge^3 \mathcal{E}_6$, and the limit of the Debarre–Voisin varieties $K_{\sigma_t}$ under a general 1-parameter degeneration $(\sigma_t)_{t \in \Delta}$ to $\sigma_0$ is the zero-locus of a general section of $\mathcal{F}$. In one other case (Section 1.1.1), the variety $K_{\sigma_0}$ is reducible of dimension 4 and the limit of the Debarre–Voisin varieties $K_{\sigma_t}$ is birationally isomorphic to the Hilbert square of a degree-2 K3 surface; it is also a 4-to-1 cover of a nonreduced component of $K_{\sigma_0}$ (very much like what happens for chordal cubics $X_{f_0}$ as described above, and $K_{\sigma_0}$ and $F_{f_0}$ are actually isomorphic! See Remark 7.10).

There is a relationship between these constructions and K3 surfaces; we actually discovered some of these special trivectors and their stabilizers starting from K3 surfaces. As explained in Section 3 few Hilbert squares of general polarized K3 surfaces can appear as limits of Debarre–Voisin varieties for polarizations of small genera $g$: only 2, 4, 6, 10, 12, and 16 occur for $g \leq 27$. We can then use Mukai’s models which exist for all these genera ([Mu2], [Mu3], [Mu4]). General K3 surfaces of genus 12 already appeared in [DV] Section 4) but they are not part of the present analysis, since the corresponding trivectors have finite stabilizers. We now list the geometric situations encountered in the remaining cases.

1.1. Stabilizers and K3 surfaces. We list here the various special trivectors $[\sigma_0] \in \mathbf{P}(\wedge^3 V_{10}^\vee)$ that we consider, their (positive-dimensional) stabilizers for the $\text{SL}(V_{10})$-action, and the corresponding limits of Debarre–Voisin varieties (which are all isomorphic to Hilbert squares of K3 surfaces with suitable polarizations) along general 1-parameter degenerations to $\sigma_0$. In most cases, the associated Plücker hypersurface $X_{\sigma_0}$ is singular and the singular locus of $X_{\sigma_0}$ gives rise to a component of $K_{\sigma_0}$, as explained in Proposition 4.4(b).

1.1.1. The group $\text{SL}(3)$ and K3 surfaces of genus 2 (degree 2) (Section 7). A general genus-2 K3 surface $(S, L)$ is a double cover of $\mathbf{P}^2$ branched along a smooth sextic curve. The Hilbert square $S[2]$ is birationally isomorphic to a moduli space $\mathcal{M}_S(0, L, 1)$ of sheaves on $S$ defined in Remark 3.2.

We take $V_{10} := \text{Sym}^3 W_3$, so that $\wedge^3 V_{10}^\vee$ is an $\text{SL}(W_3)$-representation, and we let $\sigma_0 \in \wedge^3 V_{10}^\vee$ be a generator of the 1-dimensional space of $\text{SL}(W_3)$-covariants.

The Debarre–Voisin variety $K_{\sigma_0}$ is described in Proposition 7.8; it has two 4-dimensional irreducible components $K_L$ and $K_M$ and is nonreduced along $K_L$. The Plücker hypersurface $X_{\sigma_0}$
is singular along a surface (Proposition 7.2) and the component $K_L$ of $K_{σ_0}$ is obtained from this surface by the procedure described in Proposition 4.4(b) (see Proposition 7.7(a)).

Our main result is the following (Theorem 7.20).

**Theorem 1.1.** Under a general 1-parameter deformation $(σ_t)_{t ∈ Δ}$, the Debarre–Voisin fourfolds $K_{σ_t}$ specialize, after a finite base change, to a scheme which is isomorphic to $M_{S}(0, L, 1)$, where $S$ is a general K3 surface of genus 2.

This case is the most difficult: as explained above, the limit fourfold $M_{S}(0, L, 1)$ does not sit naturally in the Grassmannian $Gr(6, V_{10})$ but maps 4-to-1 to it.

The limit on $M_{S}(0, L, 1)$ of the Plücker line bundles on $K_{σ_t}$ is the ample line bundle of square 22 and divisibility 2 described in Table 1. We show that it is globally generated for a general genus-2 $K^3$ surface $S$, but not very ample (Remark 3.2).

1.1.2. The group $Sp(4)$ and $K^3$ surfaces of genus 4 (degree 6) (Section 5.1). Let $V_4$ be a 4-dimensional vector space equipped with a nondegenerate skew-symmetric form $ω$. The hyperplane $V_5 ⊆ \Lambda^2 V_4$ defined by $ω$ is endowed with the nondegenerate quadratic form $q$ defined by wedge product and $SO(V_5, q) \cong Sp(V_4, ω)$. The form $q$ defines a smooth quadric $Q_3 ⊆ P(V_5)$ and general genus-4 $K^3$ surfaces are complete intersections of $Q_3$ and a cubic in $P(V_5)$.

There is a natural trivector $σ_0$ on the vector space $V_{10} := \Lambda^2 V_5$: view elements of $V_{10}$ as endomorphisms of $V_5$ which are skew-symmetric with respect to $q$ and define

$$σ_0(a, b, c) = Tr(a ∘ b ∘ c).$$

The associated Debarre–Voisin variety $K_{σ_0} ⊆ Gr(6, V_{10})$ was described by Hivert in [HI]: it is isomorphic to $Q_3^{[2]}$. In fact, the Plücker hypersurface $X_{σ_0}$ is singular along a copy of $Q_3$ (Lemma 5.1) and the whole of $K_{σ_0}$ is obtained from $Q_3$ by the procedure described in Proposition 4.4(b) (see Theorem 5.2).

The excess bundle analysis shows the following (Theorem 5.5).

**Theorem 1.2.** Under a general 1-parameter deformation $(σ_t)_{t ∈ Δ}$, the Debarre–Voisin fourfolds $K_{σ_t}$ specialize to a smooth subscheme of $K_{σ_0} \cong Q_3^{[2]}$ which is isomorphic to $S^{[2]}$, where $S ⊆ Q_3$ is a general genus-4 $K^3$ surface.

The restriction of the Plücker line bundle to $S^{[2]} ⊆ Q_3^{[2]} \cong K_{σ_0} ⊆ Gr(6, V_{10})$ is the ample line bundle of square 22 and divisibility 2 (see Section 2.1 for the definition of divisibility) described in Table 1. It is therefore very ample for a general genus-4 $K^3$ surface $S$.

1.1.3. The group $SL(2)$ and $K^3$ surfaces of genus 6 (degree 10) (Section 6). The variety $X ⊆ Gr(2, V_3^*) \subseteq P(Λ^2 V_3^*)$ defined by a general 3-dimensional space $W_3 \subseteq Λ^2 V_5$ of linear Plücker equations is a degree-5 Fano threefold. General genus-6 $K^3$ surfaces are quadratic sections of $X$ ([Mu2]).

The spaces $V_5$ and $W_3$ and the variety $X$ carry $SL(2)$-actions and there is an $SL(2)$-invariant decomposition $V_{10} := \Lambda^2 V_5 = V_7 ⊕ W_3$. Among the $SL(2)$-invariant trivectors, there is a natural one, $σ_0$, defined in Proposition 6.3 and the neutral component of its stabilizer is $SL(2)$.

The Debarre–Voisin $K_{σ_0}$ has one component $K_1$ which is generically smooth and birationally isomorphic to $X^{[2]}$. In fact, the Plücker hypersurface $X_{σ_0}$ is singular along a copy of the threefold $X$ and $K_1$ is obtained from $X$ by the procedure described in Proposition 4.4(b) (see Proposition 6.5).

We obtain the following (Proposition 6.8 and Theorem 6.14).
Theorem 1.3. Under a general 1-parameter deformation $(\sigma_t)_{t \in \Delta}$, the Debarre–Voisin fourfolds $K_{\sigma_t}$ specialize, after finite base change, to a smooth subscheme of $K_{\sigma_0}$ which is isomorphic to $S^{[2]}$, where $S \subseteq X$ is a general K3 surface of genus 6.

The limit on $S^{[2]}$ of the Plücker line bundles on $K_{\sigma_t}$ is the ample line bundle of square 22 and divisibility 2 described in Table 1. We show that it is not globally generated.

1.1.4. The group $G_2 \times \text{SL}(3)$ and K3 surfaces of genus 10 (degree 18) (Section 5.2). The group $G_2$ is the subgroup of $\text{GL}(V_7)$ leaving a general 3-form $\alpha$ invariant. There is a $G_2$-invariant Fano 5-fold $X \subseteq \text{Gr}(2, V_7)$ which has index 3 and general genus-10 K3 surfaces are obtained by intersecting $X$ with a general 3-dimensional space $W_3 \subseteq \bigwedge^3 V_7'$ of linear Plücker equations (Mu2).

The vector space $V_{10} := V_7 \oplus W_3$ is acted on diagonally by the group $G_2 \times \text{SL}(W_3)$ and we consider $G_2 \times \text{SL}(W_3)$-invariant trivectors $\sigma_0 = \alpha + \beta$, where $\beta$ spans $\bigwedge^3 W_3'$. The corresponding points $[\sigma_0]$ of $\mathbf{P}(\bigwedge^3 V_{10}')$ are all in the same $\text{SL}(V_{10})$-orbit and the corresponding Debarre–Voisin variety $K_{\sigma_0}$ splits as a product of a smooth variety of dimension 8 and of $\mathbf{P}(W_3')$ (Corollary 5.12).

The excess bundle analysis shows the following (Theorem 5.15).

Theorem 1.4. Under a general 1-parameter deformation $(\sigma_t)_{t \in \Delta}$, the Debarre–Voisin fourfolds $K_{\sigma_t}$ specialize to a smooth subscheme of $K_{\sigma_0}$ isomorphic to $S^{[2]}$, where $S \subseteq X$ is a general genus-10 K3 surface.

The limit on $S^{[2]}$ of the Plücker line bundles on $K_{\sigma_t}$ is the ample line bundle of square 22 and divisibility 2 described in Table 1. It is therefore very ample for a general genus-10 K3 surface $S$ (Lemma 5.10).

1.1.5. K3 surfaces of genus 16 (degree 30) (Section 8). This is the last case allowed by the numerical conditions of Section 3.3 where a projective model of a general K3 surface $S$ is known. It corresponds to the last column of Table 1. Unfortunately, the current geometric knowledge for those K3 surfaces (see Mu3) is much weaker than in the previous cases and we were not able to find a trivector vanishing on the Hilbert square $S^{[2]}$. Nevertheless, in Section 8.1 we construct on $S^{[2]}$ a rank-4 vector bundle with the same numerical properties as the restriction of the tautological quotient bundle of $\text{Gr}(6, 10)$ to a Debarre–Voisin variety. In particular, this gives a rational map from $S^{[2]}$ to $\text{Gr}(6, 10)$ with the expected polarization.

2. MODULI SPACES AND PERIOD MAP

2.1. Polarized hyperkähler fourfolds of degree 22 and divisibility 2 and their period map. Let $X$ be a hyperkähler fourfold of $K3^{[2]}$-type. The abelian group $H^2(X, \mathbb{Z})$ is free abelian of rank 23 and it carries a nondegenerate integral-valued quadratic form $q_X$ (the Beauville–Bogomolov–Fujiki form) that satisfies

$$\forall \alpha \in H^2(X, \mathbb{Z}) \quad \int_X \alpha^4 = 3q_X(\alpha)^2.$$ 

The lattice $(H^2(X, \mathbb{Z}), q_X)$ is isomorphic to the lattice

$$(\Lambda, q_\Lambda) := U^{\oplus 3} \oplus E_8(-1)^{\oplus 2} \oplus I_1(-2),$$

where $U$ is the hyperbolic plane, $E_8$ the unique positive definite even rank-8 unimodular lattice, and $I_1(-2)$ the rank-1 lattice whose generators have square $-2$. 
The divisibility \( \text{div}(\alpha) \) of a nonzero element \( \alpha \) of a lattice \((L,q_L)\) is the positive generator of the subgroup \( q_L(\alpha,L) \) of \( \mathbb{Z} \). There is a unique \( O(\Lambda) \)-orbit of primitive elements \( h \in \Lambda \) such that \( q_\Lambda(h) = 22 \) and \( \text{div}(h) = 2 \) and we fix one of these elements \( h \).

We consider pairs \((X,H)\), where \( X \) be a hyperkähler fourfold of \( K^3[2]\)-type and \( H \) is an ample line bundle on \( X \) such that \( q_X(H) = 22 \) and \( \text{div}(H) = 2 \). It follows from Viehweg’s work ([Vi]) that there is a quasi-projective 20-dimensional coarse moduli space \( M \) for these pairs and Apostolov proved in [A] that \( M \) is irreducible.

The domain \( D(h^\perp) := \{ [\alpha] \in P(\Lambda \otimes \mathbb{C}) | q_\Lambda(\alpha,\alpha) = q_\Lambda(\alpha,h) = 0, \ q_\Lambda(\alpha,\ov{\alpha}) > 0 \} \) has two connected components, both isomorphic to the 20-dimensional bounded symmetric domain of type IV associated with the lattice \( h^\perp \subseteq \Lambda \). It is acted on properly and discontinuously by the isometry group \( O(h^\perp) \) and the quotient \( F := O(h^\perp) \backslash D(h^\perp) \) is, by Baily–Borel’s theory, an irreducible 20-dimensional quasi-projective variety.

The Torelli theorem takes the following form for our hyperkähler fourfolds ([V], [GHS], Theorem 3.14], [M, Theorem 8.4]).

**Theorem 2.1** (Verbitsky, Markman). The period map \( p: M \to F \) is an (algebraic) open embedding.

Let us describe its image. Given an element \( v \in h^\perp \) of negative square, we define the associated Heegner divisor as the image by the quotient map \( D(h^\perp) \to F \) of the hypersurface

\[
\{ [\alpha] \in D(h^\perp) | q_\Lambda(\alpha,v) = 0 \}.
\]

It is an irreducible algebraic divisor in \( F \) that only depends on the even negative integer \(-2e := \text{disc}(v^\perp)\) ([DM, Proposition 4.1(2)(c)]). We denote it by \( D_{2e} \); it is nonempty if and only if \( e \) is positive and a square modulo 11 (see the end of the proof of [DM, Proposition 4.1]). The following result is [DM, Theorem 6.1].

**Proposition 2.2** (Debarre–Macrì). The image of the period map \( p: M \hookrightarrow F \) is the complement of the irreducible divisor \( D_{22} \).

### 2.2. Debarre–Voisin varieties

We now relate this material with the constructions in [DV]. Let \( V_{10} \) be a 10-dimensional vector space. As in [1], one can associate with a nonzero \( \sigma \in \bigwedge^3 V_{10}^\vee \) a subscheme \( K_\sigma \subseteq \text{Gr}(6, V_{10}) \) which, for \( \sigma \) general, is a hyperkähler fourfold of \( K^3[2]\)-type; the polarization \( H \) induced by this embedding then satisfies \( q_{K_\sigma}(H) = 22 \) and \( \text{div}(H) = 2 \).

Let us now discuss moduli spaces for Debarre–Voisin varieties. Since the Picard group of \( P(\bigwedge^3 V_{10}^\vee) \) is cyclic, we can unambiguously define the GIT quotient

\[ M_{DV} = P(\bigwedge^3 V_{10}^\vee) \sslash \text{SL}(V_{10}), \]

a projective 20-dimensional irreducible scheme.

**Proposition 2.3.** Let \( [\sigma] \in P(\bigwedge^3 V_{10}). \) If \( K_\sigma \) is smooth of dimension 4, the point \( [\sigma] \) is \( \text{SL}(V_{10})\)-semistable.

**Proof.** Let \( P(\bigwedge^3 V_{10}^\vee)_{\text{sm}} \subseteq P(\bigwedge^3 V_{10}^\vee) \) be the open subset of points \([\sigma]\) such that \( K_\sigma \) is smooth of dimension 4. The map

\[ \ov{p}: P(\bigwedge^3 V_{10}^\vee)_{\text{sm}} \to F. \]
that sends $[\sigma]$ to the period of $K_\sigma$ is regular. Let $[\sigma] \in \mathbb{P}(\Lambda^3 V_{10}^*)^\text{sm}$. Let $D$ be a nonzero effective divisor on the quasi-projective variety $\mathcal{F}$ such that $p([\sigma]) \notin D$. The closure of $p^{-1}(D)$ in $\mathbb{P}(\Lambda^3 V_{10}^*)$ is the divisor of a $\text{SL}(V_{10})$-invariant section of some power of $\mathcal{O}_{\mathbb{P}(\Lambda^3 V_{10}^*)}(1)$, which does not vanish at $[\sigma]$, hence $[\sigma]$ is $\text{SL}(V_{10})$-semistable. □

There is a modular map
\[ m: \mathcal{M}_{\text{DV}} \twoheadrightarrow \mathcal{M}, \quad [\sigma] \mapsto [K_\sigma] \]
which is regular on the open subset $\mathcal{M}_{\text{DV}}^\text{sm} \subseteq \mathcal{M}_{\text{DV}}$ corresponding to points $[\sigma]$ such that $K_\sigma$ is a smooth fourfold. This map is dominant and generically finite (see [DV, Lemma 4.6], which computes its derivative) and we expect it to be birational. Consider the Baily–Borel projective compactification $\mathcal{F} \subseteq \overline{\mathcal{F}}$ (where the boundary $\mathcal{F} \setminus \mathcal{F}$ has dimension 1) and a resolution
\[
\begin{array}{ccc}
\mathcal{M}_{\text{DV}} & \xrightarrow{\text{pom}} & \mathcal{F} \\
\uparrow \varepsilon & & \downarrow \\
\widetilde{\mathcal{M}}_{\text{DV}} & \xrightarrow{p} & \overline{\mathcal{F}}
\end{array}
\]
of the indeterminacies of $p \circ m$.

**Definition 2.4.** An HLS (for Hassett–Looijenga–Shah) divisor is an irreducible hypersurface $D \subseteq \mathcal{F}$ for which there exist a hypersurface $\tilde{D} \subseteq \widetilde{\mathcal{M}}_{\text{DV}}$ such that $p(\tilde{D}) = D$ and $\text{codim}_{\mathcal{M}_{\text{DV}}} (\varepsilon(\tilde{D})) > 1$. In other words, it is an irreducible hypersurface in $\overline{\mathcal{F}}$ that is the image by the period map of an exceptional divisor of $\varepsilon$.

The main result of this article is the following.

**Theorem 2.5.** The Heegner divisors $D_2$, $D_6$, $D_{10}$, and $D_{18}$ are HLS divisors.

This is a consequence of Theorems 1.1, 1.2, 1.3, and 1.4: in each case, we blow up a point in $\mathcal{M}_{\text{DV}}$ whose image by the period map has codimension 1 in $\mathcal{F}$ since it consists of Hilbert squares of general polarized K3 surfaces of fixed degree.

### 3. The quadratic equation and associated K3 surfaces

#### 3.1. The quadratic equation

We look for HLS divisors whose very general points are periods of polarized hyperkähler fourfolds $(X, H)$ that are birationally isomorphic to Hilbert squares of polarized K3 surfaces $(S, L)$ with $\text{Pic}(S) = \mathbb{Z}L$ and $L^2 = 2e$. We have
\[ \text{NS}(S^{[2]}) \simeq \mathbb{Z}L \oplus \mathbb{Z}\delta, \]
where $L$ is the line bundle on $S^{[2]}$ induced by $L$ and $2\delta$ is the class of the exceptional divisor of the Hilbert–Chow morphism $S^{[2]} \rightarrow S^{(2)}$ (see Section 4.1). One has
\[ q_{S^{[2]}}(L) = 2e, \quad q_{S^{[2]}}(\delta) = -2, \quad q_{S^{[2]}}(L, \delta) = 0. \]
If $\sigma: S^{[2]} \dashrightarrow X$ is a birational isomorphism, one can write $\sigma^* H = 2bL - a\delta$, where $a$ and $b$ are positive integers. Since $q_X(H) = 22$, they satisfy the quadratic equation
\[ a^2 - 4eb^2 = -11. \]
Moreover, the class $2bL - a\delta$ is movable.
3.2. The movable and ample cones. Let again \((S, L)\) be a polarized K3 surface with \(\text{Pic}(S) = \mathbb{Z}L\). The closed movable cone \(\overline{\text{Mov}}(S[2])\) was determined in [BM] (see also [DM, Example 5.3]): one extremal ray is spanned by \(L\) and the other by \(L - \mu_e \delta\), where the rational number \(\mu_e\) is determined as follows:

- if \(e\) is a perfect square, \(\mu_e = \sqrt{e}\);
- if \(e\) is not a perfect square, \(\mu_e = eb_1/a_1\), where \((a_1, b_1)\) is the minimal positive (integral) solution of the Pell equation \(x^2 - ey^2 = 1\).

Proposition 3.1. Let \(e\) be a positive integer such that the equation (4) has a solution and let \((a_2, b_2)\) be the minimal positive solution. The numbers \(e, a, b\) such that the class \(2bL - a\delta\) is movable on \(S[2]\) and of square 22 are precisely the following:

- \(e = 1\) and \((a, b) = (5, 3)\);
- \(e = 9\) and \((a, b) = (5, 1)\);
- \(e\) is not a perfect square, \(b_1\) is even, and \((a, b)\) is either \((a_2, b_2)\) or \((2eb_1b_2 - a_1a_2, a_1b_2 - \frac{1}{2}a_2b_1)\) (these pairs are equal if and only if \(11 \mid e\));
- \(e\) is not a perfect square, \(b_1\) is odd, and \((a, b) = (a_2, b_2)\).

Proof. Assume first that \(m := \sqrt{e}\) is an integer. The equation (4) is then
\[
(a - 2bm)(a + 2bm) = -11,
\]
with \(a + 2bm > |a - 2bm|\), hence \(a + 2bm = 11\) and \(a - 2bm = -1\), so that \(a = 5\) and \(bm = 3\). The only two possibilities are \(e = 1\) and \((a, b) = (5, 3)\), and \(e = 9\) and \((a, b) = (5, 1)\). In both cases, one has indeed \(a/2b < \sqrt{e}\), hence the class \(2bL - a\delta\) is movable.

Assume that \(e\) is not a perfect square. Set \(x_2 := a_2 + b_2\sqrt{e} \in \mathbb{Z}[\sqrt{e}]\) and \(\bar{x}_2 := a_2 - b_2\sqrt{e}\), so that \(x_2\bar{x}_2 = -11\) and \(0 < -\bar{x}_2 < \sqrt{11} < x_2\).

We also set \(x_1 := a_1 + b_1\sqrt{e}\) and \(\bar{x}_1 := a_1 - b_1\sqrt{e}\), so that \(x_1\bar{x}_1 = 1\) and \(0 < \bar{x}_1 < 1 < x_1\).

Let \((a'_1, b'_1)\) be the minimal positive solution of the Pell equation \(x^2 - 4ey^2 = 1\) and set \(x'_1 := a'_1 + b'_1\sqrt{e}\). If \(b_1\) is even, we have \(x'_1 = x_1\) and \(b'_1 = b_1/2\). If \(b_1\) is odd, we have \(x'_1 = x_1^2\) and \(b'_1 = a_1b_1\).

By [N, Theorem 110], all the solutions of the equation (4) are given by \(\pm x_2x_1^n\) and \(\pm \bar{x}_2x_1^n\), for \(n \in \mathbb{Z}\). Since \(x'_1 > 1\), we have \(0 < x_2x_1^{-1} < x_2\). Since \(x_2\) corresponds to a minimal solution, this implies \(x_2x_1^{-1} < \sqrt{11}\), hence \(-\bar{x}_2x'_1 > \sqrt{11}\). By minimality of \(x_2\) again, we get \(-\bar{x}_2x'_1 \geq x_2\).

It follows that the positive solutions of the equation (4) correspond to the following increasing sequence of elements of \(\mathbb{Z}[\sqrt{e}]\):
\[
(5) \quad \sqrt{11} < x_2 \leq -\bar{x}_2x'_1 < x_2x'_1 \leq -\bar{x}_2x'^2_1 < x_2x'^2_1 < \cdots
\]
By [N, Theorem 110] again, we have \(x_2 = -\bar{x}_2x'_1\) if and only if \(11 \mid e\).

Since the function \(x \mapsto x - \frac{11}{x}\) is increasing on the interval \((\sqrt{11}, +\infty)\), the corresponding positive solutions \((a, b)\) have increasing \(a\) and \(b\), hence increasing “slopes” \(a/2b = \sqrt{\frac{e}{11x}}\).

We want to know for which of these positive solutions \(a + b\sqrt{4e}\) the corresponding class \(2bL - a\delta\) is movable, that is, satisfies \(\frac{a}{2b} \leq \mu_e = \frac{eb_1}{a_1}\).

Assume first that \(b_1\) is even, so that \(x'_1 = x_1\). The inequality \(x_2 \leq -\bar{x}_2x'_1\) translates into \(a_2 \leq -a_2a_1 + 2eb_2b_1\), hence
\[
(6) \quad \frac{a_2}{2b_2} \leq \frac{eb_1}{a_1 + 1}.
\]
The class corresponding to the solution \(-\bar{x}_2 x'_1 = 2eb_1 b_2 - a_1 a_2 + (2a_1 b_2 - a_2 b_1) \sqrt{e}\) is movable if and only if we have

\[
\frac{2eb_1 b_2 - a_1 a_2}{2a_1 b_2 - a_2 b_1} \leq \frac{eb_1}{a_1} \iff a_1 (2eb_1 b_2 - a_1 a_2) \leq eb_1 (2a_1 b_2 - a_2 b_1) \iff a_2 (eb_1^2 - e b_1^2) \leq 0,
\]

which holds since \(eb_1^2 - e b_1^2 = -1\). This class is therefore movable, and so is the class corresponding to the minimal solution since it has smaller slope.

The class corresponding to the next solution \(x_2 x_1 = a_1 a_2 + 2eb_1 b_2 + (a_2 b_1 + 2a_1 b_2) \sqrt{e}\) is movable if and only if we have

\[
\frac{a_1 a_2 + 2eb_1 b_2}{a_2 b_1 + 2a_1 b_2} \leq \frac{eb_1}{a_1} \iff a_1 (a_1 a_2 + 2eb_1 b_2) \leq eb_1 (a_2 b_1 + 2a_1 b_2) \iff a_2 (a_1^2 - eb_1^2) \leq 0,
\]

which does not hold since \(a_1^2 - eb_1^2 = 1\). This class is therefore not movable.

Assume now that \(b_1\) is odd, so that \(x'_1 = x_1^2 = 2a_1^2 - 1 + 2a_1 b_1 \sqrt{e}\). The inequality \(x_2 \leq -\bar{x}_2 x'_1\) translates into \(a_2 \leq 4ea_1 b_1 b_2 - a_2 (2a_1^2 - 1)\), hence

\[
a_2 \leq \frac{eb_1}{2b_2} \leq \frac{eb_1}{a_1},
\]

which means exactly that the class corresponding to the minimal solution \(x_2 = a_2 + 2b_2 \sqrt{e}\) is movable (and it is on the boundary of the movable cone if and only if \(11 \mid e\)).

The class corresponding to the next solution \(-\bar{x}_2 x'_1 = (-a_2 + 2b_2 \sqrt{e})(2a_1^2 - 1 + 2a_1 b_1 \sqrt{e}) = -a_2 (2a_1^2 - 1) + 4ea_1 b_1 b_2 + (2b_2 (2a_1^2 - 1) - a_2 a_1 b_1) \sqrt{e}\) is movable if and only if

\[
\frac{-a_2 (2a_1^2 - 1) + 4ea_1 b_1 b_2}{2b_2 (2a_1^2 - 1) - a_2 a_1 b_1} \leq \frac{eb_1}{a_1} \iff \frac{-a_2 a_1 (2a_1^2 - 1) + 4ea_1^2 b_1 b_2}{2eb_1 b_2} \leq \frac{2eb_1 b_2 (2a_1^2 - 1) - 2ea_1 a_2 b_1}{a_1 a_2}.
\]

It follows that the class is not movable unless there is equality in (7), which happens exactly when \(-\bar{x}_2 x'_1 = x_2\). Finally, one checks that the next solution \(x_2 x'_1\) never corresponds to a movable class. \(\square\)

Let \((S, L)\) be a polarized K3 surface with \(\text{Pic}(S) = Z L\). The nef cone \(\text{Nef}(S^{[2]} )\) was determined in [BM] (see also [DM] Example 5.3): one extremal ray is spanned by \(L\), and \(\text{Nef}(S^{[2]} ) = \text{Mov}(S^{[2]} )\), unless the equation \(x^2 - 4ey^2 = 5\) has integral solutions; if the minimal positive solution of that equation is \((a_3, b_3)\), the other extremal ray of \(\text{Nef}(S^{[2]} )\) is then spanned by \(L - \nu_e \delta\), where \(\nu_e = 2eb_5/a_5 < \mu_e\).

Furthermore, in the latter case, in the decomposition ([HT] Theorem 7))

\[
\text{Mov}(S^{[2]} ) = \bigcup_{\sigma : S^{[2]} \cong X, X \text{ hyperkähler}} \sigma^*(\text{Nef}(X))
\]

into cones which are either equal or have disjoint interiors, there are only two cones (this means that there is a unique nontrivial birational map \(\sigma : S^{[2]} \cong X\), unless \(b_1\) is even and \(5 \mid e\), in which case there are three cones ([De, Example 3.18])).
3.3. Movable and nef classes of square 22 and divisibility 2. We indicate in Table 1 the positive integers \( e \leq 26 \) for which the equation (4) has solutions, the slopes of the movable and nef cones of the Hilbert square of a very general polarized K3 surface \((S, L)\) of degree \(2e\), and the classes of square 22 and divisibility 2 which are in these cones.

<table>
<thead>
<tr>
<th>( e )</th>
<th>1</th>
<th>3</th>
<th>5</th>
<th>9</th>
<th>11</th>
<th>15</th>
</tr>
</thead>
<tbody>
<tr>
<td>((a_1, b_1))</td>
<td>−</td>
<td>(2, 1)</td>
<td>(9, 4)</td>
<td>−</td>
<td>(10, 3)</td>
<td>(4, 1)</td>
</tr>
<tr>
<td>(\mu_e) (slope of movable cone)</td>
<td>(1)</td>
<td>(3/2)</td>
<td>(20/9)</td>
<td>(3)</td>
<td>(33/10)</td>
<td>(15/4)</td>
</tr>
<tr>
<td>((a_2, b_2))</td>
<td>(5, 3)</td>
<td>(1, 1)</td>
<td>(3, 1)</td>
<td>(5, 1)</td>
<td>(33, 5)</td>
<td>(7, 1)</td>
</tr>
<tr>
<td>movable classes of square 22 and div. 2</td>
<td>(6L - 5\delta)</td>
<td>(2L - \delta)</td>
<td>(2L - 3\delta)</td>
<td>(6L - 13\delta)</td>
<td>(2L - 5\delta)</td>
<td>(10L - 33\delta)</td>
</tr>
<tr>
<td>(\nu_e) (slope of ample cone)</td>
<td>(2/3)</td>
<td>(3/2)</td>
<td>(2)</td>
<td>(3)</td>
<td>(22/7)</td>
<td>(15/4)</td>
</tr>
<tr>
<td>ample classes of square 22 and div. 2</td>
<td>−</td>
<td>(2L - \delta)</td>
<td>(2L - 3\delta)</td>
<td>(2L - 5\delta)</td>
<td>−</td>
<td>(2L - 7\delta)</td>
</tr>
</tbody>
</table>

Table 1. Movable and nef classes of square 22 and divisibility 2 in \(S^{[2]}\) for \(e \leq 26\)

**Remark 3.2.** When \(e = 2\), the decomposition \([8]\) has two cones and \(S^{[2]}\) has a unique nontrivial hyperkähler birational model; it is the moduli space \(X_S := \mathcal{M}_S(0, L, 1)\) of \(L\)-semistable pure sheaves on \(S\) with Mukai vector \((0, L, 1)\).

A general element of \(X_S\) corresponds to a sheaf \(\iota_*\xi\), where \(C \in |L|\), the map \(\iota: C \hookrightarrow S\) is the inclusion, and \(\xi\) is a degree-2 invertible sheaf on \(C\) ([Mu1 Example 0.6]). The birational map \(\sigma: S^{[2]} \simeq X_S\) takes a general \(Z \in S^{[2]}\) to the sheaf \(\iota_*\mathcal{O}_C(Z)\), where \(C\) is the unique element of \(|L|\) that contains \(Z\). It is the Mukai flop of \(S^{[2]}\) along the image of the map \(\mathbb{P}^2 \hookrightarrow S^{[2]}\) induced by the canonical double cover \(\pi: S \rightarrow \mathbb{P}^2\).

The line bundle \(L - \delta\) is base-point free on \(X_S\) and defines the Lagrangian fibration \(f: X_S \rightarrow \mathbb{P}^{2v}\) that takes the class in \(X_S\) of a sheaf on \(S\) to its support. The line bundle \(3L - 2\delta\) is base-point free and not ample on both \(S^{[2]}\) and \(X_S\) ([De Exercise 3.13], [vD Lemma 2.1.12]). The square-22 class \(H := 6L - 5\delta\) is therefore also base-point free on \(X_S\) and, as we see from Table 1, it is ample. It restricts to a general fiber \(F = \text{Pic}^2(C)\) of \(f\) (where \(C \in |L|\)) as \(L|_F\), and this is twice the canonical principal polarization on \(F\). In particular, the morphism that \(H\) defines factors through the involution of \(X_S\) induced by the involution of \(S\) attached to \(\pi\) and \(H\) is not very ample.

**Remark 3.3.** When \(e = 5\), the decomposition \([8]\) has two cones and \(S^{[2]}\) has a unique nontrivial birational involution \(\iota\), which was described geometrically in [De Proposition 4.14, Example 4.15]. One has \(\iota^*(2L - 3\delta) = 6L - 13\delta\) and \(S^{[2]}\) has no nontrivial hyperkähler birational models.

**Remark 3.4.** An infinite series of examples is obtained by taking \(e = m^2 + m + 3\), so that \((a_2, b_2) = (2m + 1, 1)\). By Proposition 3.3, the class \(2L - (2m + 1)\delta\) is always movable. One checks that its slope \((2m + 1)/2\) is always less than the slope \(\nu_e\) of the nef cone, hence this class is in fact always ample.
Remark 3.5. A consequence of Proposition 3.1 is that there are always one or two movable classes of square 22 and divisibility 2 as soon as the equation (4) has a solution. As Table 1 shows, it can happen that none of these classes are ample. It can also happen that both of these classes are ample: taking \( m = 6 \) in Remark 3.4, so that \( e = 45 \), we get \( b_1 = 24 \), hence there are two such movable classes which are in fact ample since the nef and closed movable cones are equal (the equation \( x^2 - 4ey^2 = 5 \) has no integral solutions).

In the general case, the orthogonal of \( \text{NS}(S^{[2]}_1) \) in the lattice \( \Lambda \) is isomorphic to the orthogonal of \( L \) in the (unimodular) K3 lattice \( H^2(S,Z) \). Its discriminant is therefore \(-2e\) and the period of \((S^{[2]},H)\) is a general point of the Heegner divisor \( \mathcal{D}_{2e} \).

Corollary 3.6. In the moduli space \( \mathcal{M} \), there are divisors whose general points correspond to polarized hyperkähler fourfolds that are, respectively,

- isomorphic to \( (\mathcal{M}_S(0, L, 1), \sigma_1(6L - 5\delta)) \), where \((S, L)\) is a general polarized K3 surface of degree 2 (see Remark 3.3);
- isomorphic to \( (S^{[2]}, 2L - (2m + 1)\delta) \), where \((S, L)\) is a general polarized K3 surface of degree \((2m^2 + m + 3) \) (for any \( m \geq 0 \)).

In the first case, the periods dominate the Heegner divisor \( \mathcal{D}_2 \). In the second case, the periods dominate the Heegner divisor \( \mathcal{D}_{2(m^2 + m + 3)} \).

Remark 3.7. When \( \sigma \in \wedge^3 V_{10} \) is a general trivector such that the hypersurface \( X_\sigma \) is singular, the variety \( K_\sigma \) becomes singular, but, with its Plücker line bundle, birationally isomorphic to \((S^{[2]}, 10L - 33\delta)\), where \((S, L)\) is a general polarized K3 surface of degree 22 ([DV, Proposition 3.4]). As indicated in Table 1 above, the line bundle \( 10L - 33\delta \) is on the boundary of the movable cone of \( S^{[2]} \); it defines the birational map \( S^{[2]} \dashrightarrow K_\sigma \subseteq \text{Gr}(6, V_{10}) \subseteq \mathbf{P}(\wedge^6 V_{10}) \). The corresponding “periods” cover the Heegner divisor \( \mathcal{D}_{22} \).

4. Preliminary results

We collect in this section a few results that will be used repeatedly in the rest of the article.

4.1. Tautological bundles on Hilbert squares. Let \( X \) be a smooth projective variety. Consider the blow up \( \tau : \tilde{X} \times X \rightarrow X \times X \) of the diagonal and its restriction \( \tau_E : E \rightarrow X \) to its exceptional divisor \( E \). The (smooth projective) Hilbert square of \( X \) is the quotient

\[
p : \tilde{X} \times X \rightarrow X^{[2]}
\]

by the lift \( \iota \) of the involution that exchanges the two factors. It is simply ramified along \( E \) and there is a class \( \delta \in \text{Pic}(X^{[2]}) \) such that \( p^*\delta = E \). We will use the composed maps \( q_i : \tilde{X} \times X \xrightarrow{\tau} X \times X \xrightarrow{pr_1} X \).

Let \( \mathcal{F} \) be a vector bundle of rank \( r \) on \( X \). We write \( \mathcal{F} \boxplus \mathcal{F} := q_1^* \mathcal{F} \oplus q_2^* \mathcal{F} \) and \( \mathcal{F} \boxtimes \mathcal{F} := q_1^* \mathcal{F} \otimes q_2^* \mathcal{F} \); they are vector bundles on \( \tilde{X} \times X \) of respective ranks \( 2r \) and \( r^2 \). If \( \mathcal{L} \) is an invertible sheaf on \( X \), the invertible sheaf \( \mathcal{L} \boxtimes \mathcal{L} \) is \( \iota \)-invariant and descends to an invertible sheaf on \( X^{[2]} \) that we still denote by \( \mathcal{L} \). This gives an injective group morphism

\[
\text{Pic}(X) \oplus \mathbb{Z} \rightarrow \text{Pic}(X^{[2]}), \quad (\mathcal{L}, m) \mapsto \mathcal{L} + m\delta.
\]

The tautological bundle

\[
\mathcal{F}_\mathcal{F} := p_*(q_1^* \mathcal{F})
\]
is locally free of rank $2r$ on $X^{[2]}$ and there is an exact sequence ([D1] prop. 2.3, [W] (3))

$$0 \rightarrow p^*\mathcal{F} \rightarrow \mathcal{F} \otimes \mathcal{F} \rightarrow \tau_E^*\mathcal{F} \rightarrow 0,$$

of sheaves on $\widetilde{X \times X}$. In the notation of (9), we have

$$\text{det}(\mathcal{F}) = \text{det}(\mathcal{F}) - r\delta$$

and there is an isomorphism

$$H^0(X^{[2]}, \mathcal{F}) \xrightarrow{\sim} H^0(X, \mathcal{F}).$$

**Remark 4.1.** When $X \subseteq \mathbb{P}(V)$, there is a morphism $f : X^{[2]} \rightarrow \text{Gr}(2, V)$ that sends a length-2 subscheme of $X$ to the projective line that it spans in $\mathbb{P}(V)$. The rank-2 vector bundle $\mathcal{F}_{\mathcal{O}_X}$ is then the pullback by $f$ of the tautological subbundle $\mathcal{S}_2$ on $\text{Gr}(2, V)$. It is in particular generated by global sections.

We now present an analogous construction that will be used in Section 6. There is a surjective morphism

$$\text{ev}^+ : \mathcal{F} \otimes \mathcal{F} \rightarrow \tau_E^*\text{Sym}^2\mathcal{F}$$

obtained by evaluating along the exceptional divisor $E$ and then projecting onto the symmetric part of $(\mathcal{F} \otimes \mathcal{F})|_E = \tau_E^*(\mathcal{F} \otimes \mathcal{F})$.

**Lemma 4.2.** There is a locally free sheaf $\mathcal{K}_\mathcal{F}$ or rank $r^2$ on $X^{[2]}$ and an exact sequence

$$0 \rightarrow p^*\mathcal{K}_\mathcal{F} \rightarrow \mathcal{F} \otimes \mathcal{F} \xrightarrow{\text{ev}^+} \tau_E^*\text{Sym}^2\mathcal{F} \rightarrow 0,$$

Moreover, $\text{det}(\mathcal{K}_\mathcal{F}) = r\det(\mathcal{F}) - \frac{1}{2}r(r + 1)\delta$ and $H^0(X^{[2]}, \mathcal{K}_\mathcal{F}) \simeq \bigwedge^2 H^0(X, \mathcal{F})$.

**Proof.** Let $\widetilde{\mathcal{K}}_\mathcal{F}$ be the kernel of $\text{ev}^+$. It is locally free on $\widetilde{X \times X}$ and we need to show that it descends to a vector bundle on $X^{[2]}$. For that, it is enough to prove that the involution $\iota$ on $\widetilde{X \times X}$ lifts to an involution $\iota$ on $\widetilde{\mathcal{K}}_\mathcal{F}$ that acts by $-\text{Id}$ on $\widetilde{\mathcal{K}}_\mathcal{F}|_E$.

The statement is local over the diagonal of $X$. We can thus assume that $\mathcal{F}$ is trivial on $X$ with basis $(s_1, \ldots, s_r)$ and that we have local coordinates $x_1, \ldots, x_n$ on $X$ near $O \in X$. On $X \times X$, we have coordinates $x_1, \ldots, x_n, y_1, \ldots, y_n$ and the bundle $\mathcal{F} \otimes \mathcal{F}$ has basis $(s_i \otimes s_j)_{1 \leq i, j \leq r}$, where $(s_i \otimes s_j)(x_1, \ldots, x_n, y_1, \ldots, y_n) = s_i(x_1, \ldots, x_n)s_j(y_1, \ldots, y_n)$. The involution $\iota$ on $\mathcal{F} \otimes \mathcal{F}$ maps $s_i \otimes s_j$ to $s_j \otimes s_i$.

Consider a point in $X^{[2]}$ over $(O, O)$. Without loss of generality, we can assume that it corresponds to the tangent vector $\frac{\partial}{\partial y_1}$ at $x_n$. The blow up $\widetilde{X \times X}$ then has local coordinates $\tilde{x}_1, \ldots, \tilde{x}_n, \tilde{y}_1, u_2, \ldots, u_n$ at that point in which the morphism $\iota$ is given by

$$\tau^*x_i = \tilde{x}_i, \quad \tau^*y_1 = \tilde{y}_1, \quad \tau^*(y_i - x_i) = u_i(\tilde{y}_1 - \tilde{x}_1) \quad \text{for } i \geq 2.$$

The equation of the exceptional divisor $E$ is then $e := \tilde{y}_1 - \tilde{x}_1$ and the involution on $\widetilde{X \times X}$ is given by

$$\iota^*\tilde{x}_i = \tilde{y}_1, \quad \iota^*\tilde{x}_i = \tilde{x}_i + u_i(\tilde{y}_1 - \tilde{x}_1), \quad \iota^*u_i = u_i \quad \text{for } i \geq 2,$$

and satisfies $\iota^*e = -e$. The bundle $\widetilde{\mathcal{K}}_\mathcal{F}$ is thus locally generated by the sections

$$s_i \otimes s_j - s_j \otimes s_i, \quad e(s_i \otimes s_j + s_j \otimes s_i),$$

for all $i \leq j$. This shows that $\iota$ acts by $-\text{Id}$ on $\widetilde{\mathcal{K}}_\mathcal{F}|_E$.

The vector bundle $\widetilde{\mathcal{K}}_\mathcal{F}$ therefore descends to a vector bundle $\mathcal{K}_\mathcal{F}$ on $X^{[2]}$ whose determinant can be computed from the exact sequence (11).
Going back to the global situation, we see that the space of \( \tilde{\sigma} \)-antiinvariant sections of \( \mathcal{F} \otimes \mathcal{F} \) on \( \tilde{X} \times X \) that are sections of \( \mathcal{K} \) is \( \bigwedge^2 H^0(X, \mathcal{F}) \). These sections correspond exactly to the sections of \( \mathcal{K} \) on \( X[2] \). This proves the lemma.

4.2. Zero-loci of excessive dimensions and excess formula. We describe in a general context an excess computation that we will use in the proofs of Theorems 5.3, 5.15 and 6.14. Let \( M \) be a smooth variety of dimension \( n \), let \( \mathcal{E} \) be a vector bundle of rank \( r \) on \( M \), and let \( \sigma_0 \) be a section of \( \mathcal{E} \), with zero-locus \( Z \subseteq M \). The differential of \( \sigma_0 \) defines a morphism \( d\sigma_0: T_M|_Z \to \mathcal{E}|_Z \). If \( Z \) is smooth, of codimension \( s \leq r \) in \( M \), the kernel of \( d\sigma_0 \) is \( T_Z \) and we define the excess bundle \( \mathcal{F} \) to be its cokernel. It has rank \( r-s \) on \( Z \) and is isomorphic to the quotient \( \mathcal{E}|_Z/N_{Z/M} \).

Assume now that \( \mathcal{E} \) is generated by global sections and let \((\sigma_t)_{t \in \Delta} \) be a general 1-parameter deformation of \( \sigma_0 \). For \( t \in \Delta \) general, the zero-locus \( Z_t \) of the section \( \sigma_t \) is smooth of pure codimension \( r \) or empty. The bundle \( \mathcal{F} \), as a quotient of \( \mathcal{E}|_Z \), is also generated by its sections and the zero-locus of the section \( \tilde{\sigma} \) defined as the image of \( \frac{\partial \sigma_t}{\partial t}|_{t=0} \in H^0(M, \mathcal{E}) \) in \( H^0(Z, \mathcal{F}) \) is smooth of pure codimension \( r-s \) in \( Z \) or empty.

Consider the closed subset

\[
W = \{(x, t) \in M \times \Delta \mid \sigma_t(x) = 0 \}.
\]

The general fibers of the second projection \( \pi: W \to \Delta \) are smooth of pure dimension \( n-r \) or empty, and the central fiber is \( Z \). Let \( W^0 \) be the union of the components of \( W \) that dominate \( \Delta \) and assume that it is nonempty, hence of pure dimension \( n+1-r \). The central fiber of the restricted map \( \pi^0: W^0 \to \Delta \) is contained in \( Z \).

**Proposition 4.3.** For a general 1-parameter deformation \((\sigma_t)_{t \in \Delta}\), the map \( \pi^0: W^0 \to \Delta \) is smooth and its central fiber is the zero-locus of \( \tilde{\sigma} \) in \( Z \).

**Proof.** We view the family \((\sigma_t)_{t \in \Delta}\) of sections of \( \mathcal{E} \) as a section \( \tilde{\sigma} \) of the vector bundle \( \tilde{\mathcal{E}} := \text{pr}_M^* \mathcal{E} \) on \( M \times \Delta \), defining \( W \). We can write \( \tilde{\sigma} = \tilde{\sigma}_0 + t \tilde{\sigma}' + O(t^2) \) as sections of \( \tilde{\mathcal{E}} \), where \( \tilde{\sigma}_0 = \text{pr}_M^* \sigma_0 \) and

\[
\tilde{\sigma}'|_{M \times 0} = \left. \frac{\partial \sigma_t}{\partial t} \right|_{t=0}.
\]

Along \( Z \times \{0\} \subseteq W \), we have

\[
d\tilde{\sigma} = d\sigma_0 + \tilde{\sigma}' dt: T_{M \times \Delta}|_{Z \times \{0\}} \to \tilde{\mathcal{E}}|_{Z \times \{0\}}.
\]

Let \( z \in Z \) be a point where \( \tilde{\sigma}' \) does not vanish. We deduce from (13) and (14) that \( Z \times \{0\} \) and \( W \) coincide schematically around \((z, 0)\). Indeed, as \( Z \times \{0\} \) is smooth and contained in \( W \), this is equivalent to saying that their Zariski tangent spaces coincide. If they do not, since \( Z \times \{0\} \) is the fiber of \( W \) at \( 0 \), some tangent vector at \( W \) at \( 0 \) is of the type \((v, \frac{\partial}{\partial t})\). By (14), we have \( d\sigma_0|_z(v) + \tilde{\sigma}'(z) = 0 \), so that \( \tilde{\sigma}'(z) \) belongs to \( \text{Im}(d\sigma_0|_z) \). By (13), this means that the image \( \tilde{\sigma}'(z) \) vanishes in \( \mathcal{F} \), contradiction.

We thus proved that the central fiber of \( W^0 \to \Delta \) is contained set-theoretically in the zero-locus \( Z^0 \) of \( \tilde{\sigma}' \). To prove that the inclusion is scheme-theoretic, we proceed as follows. Since \( Z \subseteq M \) is smooth of codimension \( s \), we can trivialize \( \mathcal{E} \) locally along \( Z \) in such a way that in the corresponding decomposition \( \sigma = (\sigma_1, \ldots, \sigma_s) \), the \( s \) first functions have independent differentials, hence define \( Z \subseteq M \). We can write \( \tilde{\sigma} = (\tilde{\sigma}_1, \ldots, \tilde{\sigma}_s) \) and replace \( M \times \Delta \) by the vanishing locus \( M' \) of \((\tilde{\sigma}_1, \ldots, \tilde{\sigma}_s) \) which is smooth of codimension \( s \) in \( M \times \Delta \) and smooth over \( \Delta \). The central fiber of the restricted map \( \pi': M' \to \Delta \) is \( Z \) (or rather the relevant open set of \( Z \)), which means that the section \( \tilde{\sigma}|_{M'} \) vanishes along its central fiber. We then have

\[
\tilde{\sigma}|_{M'} = \frac{\tilde{\sigma}'|_{M'}}{\tilde{\sigma}'|_{M'}}.
\]
where $\sigma|_{M'}$ is the projection of $\sigma|_{M'}$ onto the $r-s$ remaining components of $\sigma$. The decomposition of $W$ into irreducible components is (near the given point of $Z$)

$$W = M_0 \cup \{\sigma|_{M'} = 0\},$$

so that $W^0$ is locally the zero-locus of the section $\sigma|_{M'}$. Finally, we observe that the restriction to $Z \subseteq M'$ of the locally defined section $\sigma|_{M'}$ is nothing but $\sigma'$. As we assumed that $\sigma'$ is general, hence transverse, it follows that $W^0$ is smooth of codimension $r-s$ in $M'$, with central fiber the zero-locus of $\sigma'$.

### 4.3. Geometry of singular trivectors.

Given a nonzero trivector $\sigma \in \bigwedge^3 V_{10}^\vee$, we relate singular points on the hypersurface $X\sigma$ to points on the Debarre–Voisin variety $K\sigma$ (see [2] and [1] for definitions). This geometric observation will allow us to describe, for the degenerate trivector points on the hypersurface $X\sigma$, the excess bundles of Section 4.2 in our specific situations.

**Proposition 4.4.** Let $\sigma \in \bigwedge^3 V_{10}^\vee$ be a nonzero trivector and let $[U_3]$ be a singular point of the hypersurface $X\sigma \subseteq \text{Gr}(3, V_{10})$.

(a) The variety $\Sigma_{U_3} := \{[W_6] \in K\sigma \mid W_6 \supseteq U_3\}$ is nonempty of dimension everywhere at least 2 and for all $[W_6] \in \Sigma_{U_3}$, one has $\dim(T_{K\sigma}[W_6]) \geq 4$. In particular, if $K\sigma$ has (expected) dimension 4 at $[W_6]$, it is singular at that point.

(b) If $[U'_3]$ is another singular point of $X\sigma$ such that $W_6 := U_3 + U'_3$ has dimension 6, the point $[W_6]$ is in $K\sigma$.

**Proof.** Let us prove (a). Let $[U_3] \in \text{Sing}(X\sigma)$ and let $[W_6] \in \Sigma_{U_3}$. We will show that the differential $d\sigma$ of the section $\sigma$ of $\bigwedge^3 E_6$ defining $K\sigma$ does not have maximal rank at $[W_6]$.

As explained in the proof of [DV, Proposition 3.1], this differential

$$d\sigma : T_{\text{Gr}(6, V_{10})}[W_6] = \text{Hom}(W_6, V_{10}/W_6) \rightarrow \bigwedge^3 W_6^\vee$$

maps $u \in \text{Hom}(W_6, V_{10}/W_6)$ to the 3-form

$$d\sigma(u)(w_1, w_2, w_3) = \sigma(u(w_1), w_2, w_3) + \sigma(w_1, u(w_2), w_3) + \sigma(w_1, w_2, u(w_3)).$$

Since $[U_3]$ is singular on $X\sigma$, the trivector $\sigma$ vanishes on $\bigwedge^2 U_3 \wedge V_{10}$ ([DV, Proposition 3.1]), hence $d\sigma(u)$ vanishes on $\bigwedge^3 U_3$. The composite

$$\text{Hom}(W_6, V_{10}/W_6) \xrightarrow{d\sigma} \bigwedge^3 W_6^\vee \rightarrow \bigwedge^3 U_3^\vee$$

is therefore zero, hence $d\sigma$ does not have maximal rank.

It remains to prove that the variety $\Sigma_{U_3}$ is nonempty of dimension everywhere $\geq 2$. This follows from the fact that it is defined in the smooth 12-dimensional variety $\text{Gr} := \{[W_6] \in \text{Gr}(6, V_{10}) \mid W_6 \supseteq U_3\} \simeq \text{Gr}(3, V_{10}/U_3)$ as the zero-locus of a section of the rank-10 vector bundle $(U_3^\vee \otimes \bigwedge^2 E_3) \oplus \bigwedge^3 E_3$, whose top Chern class is nonzero.

Let us prove (b). Since $[U_3]$ and $[U'_3]$ are singular points of $X\sigma$, the trivector $\sigma$ vanishes on $\bigwedge^2 U_3 \wedge V_{10}$ and $\bigwedge^2 U'_3 \wedge V_{10}$, hence also on $\bigwedge^3 (U_3 + U'_3)$. In particular, if $U_3 + U'_3$ has dimension 6, it defines a point of $K\sigma$. \qed

The proof above also gives the following information which will be useful when we compute the excess bundles of Section 4.2 in our specific situations.

**Lemma 4.5.** In Proposition 4.4(a), the restriction map $\bigwedge^3 W_6^\vee \rightarrow \bigwedge^3 U_3^\vee$ vanishes on $\text{Im}(d\sigma)$.

In Proposition 4.4(b), the restriction map $\bigwedge^3 W_6^\vee \rightarrow \bigwedge^3 U_3^\vee \oplus \bigwedge^3 U'_3^\vee$ vanishes on $\text{Im}(d\sigma)$. 


Remark 4.6. In Sections 5.1 and 5.2 we will work with a generically smooth component \( K_0 \) of a Debarre–Voisin variety \( K_{\sigma_0} \) of excessive dimension 6, so that the image of \( d\tilde{\sigma}_0 \) has codimension 2 along its smooth locus. Furthermore, a general point of \( K_0 \) is of the form \([U_3 \oplus U'_3]\), with \([U_3], [U'_3]\) in some smooth subvariety \( W \) of \( \text{Sing}(X_{\sigma_0}) \). By Proposition 4.4(b), there is a rational dominant map

\[
f : W^{[2]} \rightarrow K_0
\]

\(([U_3], [U'_3]) \mapsto [U_3 + U'_3].\)

Lemma 4.5 then tells us that the image of \( d\tilde{\sigma}_0 \) vanishes in the two-dimensional space \( \bigwedge^3 U'_3 \oplus \bigwedge^3 U''_3 \). This identifies, on a Zariski open subset of \( W^{[2]} \), the pullback by \( f \) of the excess bundle on \( K_0 \) with the tautological bundle \( \mathcal{O}_{\tilde{W}(1)} \), where \( \mathcal{O}_{\tilde{W}(1)} \) is the Plücker line bundle on \( W \subseteq \text{Gr}(3, V_{10}) \). By Remark 4.1 it is generated by its global sections.

5. The HLS divisors \( \mathcal{D}_6 \) and \( \mathcal{D}_{18} \)

We describe in this section two polystable (semistable with closed orbit in the semistable locus) trivectors in the moduli space \( \mathcal{M}_{DV} = \mathcal{P} \langle \bigwedge^3 V_{10}' \rangle / \text{SL}(V_{10}) \) whose total images by the moduli map

\[
m : \mathcal{M}_{DV} \rightarrow \mathcal{M}
\]

are the hypersurfaces in \( \mathcal{M} \) whose general points are pairs \((S^{[2]}, 2L-\delta)\), where \((S, L)\) is a general polarized K3 surface of degree 6 (resp. pairs \((S^{[2]}, 2L-5\delta)\)), where \((S, L)\) is a general polarized K3 surface of degree 18) (see Table I). As explained in Section 3 their total images by the composition

\[
p \circ m : \mathcal{M}_{DV} \rightarrow \mathcal{F}
\]

are therefore the Heegner divisors \( \mathcal{D}_6 \) (resp. \( \mathcal{D}_{18} \)). A common feature of these two specific trivectors \( \sigma_0 \), which makes the specialization analysis quite easy, is that the corresponding Debarre–Voisin varieties \( K_{\sigma_0} \) are smooth but of larger-than-expected dimension. The limit of the Debarre–Voisin varieties along a 1-parameter degeneration to \( \sigma_0 \) is then a smooth fourfold obtained as the zero-locus of a general section of the excess bundle on \( K_{\sigma_0} \) associated with this situation (see Section 4.2).

5.1. The HLS divisor \( \mathcal{D}_6 \). We construct a trivector \( \sigma_0 \) whose Debarre–Voisin variety \( K_{\sigma_0} \) is smooth but has excessive dimension 6. The neutral component of the stabilizer of \( \sigma_0 \) is \( \text{Sp}(4) \) and \( \sigma_0 \) associated with this situation (see Table I). The total image in \( \mathcal{F} \) of the point \([\sigma_0]\) is the Heegner divisor \( \mathcal{D}_6 \). The main result of this section is Theorem 5.5.

5.1.1. The \( \text{Sp}(4) \)-invariant trivector. Let \( V_4 \) be a 4-dimensional vector space equipped with a symplectic form \( \omega \) and let \( V_5 \subseteq \bigwedge^2 V_4 \) be the hyperplane defined by \( \omega \), endowed with the non-degenerate quadratic form \( q \) defined by \( q(x, y) = (\omega \wedge \omega)(x \wedge y) \). The form \( q \) defines a smooth quadric \( Q_3 \subseteq \mathcal{P}(V_5) \).

The 10-dimensional vector space \( V_{10} := \bigwedge^2 V_5 \cong \text{Sym}^2 V_4 \) can be identified with the space of endomorphisms of \( V_5 \) which are skew-symmetric with respect to \( q \) and we define a trivector \( \sigma_0 \) on \( V_{10} \) as in (3) by \( \sigma_0(a, b, c) = \text{Tr}(a \circ b \circ c) \). It is invariant for the canonical action of the group \( \text{Sp}(V_4, \omega) = \text{SO}(V_5, q) \) on \( \bigwedge^3 V_{10}' \).

This is a particular case of a general situation studied by Hivert, who proved in particular that the Debarre–Voisin variety \( K_{\sigma_0} \) is smooth of dimension 6 ([III Definition 1.2 and Theorem 4.1]). He moreover gave a very concrete description of this variety. We will use the hypersurface \( X_{\sigma_0} \subseteq \text{Gr}(3, V_{10}) \) defined in (2).
Proposition 5.1. (a) The image of the morphism
\[ j: Q_3 \rightarrow \text{Gr}(3, V_{10}) \]
\[ x \mapsto [x \wedge x^{+\eta}] \]
is contained in the singular locus of the hypersurface \( X_{\sigma_0} \subseteq \text{Gr}(3, V_{10}) \).

(b) The morphism \( j \) is an embedding and \( j^* \mathcal{O}_{\text{Gr}(3, V_{10})}(1) \cong \mathcal{O}_{Q_3}(3) \).

Proof. Let \( x \in Q_3 \). If \( z \in x^{+\eta} \), the skew-symmetric endomorphism \( a_z \) of \( V_5 \) associated with \( x \wedge z \) is
\[ a_z(u) = q(x, u)z - q(z, u)x, \]
and thus, if \( z, z' \in x^{+\eta} \), we have
\[ a_{z'} \circ a_z(u) = q(x, u)q(x, z)z' - q(z, u)q(x, x)z' - q(x, u)q(z', z)x + q(z, u)q(z', x)x \]
which is symmetric in \( z \) and \( z' \), proving that \( a_z \) and \( a_{z'} \) commute. The endomorphism \( a_{z'} \circ a_z \) is then symmetric, hence \( \text{Tr}(a_{z'} \circ a_z \circ c) = 0 \) for any skew-symmetric endomorphism \( c \in V_{10} \). By [DV, Proposition 3.1], this implies item (a).

We now prove (b). The morphism \( j \) is injective because \( x^{+\eta} \) is the tangent space to \( Q_3 \) at \([x] \) and this hyperplane is tangent only at \([x] \). Since \( j \) is \( O(V_5, q) \)-equivariant, it is an embedding. Consider now the exact sequence
\[ 0 \rightarrow \mathcal{K} \rightarrow V_5 \otimes \mathcal{O}_{Q_3} \xrightarrow{q} \mathcal{O}_{Q_3}(1) \rightarrow 0 \]
defining the rank-4 vector bundle \( \mathcal{K} \cong \Omega_{\mathbb{P}(V_5)}(1)|_{Q_3} \) with fiber \( x^{+\eta} \) at \([x] \) and the exact sequence
\[ 0 \rightarrow \mathcal{O}_{Q_3}(-2) \rightarrow \mathcal{K} \otimes \mathcal{O}_{Q_3}(-1) \xrightarrow{\wedge} j^* \mathcal{J}_3 \rightarrow 0, \]
which implies \( j^* \mathcal{J}_3 \cong \Omega_{Q_3} \). We obtain the desired isomorphism \( j^* \mathcal{O}_{\text{Gr}(3, V_{10})}(1) \cong \mathcal{O}_{Q_3}(3) \) by taking determinants. \qed

By Propositions 4.4 and 5.1, we have a rational map \( f: Q_3^{[2]} \rightarrow K_{\sigma_0} \) which is \( \text{Sp}(4) \)-equivariant. The following result is [Hi, Theorem 6.3].

Theorem 5.2 (Hivert). The map \( f: Q_3^{[2]} \rightarrow K_{\sigma_0} \) is an isomorphism.

Proof. Any point in \( Q_3^{[2]} \) spans a line in \( \mathbb{P}(V_5) \), hence defines an element of \( \text{Gr}(2, V_5) \). The corresponding morphism \( \varepsilon: Q_3^{[2]} \rightarrow \text{Gr}(2, V_5) \) has a rational inverse: the intersection of a line in \( \mathbb{P}(V_5) \) with \( Q_3 \) is a subscheme of length 2 of \( Q_3 \), except when the line is contained in \( Q_3 \). The morphism \( \varepsilon \) is therefore the blow up of the scheme of lines contained in \( Q_3 \) (which is the image of the Veronese embedding \( v_2: \mathbb{P}(V_4) \hookrightarrow \mathbb{P}(\text{Sym}^2 V_4) = \mathbb{P}(\Lambda^2 V_5) \); see [Hi, Section 6.2]).

Hivert proves in [Hi, Section 6.2] that the linear system \( |\mathcal{J}_2(\mathbb{P}(V_4))(3)| \) embeds \( Q_3^{[2]} \) into the linear span of \( K_{\sigma_0} \) in the Plücker embedding of \( \text{Gr}(6, V_{10}) \) and that its image coincides with \( K_{\sigma_0} \). \qed

5.1.2. Orbit and stabilizer. The decomposition of \( \Lambda^3 V_{10}^\vee \) into irreducible \( \text{Sp}(4) \)-representations is ([Hi, Section 6.2])
\[ \Lambda^3 V_{10}^\vee = V_{4\omega_1} \oplus V_{3\omega_2} \oplus V_{2\omega_1+\omega_2} \oplus V_{2\omega_2} \oplus V_{\omega_2} \oplus \mathbb{C}. \]
The last term is the space of \( \text{Sp}(4) \)-invariants; it is generated by our trivector \( \sigma_0 \) defined in (3). The first term is \( \text{Sym}^3 V_4 \) and the second term is \( H^0(Q_3, \mathcal{O}_{Q_3}(3)) \). Since \( sp(4) = \text{Sym}^2 V_4 = V_{2\omega_1} \) and
\[ \text{End}(V_{10}) = V_{4\omega_1} \oplus V_{2\omega_1} \oplus V_{2\omega_1+\omega_2} \oplus V_{2\omega_2} \oplus V_{\omega_2} \oplus \mathbb{C}, \]
there is an exact sequence
\[ 0 \to \mathfrak{sp}(4) \to \text{End}(V_{10}) \to \wedge^3 V_{10}^\vee \to H^0(Q_3, \mathcal{O}_{Q_3}(3)) \to 0. \]

We prove that the tangent space to the stabilizer of \( \sigma_0 \) is \( \mathfrak{sp}(4) \), hence the normal space to the GL(\( V_{10} \))-orbit of \( \sigma_0 \) is \( H^0(Q_3, \mathcal{O}_{Q_3}(3)) \).

**Proposition 5.3.** The neutral component of the stabilizer of \( \sigma_0 \) for the SL(\( V_{10} \))-action is \( \text{Sp}(V_4) = \text{SO}(V_5) \) and the point \([\sigma_0] \) of \( \mathbb{P}(\wedge^3 V_{10}) \) is polystable for the SL(\( V_{10} \))-action.

**Proof.** The neutral component of the stabilizer acts on the Debarre–Voisin variety \( K_{\sigma_0} \), which is isomorphic to \( Q_3[2] \). Since it is connected, it acts trivially on the Néron–Severi group, hence preserves the exceptional divisor of the Hilbert–Chow morphism \( Q_3[2] \to Q_3(2) \). It therefore acts on \( Q_3[2] \), hence on \( Q_3 \). It is therefore in SO(\( V_5 \)).

To show that \([\sigma_0] \) is polystable, we will use a result of Luna. By Proposition 5.4 below, the stabilizer SO(\( V_5 \)) has finite index in its normalizer in SL(\( V_{10} \)). By [Lun Corollaire 3] (applied to the group SL(\( V_{10} \)) acting on \( \wedge^3 V_{10}^\vee \)), the orbit of \( \sigma_0 \) is closed in \( \wedge^3 V_{10}^\vee \), hence \([\sigma_0] \) is polystable. \( \square \)

We prove the classical result used in the proof above.

**Proposition 5.4.** Let \( G \) be a semisimple algebraic group with a faithful irreducible representation \( G \to \text{SL}(V) \). The group \( G \) has finite index in its normalizer in \( \text{SL}(V) \).

**Proof.** According to the discussion after [3] Lemma 16.3.8], the group of outer automorphisms of \( G \) is finite. The kernel of the operation \( N := N_{\text{SL}(V)}(G) \to \text{Aut}(G) \) of the normalizer by conjugation is contained in the centralizer \( C := C_{\text{SL}(V)}(G) \) and the kernel of the induced morphism \( N/G \to \text{Out}(G) \) is contained in the image of \( C \) in \( N/G \). It is therefore sufficient to show that \( C \) is a finite group. But this follows from Schur’s lemma: any eigenspace of an element of \( C \) is stable by \( G \), hence equal to \( V \). Therefore, \( C \) consists of homotheties, hence is finite. \( \square \)

5.1.3. Degenerations and excess bundles. Consider a general 1-parameter deformation \( (\sigma_t)_{t \in \Delta} \). The derivative \( \frac{\partial}{\partial t} |_{t=0} \) provides, by the discussion in Section 5.1.2, a general section of \( \mathcal{O}_{Q_3}(3) \) which defines a K3 surface \( S \subseteq Q_3 \subseteq \mathbb{P}(V_3) \) of degree 6.

**Theorem 5.5.** Let \( (\sigma_t)_{t \in \Delta} \) be a general 1-parameter deformation. Let \( \mathcal{X} \to \Delta \) be the associated family of Debarre–Voisin varieties and let \( \mathcal{X}^0 \to \Delta \) be the irreducible component of \( \mathcal{X} \) that dominates \( \Delta \). Then \( \mathcal{X}^0 \to \Delta \) is smooth and it central fiber is isomorphic to \( S[2] \), embedded in \( \text{Gr}(6,10) \) as \( S[2] \subseteq Q_3[2] \cong K_{\sigma_0} \subseteq \text{Gr}(6,V_{10}) \), where \( S \) is a general genus-4 K3 surface.

The proof of the theorem will be based on the excess computation presented in Section 4.2, we want to apply Proposition 4.3 with \( M = \text{Gr}(6, \wedge^3 V_5) \) and \( \mathcal{E} = \wedge^3 \mathcal{E}_6 \), where \( \mathcal{E}_6 \) is the dual of the tautological rank-6 subbundle on \( \text{Gr}(6, \wedge^3 V_5) \). For this, we need to identify the rank-2 excess bundle \( \mathcal{F} \) on \( K_{\sigma_0} \cong Q_3[2] \). We use the notation of Section 4.1.

**Proposition 5.6.** The excess bundle \( \mathcal{F} \) on \( Q_3[2] \) is isomorphic to the tautological bundle \( \mathcal{F}_{\mathcal{O}_{Q_3}(3)} \).

**Proof.** By definition, \( \mathcal{F} \) is a rank 2-quotient bundle of \( \wedge^3 \mathcal{E}_6 \vert_{Q_3[2]} \), hence of \( \wedge^3 V_{10}^\vee \otimes \mathcal{O}_{Q_3[2]} \).

Since \( j \) is an embedding (Proposition 5.1), the rank-2 vector bundle \( \mathcal{F}_{\mathcal{O}_{Q_3}(3)} \) is generated by the space \( \wedge^3 V_{10}^\vee \) of global sections by Remark 4.6. More precisely, on the dense open set \( U \subseteq Q_3[2] \) of pairs \( \{x, y\} \) such that \( (x \wedge x^\vee) \cap (y \wedge y^\vee) = \{0\} \), the evaluation map
\[ \wedge^3 V_{10}^\vee \otimes \mathcal{O}_{Q_3[2]} \to \mathcal{F}_{L^{\otimes 3}} \tag{18} \]
factors through the composite map
\[(19) \quad \wedge^3 V_{10}^\vee \otimes \mathcal{O}_{Q_3} \to \wedge^3 \mathcal{E}_0|_{Q_3} \to \mathcal{F}.
\]
The bundles \(\mathcal{F}\) and \(\mathcal{S}_{L3}\) therefore coincide as quotients of \(\wedge^3 V_{10}^\vee \otimes \mathcal{O}_{Q_3}\): the morphisms \(Q_3 \to \text{Gr}(2, \wedge^3 V_{10})\) that they define coincide on the dense set \(U\), hence they are the same. \(\square\)

**Proof of Theorem 5.3.** We apply Proposition 4.3 by Theorem 5.2 the locus \(Z = K_{\sigma_0}\) is smooth of codimension 18 in \(M\), isomorphic to \(Q_3^{[2]}\), and, by Proposition 5.6 the rank-2 excess bundle \(\mathcal{F}\) on \(Q_3^{[2]}\) is isomorphic to \(\mathcal{S}_{\mathcal{E}_Q(3)}\). The 5-dimensional variety \(\mathcal{X}^0\) is therefore smooth with fiber over 0 the smooth zero-locus of the section \(\sigma^0\) of \(\mathcal{F}\).

More precisely, the proof of Proposition 5.6 shows that the composite map \((19)\) can be identified with the map \((18)\) induced by the (composed) evaluation map
\[\wedge^3 V_{10}^\vee \otimes \mathcal{O}_{Q_3} \to H^0(Q_3, \mathcal{O}(3)) \otimes \mathcal{O}_{Q_3} \to \mathcal{O}_{Q_3}(3).
\]
The derivative \(\frac{\partial a}{\partial \alpha}|_{\alpha=0}\) provides via the surjective map \(a\) a section of \(\mathcal{O}_{Q_3}(3)\) that defines a general \(K3\) surface \(S \subseteq Q_3\) of genus 6 and the zero-locus of \(\sigma^0\) can be identified with \(S^{[2]} \subseteq Q_3^{[2]}\). \(\square\)

### 5.2. The HLS divisor \(\mathcal{D}_{18}\).

We now construct a trivector \(\sigma_0\) whose Debarre–Voisin variety \(K_{\sigma_0}\) is smooth but has excessive dimension 10 (Corollary 5.12). The space \(V_{10}\) decomposes as \(V_7 \oplus W_3\) and \(\sigma_0\) as \(\alpha + \beta\), with \(\alpha \in \wedge^3 V_7^\vee\) and \(\beta \in \wedge^3 W_3^\vee\). For the \(\text{SL}(V_{10})\)-action, the point \([\sigma_0]\) of \(\mathbb{P}(\wedge^3 V_{10}^\vee)\) has stabilizer \(G_2 \times \text{SL}(3)\) and is polystable (Corollary 5.13). The main result of this section is Theorem 5.15.

#### 5.2.1. K3 surfaces of genus 10 (degree 18).

A general polarized \(K3\) surface \((S, L)\) of degree 18 carries a rank-2 Lazarsfeld–Mukai bundle \(\mathcal{E}\) that satisfies \(\det(\mathcal{E}) = L\) and \(c_2(\mathcal{E}) = 6\). The vector space \(V_7 := H^0(S, \mathcal{E}_2)\) has dimension 7, the sections of \(\mathcal{E}_2\) embed \(S\) into \(\text{Gr}(2, V_7)\), and via this embedding, \(S\) can be described as follows (\([\text{Mu2}]\)).

Let \(\alpha \in \wedge^3 V_7^\vee\) be general. The 7-dimensional space \(I_X \subseteq \wedge^2 V_7^\vee\) of Plücker linear sections given by \(u \cdot \alpha\), for \(u \in V_7\), cuts out a smooth fivefold \(X \subseteq \text{Gr}(2, V_7)\). We have \(K_X = \mathcal{O}_X(-3)\) and one gets a general \(K3\) surface \(S\) of degree 18 by intersecting \(X\) with a projective space \(\mathbb{P}(W_3^\perp)\) cut out by three extra general Plücker linear sections. The subspace \(I_S = I_X \oplus W_3 \subseteq \wedge^2 V_7^\vee\) of Plücker linear sections vanishing on \(S\) has dimension 10.

Recall from Section 3 that we are looking for a rank-6 vector bundle \(\mathcal{I}_6\) with determinant \(-2L + 5\delta\) on \(S^{[2]}\), in order to embed \(S^{[2]}\) in a Debarre–Voisin variety in \(\text{Gr}(6, 10)\). We will construct it as a direct sum
\[
\mathcal{I}_6 = \mathcal{I}_4 \oplus \mathcal{I}_2.
\]
We first construct the vector bundle \(\mathcal{I}_4\) as follows. The surjective evaluation map \(V_7^\vee \otimes \mathcal{O}_S \to \mathcal{E}_2\) induces, with the notation of Section 4.1, a surjective evaluation map
\[
ev: V_7^\vee \otimes \mathcal{O}_{S^{[2]}} \to \mathcal{I}_6.
\]
Indeed, the nonsurjectivity of \(\ev\) at a point \(([V_3], [V_2])\) of \(S^{[2]}\) means that the subspace \(V_3 := \langle V_2, V_2^\vee \rangle\) of \(V_7\) has dimension 3. Then, \(S \cap \text{Gr}(2, V_3)\) contains a subscheme of length 2. Since \(S\) is defined by linear Plücker equations in \(\text{Gr}(2, 7)\), it contains a line, which contradicts the fact that it is general.

Set
\[(20) \quad \mathcal{I}_4 := \mathcal{I}_{62} \subseteq V_7 \otimes \mathcal{O}_{S^{[2]}}.
\]
The following lemma will be used later on.
Lemma 5.7. The morphism $S^{[2]} \to \text{Gr}(4, V_7)$ associated with the bundle $\mathcal{J}_4$ takes value in the set of 4-dimensional vector subspaces that are totally isotropic for the 3-form $\alpha$ on $V_7$.

Proof. It is enough to check the conclusion at a general point $([V_2], [V_2'])$ of $S^{[2]}$. Then $V_2$ and $V_2'$ are transverse vector subspaces of $V_7$ which belong to $X$, hence satisfy $(\bigwedge^2 V_2) \cdot \alpha = (\bigwedge^2 V_2') \cdot \alpha = 0$ in $V_7$. The space $V_4 := \langle V_2, V_2' \rangle \subseteq V_7$ is the fiber of $\mathcal{J}_4$ at $([V_2], [V_2'])$. The restriction $\alpha' := \alpha|_{V_4}$ is a 3-form which is either decomposable with one-dimensional kernel or 0. If it is nonzero, all the elements $[U_2] \in \text{Gr}(2, V_4)$ that satisfy $U_2 \cdot \alpha' = 0$ must contain the kernel of $\alpha'$ and this contradicts the equality $V_2 \cap V_2' = \{0\}$. □

Turning to the construction of $\mathcal{J}_2$, we now show the following.

Lemma 5.8. Let $z$ be a point of $S^{[2]}$ and set $V_4 := \mathcal{J}_{4,z} \subseteq V_7$. Consider the composition

$$r_z : I_S \hookrightarrow \bigwedge^2 V_4^\vee \to \bigwedge^2 V_4^\vee.$$

Then,

(a) the kernel of $r_z$ intersects $I_X$ along a 4-dimensional vector space;
(b) the map $r_z$ has rank 4;
(c) the cokernel of $r_z$ can be identified with the fiber $\mathcal{I}_{L,z}$.

Proof. We know from the proof of Lemma 5.7 that $\alpha|_{V_4} = 0$, which implies that the 2-forms $u \cdot \alpha$, for $u \in V_4$, vanish on $V_4$. They all belong to $I_X$, so we have $\dim(\text{Ker}(r_z) \cap I_X) \geq 4$. If the inequality is strict, there is a 5-dimensional subspace $V_5$ of $V_7$, containing $V_4$ such that $u \cdot \alpha$ vanishes on $V_4$ for $u \in V_5$. But $\alpha$ then vanishes identically on $V_5$, which contradicts the fact that $\alpha \in \bigwedge^3 V_7$ is general so has no 5-dimensional totally isotropic subspace. This proves (a).

Turning to the proof of (b) and (c), the image of $r_z$ is contained in the space of sections of the Plücker line bundle on $\text{Gr}(2, V_4)$ vanishing on the length-2 subscheme $z$, and this space is 4-dimensional. It remains to see that the rank of $r_z$ is at least 4. By (a), the restriction of $r_z$ to $I_X \subseteq I_S$ has rank 3. The image $r_z(I_X)$ defines a conic in $\text{Gr}(2, V_4) \subseteq \text{Gr}(2, V_7)$ which is contained in $X$ by definition. If $r_z$ has rank only 3, this conic is contained in $S$, which contradicts the fact that $S$ is general. □

By Lemma 5.8, we have an exact sequence

$$0 \to \mathcal{J}_4'/I_S \otimes O_{S^{[2]}} \to \bigwedge^2 \mathcal{J}_4' \to \mathcal{I}_L \to 0$$

of vector bundles on $S^{[2]}$. The rank-6 vector bundle $\mathcal{J}_4'$ that it defines contains the rank-4 bundle $\mathcal{J}_4 \subseteq I_X \otimes O_{S^{[2]}}$ (see [20]) and we thus get a rank-2 bundle

$$\mathcal{J}_2 := \mathcal{J}_4'/\mathcal{J}_4 \subseteq W_3 \otimes O_{S^{[2]}}.$$

Lemma 5.9. The vector bundle $\mathcal{J}_2$ has determinant $-L + 3\delta$, the vector bundle $\mathcal{J}_4$ has determinant $-L + 2\delta$, and the vector bundle $\mathcal{J}_6$ has determinant $-2L + 5\delta$.

Proof. By (10), the determinant of $\mathcal{J}_4' = \mathcal{J}_6'$ equals $L - 2\delta$, hence $\det(\bigwedge^2 \mathcal{J}_4') = 3L - 6\delta$, while $\det(\mathcal{I}_L) = L - \delta$. Together with the exact sequence (21), this implies

$$\det(\mathcal{J}_6') = L - \delta - (3L - 6\delta) = -2L + 5\delta.$$

We then get

$$\det(\mathcal{J}_2) = \det(\mathcal{J}_6') - \det(\mathcal{J}_4) = -2L + 5\delta - (-L + 2\delta) = -L + 3\delta,$$

which proves the lemma. □
Set $\mathcal{D}_6 := \mathcal{D}_4 \oplus \mathcal{D}_2$. It is a subbundle of the trivial rank-10 bundle on $S^{[2]}$ with fiber $I_X \oplus W_3$, and this defines a morphism

\begin{equation}
\varphi = (\varphi_1, \varphi_2): S^{[2]} \longrightarrow \text{Gr}(4, V_7) \times \text{Gr}(2, W_3) \subseteq \text{Gr}(6, V_7 \oplus W_3).
\end{equation}

Lemma 5.10. If the surface $S$ is general, the morphism $\varphi$ is injective and the Plücker line bundle restricts to $2L - 5\delta$ on $S^{[2]}$.

Proof. It suffices to show that the first component $\varphi_1$ of $\varphi$ is injective. Let $z \in S^{[2]}$ and let $[V_4] := \varphi_1(z) = \mathcal{D}_{4,z} \subseteq V_7$. As we saw in the proof of Lemma 5.8, the data $V_4 \subseteq V_7$ determine a (possibly singular) conic $C$ in $\text{Gr}(2, V_4) \subseteq X$ and the image of the map $I_S \to H^0(C, \mathcal{O}_C(2))$ has rank at least 1, as otherwise the rank of the map $I_S \to \bigwedge^2 V_7^{\vee}$ would be only 3. A nonzero linear form on a conic vanishes on a line contained in the conic or along a subscheme of length 2. Since a general $S$ contains no lines, there is at most one length-2 subscheme of $S$ on this conic.

The pullback of the Plücker line bundle to $S^{[2]}$ was computed in Lemma 5.9 \hfill \Box

We will see in Proposition 5.16 that $\varphi$ is actually an embedding.

The tautological quotient bundle on the Grassmannian $\text{Gr}(6, V_7 \oplus W_3)$ pulls back via $\varphi$ to a rank-4 vector bundle on $S^{[2]}$ generated by 10 sections and with determinant $2L - 5\delta$ (Lemma 5.9).

5.2.2. The $G_2 \times \text{SL}(3)$-invariant trivector. We let $V_{10} := V_7 \oplus W_3$ and we take as before $\alpha \in \bigwedge^3 V_7^{\vee}$ general. If $\beta$ is a generator of $\bigwedge^3 W_3^{\vee}$, we let $\sigma_0 := \alpha + \beta$.

If $S$ is a K3 surface as above, the image $\varphi(S^{[2]})$ (see (23)) is, by Lemma 5.7, contained in the Debarre–Voisin variety $K_{\sigma_0}$. We first determine this variety.

Proposition 5.11. Let $V_{10}$ and $\sigma_0 = \alpha + \beta$ be as above. Any 6-dimensional subspace $W_6 \subseteq V_{10}$ which is totally isotropic for $\sigma_0$ is of the form $W_4 \oplus W_2$, where $W_4 \subseteq V_7$ is totally isotropic for $\alpha$ and $W_2 \subseteq W_3$ is of dimension 2 (hence totally isotropic for $\beta$).

Conversely, any such space is obviously totally isotropic for $\sigma_0$.

Proof. Denote by $p_1: W_6 \to V_7$ and $p_2: W_6 \to W_3$ the two projections. We first claim that $\text{rank}(p_1) \leq 5$. Indeed, on $W_6$, we have $p_1^* \alpha = p_2^* \beta$ and, as $\beta$ is decomposable, $p_2^* \beta$ vanishes on a hyperplane of $W_6$. But $\alpha$ does not vanish on any 5-dimensional subspace of $V_7$, which shows that $p_1$ must have a nontrivial kernel.

We next claim that $p_1$ cannot have rank 5. Indeed, if it does, $p_1^* \alpha$ is nonzero, so $p_2^* \beta$ is nonzero. But the kernel of $p_2^* \beta$ is then $\text{Ker}(p_2)$ and it must be equal to the kernel of $p_1^* \alpha$, that is, $p_1^{-1}(\text{Ker}(\alpha|_{\text{Im}(p_1)}))$. As $p_1$ has rank $\leq 5$, it follows that there is a nonzero $u$ in $\text{Ker}(p_1) \cap \text{Ker}(p_2)$, which is absurd. From these two facts, we conclude that $p_1$ has rank at most 4. A similar argument shows that $p_2$ has rank at most 2, that is, $p_2^* \beta = 0$, and thus $p_1^* \alpha = 0$, that is, $\alpha|_{\text{Im}(p_1)} = 0$. Finally, as $W_6 \subseteq p_1(W_6) + p_2(W_6)$, we conclude that we must haveequality. \hfill \Box

Corollary 5.12. The Debarre–Voisin variety $K_{\sigma_0}$ is smooth of dimension 10 and splits as a product $K_{\alpha} \times \mathbf{P}(W_3^{\vee})$.

Proof. Let $K_{\alpha}' \subseteq \text{Gr}(4, V_7)$ be the variety of subspaces $V_4 \subseteq V_7$ that are totally isotropic for $\alpha$. It is the zero-locus of a general section of the globally generated, rank-4, bundle $\bigwedge^3 E_4$, hence it is smooth of dimension 8. Finally, Proposition 5.11 implies $K_{\sigma_0} \simeq K_{\alpha}' \times \mathbf{P}(W_3^{\vee})$. \hfill \Box
5.2.3. **Stabilizer.** The determination of the stabilizer of our trivector $\sigma_0$ is a consequence of Proposition 5.11.

**Corollary 5.13.** The stabilizer of the trivector $\sigma_0$ in $\text{SL}(V_{10})$ is $G_2 \times \text{SL}(3)$, where $G_2$ is the stabilizer of $\alpha$ and $\text{SL}(3)$ is the stabilizer of $\beta$, and the point $[\sigma_0]$ of $\mathbb{P}(\wedge^3 V'_{10})$ is polystable for the $\text{SL}(V_{10})$-action.

**Proof.** The stabilizer $G_{\sigma_0}$ of $[\sigma_0]$ obviously contains $G_2 \times \text{SL}(3)$. For the reverse inclusion, it suffices to show that $G_{\sigma_0}$ preserves the decomposition
\[
V_{10} = V_7 \oplus W_3.
\]
Now $G_{\sigma_0}$ acts on $\text{Gr}(6, V_{10})$ preserving the Debarre–Voisin variety $K_{\sigma_0}$, which is a product $K'_\alpha \times P(W'_3)$ by Proposition 5.11. But the connected component of the automorphisms group of a product of projective varieties is the product of the connected components of its factors. Thus $G_{\sigma_0}$ acts on each factor $K'_\alpha$ and $P(W'_3)$. This implies that it preserves the direct sum decomposition (24).

To prove the polystability of $[\sigma_0]$, we invoke as before Luna’s results. Any element of the normalizer of $G_2 \times \text{SL}(3)$ in $\text{SL}(V_{10})$ must preserve the direct sum decomposition $V_{10} = V_7 \oplus W_3$, hence can be written as $\lambda g \cdot \lambda g'$, with $g, g' \in N_{\text{SL}(V_7)}(G_2)$, $g' \in \text{SL}(3)$, and $\lambda^7 \lambda'^3 = 1$.

By [Lu, Corollaire 1], the $\text{SL}(V_{10})$-orbit of $\sigma_0$ in $\wedge^3 V'_{10}$ is closed if and only if its $G_{\alpha+\beta}$-orbit is closed. Since $G_2$ has finite index in its normalizer $N_{\text{SL}(V_7)}(G_2)$ (Proposition 5.4), the latter is equivalent to the closedness of the orbit for the $C^*$-action $t \cdot (\alpha + \beta) = t^3 \alpha + t^{-1} \beta$, and this orbit is closed in $\wedge^3 V'_{10}$ because neither $\alpha$ nor $\beta$ is 0. This proves that $[\sigma_0]$ is polystable. \hfill $\square$

5.2.4. **Degenerations and excess bundles.** The Debarre–Voisin variety $K_{\sigma_0}$ is, by Corollary 5.12 smooth of codimension 14 in $\text{Gr}(6, V_{10})$ and isomorphic to $K'_\alpha \times P(W'_3)$. It is the zero-locus of a section of the rank-20 vector bundle $\wedge^3 \mathcal{E}_6$ on $\text{Gr}(6, V_{10})$, hence it carries an excess bundle $\mathcal{F}$ of rank 6, described in the following proposition.

**Proposition 5.14.** One has an isomorphism $\mathcal{F} \simeq \mathcal{Q}_2 \otimes ((\wedge^2 \mathcal{E}_4)/\mathcal{Q}_3)$ between vector bundles on $K_{\sigma_0} \simeq K'_\alpha \times P(W'_3)$, where

- the bundle $\mathcal{Q}_2$ is the pullback of the rank-2 quotient bundle on $P(W'_3)$,
- the bundle $\mathcal{E}_4$ is the pullback of the dual of the tautological rank-4 subbundle on $K'_\alpha \subseteq \text{Gr}(4, V_7)$,
- the bundle $\mathcal{Q}_3$ is the pullback of the rank-3 quotient bundle on $K'_\alpha \subseteq \text{Gr}(4, V_7)$,
- the injective map $\mathcal{Q}_3 \hookrightarrow \wedge^3 \mathcal{E}_4$ is induced by the composite map
\[
V_7 \otimes \mathcal{O}_{K_{\sigma_0}} \xrightarrow{\alpha \cdot j} \wedge^2 V'_7 \otimes \mathcal{O}_{K_{\sigma_0}} \to \wedge^2 \mathcal{E}_4.
\]

**Proof.** The excess bundle $\mathcal{F}$ is by definition the cokernel of
\[
d\sigma_0: T_{\text{Gr}(6, V_{10})} \longrightarrow \wedge^3 \mathcal{E}_6.
\]
Along $K_{\sigma_0}$, Proposition 5.11 tells us that $\mathcal{E}_6 = \mathcal{E}_4 \oplus \mathcal{Q}_2$, so that
\[
\wedge^3 \mathcal{E}_6 = \wedge^3 \mathcal{E}_4 \oplus (\wedge^2 \mathcal{E}_4 \otimes \mathcal{Q}_2) \oplus (\mathcal{E}_4 \otimes \wedge^2 \mathcal{Q}_2).
\]
On the other hand, the tangent bundle $T_{\text{Gr}(6, V_{10})}$ is isomorphic to $\mathcal{E}_6 \otimes \mathcal{E}_4$ and $d\sigma_0$ is the composition
\[
\mathcal{E}_6 \otimes \mathcal{E}_4 \to \mathcal{E}_6 \otimes \wedge^2 \mathcal{E}_6 \to \wedge^3 \mathcal{E}_6,
\]
where the second map is the wedge product map and the first one is induced by the factorization
\[
\mathcal{E}_4 \to \wedge^2 \mathcal{E}_6.
of \((\sigma_0) : V_{10} \otimes \mathcal{O}_{K_{\sigma_0}} \to \Lambda^2 \mathcal{E}_6\). We now decompose \(T_{\text{Gr}(6,V_{10})} = \mathcal{E}_6 \otimes \mathcal{E}_4\) along \(K_{\sigma_0}\) as
\[
T_{\text{Gr}(6,V_{10})} = (\mathcal{E}_4 \oplus \mathcal{Q}_2) \otimes (\mathcal{Q}_3 + \mathcal{Q}_4) = (\mathcal{E}_4 \otimes \mathcal{Q}_3) \oplus (\mathcal{Q}_2 \otimes \mathcal{Q}_3) \oplus (\mathcal{E}_4 \otimes \mathcal{Q}_1) \oplus (\mathcal{Q}_2 \otimes \mathcal{Q}_1).
\]
The composite map \((26)\) maps \((27)\) to \((25)\) preserving the decompositions and it is easy to see that the only piece with a nontrivial quotient is
\[
\mathcal{Q}_2 \otimes \mathcal{Q}_4 \longrightarrow \Lambda^2 \mathcal{E}_4 \otimes \mathcal{Q}_2,
\]
where the map is induced by \(\alpha \downarrow\). This completes the proof.

The following theorem is the main result of this section.

**Theorem 5.15.** Let \((\sigma_t)_{t \in \Delta}\) be a general 1-parameter deformation. Let \(\mathcal{H} \to \Delta\) be the associated family of Debarre–Voisin varieties and let \(\mathcal{H}^0\) be the irreducible component of \(\mathcal{H}\) that dominates \(\Delta\). Then \(\mathcal{H}^0 \to \Delta\) is smooth and its central fiber is isomorphic to \(S^{[2]}\), embedded in \(\text{Gr}(6,10)\) as in Lemma 5.10, where \(S\) is a general genus-10 K3 surface.

**Proof.** The proof follows the same line as the proof of Theorem 5.3. We apply Proposition 4.3 and conclude that the central fiber is the zero-locus of a general section of the excess bundle \(\mathcal{F}\) on \(K_{\sigma_0}\). It is in particular smooth since the excess bundle is generated by its sections. The proof is completed using Proposition 5.14 and the following proposition.

**Proposition 5.16.** Let \(S \subseteq X \subseteq \text{Gr}(2,V_7)\) be a general genus-10 K3 surface. The morphism \(\varphi\) from Lemma 5.10 induces an isomorphism between \(S^{[2]}\) and the zero-locus of \(\mathcal{F}\) on \(K_{\sigma_0}\) as in Lemma 5.10.

**Proof.** The space of global sections of \(\mathcal{F}\) is equal to \(W_3^\vee \otimes (\Lambda^2 V_7^\vee / V_7)\). We identify \(V_7\) with \(I_X\). Choosing a general section \(s\) of \(\mathcal{F}\), we thus get a K3 surface \(\tilde{S} \subseteq X\) defined by the three-dimensional space of sections \(\text{Im}(W_3 \to H^0(X, \mathcal{O}_X(1)))\).

Lemma 5.10 and the lemma below imply that \(\varphi\) is an injective morphism between \(S^{[2]}\) and the smooth zero-locus of \(s\). By Zariski’s Main Theorem, it is an isomorphism, which proves the proposition.

**Lemma 5.17.** The zero-locus of \(s\) coincides with the image \(\varphi(S^{[2]}) \subseteq K_{\sigma_0}\).

**Proof.** Let \([V_4] \in K_{\sigma_0}\) and let \(W_2 \subseteq V_3\) be of dimension 2. Assume that the section \(s\) of \(\mathcal{F}\) vanishes at \(([V_3],[W_2])\). Lifting \(s\) to an element of \(\text{Hom}(W_3, \Lambda^2 V_7^\vee)\), this means by the description of \(\mathcal{F}\) given in Proposition 5.14, that the image of the two-dimensional space \(s(W_2) \subseteq \Lambda^2 V_7^\vee\) in \(\Lambda^2 V_4^\vee\) is contained in the image \(V_3 \subseteq \Lambda^2 V_4^\vee\) of the natural map \(\alpha \downarrow\bullet:\ V_7 / V_4 \to \Lambda^2 V_4^\vee\).

The intersection of \(X\) with the Grassmannian \(\text{Gr}(2,V_4)\) is defined by the three Plücker equations given by \(V_3\). The existence of \(W_2\) as above is equivalent to saying that \(V_3\) and \(V_3\) span only a subspace of dimension 4 of \(\Lambda^2 V_7^\vee\), or, equivalently, that the length of the subscheme of \(\text{Gr}(2,V_4)\) defined by \(V_3\) and \(V_3\) is at least 2. This subscheme is equal to \(S \cap \text{Gr}(2,V_4)\). Furthermore, the space \(W_2\) is contained in the subspace of \(W_3\) vanishing on the conic defined by \(X \cap \text{Gr}(2,V_4)\). Looking at the construction of the injective morphism \(\varphi:\ S^{[2]} \to K_{\sigma_0}\) given in Lemma 5.10, we conclude that \(\varphi(S^{[2]})\) is contained in the vanishing locus of \(s\). As both are fourfolds of the same degree, they must agree. This proves the lemma.

6. The HLS divisor \(\Omega_{10}\)

Let \((S,L)\) be a general K3 surface of degree 10. As we saw in Section 3, the Hilbert square \(S^{[2]}\) with the polarization \(2L - 3\delta\) is a limit of Debarre–Voisin varieties. We will first construct a rank-4 vector bundle on \(S^{[2]}\) mapping it to \(\text{Gr}(6,10)\) and then construct a trivector \(\sigma_0\) vanishing...
on the image. It turns out that $\sigma_0$ is $\text{SL}(2)$-invariant and that the Debarre–Voisin variety $K_{\sigma_0}$ only depends on a certain $\text{SL}(2)$-invariant Fano threefold $X \subseteq \text{Gr}(2, 5)$ in which $S$ naturally sits. The rank-4 vector bundle is not globally generated and $K_{\sigma_0}$ is not irreducible in this case, but we nevertheless conclude in Theorem 6.1.4 that a 1-parameter degeneration to $\sigma_0$ expresses a general pair $(S^{[2]}, 2L - 3\delta)$ as a limit of Debarre–Voisin varieties.

### 6.1. The Fano threefold $X$ and $K3$ surfaces of genus 6

Let $V_5$ be a 5-dimensional vector space and let $W_3 \subseteq \bigwedge^2 V_5$ be a general 3-dimensional vector subspace. Let $X \subseteq \text{Gr}(2, V_5^\vee)$ be the Fano threefold of index 2 and degree 5 defined by the Plücker equations in $W_3$. It has no moduli, the variety of lines contained in $X$ is a smooth surface isomorphic to $\mathbb{P}^2$ ([10 Corollary (6.6)(ii)]), and the automorphism group of $X$ is $\text{PGL}(2)$. In fact, if $U_2$ is the standard self-dual irreducible representation of $\text{SL}(2)$ and $Q := \text{Sym}^4 U_2$, there is a direct sum decomposition

$$V_{10} := \bigwedge^2 V_5 = V_7 \oplus W_3$$

into irreducible representations, with $V_7 = \text{Sym}^6 U_2$ and $W_3 = \text{Sym}^2 U_2$, so that $X$ is the unique $\text{SL}(2)$-invariant section of $\text{Gr}(2, V_5^\vee)$ by a linear subspace of codimension 3 ([CC, Section 7.1]).

A general polarized $K3$ surface $(S, L)$ of degree 10 is obtained as a quadratic section of $X$ ([Mu2]). Let $E_2$ be the restriction to $X$ of the dual of the tautological subbundle on $\text{Gr}(2, V_5^\vee)$. Lemma 4.2 gives us a rank-4 vector bundle $\mathcal{X}_{E_2}$ on $X^{[2]}$ whose restriction $\mathcal{Q}_4$ to $S^{[2]}$ satisfies $H^0(S^{[2]}, \mathcal{Q}_4) \cong \bigwedge^2 V_5$ and $\text{det}(\mathcal{Q}_4) = 2L - 3\delta$.

**Remark 6.1.** Using the package Schubert2 of Macaulay2 ([GS]; the code can be found in [X]), one checks that the vector bundle $\mathcal{Q}_4$ has the same Segre numbers

$$s_1^4 = 1452, \ s_1^2s_2 = 825, \ s_1s_3 = 330, \ s_2^2 = 477, \ s_4 = 105$$

as the rank-4 tautological quotient bundle on Debarre–Voisin varieties $K_4 \subseteq \text{Gr}(6, 10)$, computed in [DV] (11)). The pair $(S^{[2]}, \mathcal{Q}_4)$ is therefore a candidate to be a limit of Debarre–Voisin varieties (as a subvariety of $\text{Gr}(6, 10)$). One difficulty in the present case is that the vector bundle $\mathcal{Q}_4$ is not generated by its sections (Proposition 6.2(b)). This explains why in Theorem 1.3 the central fiber is only birationally isomorphic to $S^{[2]}$.

Since $W_3$ has no rank-2 elements, for any $[x] \in \mathbb{P}(V_5)$, the subspace

$$x \wedge W_3 \subseteq \bigwedge^3 V_5 \cong \bigwedge^2 V_5^\vee$$

has dimension 3. Set

$$V_{4,[x]} := x \wedge V_5 \subseteq \bigwedge^2 V_5.$$ 

We have $\langle V_{4,[x]}, x \wedge W_3 \rangle = 0$. Setting

$$V_{7,[x]} := (x \wedge W_3)^\perp \subseteq \bigwedge^2 V_5,$$

we thus have $V_{4,[x]} \subseteq V_{7,[x]} \subseteq \bigwedge^2 V_5$. Finally, we set

$$K_1 := \{[W_6] \in \text{Gr}(6, \bigwedge^2 V_5) \mid \exists [x] \in \mathbb{P}(V_5), \ V_{4,[x]} \subseteq W_6 \subseteq V_{7,[x]}\}.$$ 

We observe that $K_1$ is smooth of dimension 6.

**Proposition 6.2.** (a) The space $\bigwedge^2 V_5$ of global sections of the rank-4 vector bundle $\mathcal{X}_{E_2}$ on $X^{[2]}$ induces a birational map

$$\varphi: X^{[2]} \dasharrow K_1 \subseteq \text{Gr}(6, \bigwedge^2 V_5)$$

which is regular outside the 4-dimensional locus in $X^{[2]}$ consisting of length-2 subschemes contained in a line contained in $X$.

(b) If $S$ is general, the restriction of $\varphi$ to $S^{[2]}$ is the map induced by global sections of $\mathcal{Q}_4$ and it is regular outside a smooth surface isomorphic to the surface of lines in $X$. 
Proof. At a point of $X^{[2]}$ corresponding to different vector subspaces $V_2, V_2' \subseteq V_5'$, the evaluation map of $\mathcal{H}_{d_2}$ is the restriction
\[ \Lambda^2 V_5 \longrightarrow V_2' \otimes V_2'. \]
It is surjective if and only if $V_2 \cap V_2' = \{0\}$, which means exactly that the line joining $[V_2]$ and $[V_2']$ is not contained in $\text{Gr}(2, V_5')$ or, equivalently, in $X$.

A nonreduced point $z = ([V_2], u)$, where $u \in \text{Hom}(V_2, V_5'/V_2)$, the fiber $\mathcal{H}_{d_2, z}$ appears in an extension
\[ 0 \rightarrow \text{Sym}^2 V_2' \rightarrow \mathcal{H}_{d_2, z} \overset{\alpha}{\longrightarrow} \Lambda^2 V_2' \rightarrow 0 \]
The composition $r: \Lambda^2 V_5 \rightarrow \Lambda^2 V_2'$ of the evaluation map $\Lambda^2 V_5 \rightarrow \mathcal{H}_{d_2, z}$ at $z$ with $\alpha$ is given by restriction, hence is surjective, and its kernel maps to $\text{Sym}^2 V_2'$ via the composite map
\[ \text{Ker}(r) \rightarrow (V_5'/V_2)^\vee \otimes V_2' \overset{u^\vee \otimes \text{id}}{\longrightarrow} V_5' \otimes V_2' \rightarrow \text{Sym}^2 V_2'. \]
This composite map (hence also the evaluation map at $z$) is surjective if and only if $u$ has (maximal) rank 2, which means exactly that the line spanned by $z$ is contained in $\text{Gr}(2, V_5')$ or, equivalently, in $X$. This proves the first part of (a), and also (b), since a general $S$ contains no lines.

It remains to prove that $\varphi$ is birational onto $K_1$. Let $[W_6] = \varphi([V_2], [V_2'])$. If $V_2$ and $V_2'$ are complementary, they span a subspace $V_5' \subseteq V_5''$ of dimension 4. Denoting by $x \in V_5$ a linear form defining $V_5'$, one has $V_{4,[x]} \subseteq W_6$. Next, $W_3|V_5'$ vanishes on $\Lambda^2 V_2$ and $\Lambda^2 V_2'$, hence
\[ W_3|V_5' \subseteq V_2' \otimes V_2''. \]
The vanishing of $W_6$ in $V_5'' \otimes V_2''$ thus implies that $W_6|V_5'$ is orthogonal to $W_3|V_5'$ for the natural pairing on $\Lambda^2 V_4$. Equivalently, $W_6$ is orthogonal to $x \wedge W_3$ for the pairing between $\Lambda^2 V_5$ and $\Lambda^3 V_5$. This shows that $\text{Im}(\varphi)$ is contained in $K_1$.

Conversely, let $[W_6]$ be a general element of $K_1$. Then
\[ V_{4,[x]} \subseteq W_6 \subseteq V_{7,[x]} \]
for some $[x] \in P(V_5)$, so that $W_6|V_5'$ has dimension 2, where $V_4'$ is defined by $x$.

Since $W_6$ is orthogonal to $x \wedge W_3$, it follows that $W_6|V_4'$ is orthogonal to $W_3|V_5'$. The 3-dimensional space $W_3|V_5' \subseteq \Lambda^2 V_4$ defines a conic $X \cap \text{Gr}(2, V_4')$ in the Grassmannian $\text{Gr}(2, V_4')$ and it is easy to check that a 2-dimensional subspace $W_2' \subseteq \Lambda^2 V_4$ cuts out two points on this conic if and only $W_2' \perp W_3|V_5'$. This shows that $K_1$ is contained in $\text{Im}(\varphi)$.

The proof that $\varphi$ is birational follows from the last argument. Indeed, pairs of points in the conic above correspond bijectively to two-dimensional subspaces of $W_3|V_5'$, at least if the conic is nonsingular. \hfill \Box

6.2. The $\text{SL}(2)$-invariant trivector. We now construct a trivector $\sigma_0$ on $V_{10} = \Lambda^2 V_5$ such that $K_1$ is a generically smooth component of the Debarre–Voisin variety $K_{\sigma_0}$.

Proposition 6.3. There exists a unique trivector $\sigma_0 \in \Lambda^3 V_{10}'$ such that, for any $[x] \in P(V_5)$, the restriction $\sigma_0|V_{7,[x]}$ comes from a nonzero element of $\Lambda^3(V_{7,[x]}/V_{4,[x]})'$. This trivector is invariant under $\text{SL}(2)$-action described in Section 6.1.

Proof. Let $\mathcal{V}_4$ be the rank-4 vector bundle on $P(V_5)$ image of the bundle map $V_5 \otimes \mathcal{O}_{P(V_5)}(-1) \rightarrow \Lambda^2 V_5 \otimes \mathcal{O}_{P(V_5)}$ given by wedge product. We define another vector bundle $\mathcal{V}_7$ on $P(V_5)$ by the exact sequence
\[ 0 \rightarrow \mathcal{V}_7 \rightarrow \Lambda^2 V_5 \otimes \mathcal{O}_{P(V_5)} \overset{\alpha}{\longrightarrow} W_3' \otimes \mathcal{O}_{P(V_5)}(1) \rightarrow 0, \]
(30)
where the map \( a \) at the point \([x]\) is the wedge product map with \( x \) with value in \( \Lambda^2 V_5 \), followed by the natural map \( \Lambda^3 V_5 \cong \Lambda^2 V_5^\vee \rightarrow W_3^\vee \). The fibers of \( \mathcal{V}_4 \) and \( \mathcal{V}_7 \) at \([x]\) \( \in \mathbb{P}(V_5) \) are the vector subspaces 
\[
V_{4,[x]} \subseteq V_{7,[x]} \subseteq \Lambda^2 V_5
\]
defined previously. There is an exact sequence
\[
(31) 0 \rightarrow \mathcal{O}_{\mathbb{P}(V_5)}(-2) \rightarrow V_5 \otimes \mathcal{O}_{\mathbb{P}(V_5)}(-1) \rightarrow \mathcal{V}_4 \rightarrow 0
\]
from which, together with \((30)\), we deduce \( \det(\mathcal{V}_4) \cong \det(\mathcal{V}_7) \cong \mathcal{O}_{\mathbb{P}(V_5)}(-3) \), hence
\[
\det(\mathcal{V}_7/\mathcal{V}_4) = \mathcal{O}_{\mathbb{P}(V_5)}.
\]
The line bundle \( \Lambda^3(\mathcal{V}_7/\mathcal{V}_4)^\vee \) thus has a nowhere vanishing section \( \omega \).

We set \( \mathcal{E}_7 := \mathcal{V}_7^\vee \). Via the inclusion \( \Lambda^3(\mathcal{V}_7/\mathcal{V}_4)^\vee \subseteq \Lambda^3 \mathcal{E}_7 \), the section \( \omega \) provides a section of \( \Lambda^3 \mathcal{E}_7 \). By Lemma 6.4 below, this section defines a unique trivector \( \sigma_0 \) with the desired properties, which proves the proposition. \( \square \)

**Lemma 6.4.** The restriction map 
\[
\Lambda^3(\Lambda^2 V_5^\vee) \otimes \mathcal{O}_{\mathbb{P}(V_5)} \rightarrow \Lambda^3 \mathcal{E}_7
\]
induces an isomorphism on global sections.

**Proof.** The dual
\[
(32) 0 \rightarrow W_3 \otimes \mathcal{O}_{\mathbb{P}(V_5)}(-1) \rightarrow \Lambda^2 V_5^\vee \otimes \mathcal{O}_{\mathbb{P}(V_5)} \rightarrow \mathcal{E}_7 \rightarrow 0
\]
of the exact sequence \((30)\) implies that the bundle \( \mathcal{G} \) defined by the exact sequence
\[
0 \rightarrow \mathcal{G} \rightarrow \Lambda^3(\Lambda^2 V_5^\vee) \otimes \mathcal{O}_{\mathbb{P}(V_5)} \rightarrow \Lambda^3 \mathcal{E}_7 \rightarrow 0
\]
has a filtration with graded pieces
\[
W_3 \otimes \Lambda^2 \mathcal{E}_7(-1), \ \Lambda^2 W_3 \otimes \mathcal{E}_7(-1), \ \Lambda^3 W_3 \otimes \mathcal{O}_{\mathbb{P}(V_5)}(-3).
\]
It thus suffices to show that these three bundles have vanishing \( H^0 \) and \( H^1 \).

This is obvious for the last bundle. For the second bundle, this follows from \((32)\). For the first bundle, we take the second exterior power of \((32)\) tensored by \( \mathcal{O}_{\mathbb{P}(V_5)}(-1) \) and get
\[
0 \rightarrow \mathcal{G}' \rightarrow \Lambda^2(\Lambda^2 V_5^\vee) \otimes \mathcal{O}_{\mathbb{P}(V_5)}(-1) \rightarrow (\Lambda^2 \mathcal{E}_7)(-1) \rightarrow 0,
\]
where the bundle \( \mathcal{G}' \) is an extension
\[
(33) 0 \rightarrow \Lambda^2 W_3 \otimes \mathcal{O}_{\mathbb{P}(V_5)}(-3) \rightarrow \mathcal{G}' \rightarrow W_3 \otimes \mathcal{E}_7(-2) \rightarrow 0.
\]
We then get the desired vanishing
\[
H^0(\mathbb{P}(V_5), \Lambda^2 \mathcal{E}_7(-1)) = 0 = H^1(\mathbb{P}(V_5), \Lambda^2 \mathcal{E}_7(-1))
\]
from the vanishings \( H^1(\mathbb{P}(V_5), \mathcal{G}') = H^2(\mathbb{P}(V_5), \mathcal{G}') = 0 \) which follow from \((33)\) and the similar vanishing for \( \mathcal{E}_7(-2) \). \( \square \)

The threefold \( X \) discussed in Section 6.1 embeds in \( \text{Gr}(3, \Lambda^2 V_5) \) as follows: a point \([V_2]\) \( \in X \) parametrizes a vector subspace \( V_2 \subseteq V_5 \) of dimension 2. Let \( V_3 \subseteq V_5 \) be the kernel of the restriction map \( V_5 \rightarrow V_2^\vee \). Then \( U_3 := \Lambda^3 V_3 \subseteq \Lambda^2 V_5 \) has dimension 3 and it determines \( V_2 \).

**Proposition 6.5.** (a) The threefold \( X \subseteq \text{Gr}(3, \Lambda^2 V_5) \) is contained in the singular locus of the Plücker hypersurface \( X_{\sigma_0} \).

(b) The rational map \( \varphi: X^{|2|} \rightarrow \text{Gr}(6, \Lambda^3 V_5) \) defined in Proposition 6.2 sends a general pair \(([V_2], [V_2'])\) to the subspace \( \langle U_3, U_3' \rangle \subseteq \Lambda^2 V_5 \).

(c) The variety \( K_1 \) is contained in the Debarre–Voisin variety \( K_{\sigma_0} \).
Proof. We first observe the following.

Lemma 6.6. Let $[V_2] \in X$ and let $V_3$ and $U_3 = \bigwedge^2 V_3$ be as above. For any $[x] \in \mathbb{P}(V_3)$, we have $U_3 \subseteq V_7,[x]$ and $\dim(U_3 \cap V_4,[x]) = 2$.

Proof. We want to show that $x \wedge W_3$ is orthogonal to $U_3$, which means that for any $w \in W_3$ and any $u \in U_3$, one has $x \wedge w \wedge u = 0$ in $\bigwedge^3 V_5$. This is clear as $x \wedge u \in \bigwedge^3 V_3$ and $w$ vanishes on $V_2$, hence belongs to $V_3 \wedge V_5$. The second statement is obvious because $U_3 \cap V_4,[x] = x \wedge V_3$. \qed

Lemma 6.6 implies that $X$ is contained in the singular locus of $X_{\sigma_0}$. Indeed, let again $[V_2] \in X$ and let $V_3$ and $U_3$ be as above. For $[x] \in \mathbb{P}(V_3)$, the 3-form $\sigma_0|_{V_7,[x]}$ is, by construction, the wedge product of 3 linear forms that vanish on $V_4,[x]$. Thus the intersection $X_{\sigma_0} \cap \text{Gr}(3,V_7,[x])$ is singular along the set of 3-dimensional subspaces of $V_7,[x]$ that intersect $U_3,[x]$ in dimension 2. Lemma 6.6 says that $U_3 \subseteq V_7,[x]$ satisfies this condition. We thus proved that $X_{\sigma_0} \cap \text{Gr}(3,V_7,[x])$ is singular at the point $[U_3]$, for any $[x] \in \mathbb{P}(V_3)$. This means that the Zariski tangent space $T_{X_{\sigma_0},[U_3]}$ contains $T_{\text{Gr}(3,V_7,[x]),[U_3]}$ for any $[x] \in \mathbb{P}(V_3)$. We then use the following fact.

Lemma 6.7. The vector subspaces $T_{\text{Gr}(3,V_7,[x]),[U_3]} \subseteq T_{\text{Gr}(3,\bigwedge^2 V_3)[U_3]}$, for $[x] \in \mathbb{P}(V_3)$, span the tangent space $T_{\text{Gr}(3,\bigwedge^2 V_3)[U_3]}$.

Proof. We have $T_{\text{Gr}(3,V_7,[x]),[U_3]} = \text{Hom}(U_3,V_7,[x]/U_3)$ and $T_{\text{Gr}(3,\bigwedge^2 V_3)[U_3]} = \text{Hom}(U_3,\bigwedge^2 V_3/U_3)$, so the lemma is equivalent to the fact that the $V_7,[x]$, for $[x] \in \mathbb{P}(V_3)$, span $\bigwedge^2 V_5$. As $V_7,[x] = x \wedge W_3^4$, the statement is equivalent to $\bigcap_{x \in V_3} (x \wedge W_3) = 0$, which is obvious. \qed

We thus proved item (a) of the proposition: $X$ is contained in the singular locus of $X_{\sigma_0}$.

By Proposition 4.4(b), there is a rational map $f : X^{[2]} \dashrightarrow K_{\sigma_0}$. Let us compare $\varphi$ and $f$. The map $\varphi$ sends $([V_2],[V_2'])$ to the kernel of the map $\bigwedge^2 V_5 \to V_2^\vee \otimes V_2'^\vee$. Since $V_3$ vanishes in $V_2^\vee$, the image of $U_3 = \bigwedge^2 V_3$ vanishes in $V_2^\vee \otimes V_2'^\vee$ and similarly for $U_3'$. It follows that

$$U_3 + U_3' = \text{Ker}(\bigwedge^2 V_5 \to V_2^\vee \otimes V_2'^\vee)$$

when both spaces have the same expected dimension 6. This proves items (b) and (c). \qed

6.3. Stabilizer, degenerations, and excess bundles. Recall that $X \subseteq \text{Gr}(2, V_5^\vee)$ is a Fano threefold of index 2 and degree 5. We have defined a trivector $\sigma_0$ on $V_{10} = \bigwedge^2 V_5$ such that the smooth sixfold $K_1$ defined in (29) is contained in $K_{\sigma_0}$ (Proposition 6.5(c)).

The birational map $\varphi : X^{[2]} \dashrightarrow K_1$ defined in Proposition 6.2 induces an isomorphism between a dense open subset $U \subseteq X^{[2]}$ and its open image. We identify $U$ with $\varphi(U)$.

Proposition 6.8. (a) The Debarre–Voisin variety $K_{\sigma_0}$ is smooth of dimension 6 along $U$, hence $K_1$ is a generically smooth irreducible component of $K_{\sigma_0}$.

(b) On $U$, the excess bundle $\mathcal{F}$ and the tautological bundle $\mathcal{O}_{X^{(2)}}$ coincide as quotients of $\bigwedge^3 V_{10} \otimes \mathcal{O}_U$.

Before giving the proof, let us note the following consequence.

Corollary 6.9. The neutral component of the stabilizer of $\sigma_0$ for the $\text{SL}(V_{10})$-action is the group $\text{SL}(2)$.

We do not prove that the point $[\sigma_0]$ is polystable.
Proof. An element \(g\) of this stabilizer acts on the Debarre–Voisin variety \(K_{\sigma_0}\) and the neutral component acts preserving the irreducible components. By Proposition 6.8, it acts on \(K_1\). But \(K_1\) is a \(\mathbb{P}^2\)-bundle over \(\mathbb{P}(V_5)\), so \(g\) (via its action on \(\text{Gr}(6, \Lambda^2 V_5)\)) has to act on the base \(\mathbb{P}(V_5)\) and this action lifts to the projective bundle \(K_1\). One easily concludes that \(g\) defines an automorphism of \(\mathbb{P}(V_5)\) whose induced action on \(\text{Gr}(2, V_5^*)\) preserves \(X\). \(\Box\)

The proof of Proposition 6.8 will use a few more preparatory steps. We start with the following easy lemma.

**Lemma 6.10.** For any \([W_6] \in K_{\sigma_0}\) and any \([x] \in \mathbb{P}(V_5)\), the vector space \(W_6 \subseteq \Lambda^2 V_5\) intersects \(V_{4,[x]}\) nontrivially; it follows that \(\dim(\mathbb{P}(W_6) \cap \text{Gr}(2, \Lambda^2 V_5)) \geq 3\).

**Proof.** The assumption is that \(\sigma_0\) vanishes on \(W_6\). The space \(V := W_6 \cap V_{7,[x]}\) is of dimension at least 3. By construction (see Proposition 6.3), the restriction of \(\sigma_0\) to \(V_{7,[x]}\) is a generator of \(\Lambda^3(V_{7,[x]}/V_{4,[x]})^\vee\), hence the vanishing of \(\sigma_0|_V\) means \(V \cap V_{4,[x]} \neq \{0\}\). Hence \(W_6 \cap V_{4,[x]} \neq \{0\}\).

For the second statement, observe that the set of \([x] \in \mathbb{P}(V_5)\) such that \(W_6 \cap V_{4,[x]} \neq \{0\}\) is the image in \(\mathbb{P}(V_5)\) of the universal \(\mathbb{P}^1\)-bundle over \(\mathbb{P}(W_6) \cap \text{Gr}(2, \Lambda^2 V_5)\). Since all \([x] \in \mathbb{P}(V_5)\) have this property, the dimension of this bundle must be at least 4. \(\Box\)

Let us show the following consequence.

**Corollary 6.11.** Let \(K'_{1}\) be an irreducible component of \(K_{\sigma_0}\) containing \(K_1\). For any \([W_6] \in K'_{1}\), there is a unique \([x] \in \mathbb{P}(V_5)\) such that \(V_{4,[x]}\) is contained in \(W_6\).

**Proof.** The uniqueness is clear, as \(x \wedge V_5 + y \wedge V_5\) has dimension 7 for nonproportional \(x, y\). For the existence, we observe that for a general \([W_6] \in K'_{1}\), the intersection \(\mathbb{P}(W_6) \cap \text{Gr}(2, \Lambda^2 V_5)\) is equal to \(\mathbb{P}(V_{4,[x]})\) with its reduced structure. We now deform \([W_6]\) to a general element \([W_6]_{\Delta}\) of the component \(K'_{1,\Delta}\), say along a family \((\mathbb{P}(W_6), t \in \Delta) \subseteq \Lambda^2 V_5\) of 6-dimensional vector subspaces. By Lemma 6.10, we know that for any \(t \in \Delta\), the intersection \(\mathbb{P}(W_6) \cap \text{Gr}(2, \Lambda^2 V_5)\) remains of dimension \(\geq 3\). As for \(t = 0\), it is reduced, of dimension 3 and degree 1, the same holds for \(t \in \Delta\) general. As the only 3-dimensional projective subspaces of \(\text{Gr}(2, \Lambda^2 V_5)\) are of the form \(\mathbb{P}(V_{4,[x]}),\) we obtain that \(W_6 = \mathbb{P}(W_6)\) for \(t\) general, contains a space \(V_{4,[x]}\).

**Proof of Proposition 6.8(a).** Let as above \(K'_{1}\) be an irreducible component of \(K_{\sigma_0}\) containing \(K_1\) and let \([W_6] \in K'_{1}\). We know by Corollary 6.11 that there exists \([x] \in \mathbb{P}(V_5)\) such that \(V_{4,[x]}\) is contained in \(W_6\). We also note from the proof of Corollary 6.11 that the point \([x] \in \mathbb{P}(V_5)\) is general. There is a short exact sequence

\[
0 \rightarrow V_{4,[x]} \rightarrow \Lambda^2 V_5 \rightarrow \Lambda^2 V_{4,[x]} \rightarrow 0.
\]

Here, \(V_{4,[x]} = x \wedge V_5\) is seen on the left as a subspace of \(\Lambda^2 V_5\) and on the right as the quotient \(V_5/\mathbb{C}x\).

The trivector \(\sigma_0 \in \Lambda^3(\Lambda^2 V_5)^\vee\) vanishes in the first quotient \(\Lambda^3 V_{4,[x]}^\vee\), hence it has an image \(\sigma_{0,x}\) in the next step of the filtration on \(\Lambda^3(\Lambda^2 V_5)^{\vee}\) associated with (34), namely \(\Lambda^2 V_{4,[x]}^\vee \otimes \Lambda^2 V_{4,[x]}^\vee\). From the construction of \(\sigma_0\), we know that \(\sigma_{0|V_{7,[x]}}\) comes from \(\Lambda^1(V_{7,[x]}/V_{4,[x]})^\vee\), which implies that \(\sigma_{0|V_{7,[x]}}\) vanishes in \((V_{7,[x]}/V_{4,[x]})^\vee \otimes \Lambda^2 V_{4,[x]}^\vee\), or equivalently belongs to \((x \wedge W_3) \otimes \Lambda^2 V_{4,[x]}^\vee\), where we identify \(x \wedge W_3 \subseteq \Lambda^2 V_{4,[x]}^\vee\) as defining \(V_{7,[x]}\) (so that its image in \(\Lambda^2 V_{4,[x]}^\vee\) defines \(V_{7,[x]}/V_{4,[x]}\)).

Let us examine \(\sigma_{0,x} \in (x \wedge W_3) \otimes \Lambda^2 V_{4,[x]}^\vee\). We claim the following.

**Lemma 6.12.** For \([x] \in \mathbb{P}(V_5)\) general, the rank of \(\sigma_{0,x}\) is 3.
Proof. Recall that $V_5$ and $W_3$ are irreducible representations of $SL(2)$ (Section 6.1). The trivector $\sigma_0$ is invariant under the induced $SL(2)$-action on $\Lambda^3 V_5^\vee = \Lambda^3 (\Lambda^2 V_5)^\vee$.

From (30), we see that $V_{4,[x]}$, seen as a quotient of $V_5$, is the fiber at $[x]$ of the vector bundle $\mathcal{V}'_4 := \mathcal{V}_4(1)$. Since $H^0(P(V_5), \Lambda^2 \mathcal{V}') \simeq \Lambda^2 V_5$ and $W_3 \subseteq \Lambda^2 V_5$ is general, there is an injection

$$W_3 \otimes \mathcal{O}(P(V_5)) \hookrightarrow \Lambda^2 \mathcal{V}'_4$$

whose dual is a surjection $\Lambda^2 \mathcal{V}'_4 \twoheadrightarrow W_3 \otimes \mathcal{O}(P(V_5))$. The tensors $\sigma_{0,x}$ globalize to a section $\sigma_0$ of the bundle $W_3 \otimes \Lambda^2 \mathcal{V}'_4 \otimes \mathcal{O}(P(V_5))$. Since $\det(\mathcal{V}'_4) = \mathcal{O}(P(V_5))$, we have

$$\Lambda^2 \mathcal{V}'_4 \otimes \mathcal{O}(P(V_5)) \simeq \Lambda^2 (\mathcal{V}_4(1)),$$

hence $\sigma_0$ is a section of the bundle $W_3 \otimes \Lambda^2 \mathcal{V}'_4$. We also have

$$H^0(P(V_5), W_3 \otimes \Lambda^2 \mathcal{V}'_4 = W_3 \otimes H^0(P(V_5), \Lambda^2 \mathcal{V}_4) = W_3 \otimes \Lambda^2 V_5.$$  

It follows that $\sigma_0$ provides an element of $W_3 \otimes \Lambda^2 V_5$ which must be $SL(2)$-invariant. The decomposition $[28]$ tells us that there is exactly one such element, $\text{Id}_{W_3}$ (we use the isomorphism $W_3 \simeq W_5^\vee$ given by the $SL(2)$-action). The conclusion of this analysis is that either $\sigma_0 = 0$ or the rank of $\sigma_{0,x}$ is 3.

To finish the proof of the lemma, we just have to exclude the case $\sigma_0 = 0$. If this vanishing holds, $\sigma_0$ vanishes on any 3-dimensional subspace of $\Lambda^2 V_5$ that intersects one $x \wedge V_5$ along a 2-dimensional subspace. It is easy to exclude this possibility: the condition says that $\sigma_0 \in \Lambda^3 (\Lambda^2 V_5^\vee)$ vanishes on all elements of the form

$$(x \wedge y) \wedge (x \wedge z) \wedge (v \wedge w) \in \Lambda^3 (\Lambda^2 V_5)$$

for $x, y, z, v, w \in V_5$. But this would force $\sigma_0 = 0$, because these elements span $\Lambda^3 (\Lambda^2 V_5)$. Indeed, this space is generated by general decomposable elements of the form $m = (x \wedge y) \wedge (t \wedge z) \wedge (v \wedge w)$. By generality, we have $v = \alpha x + \beta y + \gamma t + \delta z + \varepsilon w$ and expanding $m$, we get a sum of terms of type (35). \qed

Let us go back to the point $[W_6]$ of $K'_1$, where $W_6$ contains $V_{4,[x]}$ for some general $[x] \in P(V_5)$. Since $\sigma_0|_{W_6} = 0$, the tensor $\sigma_{0,x}$ vanishes in $(W_6/V_{4,[x]})^\vee \otimes \Lambda^2 V_{4,[x]}$. By Lemma 6.12, we conclude that $x \wedge V_3$ has to vanish on $W_6$, that is $W_6 \subseteq V_{7,[x]}$. Thus $[W_6] \in K_1$ and we proved that $K_1$ is an irreducible component of $K_{\sigma_0}$.

In order to prove that $K_1$ and $K_{\sigma_0}$ are equal as schemes generically along $K_1$, we observe that the argument just given is of an infinitesimal nature, hence proves that $K_1$ and $K_{\sigma_0} \cap \text{Gr}(6, x, \Lambda^2 V_5)$ are equal as schemes generically along $K_1$, where $\text{Gr}(6, x, \Lambda^2 V_5) \subseteq \text{Gr}(6, \Lambda^2 V_5)$ is the set of $W_6 \subseteq \Lambda^2 V_5$ such that $x \wedge V_5 = V_{4,x} \subseteq W_6$ for some $x \in P(V_5)$. In order to conclude, we thus just need to show that $K_{\sigma_0}$ is schematically contained in $\text{Gr}(6, x, \Lambda^2 V_5)$ generically along $K_1$. This is a consequence of the following infinitesimal version of Corollary 6.11. \qed

Lemma 6.13. Let $[W_6] \in K_1$ be general and let $x \in P(V_5)$ be such that $V_{4,x} \subseteq W_6$. For any first order deformation $[W_{6,x}]$ of $[W_6]$ in $K_{\sigma_0}$, there exists a first order deformation $x_e$ of $x$ such that, at first order, $V_{4,x_e} = x_e \wedge V_5 \subseteq W_{6,x}$.  

Proof. Let $x \wedge y \in P(V_{4,x})$ be such that

$$(x \wedge y) \subseteq V_5 \subseteq (x \wedge y).$$

The proof of Lemma 6.12 shows that there exists a unique first order deformation $y_e \in P(V_{4,y}) \subseteq \text{Gr}(2, V_5)$ such that $W_{6,x} \cap (y \wedge V_5) = (y \wedge y_e)$. Since $[W_6]$ is a general point of $K_1$, the set of points $y$ satisfying (36) is the complement of a closed algebraic subset of codimension $\geq 2$ in $P(V_{4,x})$. The collection of $y_e$ thus extends to a first order deformation of $P(V_{4,x})$ in $\text{Gr}(2, V_5)$. But the latter are in bijection with the first order deformations of $x \in P(V_5)$. \qed
Proof of Proposition 6.8(b). We are exactly in the setting of Lemma 4.5 and Remark 4.6 by Proposition 6.5(a), there is an embedding $j: X \hookrightarrow \text{Sing}(X_{\sigma_0}) \subseteq \text{Gr}(3, V_{10})$: it induces a map $\varphi: X^{[2]} \rightarrow K_1$, where $K_1$ is a generically reduced 6-dimensional component of $K_{\sigma_0}$ (Proposition 6.8(a)). The map $\varphi$ is birational by Proposition 6.2 and $j^*\mathcal{O}_{\text{Gr}(3, V_{10})}(1) = \mathcal{O}_X(2)$.

On $U$, the vector bundles $\mathcal{F}_t^{\sigma_0(3, V_{10})}(1) = \mathcal{F}_{\bar{U}(2)}$ and $\mathcal{F}$ both have rank 2 and are quotients of $\Lambda^3 V_{10}^\vee \otimes \mathcal{O}_U$; furthermore, Lemma 4.5 says that the evaluation map
$$\text{ev}: \Lambda^3 V_{10}^\vee \otimes \mathcal{O}_U \to \mathcal{F}_{\bar{U}(2)}$$
factors through $\mathcal{F}$. This proves that they are the same. \qed

We finally prove our main result.

**Theorem 6.14.** Let $(\sigma_t)_{t \in \Delta}$ be a very general 1-parameter deformation. Over a finite cover $\Delta' \rightarrow \Delta$, there is a family of smooth polarized hyperkähler fourfolds $\mathcal{X}' \rightarrow \Delta'$ such that a general fiber $\mathcal{X}'_t$ is isomorphic to $K_{\sigma_t}$ and the central fiber is isomorphic to $S^{[2]}$, where $(S, L)$ is a general K3 surface of genus 6, with the polarization $2L - 3\delta$.

**Proof.** Let $\mathcal{X} \rightarrow \Delta$ be the associated family of Debarre–Voisin varieties, let $\mathcal{X}^0$ be the irreducible component of $\mathcal{X}$ that dominates $\Delta$, and let $U \subseteq K_{\sigma_0} = \mathcal{X}_0$ be the Zariski open set of Proposition 6.8. Then $\mathcal{X}_0$ is smooth of dimension 6 along $U$, so that the analysis of Section 4.2 applies.

By Proposition 6.8(b), on $U$, the excess bundle $\mathcal{F}$ can be identified with $\mathcal{F}_{\bar{U}(2)}$ as quotients of $\Lambda^3 V_{10}^\vee$. The element $\sigma_0$ thus gives a section $f$ of $\mathcal{F}_{\bar{U}(2)}$ and we conclude that if $\sigma_0$ is general enough, the zero-locus of $\sigma_0$ is equal to $S^{[2]} \cap U$, where $S \subseteq X$ is the K3 surface defined by $f$.

Moreover, the open subset $S^{[2]} \cap U$ is then dense in $S^{[2]}$ and we thus proved that the central fiber of $\mathcal{X}^0$ has one reduced component which is birationally isomorphic to $S^{[2]}$. By [KLSV], it follows that after base change $\Delta' \rightarrow \Delta$ and shrinking, there exists a family $\pi': \mathcal{X}' \rightarrow \Delta'$ that is fiberwise birationally isomorphic to $\mathcal{X}^0 \times_{\Delta} \Delta'$, all of whose fibers are smooth hyperkähler fourfolds, with (smooth) central fiber birationally isomorphic to $S^{[2]}$. Since $S^{[2]}$ has no nontrivial hyperkähler birational models (Section 3.3), the central fiber is in fact isomorphic to $S^{[2]}$.

The varieties $\mathcal{X}_t$, for $t$ very general, have Picard number 1, hence no nontrivial smooth hyperkähler birational models. It follows that $\mathcal{X}_t' \simeq \mathcal{X}_t$ and this holds for all $t \neq 0$. \qed

**Remark 6.15.** From the viewpoint of subvarieties of $\text{Gr}(6, V_{10})$, the situation is not completely explained. The varieties $\mathcal{X}_t$ are smooth subvarieties of $\text{Gr}(6, V_{10})$ of degree 1452. The variety $S^{[2]}$ is mapped to $\text{Gr}(6, V_{10})$ via the rational map $\varphi$ described in Proposition 6.2 but since this map is not regular, its image $\varphi(S^{[2]}) \subseteq \text{Gr}(6, V_{10})$ has degree $< 1452$. The limit (in the Hilbert scheme) of the subvarieties $\mathcal{X}_t \subseteq \text{Gr}(6, V_{10})$ must therefore have another irreducible component.

7. **The HLS divisor $D_2$**

We describe a polystable point in the moduli space $\mathcal{M}_{DV} = \mathbb{P}(\Lambda^3 V_{10}^\vee)/\text{SL}(V_{10})$ whose total image by the moduli map
$$m: \mathcal{M}_{DV} \rightarrow \mathcal{M}$$
is the divisor whose general points are the fourfolds $\mathcal{M}(S(0, L, 1)$ described in Remark 3.2 where $(S, L)$ is a general polarized K3 surface of degree 2. As explained in Section 3 this divisor is therefore the Heegner divisor $D_2$. 

7.1. The SL(3)-invariant trivector. We take $V_{10} := \text{Sym}^3 W_3$. The $\text{SL}(W_3)$-representation $\wedge^3 V_{10}^\vee$ decomposes as

$$\wedge^3 V_{10}^\vee = \wedge^3 (\text{Sym}^3 W_3^\vee) = \Gamma_{0,6} \oplus \Gamma_{3,3} \oplus \Gamma_{2,2} \oplus \Gamma_{0,0},$$

where $\Gamma_{a,b}$ is the irreducible representation given by the kernel of the contraction map $\text{Sym}^a W_3 \otimes \text{Sym}^b W_3^\vee \to \text{Sym}^{a-1} W_3 \otimes \text{Sym}^{b-1} W_3^\vee$. The first term is $\text{Sym}^6 W_3^\vee = H^0(\mathbb{P}(W_3), \mathcal{O}_{\mathbb{P}(W_3)}(6))$. The last term is the (1-dimensional) space of $\text{SL}(W_3)$-invariants and we pick a generator $\sigma_0$.

This trivector $\sigma_0$ can be constructed via the “symbolic method” as follows (thanks to Claudio Procesi). Choose a generator $\eta$ for $\wedge^3 W_3$ and write $a \wedge b \wedge c := \text{det}(a, b, c)\eta$ for all $a, b, c \in W_3$. Then $\sigma_0$ is the unique trivector on $V_{10}$ such that

$$\forall x, y, z \in W_3 \quad \sigma_0(x^3, y^3, z^3) = \text{det}(x, y, z)^3$$

(it is alternating and $\text{SL}(W_3)$-invariant because it is so when the entries are cubes). Let $(x, y, z)$ be a basis for $W_3$ and write $\alpha \in \text{Sym}^3 W_3$ as

$$\alpha = \alpha_{000} x^3 + \alpha_{030} y^3 + \alpha_{003} z^3 + 3(\alpha_{012} x^2 y + \alpha_{012} x z^2 + \alpha_{012} y^2 z + \alpha_{021} xy^2 + \alpha_{021} x^2 z + \alpha_{021} y z^2) + 6\alpha_{111} xyz.$$

A straightforward computation (umbral calculus) shows that

$$\sigma_0(\alpha, \beta, \gamma) = \sum_{\tau \in \mathcal{P}} \varepsilon(\tau) \alpha_{\tau(3,0,0)} \beta_{\tau(0,3,0)} \gamma_{\tau(0,0,3)} - 3 \sum_{\tau \in \mathcal{P}} \varepsilon(\tau) \alpha_{\tau(0,3,0)} \beta_{\tau(2,0,1)} \gamma_{\tau(0,1,2)} - 3 \sum_{\tau \in \mathcal{P}} \varepsilon(\tau) \alpha_{\tau(0,0,3)} \beta_{\tau(2,1,0)} \gamma_{\tau(1,0,2)} - 3 \sum_{\tau \in \mathcal{P}} \varepsilon(\tau) \alpha_{\tau(0,2,1)} \beta_{\tau(2,1,0)} \gamma_{\tau(0,1,2)} - 6 \sum_{\tau \in \mathcal{P}} \varepsilon(\tau) \alpha_{\tau(2,1,0)} \beta_{\tau(1,0,2)} \gamma_{\tau(1,1,1)} - 6 \sum_{\tau \in \mathcal{P}} \varepsilon(\tau) \alpha_{\tau(2,0,1)} \beta_{\tau(2,0,1)} \gamma_{\tau(1,1,1)}.$$

In each sum above, $\mathcal{P}$ denotes the permutation group of the relevant subset of the family of indices. In particular, we get the following.

**Lemma 7.1.** For each $r \in \{1, 2, 3\}$, let $x^{i_r} y^{j_r} z^{k_r}$ be a degree-$3$ monomial. Then

$$\sigma_0(x^{i_1} y^{j_1} z^{k_1}, x^{i_2} y^{j_2} z^{k_2}, x^{i_3} y^{j_3} z^{k_3}) \neq 0$$

if and only if $i_1 + i_2 + i_3 = j_1 + j_2 + j_3 = k_1 + k_2 + k_3 = 3$ and not all monomials are equal to $xyz$.

7.2. The hypersurface $X_{\sigma_0}$. The equation of the hypersurface $X_{\sigma_0} \subseteq \text{Gr}(3, \text{Sym}^3 W_3)$ defined in (2) is given by (39). More precisely, order the multiindices as in Table 2 and denote the corresponding Plücker coordinates on $\wedge^3 (\text{Sym}^3 W_3)$ by $q_{012}, \ldots, q_{789}$; then $X_{\sigma_0}$ is the intersection of $\text{Gr}(3, \text{Sym}^3 W_3)$ with the hyperplane

$$q_{012} - 3(q_{058} + q_{147} + q_{236} + q_{345} + q_{678}) - 6(q_{389} + q_{469} + q_{579}) = 0.$$

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7.2.1. The singular locus of \(X_{s_0}\). We show in this section that the hypersurface \(X_{s_0}\) is singular along a surface which we first describe. Let

\[
V \subseteq \mathbb{P}(\text{Sym}^3 W_3)
\]

be the Veronese embedding and let \(V \subseteq \mathbb{P}(\text{Sym}^3 W_3)\) be its image. The projective tangent space to \(V\) at \([x^3]\) is \(\mathbb{P}(x^2 \cdot W_3)\), hence the Gauss map of \(V\) is

\[
\nu : V \to \mathbb{P}(\text{Sym}^3 W_3)
\]

(41)

Proof. Let \(u_3\) be a basis of \(x, y, z\) spanned by monomials in \(x, y, z\) and \(r, s\) such that

\[
\langle r \rangle \supseteq \langle s \rangle > \langle m_3 \rangle > \langle m_4 \rangle.
\]

(42)

Assume that \([0] \neq [\mathbb{P}(W_3)]\). Given \([W_3] \neq [\mathbb{P}(W_3)]\), we denote by \(V(\varphi) \subseteq \mathbb{P}(W_3)\) its zero-locus and, given a vector subspace \(U \subseteq \text{Sym}^3 W_3\), we let

\[
L(U) := \{ \varphi \in \mathbb{P}(U) \mid [\varphi] \in \mathbb{P}(U) \} \subseteq [\nu_{W_3}] (3)
\]

be the associated linear system.

Lemma 7.4. Let \(U_3 \subseteq \text{Sym}^3 W_3\) be a 3-dimensional subspace. Suppose that one of the following holds:

(a) there exists \([\varphi] \in \mathbb{P}(U_3)\) such that \(V(\varphi)\) is singular at a point \(p \in \mathbb{P}(W_3)\) not contained in the base-locus of \(L(U_3)\);

(b) there exists an element of \(L(U_3)\) with an ordinary node.

Then \([U_3]\) is not a singular point of \(X_{s_0}\).

Proof. Assume that \([U_3]\) is a singular point of \(X_{s_0}\). We will reach a contradiction in both cases. Suppose that (a) holds. Let \((x, y, z)\) be a basis of \(W_3\) such that \(p = (0,0,1)\). Then \(\varphi = f_2(x, y)z + \ldots\)
Let \( f_3(x, y) \), where \( f_2, f_3 \) are homogeneous of respective degrees 2 and 3, not both zero. By assumption, there exists \( [ψ] \in \mathbf{P}(U_3) \) such that \( p \notin V(ψ) \). Thus \( ψ = z^3 + f_1(x, y)z^2 + f_2(x, y)z + f_3(x, y) \). Let \( λ \) be the 1-parameter subgroup of \( \text{GL}(W_3) \) given (in the chosen basis) by

\[
λ(t) = \text{diag}(t^{n+1}, t^n, 1), \quad n \geq 3.
\]

Let \( \overline{U}_3 := \lim_{t \to 0} λ(t)U_3 \). The hypersurface \( X_{σ_0} \) is mapped to itself by \( \text{SL}(W_3) \), hence it is singular at \( λ(t)U_3 \) for all \( t \in \mathbb{C}^* \), hence also at \( \overline{U}_3 \). A simple computation shows that if \( f_2 \neq 0 \), then \( \lim_{t \to 0} λ(t)[ψ] = [x'y^3z] \), where \( x'y^3 \) is the monomial with highest power of \( y \) appearing in \( f_2 \), and that if \( f_2 = 0 \), then \( \lim_{t \to 0} λ(t)[ψ] = [x'y^3] \), where \( x'y^3 \) is the monomial with highest power of \( y \) appearing in \( f_3 \).

On the other hand, \( \lim_{t \to 0} λ(t)[ψ] = [z^3] \). The subspace \( \overline{U}_3 \) is generated by monomials in \( x, y, z \), because the weights of the action of \( λ \) on \( \text{Sym}^3W_3 \) are pairwise distinct. Thus \( \overline{U}_3 \) is generated by monomials in \( x, y, z \) and contains \( z^3 \) and one of \( x'y^3z, x'y^3 \). By Lemma 7.3 \( \overline{U}_3 \) is not contained in \( \text{Sing}(X_{σ_0}) \). This is a contradiction.

Suppose now that (b) holds. By assumption, there exist a basis \((x, y, z) \) of \( W_3 \) and \( [ψ] \in \mathbf{P}(U_3) \) such that \( ψ = xyz + f_3(x, y) \). Let \( λ \) be the 1-parameter subgroup in (45) and set \( \overline{U}_3 := \lim_{t \to 0} λ(t)U_3 \). Arguing as above, we get that \( X_{σ_0} \) is singular at \( \overline{U}_3 \). A simple computation shows that \( \lim_{t \to 0} λ(t)[xyz + f_3(x, y)] = [xyz] \). Since \( \overline{U}_3 \) is generated by monomials in \( x, y, z \), this contradicts Lemma 7.3.

We now prove the reverse inclusion \( \text{Sing}(X_{σ_0}) \subseteq g(V) \). Let \([U_3] \in \text{Sing}(X_{σ_0}) \). One of the following holds:

(a) there exists \([ψ] \in \mathbf{P}(U_3) \) such that \( V(ψ) \) is singular at a point not contained in the base-locus of \( L(U_3) \);
(b) the base-locus of \( L(U_3) \) is zero-dimensional and all curves in \( L(U_3) \) are smooth outside the base-locus;
(c) the base-locus of \( L(U_3) \) is one-dimensional and all curves in \( L(U_3) \) are smooth outside the base-locus.

If (a) holds, \([U_3] \) is not a singular point of \( X_{σ_0} \) by Lemma 7.4. This is a contradiction.

Suppose that (b) holds. We claim that there exists \( p \in \mathbf{P}(W_3^γ) \) such that all elements of \( L(U_3) \) are singular at \( p \). The set

\[
Σ := \{(p, [ψ]) \in \mathbf{P}(W_3^γ) \times L(U_3) \mid p \text{ is a singular point of } V(ψ)\}
\]

is the intersection of 3 divisors in \( |O_{\mathbf{P}(W_3^γ)}(2) \boxtimes O_{L(U_3)}(1)| \). If \( Σ \) has (pure) dimension 1, its projection to \( \mathbf{P}(W_3^γ) \) is a sextic curve, which contradicts (b). Hence \( \dim(Σ) > 1 \) and there exists a point \( p \) such that all curves in \( L(U_3) \) are singular at \( p \). By Lemma 7.4(b), no element of \( L(U_3) \) has an ordinary node at \( p \). It follows that there are linearly independent \([ψ_1], [ψ_2] \in \mathbf{P}(U_3) \) such that \( V(ψ_1) \) and \( V(ψ_2) \) have multiplicity 3 at \( p \). Thus, there exists a nonzero linear combination \( c_1ψ_1 + c_2ψ_2 \) such that \( V(c_1ψ_1 + c_2ψ_2) \) is singular along a line. This contradicts our assumption (b).

Lastly, suppose that (c) holds. The base-locus of \( L(U_3) \) is either a line or a conic (possibly degenerate). Assume that it is a line \( R \). By Lemma 7.4(b), no element of \( L(U_3) \) has an ordinary node. This forces \( L(U_3) \) to be \( R + L_0 \), where \( L_0 \subseteq |O_{\mathbf{P}(W_3^γ)}(2)| \) is one of the following:

(α) the linear system of conics tangent to \( R \) at a fixed \( p \in R \) and containing a fixed \( q \in \mathbf{P}(W_3^γ) \setminus R \);
(β) the linear system of conics with multiplicity of intersection at least 3 with a fixed smooth conic tangent to \( R \) at a fixed \( p \in R \);
(γ) the linear system of conics singular at a fixed \( p \in R \).
If (α) holds, there exists a basis \((x, y, z)\) of \(W_3\) such that \(U_3 = \langle x^2 y, x y^2, y^2 z \rangle\). This contradicts Lemma 7.3.

If (β) holds, there exists a basis \((x, y, z)\) of \(W_3\) such that \(U_3 = \langle x^3 + y^2 z, x y^2, y^3 \rangle\). Let \(\lambda\) be the 1-parameter subgroup of \(\text{GL}(W_3)\) given by \(\lambda(t) = \text{diag}(t^{-1}, t^{-3}, 1)\). Then \(\lim_{t \to 0} U_3 = \langle x^3, x y^2, y^3 \rangle\), which contradicts Lemma 7.3.

If (γ) holds, there exists a basis \((x, y, z)\) of \(W_3\) such that \(U_3 = \langle x^2 y, x y^2, y^3 \rangle\) and this contradicts Lemma 7.3.

This proves that the base-locus of \(L(U_3)\) is not a line, hence it is a conic. If the conic has rank at least 2, there are elements of \(L(U_3)\) with an ordinary node and this contradicts Lemma 7.4. Hence the base-locus of \(L(U_3)\) is a double line, that is, \([U_3] \in g(V)\). □

7.2.2. The germ of \(X_{\sigma_0}\) at its singular points. The local structure of \(X_{\sigma_0}\) at its singular points will be needed in the proof of Theorem 7.20.

Lemma 7.5. Let \(p\) be a singular point of \(X_{\sigma_0}\). The (analytic) germ \((X_{\sigma_0}, p)\) is isomorphic to the germ \((\Delta^2 \times (\sum_{i=1}^{19} c_i^2 = 0)), 0\).

Proof. Let \(p := [U_3]\) and let \((x, y, z)\) be a basis of \(W_3\) such that \(U_3 = \langle x^3, x y^2, y^2 z \rangle\). We write a local equation of \(X_{\sigma_0}\) in a neighborhood of \(p\), adopting the notation in Sections 7.1 and 7.2. In particular, coordinates on \(\text{Sym}^3 W_3\) are defined by (38) and we order them as in Table 2. Now \(p\) has coordinates \(q_{037} = 1\) and \(q_{ijk} = 0\) for \(\{i, j, k\} \neq \{0, 3, 7\}\). Affine coordinates on the open subset

\[\text{Gr}(3, \text{Sym}^3 W_3)_{q_{037}} \subseteq \text{Gr}(3, \text{Sym}^3 W_3)\]

defined by \(q_{037} \neq 0\) are given by \(q'_{ijk} := q_{ijk}/q_{037}\) for all \(0 \leq i < j < k \leq 9\) such that exactly two of the indices \(i, j, k\) belong to \(\{0, 3, 7\}\). By (40), \(X_{\sigma_0} \cap \text{Gr}(3, \text{Sym}^3 W_3)_{q_{037}}\) has equation

\[
0 = q'_{013}q'_{027} - q'_{017}q'_{023} - 3(q'_{035}q'_{057} - q'_{038}q'_{057} + q'_{017}q'_{347} + q'_{047}q'_{137} - q'_{020}q'_{367}) + 3(q'_{036}q'_{027} - q'_{034}q'_{357} + q'_{065}q'_{347} + q'_{067}q'_{357} - q'_{076}q'_{367}) + \text{cubic term}.
\]

The tangent cone of \(X_{\sigma_0}\) at \(p\) is defined by the vanishing (in \(\mathbb{C}^2\)) of this quadratic term. A computation gives

\[T_{g(V),p} = \left< \frac{\partial}{\partial q_{039}} + 2 \frac{\partial}{\partial q'_{067}}, 2 \frac{\partial}{\partial q'_{034}} + \frac{\partial}{\partial q'_{079}} \right>.
\]

Another computation shows

\[T_{g(V),p} = \text{Ker}(\varphi).
\]

This proves the lemma. □

7.3. The variety \(K_{\sigma_0}\). We describe in Proposition 7.8 the Debarre–Voisin variety \(K_{\sigma_0}\) associated with the trivector \(\sigma_0\) on \(V_{10} = \text{Sym}^3 W_3\) defined in Section 7.1.

7.3.1. Two distinguished subvarieties of \(K_{\sigma_0}\).

Definition 7.6. (a) Given \([a] \in \text{P}(W_3^*)\) and a codimension 1 subspace \(H \subseteq \text{Sym}^2(a^\perp)\), let

\[I(a, H) := \text{image of } H \text{ via the inclusion } (\text{Sym}^2(a^\perp) \hookrightarrow \text{Sym}^3 W_3),\]

\[L(a, H) := (a \cdot I(a, H))^\perp \subseteq \text{Sym}^3 W_3.
\]

Note that \(\dim I(a, H) = 2\) and \(\dim L(a, H) = 6\).

(b) Given \([a] \in \text{P}(W_3^*)\) and \([x] \in \text{P}(W_3)\), let

\[J(a, x) := x \cdot \text{Ker}(a) \subseteq \text{Sym}^2 W_3,
\]

\[M(a, x) := (a \cdot J(a, x))^\perp \subseteq \text{Sym}^3 W_3
\]
Note that \( \dim J(a, x) = 2 \) and \( \dim M(a, x) = 6 \).

(c) Finally, define two irreducible subvarieties of \( \text{Gr}(6, V_{10}) \) by setting

\[
K_L := \{ [L(a, H)] \mid [a] \in \mathbb{P}(W_3^\vee), \ H \subseteq \text{Sym}^2(a^\perp) \text{ hyperplane} \},
\]

\[
K_M := \{ [M(a, x)] \mid [a] \in \mathbb{P}(W_3^\vee), \, [x] \in \mathbb{P}(W_3) \}.
\]

We list the subspaces \( a \cdot I(a, H)^\perp \) and \( a \cdot J(a, x)^\perp \) up to isomorphism. First notice that there exist linearly independent \( x, y \in W_3 \) such that

\[ H = \langle x^2, y^2 \rangle \text{ or } H = \langle x^2, xy \rangle. \]

As is easily checked, there exists a basis \((a, b, c)\) of \( W_3^\vee \) such that

\[
a \cdot I(a, H)^\perp = \begin{cases} a \cdot \langle a^2, ab, ac, bc \rangle & \text{if } H = \langle x^2, y^2 \rangle, \\ a \cdot \langle a^2, ab, ac, c^2 \rangle & \text{if } H = \langle x^2, xy \rangle, \end{cases}
\]

\[
a \cdot J(a, H)^\perp = \begin{cases} a \cdot \langle a^2, b^2, bc, c^2 \rangle & \text{if } a(x) \neq 0, \\ a \cdot \langle a^2, ab, ac, c^2 \rangle & \text{if } a(x) = 0. \end{cases}
\]

We now show that the varieties \( K_L \) and \( K_M \) are both contained in \( K_{\sigma_0} \).

**Proposition 7.7.** (a) The subvariety of \( K_{\sigma_0} \) obtained from the surface \( g(V) \subseteq \text{Sing}(X_{\sigma_0}) \) by the procedure described in Proposition 4.4(b) is \( K_L \).

(b) The variety \( K_M \) is contained in \( K_{\sigma_0} \).

**Proof.** By Proposition 4.4(b), for \( x, y \in W_3 \) linearly independent, the 6-dimensional subspace \( x^2 \cdot W_3 + y^2 \cdot W_3 \subseteq \text{Sym}^3 W_3 \) corresponds to a point of \( K_{\sigma_0} \). This is exactly \( L(a, H) \), where \( a^\perp = \langle x, y \rangle \) and \( H = \langle x^2, y^2 \rangle \). Since \( K_L \) is irreducible of dimension at most 4, this proves (a).

By (46), if \( a(x) \neq 0 \), then \( M(a, x) = \langle x^2 y, x^2 z, y^3, y^2 z, y z^2, z^3 \rangle \) in a suitable basis \((x, y, z)\) of \( W_3 \). By Lemma 7.1 this is a point of \( K_{\sigma_0} \), which proves (b).

The rest of Section 7.3 will be devoted to the proof of the following result.

**Proposition 7.8.** One has \((K_{\sigma_0})_{\text{red}} = K_L \cup K_M\).

As an addition to this statement, note that \( K_{\sigma_0} \) is nonreduced along its component \( K_L \): this follows from Propositions 7.7(a) and 4.4(a).

The following remark (which adds to the description of \( K_L \) in Proposition 7.7(a)) will be useful in the proof of Theorem 7.20.

**Remark 7.9.** If \([U_6] \subseteq K_{\sigma_0}\), one of the following holds:

(a) either \([U_6] \subseteq K_L \setminus K_M\) and the scheme-theoretic intersection \( \text{Gr}(3, U_6) \cap g(V) \) is the union of two reduced (distinct) points;

(b) or \([U_6] \subseteq K_M \setminus K_L\) and \( \text{Gr}(3, U_6) \cap g(V) = \emptyset\);

(c) or \([U_6] \subseteq K_L \cap K_M\) and the scheme-theoretic intersection \( \text{Gr}(3, U_6) \cap g(V) \) has length 2.

**Remark 7.10.** Let \( F_I \subseteq \text{Gr}(2, \text{Sym}^2 W_3) \) be the set of all \( I(a, H) \) and let \( F_J \subseteq \text{Gr}(2, \text{Sym}^2 W_3) \) be the set of all \( J(a, x) \). The variety of lines on the chordal cubic in \( \mathbb{P}(\text{Sym}^2 W_3) \) is equal to \( F_I \cap F_J \), both \( F_I \) and \( F_J \) are smooth of dimension 4, and their intersection is smooth of dimension 3 (VD Proposition 3.2.4)). Thus, by Proposition 7.8 \( K_{\sigma_0} \) is isomorphic to the variety of lines on the chordal cubic.
Proposition 7.12. We view

\[ \rho \in \text{GL}(H^0(\mathbb{P}(W_3^\gamma)^*, \mathcal{O}(W_3^\gamma)(3))) \]

be the 1-parameter subgroup of \( GL(3) \). The isotypic components, ordered in increasing order, are \( \text{Sym}^3 \lambda : \mathbb{C}^* \to \text{Sym}^3 W_3 \) has isotypic components of dimension 1. Generators of the isotypic components, ordered in increasing order, are

\[ z^3, yz^2, xz^2, y^2z, xyz, x^2z, y^3, xy^2, x^2y, x^3. \]

It follows that \( U_3 = \langle x^2z, xyz, y^2z \rangle \). By Lemma 7.11, one gets \( \overline{U_3} \notin X_{\sigma_0} \), a contradiction. \( \square \)

Proposition 7.12. Let \( U_6 \in K_{\sigma_0} \). For every \( [a] \in \mathbb{P}(W_3^\gamma) \), we have

\[ (a \cdot \text{Sym}^2 W_3^\gamma) \cap U_6^\perp \neq \{0\}. \]

Proof. We view \( U_6 \) as a subspace of \( H^0(\mathbb{P}(W_3^\gamma), \mathcal{O}(W_3^\gamma)(3)) \). Let \( p \in \mathbb{P}(W_3^\gamma) \). If the natural map

\[ U_6 \to (\mathcal{O}(W_3^\gamma)_{U_6}/m_0^3) \otimes \mathcal{O}(W_3^\gamma)(3) \]

is surjective, or equivalently bijective since both spaces have dimension 6, the kernel of the map \( \mathcal{O}_U(\mathbb{P}(W_3^\gamma)) \to (\mathcal{O}(W_3^\gamma)_{U_6}/m_0^3) \otimes \mathcal{O}(W_3^\gamma)(3) \) is a 3-dimensional subspace \( U_3 \subseteq U_6 \cap H^0(\mathbb{P}(W_3^\gamma), m_0^3(3)) \) such that the natural map \( U_3 \to (m_0^3/\Lambda^3) \otimes \mathcal{O}_U(\mathbb{P}(W_3^\gamma)(3)) \) is an isomorphism. By Lemma 7.11, \( [U_3] \notin X_{\sigma_0} \), but this is absurd because \( [U_6] \in K_{\sigma_0} \). The map (49) is therefore not surjective.

Assume first that \( p = [a] \) is not in the base-locus of the linear system \( \mathcal{P}(U_6) \). The map \( \mathcal{P}(W_3^\gamma) \to \mathcal{P}_2 U_6^\perp \) defined by \( \mathbb{P}(U_6) \) is the composition

\[ \mathcal{P}(W_3^\gamma) \to \mathcal{P}(\text{Sym}^3 W_3^\gamma) \to \mathcal{P}(U_6^\perp) \]

of the Veronese map \( \nu_3 \) and the projection with center \( \mathbb{P}(U_6^\perp) \). If (48) does not hold, the second-order osculating plane \( \mathcal{P}(a \cdot \text{Sym}^2 W_3^\gamma) \) to the Veronese surface \( \nu_3(\mathbb{P}(W_3^\gamma)) \) does not meet the center of projection \( \mathbb{P}(U_6^\perp) \), hence (49) is bijective, which we just prove does not hold. It follows that (48) holds if \( [a] \) is not in the base-locus of \( \mathbb{P}(U_6) \). Since the property (48) is closed, it holds for all \( [a] \in \mathbb{P}(W_3^\gamma) \). \( \square \)

### 7.3.3. Three-dimensional linear system of plane cubics containing many reducible cubics

Let \( [U_6] \in K_{\sigma_0} \). Then \( \mathbb{P}(U_6^\perp) \subseteq \mathbb{P}(\text{Sym}^3 W_3^\gamma) \) is a 3-dimensional linear systems of cubics in \( \mathbb{P}(W_3) \). By Proposition 7.12, given any line \( R \subseteq \mathbb{P}(W_3) \), there exists a cubic in \( \mathbb{P}(U_6^\perp) \) containing \( R \). We prove here the following result.

Proposition 7.13. Let \( \Lambda \subseteq |\mathcal{O}_2(3)| \) be a 3-dimensional linear system such that, for each line \( R \subseteq \mathbb{P}^2 \), there exists a cubic in \( \Lambda \) containing \( R \). One of the following holds:

(a) the base-locus of \( \Lambda \) contains a line;
(b) there exists a (possibly degenerate) conic \( C \subseteq \mathbb{P}^2 \) such that \( \Lambda \) contains \( C + |\mathcal{O}_2(1)| \);
(c) in a suitable basis \( (x, y, z) \) of \( H^0(\mathbb{P}^2, \mathcal{O}_2(1)) \), one of the following holds:

\[ \Lambda = \mathbb{P}((x^3, y^3, z^3, xyz)), \]
(c2) \( \Lambda \subseteq P((xz^2, yz^2, x^3, x^2y, xy^2, y^3)) \),
(c3) \( \Lambda = P((xyz, x^2y + yz^2, x^2z + yz^2, xy^2 + xz^2)) \),
(c4) \( \Lambda \subseteq P((xyz, x^2y + xz^2, xy^2 + yz^2, x^2z, y^2z)) \),
(c5) \( \Lambda \subseteq P((x^2z, xyz, x^2y^2 + xz^2, y^3, yz^2)) \),
(c6) \( \Lambda \subseteq P((x^2z - xy^2, y^3, y^2z, yz^2, z^3)) \).

Here is the corollary of interest to us.

**Corollary 7.14.** Let \([U_6] \in K_{\sigma_0}\). One of the following holds:

(a) \( U^+_6 = f_1 \cdot U_4 \), where \( f_1 \in W_3^\vee \) and \( U_4 \subseteq \text{Sym}^2 W_3^\vee \) is a 4-dimensional subspace;

(b) \( U^+_6 \supseteq f_2 \cdot W_3^\vee \), where \( f_2 \in \text{Sym}^2 W_3^\vee \).

**Proof.** As noted above, \( \Lambda := P(U^+_6) \) is a 3-dimensional linear system of cubics satisfying the hypothesis of Proposition 7.13. Hence one of the following items (a), (b), (c) of that proposition holds. If (a) holds, then (a) holds; if (b) holds, then (β) holds. One checks that if (c) holds, \([U_6]\) is not in \( K_{\sigma_0} \). For example, suppose that (c6) holds and let \((a, b, c)\) be the basis of \( W_3^\vee \) dual to the basis \((x, y, z)\) of \( W_3 \). Then \( U^+_6 \supseteq \langle a^2c + ab^2, a^3, a^2b, abc, ac^2 \rangle \) and this is absurd, because \( \sigma_0(a^2c + ab^2, abc, ac^2) \neq 0 \) by Lemma 7.1. □

Before proving Proposition 7.13 we go through two preliminary results. The first is an easy exercise which we leave to the reader.

**Lemma 7.15.** Let \( \Lambda \subseteq |\mathcal{O}_{P^2}(3)| \) be a linear system all of whose elements are reducible. Then, either \( \Lambda \) has a 1-dimensional base-locus or all cubics in \( \Lambda \) have multiplicity 3 at a fixed point.

**Proposition 7.16.** Let \( \Lambda \subseteq |\mathcal{O}_{P^2}(3)| \) be a 2-dimensional linear system. Suppose that, given an arbitrary line \( R \subseteq P^2 \), there exists a cubic in \( \Lambda \) containing \( R \). Then, there exists a conic \( C \) such that \( \Lambda = C + |\mathcal{O}_{P^2}(1)| \).

**Proof.** By our hypothesis, the variety of reducible cubics in \( \Lambda \) has dimension 2, hence every cubic in \( \Lambda \) is reducible. Since all cubics in \( \Lambda \) cannot have multiplicity 3 at a fixed point, Lemma 7.15 implies that the base-locus of \( \Lambda \) contains a line \( R \) or a conic \( C \). If the latter holds, we are done because \( \text{dim} (\Lambda) = 2 \). If the former holds, \( \Lambda = R + \Lambda' \), where \( \Lambda' \) is a 2-dimensional linear system of conics such that, given any line \( R \subseteq P^2 \), there exists a conic in \( \Lambda' \) containing \( R \). In particular all conics in \( \Lambda' \) are reducible. It follows that there exists a line \( R' \) such that \( \Lambda' = R' + |\mathcal{O}_{P^2}(1)| \). Thus \( \Lambda = (R + R') + |\mathcal{O}_{P^2}(1)| \). □

**Proof of Proposition 7.13.** If the base-locus of \( \Lambda \) has dimension 1, item (a) holds. From now on, we assume that the base-locus of \( \Lambda \) is finite. Let \( f: P^2 \dashrightarrow \Lambda' \simeq P^3 \) be the natural map. Let \( B \subseteq P^2 \) be the (schematic) base-locus of \( \Lambda \), so that \( \Lambda \subseteq |\mathcal{I}_B(3)| \). Let \( f_B: P^2 \dashrightarrow |\mathcal{I}_B(3)|^\vee \) be the natural rational map. Then \( f \) is the composition \( \pi \circ f_B \), where \( \pi: |\mathcal{I}_B(3)|^\vee \dashrightarrow \Lambda' \) is a projection whose center does not intersect the (closed) image \( f_B(P^2) \).

The (closed) image \( f(P^2) \) is either a curve or a surface. If it is a curve, \( \Lambda \) is the linear system of cubics in \( P^2 \) which have multiplicity 3 at a fixed point. This contradicts our hypothesis. Hence \( f \) has finite positive degree onto the surface \( \Sigma := f(P^2) \). As one easily checks,

(i) either \( B \) is the complete intersection of a (possibly degenerate) conic \( C \) and a cubic,
(ii) or the restriction of \( f_B \) to a subscheme \( Z \subseteq P^2 \smallsetminus B \) of length 2 is not injective if and only if the schematic intersection \( (Z) \cap B \) has length 3.

If (i) holds, \( \Lambda = |\mathcal{I}_B(3)| \), hence \( \Lambda \supseteq C + |\mathcal{O}_{P^2}(1)| \). Thus item (b) of Proposition 7.13 holds. From now on, we assume that (ii) holds.
Assume first that \( f \) has degree 1 onto its image. If \( R \subseteq \mathbb{P}^2 \setminus B \) is a line, \( f_B(R) \) is a twisted cubic by item (ii). A dimension count shows that

\( (\alpha) \) either \( f(R) \) is also a twisted cubic for a general line \( R \subseteq \mathbb{P}^2 \setminus B \),

\( (\beta) \) or the projection \( \pi : \mathcal{V}_B(3) |^\vee \to \mathcal{V}_B(3) |^\vee \) maps to the same point \( f_B(R_1 \setminus B) \) and \( f_B(R_2 \setminus B) \),

where \( R_1, R_2 \subseteq \mathbb{P}^2 \) are distinct lines such that \( \text{length}(R_1 \cap B) = 3 \),

\( (\gamma) \) or the differential of \( f \) vanishes at all points of \( R \setminus B \), where \( R \subseteq \mathbb{P}^2 \) is a line such that \( \text{length}(R \cap B) = 3 \).

If \( (\alpha) \) holds, no cubics in \( \Lambda \) contain a general line \( R \subseteq \mathbb{P}^2 \), because \( f(R) \subseteq \mathcal{V}_B(3) |^\vee \) is a twisted cubic, and this contradicts the hypothesis of Proposition 7.13. If \( (\beta) \) holds, \( \dim(\Lambda) = 4 \), length(\( B \)) = 5, and \( B \) is a subscheme of \( R_1 \cup R_2 \). It follows that \( \Lambda \supseteq R_1 + R_2 + |\mathcal{O}_{\mathbb{P}^2}(1)| \), hence item (b) of Proposition 7.13 holds. If \( (\gamma) \) holds, \( \Lambda \supseteq 2R + |\mathcal{O}_{\mathbb{P}^2}(1)| \) and item (b) holds again.

Assume now that \( f \) has degree greater than 1 onto its image. Suppose that the surface \( \Sigma \subseteq \mathcal{V}_B(3) |^\vee \) has degree 2. Let \( \tilde{\mathbb{P}}^2 \to \mathbb{P}^2 \) be a smooth blow up such that \( \tilde{f} : \tilde{\mathbb{P}}^2 \to \mathbb{P}^2 \to \Sigma \) is a morphism. Let \( V \subseteq \Sigma \) be the union of the set of singular points of the branch divisor of \( f \) (this includes the points over which the fiber is not finite) and the vertex of \( \Sigma \) if \( \Sigma \) is a cone.

The linear system \( \Lambda \) contains a 2-dimensional family of reducible cubics that contain a general line and these cubics correspond to planes in \( \mathcal{V}_B(3) |^\vee \) either meet \( V \) or are tangent to \( \Sigma \) at a smooth point of \( \Sigma \). If these planes all pass through a point of \( \gamma \), Proposition 7.13 holds. If \( \gamma \) is a subscheme of \( R_1 \cup R_2 \) and \( \Lambda \) contains a 2-dimensional family of reducible cubics that contain a general line, we can apply Proposition 7.16 and item (b) holds. Otherwise, given a general line \( R \subseteq \mathbb{P}^2 \), there exists a plane tangent to \( \Sigma \) at a smooth point such that the corresponding cubic contains \( R \). If \( \Sigma \) is smooth, the cubics corresponding to tangent planes are of the form \( C_1 + C_2 \), where \( C_1 \) and \( C_2 \) belong to two fixed pencils of curves corresponding to the two pencils of lines on \( \Sigma \) and this is absurd because they do not contain a general line. If \( \Sigma \) is a cone, the set of tangent planes is the linear system of planes through the vertex and we are reduced to the first case.

We may therefore assume \( \deg(\Sigma) \geq 3 \). We claim that the (schematic) base-locus \( B \) of \( \Lambda \) is curvilinear. It is not, there is a (single) point \( p \) in the support of \( B \) such that, in a neighborhood of \( p \), we have \( \mathcal{I}_B = m_p^2 \). This implies \( \deg(f) \deg(\Sigma) \leq 5 \), hence \( \deg(\Sigma) = 2 \), which is absurd.

Since \( B \) is curvilinear, it is locally a complete intersection, hence \( \deg(f) \deg(\Sigma) + \text{length}(\mathcal{O}_B) = 9 \). Since \( \deg(f) \geq 2 \) and \( \deg(\Sigma) \geq 3 \), one of the following holds:

(I) \( B \) is empty and \( \deg(f) = \deg(\Sigma) = 3 \);

(II) \( B \) is a single reduced point and \( \deg(f) = 2 \);

(III) \( B \) has length 3 and \( \deg(f) = 2 \).

Suppose that (I) holds. In particular, \( f : \mathbb{P}^2 \to \Sigma \) is regular. Let us show that item (c1) of Proposition 7.13 holds. First, we claim that \( \Sigma \) has isolated singularities. In fact, if \( \Sigma \) is a cone, one gets a contradiction arguing as in the proof that \( \Sigma \) cannot be a quadric. If \( \Sigma \) is a nonnormal cubic (and not a cone), its normalization \( \tilde{\Sigma} \) is the Hirzebruch surface \( F_1 \) and we get a contradiction because the dominant map \( \mathbb{P}^2 \to \Sigma \) lifts to a dominant map \( \mathbb{P}^2 \to F_1 \), and \( \rho(F_1) > \rho(\mathbb{P}^2) \). We have proved that \( \Sigma \) has isolated singularities.

The map \( f : \mathbb{P}^2 \to \Sigma \) is finite and \( f^* \omega_{\Sigma} \equiv \omega_{\mathbb{P}^2} \), hence \( f \) is unramified in codimension 1. Hence, if \( C \in \Lambda \) is general, the map \( C \to f(C) \) is the quotient map for the action of a subgroup of \( \text{Pic}^0(C) \) of order 3. This action is the restriction of an automorphism \( \varphi_C \) of \( \mathbb{P}^2 \) of order 3. We prove that \( \varphi_C \) does not depend on \( C \). Let \( C' \in \Lambda \) be another general cubic and let \( H, H' \subseteq \mathcal{V}_B(3) \) be the planes corresponding to \( C, C' \). The 9 points in \( C \cap C' \) are partitioned into the union of the three fibers (each of cardinality 3) of the three points of intersection of the line \( H \cap H' \) with \( \Sigma \). It follows that \( \varphi_C \) and \( \varphi_{C'} \) agree on the 9 points in \( C \cap C' \), hence are equal. The upshot is
that there exists an order 3 automorphism \( \varphi \) of \( \mathbb{P}^2 \) such that \( f : \mathbb{P}^2 \to \Sigma \) is the corresponding quotient map and \( f^* \mathcal{O}_\Sigma(1) \cong \mathcal{O}_\mathbb{P}^2(3) \). It follows that (c1) holds.

Suppose that (II) holds. Let \( \hat{\mathbb{P}}^2 \to \mathbb{P}^2 \) be the blow up of the base point of \( \Lambda \). Then \( f \) induces a regular finite map \( \hat{f} : \hat{\mathbb{P}}^2 \to \Sigma \) of degree 2. Since the exceptional divisor of \( \hat{\mathbb{P}}^2 \to \mathbb{P}^2 \) is the unique \((-1)\)-curve of \( \hat{\mathbb{P}} \), the covering involution of \( \hat{f} \) descends to an involution \( : \mathbb{P}^2 \to \mathbb{P}^2 \) leaving invariant the cubics in \( \Lambda \). In suitable coordinates, we have \( (x, y, z) = (x, x, -z) \). Since the cubics in \( \Lambda \) are \( \iota \)-invariant, we have \( \Lambda \subseteq \mathbb{P}( \langle xz^2, yz^2, x^2y, xy^2, y^3 \rangle ) \) and (c2) holds.

Suppose that (III) holds. The blow up \( \text{Bl}_B \mathbb{P}^2 \) of \( B \) is a weak Del Pezzo surface (the anticanonical bundle is big and nef) with DuVal singularities. The anticanonical system \( |\mathcal{I}_B(3)| \) defines a map \( \text{Bl}_B \mathbb{P}^2 \to |\mathcal{I}_B(3)|^\vee \cong \mathcal{O}_B^6 \) whose image is a Del Pezzo surface \( S \) with DuVal singularities. The rational map \( f : \mathbb{P}^2 \dashrightarrow \Lambda^\vee \) is the composition of the natural rational map \( \mathbb{P}^2 \dashrightarrow S \) and the restriction to \( S \) of a projection \( |\mathcal{I}_B(3)|^\vee \dashrightarrow \Lambda^\vee \) with center disjoint from \( S \). The latter is a map \( f : S \to \Sigma \) which is finite, of degree 2. If \( \hat{\tau} : S \to S \) is its covering involution, \( \Lambda \) is contained in the projectivization of \( \hat{\tau} \)-invariant subspace of \( H^0(S, \omega_{S}^{-1}) \).

If the involution \( \hat{\tau} \) descends to a regular involution of \( \mathbb{P}^2 \), item (c2) holds by the argument given above. Thus we assume that \( \hat{\tau} \) is a birational nonregular involution of \( \mathbb{P}^2 \); in particular, \( B \) is not contained in a line and there exist coordinates \( x, y, z \) such that

\[
\begin{align*}
(a) & \quad |\mathcal{I}_B(3)| = \mathbb{P}( \langle x^2y, x^2z, xy^2, xyz, xz^2, yz^2, y^2z, yz^2 \rangle ) \quad \text{and} \quad B = \{ (1, 0, 0), (0, 1, 0), (0, 0, 1) \}, \\
(b) & \quad |\mathcal{I}_B(3)| = \mathbb{P}( \langle x^2z, y^2z, xyz, xz^2, y^2z, yz^2, y^2z, yz^2 \rangle ) \quad \text{and} \quad B \text{ supported at } (1, 0, 0) \text{ and } (0, 0, 1), \\
(c) & \quad |\mathcal{I}_B(3)| = \mathbb{P}( \langle x^2z - xy^2, x^2z, x^2z, y^2z, yz^2, y^2z, yz^2, y^2z \rangle ) \quad \text{and} \quad B \text{ is curvilinear (nonlinear) supported at } (1, 0, 0) \text{ with tangent line } z = 0.
\end{align*}
\]

The standard Cremona quadratic map and the first and second standard degenerate quadratic maps (see [Do, Example 7.1.9]) provide examples of such an involution in each of these cases

\[
\tau_a(x, y, z) = (yz, xz, yz), \quad \tau_b(x, y, z) = (xz, yz, y^2), \quad \tau_c(x, y, z) = (-xz + y^2, yz, z^2).
\]

Suppose that (a) holds. Every involution \( \tau \) of \( S \) which does not descend to \( \mathbb{P}^2 \) is given by \( \tau_a \circ h \), where \( h \in \text{PGL}(3) \) permutes the points of \( B \). If \( h \) fixes the points of \( B \), we get (after rescaling \( x, y, z \)) \( \tau = \tau_a \), while if \( h \) defines a transposition of \( B \), we have \( \tau([x, y, z]) = [xz, yz, xy] \) in suitable coordinates. The \( \tau \)-invariant subspace of \( H^0(S, \omega_{S}^{-1}) \) is equal to \( \langle x^2z, xy^2 + y^2z, x^2z + y^2z, xy^2 + x^2z \rangle \) if the former holds, and to \( \langle xy^2 + x^2z, x^2z + y^2z, y^2z, y^2z \rangle \) if the latter holds. Hence if the former holds, (c3) holds; if the latter holds, (c4) holds.

Suppose that (b) holds. The relevant birational involutions of \( \mathbb{P}^2 \) are given by \( \tau_b \circ h \), where \( h \in \text{PGL}(3) \) is given by \( h(x, y, z) = (\alpha x + \beta y, -\alpha y, -\alpha^{-2}z) \) or by \( h(x, y, z) = (\alpha x, \alpha y, \alpha^{-2}z) \) with \( \alpha \in \mathbb{C}^* \) and \( \beta \in \mathbb{C} \). In a suitable coordinate system, \( \tau = \tau_b \). The \( \tau_b \)-invariant subspace of \( H^0(S, \omega_{S}^{-1}) \) is \( \langle x^2z, xy^2 + x^2z, y^2z, y^3 + y^2z \rangle \), hence (c5) holds.

Lastly, suppose that (c) holds. The relevant birational involutions of \( \mathbb{P}^2 \) are given by \( \tau_c \circ h \), where \( h \in \text{PGL}(3) \) is given by \( h([x, y, z]) = [x + \beta y + \gamma z, y, z] \). In a suitable coordinate system, such a birational involution is equal to \( \tau_c \). The \( \tau_c \)-invariant subspace of \( H^0(S, \omega_{S}^{-1}) \) is \( \langle x^2z - xy^2, y^3, y^2z, yz^2, z^3 \rangle \), hence (c6) holds.

### 7.3.4. Description of \( K_{\sigma_0} \)

Let \( [U_6] \in K_{\sigma_0} \) and let \( T_1 := U_6^+ \). By Corollary 7.14 either \( T_1 \equiv f_1 \cdot U_4 \), where \( U_4 \subseteq \text{Sym}^2 W_3^\gamma \) is a 4-dimensional subspace, or \( T_1 \supseteq f_2 \cdot W_3^\gamma \), where \( f_2 \in \text{Sym}^2 W_3^\gamma \). Hence, by (46), Propositions 7.17 and 7.18 below finish the proof of Proposition 7.8.

### Proposition 7.17

Let \( T_1 \subseteq \text{Sym}^2 W_3^\gamma \) be a 4-dimensional subspace such that \( T_1 = f_1 \cdot U_4 \), where \( 0 \neq f_1 \in W_3^\gamma \) and \( U_4 \subseteq \text{Sym}^2 W_3^\gamma \) is a 4-dimensional subspace. Then \( T_1^+ \in K_{\sigma_0} \) if and only if...
there exists a basis \((a, b, c)\) of \(W_3^\vee\) such that
\[
T_4 = \begin{cases}
  a \cdot \langle a^2, ab, ac, bc \rangle, & \text{or} \\
  a \cdot \langle a^2, ab, ac, c^2 \rangle, & \text{or} \\
  a \cdot \langle a^2, b^2, bc, c^2 \rangle.
\end{cases}
\]

Proof. Let \(R_2 := U_4^\perp \subseteq \text{Sym}^2 W_3\). Up to the action of \(\text{GL}(W_3)\), there are 8 possibilities for \(R_2\), described as follows in a basis \((x, y, z)\) of \(W_3\). In the case where the general conic (in \(\text{P}(W_3^\vee)\)) defined by \(\text{P}(R_2)\) is smooth, hence the base-locus is a 0-dimensional curvilinear scheme, we have

1. \(R_2 = \langle xy, (x + y + z)z \rangle\), that is, the base-locus of the pencil of conics defined by \(\text{P}(R_2)\) consists of 4 distinct points;
2. \(R_2 = \langle xy, (x + z)z \rangle\), that is, the base-locus of the pencil of conics defined by \(\text{P}(R_2)\) consists of two reduced points and a point of multiplicity 2;
3. \(R_2 = \langle xy, z^2 \rangle\), that is, the base-locus of the pencil of conics defined by \(\text{P}(R_2)\) consists of two points of multiplicity 2;
4. \(R_2 = \langle xy, x^2 + yz \rangle\), that is, the base-locus of the pencil of conics defined by \(\text{P}(R_2)\) consists of one point of multiplicity 3 and a reduced point;
5. \(R_2 = \langle y^2, x^2 + yz \rangle\), that is, the base-locus of the pencil of conics defined by \(\text{P}(R_2)\) consists of one point of multiplicity 4.

The remaining \(R_2\) are those for which all the conics parametrized by \(\text{P}(R_2)\) are singular:

(a) \(R_2 = \langle y^2, z^2 \rangle\);
(b) \(R_2 = \langle y^2, yz \rangle\);
(c) \(R_2 = \langle xy, xz \rangle\).

Correspondingly, we get the following lists of 4-dimensional subspaces \(U_4 \subseteq \text{Sym}^2 W_3^\vee\):
\[
U_4 = \begin{cases}
  \langle a^2, b^2, ac - c^2, bc - c^2 \rangle, \\
  \langle a^2, b^2, ac - c^2, bc \rangle, \\
  \langle a^2, b^2, ac, bc \rangle, \\
  \langle ac, b^2, c^2, a^2 - bc \rangle, \\
  \langle ab, ac, c^2, a^2 - bc \rangle.
\end{cases}
\]

and
\[
U_4 = \begin{cases}
  \langle a^2, ab, ac, bc \rangle, \\
  \langle a^2, ab, ac, c^2 \rangle, \\
  \langle a^2, b^2, bc, c^2 \rangle.
\end{cases}
\]

Every 4-dimensional \(U_4 \subseteq \text{Sym}^2 W_3^\vee\) is equivalent modulo \(\text{GL}(W_3)\) to one and only one of the spaces \(U_4\) given above. Let \(f_1 \in W_3^\vee\) be nonzero and let \(U_4\) be one of the subspaces in (51).

We claim that \((f_1 \cdot U_4)^\perp\) does not belong to \(K_{\sigma_0}\). To see this, first note that there exists a 1-parameter subgroup of \(\text{GL}(W_3)\) such that \(\text{lim}_{t \to 0} \lambda(t)U_4\) is equal to the subspace in the last line of (51) (this is clear since \(U_4 = R_4^\perp\)). Hence it suffices to prove that \((f_1 \cdot U_4)^\perp\) does not belong to \(K_{\sigma_0}\) for \(U_4\) as in the last line of (51). Next, by acting with a 1-parameter subgroup of \(\text{GL}(W_3)\) given by \(\text{diag}(t^r, t^s, t^t)\) (in the given basis), with \(2q = r + s\), we may assume \(f_1 \in \{a, b, c\}\). An explicit computation then gives
\[
\langle a \cdot \langle ab, ac, c^2, a^2 - bc \rangle \rangle^\perp = \langle ab^2, b^2c, bc^2, c^3, a^3 + abc \rangle,
\]
\[
\langle b \cdot \langle ab, ac, c^2, a^2 - bc \rangle \rangle^\perp = \langle a^3, a^2c, ac^2, b^3, c^3, a^2b + b^2c \rangle,
\]
\[
\langle c \cdot \langle ab, ac, c^2, a^2 - bc \rangle \rangle^\perp = \langle a^3, a^2b, ab^2, b^3, b^2c, a^2c + bc^2 \rangle.
\]
By Lemma 7.1 we have $\sigma_0(b^3,c^3,a^3 + abc) \neq 0$, $\sigma_0(a^3,b^3,c^3) \neq 0$, and $\sigma_0(a^3,b^2c,ac^2 + bc^2) \neq 0$. It follows that the first, second, and third spaces are not in $K_{\sigma_0}$.

We are left with $U_4$ as in (52). We know that $(a \cdot U_4)^\perp \subseteq K_{\sigma_0}$. It remains to prove that if $f_1 \notin (x)$, then $(f_1 \cdot U_4)^\perp \subseteq K_{\sigma_0}$. Acting with a suitable 1-parameter subgroup of $GL(W_3)$, we may assume $f_1 \in \{b,c\}$. An explicit computation similar to the one presented above finishes the proof. □

**Proposition 7.18.** Let $T_4 \subseteq \text{Sym}^3W_3^\vee$ be a 4-dimensional subspace. Suppose that there exists a nonzero $f_2 \in \text{Sym}^2W_3^\vee$ such that $T_4 \supseteq (f_2 \cdot W_3^\vee)$. Then $[T_4^\perp] \in K_{\sigma_0}$ if and only if there exists a basis $(a,b,c)$ of $W_3^\vee$ such that

$$T_4 = \begin{cases} a \cdot \langle a^2, ab, ac, bc \rangle, & \text{or} \\ a \cdot \langle a^2, ab, ac, c^2 \rangle. \end{cases}$$

**Proof.** There exists a basis $(a,b,c)$ of $W_3^\vee$ and $g \in \text{Sym}^3W_3^\vee$ such that (according to the rank of $f_2$)

$$T_4 = \begin{cases} \langle a^2b + ac^2, ab^2 + bc^2, abc + c^2, g \rangle, & \text{or} \\ \langle a^2b, ab^2, abc, g \rangle, & \text{or} \\ \langle a^3, a^2b, a^2c, g \rangle. \end{cases}$$

Suppose that $T_4$ is as in the first line. Let $\lambda$ be the 1-parameter subgroup, diagonal in the basis $(a,b,c)$, given by $\text{diag}(1,t^r, t^s)$. Then $\text{lim}_{t \to 0} \lambda(t)T_4$ is as in the second line. We show that for $T_4$ as in the second line, the orthogonal $T_4^\perp$ is not in $K_{\sigma_0}$. Let $\lambda$ be any 1-parameter subgroup diagonal in the basis $(a,b,c)$, with pairwise distinct weights of the action on $\text{Sym}^3W_3^\vee$. Then $T_4 := \text{lim}_{t \to 0} \lambda(t)T_4$ is monomial and it contains $a^2b$, $ab^2$, and $abc$. Hence the orthogonal $T_4^\perp$ is monomial, of dimension 6, contained in

$$\langle a^3, a^2b, a^2c, b^2c, bc^2, c^3 \rangle.$$ 

A direct check shows that the above subspace contains no monomial subspace of dimension 6 on which $\sigma_0$ vanishes. It follows that $[T_4^\perp]$ is not in $K_{\sigma_0}$.

Suppose now that $T_4$ is as in the third line. We prove by contradiction that $a \mid g$ (once that is known, we might need to rename $b,c$). Let $\lambda$ be a 1-parameter subgroup, diagonal in the basis $(a,b,c)$, given by $\text{diag}(1,t^r, t^s)$, where $r > 3s$. Then $T_4 := \text{lim}_{t \to 0} \lambda(t)T_4$ is monomial and by our assumption $a \notmid g$, there exist $i,j$ such that $T_4 = \langle a^3, a^2b, a^2c, b^ic \rangle$. Hence $T_4^\perp$ contains $\langle ab^2, abc, ac^2 \rangle$ and is therefore not in $K_{\sigma_0}$. It follows that $[T_4^\perp]$ is not in $K_{\sigma_0}$. □

### 7.4. Orbit and stabilizer

Recall that $V_{10} = \text{Sym}^3W_3$. Since $\mathfrak{sl}(3) = \Gamma_{1,1}$ and

$$\text{End}(V_{10}) = \Gamma_{3,3} \oplus \Gamma_{2,2} \oplus \Gamma_{1,1} \oplus \Gamma_{0,0},$$

it follows from the decomposition (37) that there is an exact sequence

$$0 \to \mathfrak{sl}(3) \to \text{End}(V_{10}) \to \bigwedge^3 V_{10}^\vee \xrightarrow{a} \Gamma_{0,6} \to 0.$$  

We prove below that the stabilizer of $[\sigma_0]$ is $\text{SL}(3)$. The normal space at $[\sigma_0]$ to the $\text{SL}(V_{10})$-orbit of $[\sigma_0]$ is therefore $\Gamma_{0,6} = H^0(\mathcal{P}(W_3), \mathcal{O}_{\mathcal{P}(W_3)}(6))$. The map $a$ was given a geometric interpretation in (43).

**Proposition 7.19.** The stabilizer of $[\sigma_0]$ in $\text{SL}(V_{10})$ is equal to the image of $\text{SL}(W_3) \to \text{SL}(V_{10})$ and the point $[\sigma_0] \in \mathcal{P}(\bigwedge^3 V_{10}^\vee)$ is polystable for the $\text{SL}(V_{10})$-action.
Proof. The stabilizer contains $\text{SL}(W_3)$ by choice of $\sigma_0$. Conversely, if $g \in \text{SL}(V_{10})$ stabilizes $[\sigma_0]$, it maps $X_{\sigma_0}$ to itself, hence the singular locus of $X_{\sigma_0}$ to itself. By Proposition 7.2, this singular locus is equal to $g(V) \subseteq P(\text{Sym}^3W_3)$. Thus $g$ maps to itself the subvariety of $P(\text{Sym}^3W_3)$ swept out by projective tangent planes to the Veronese surface $V$. Since the singular locus of this subvariety is $V$, the automorphism $g$ maps $V$ to itself, hence belongs to $\text{SL}(W_3)$.

It follows from Proposition 5.4 that this stabilizer has finite index in its normalizer, hence $[\sigma_0]$ is polystable by [Lu, Corollaire 3].

7.5. Degenerations. The following theorem is the main result of Section 7. We consider a general 1-parameter deformation $(\sigma_t)_{t \in \Delta}$ of our trivector $\sigma_0$. By the exact sequence (55), we obtain a general element of $H^0(P(W_3), \mathcal{O}_{P(W_3)}(6))$, hence a double cover $S \rightarrow P(W_3)$ branched along the cubic curve that it defines, where $S$ is a K3 surface of genus 2. The moduli space $\mathcal{M}_S(0, L, 1)$, a hyperkähler fourfold birationally isomorphic to $S[2]$, was defined in Remark 3.2.

Theorem 7.20. Let $(\sigma_t)_{t \in \Delta}$ be a general 1-parameter deformation. Over a finite cover $\Delta' \rightarrow \Delta$, there is a family of smooth polarized hyperkähler fourfolds $\mathcal{X}' \rightarrow \Delta'$ such that a general fiber $\mathcal{X}'_0$ is isomorphic to $K_{\sigma_0}$, and the central fiber is isomorphic to $\mathcal{M}_S(0, L, 1)$, where $S$ is a general K3 surface of genus 2, with the polarization $6L - 5\delta$.

The proof will be given at the very end of this section.

Set $\mathcal{G} := \text{Gr}(3, V_{10}) \times \Delta$ and consider the blow up

$$
\varphi : \widetilde{\mathcal{G}} := \text{Bl}_{g(V) \times \{0\}} \mathcal{G} \rightarrow \mathcal{G}
$$

(see (42) for the definition of the surface $g(V)$). The exceptional divisor $E \rightarrow g(V)$ is a bundle of 19-dimensional projective spaces. We view $\widetilde{\mathcal{G}} \rightarrow \Delta$ as a degeneration of $\text{Gr}(3, V_{10})$ with central fiber $\text{Bl}_{g(V)} \text{Gr}(3, V_{10}) \cup E$.

Write the deformation in Theorem 7.20 as $\sigma_t = \sigma_0 + t \sigma + O(t^2)$, where, by the analysis of Section 7.4, we may assume that $\sigma$ is very general in $\text{Sym}^9W'_3 \subseteq \wedge^3V'_{10}$. Consider the strict transform $\mathcal{X} \subseteq \widetilde{\mathcal{G}}$ of

$$
\{([U_3], t) \in \mathcal{G} \mid \sigma^2|_{U_3} \equiv 0\},
$$

with projection $\pi : \widetilde{\mathcal{X}} \rightarrow \Delta$. By (43), the hypersurface $X_\sigma$ intersects transversely $g(V)$ and $\text{div}(\sigma)$ is identified with $C := X_\sigma \cap g(V)$. Hence

$$
\mathcal{X}_t := \pi^{-1}(t) \simeq \begin{cases} X_{\sigma^2} & \text{if } t \neq 0, \\ \text{Bl}_{g(V)} X_{\sigma^2} \cup Q & \text{if } t = 0, \end{cases}
$$

where $Q \subseteq E$ is a bundle of 18-dimensional quadrics over $g(V)$, with smooth fibers over $g(V) \setminus C$ and fibers of corank 1 over $C$ (this follows from Lemma 7.5 and holds because we performed a degree-2 base change in (56)).

We identify $K_\sigma$ with the closed subset of the Hilbert scheme of $X_\sigma$ defined by

$$
\{[U_6] \in \text{Gr}(6, V_{10}) \mid \text{Gr}(3, U_6) \subseteq X_\sigma\}.
$$

This defines a subscheme $\mathcal{X} \rightarrow \Delta^*$ of the relative Hilbert scheme $\text{Hilb}(\mathcal{X}/\Delta)$, with fiber $K_{\sigma^2 + t^2 \sigma}$ at $t$, and we take its schematic closure $\rho : \mathcal{X} \rightarrow \Delta$.

Proposition 7.21. There exists an irreducible component $K'_t$ of $\mathcal{X}_0$ which is birationally isomorphic to $S[2]$, where $S$ is the genus-2 K3 surface of Theorem 7.20.
Proof. Let \([U_6] \in K_L \setminus K_M\). By Remark 7.9, the scheme-theoretic intersection \(\text{Gr}(3, U_6) \cap g(V)\) is two reduced points \(p_1, p_2\). Let \(\tilde{\text{Gr}}(3, U_6) \subseteq \tilde{\mathcal{F}}_0\) be the strict transform of \(\text{Gr}(3, U_6)\), that is, the blow up of \(\text{Gr}(3, U_6)\) at \(p_1, p_2\). We have \(\tilde{\text{Gr}}(3, U_6) \cap \tilde{Q} = \{A_1, A_2\}\), where \(A_i\), for \(i \in \{1, 2\}\), is an 8-dimensional linear subspace of the fiber \(E_{p_i}\) of \(E\) over \(p_i\), contained in the fiber \(Q_{p_i}\) of \(Q\) over \(p_i\). Every subscheme of \(\tilde{\mathcal{F}}_0\) given by

\[
\tilde{\text{Gr}}(3, U_6) \cup R_1 \cup R_2, \quad A_i \subseteq R_i \subseteq Q_{p_i}, \quad [R_i] \in \text{Gr}(9, E_{p_i})
\]
corresponds to a point of \(\tilde{\mathcal{F}}_0\). Moreover, by Proposition 7.8, these subschemes are parametrized by an open subset of the fiber \(\text{Hilb}(\tilde{\mathcal{F}}/\Delta)_0\), whose closure in \(\text{Hilb}(\tilde{\mathcal{F}}/\Delta)\) (equivalently, in \(\tilde{\mathcal{F}}\)) is therefore an irreducible component of \(\tilde{\mathcal{F}}_0\); we denote it by \(K_L^0\). Now \(Q_{p_i}\) is an 18-dimensional quadric, either smooth or of corank 1, which is smooth at each point of \(A_i\) (Lemma 7.3). It follows that there are exactly two 9-dimensional linear subspaces of \(Q_{p_i}\) containing \(A_i\) if \(Q_{p_i}\) is smooth (that is, if \(p_i \notin C\)) and one such linear subspace if \(Q_{p_i}\) is singular (that is, if \(p_i \in C\)).

By construction, an open dense subset \(K_L^0\) of \(K_L^\prime\) parametrizes subschemes as in (57), where \([U_6] \in K_L\) is such that \(\text{Gr}(3, U_6) \cap g(V)\) is reduced (of length 2). The set of such \([U_6]\) is exactly \(K_L \setminus K_M\). We have a forgetful map

\[
K_L^0 \rightarrow K_L \setminus K_M)
\]

(58)

(59)

Let \(\rho: S^{(2)} \rightarrow P(W_3)^{(2)}\) be the map induced by the double cover \(S \rightarrow P(W_3)\). By definition of \(R_1, R_2\), the map in (58) can be identified with the map

\[
S^{(2)} \setminus \{\rho^{-1}(2x) \mid x \in P(W_3)\} \rightarrow P(W_3)^{(2)} \setminus \{2x \mid x \in P(W_3)\}
\]

obtained by restricting \(\rho\). In particular, \(K_L^\prime\) is birationally isomorphic to \(S^{(2)}\) and the forgetful map \(K_L^\prime \rightarrow K_L\) has degree 4.

**Proposition 7.22.** The irreducible component \(K_L^\prime\) has multiplicity one in \(\tilde{\mathcal{F}}_0\).

**Proof.** A point \(x\) of \(K_L^0\) (notation as in the proof of Proposition 7.21) parametrizes a scheme \(Z := \tilde{\text{Gr}}(3, U_6) \cup R_1 \cup R_2\) as in (57), where the scheme-theoretic intersection \(\text{Gr}(3, U_6) \cap g(V)\) is the union of two reduced points \(p_1 = [U_{3,1}]\) and \(p_2 = [U_{3,2}]\), neither of which is contained in \(X_\sigma\).

The scheme \(Z\) is locally a complete intersection in \(Y_0\). Hence there is a well-defined normal bundle \(N_{Z/Y_0}\) and it suffices to prove \(H^1(Z, N_{Z/Y_0}) = 0\) (because \(K_L^\prime\) is an open neighborhood of \(x\) in the fiber \(\text{Hilb}_P(\tilde{\mathcal{F}}/\Delta)_0\)). In order to prove \(H^1(Z, N_{Z/Y_0}) = 0\), it therefore suffices to show

\[
H^1(\tilde{\text{Gr}}(3, U_6), N_{\tilde{\text{Gr}}(3, U_6)/\tilde{X}_0}) = 0.
\]

(60)

Let \(\tilde{\text{Gr}}(3, V_{10}) := \text{Bl}_g(V) \text{ Gr}(3, V_{10})\). We have the normal exact sequence

\[
0 \rightarrow N_{\tilde{\text{Gr}}(3, U_6)/\tilde{X}_0} \rightarrow N_{\tilde{\text{Gr}}(3, U_6)/\tilde{\text{Gr}}(3, V_{10})} \rightarrow \mathcal{O}_{\tilde{\text{Gr}}(3, V_{10})}(\tilde{X}_0)|_{\tilde{\text{Gr}}(3, U_6)} \rightarrow 0.
\]

(61)

We claim that

\[
H^0(\tilde{\text{Gr}}(3, U_6), \mathcal{O}_{\tilde{\text{Gr}}(3, V_{10})}(\tilde{X}_0)|_{\tilde{\text{Gr}}(3, U_6)}) = 0.
\]

(62)
In fact, the natural map $\psi: \tilde{\text{Gr}}(3,V_{10}) \to \text{Gr}(3,V_{10})$ is the blow up of the points $p_1$ and $p_2$. Let $A = A_1 + A_2$ be the exceptional divisor of $\psi$ and let $\mathcal{O}_{\text{Gr}}(1)$ be the Plücker line bundle on $\text{Gr}(3,V_{10})$. Since $X_0$ is a divisor in $|\mathcal{O}_{\text{Gr}}(1)|$ with multiplicity 2 along $g(V)$, we have

\begin{equation}
\mathcal{O}_{\text{Gr}(3,V_{10})}(\tilde{X}_0)|_{\text{Gr}(3,U_6)} \simeq \mathcal{O}_{\text{Gr}(3,U_6)}(\psi^* \mathcal{O}_{\text{Gr}}(1) - 2A).
\end{equation}

Let $x$ be a general point in $\tilde{\text{Gr}}(3,U_6)$ and set $[U_3] := \psi(x) \in \text{Gr}(3,U_6)$. We may assume that $U_3$ is transverse to $U_{3,1}$ and $U_{3,2}$, hence there exists a Segre embedding $\Phi: \mathbb{P}^1 \times \mathbb{P}^2 \to \mathbb{P}(U_6)$ such that $\Phi((0,1) \times \mathbb{P}^2) = \mathbb{P}(U_{3,1})$, $\Phi((1,0) \times \mathbb{P}^2) = \mathbb{P}(U_{3,2})$, and $\Phi((1,1) \times \mathbb{P}^2) = \mathbb{P}(U_3)$. Let $\varphi: \mathbb{P}^1 \to \text{Gr}(3,U_6)$ be the map defined by $\Phi$ and let $\Gamma \subseteq \tilde{\text{Gr}}(3,U_6)$ be the strict transform of $\varphi(\mathbb{P}^1)$. Then $\Gamma \cdot \psi^* \mathcal{O}_{\text{Gr}}(1) = 3$ and $\Gamma \cdot A = 2$, hence $\Gamma \cdot (\psi^* \mathcal{O}_{\text{Gr}}(1) - 2A) = -1$. It follows that any section of the right side of (63) vanishes at general points of $\tilde{\text{Gr}}(3,U_6)$ hence is the zero section. This proves (62).

By (62) and (61), it suffices, in order to prove (60), to prove

\[ H^1(\tilde{\text{Gr}}(3,U_6), N_{\tilde{\text{Gr}}(3,U_6)/\text{Gr}(3,V_{10})}) = 0. \]

The differential of $\psi$ defines an exact sequence

\[ 0 \to N_{\text{Gr}(3,U_6)/\text{Gr}(3,V_{10})} \to \psi^* N_{\text{Gr}(3,U_6)/\text{Gr}(3,V_{10})} \xrightarrow{a} \mathcal{O}_{A_1}^{\oplus 10} \oplus \mathcal{O}_{A_2}^{\oplus 10} \to 0. \]

The map induced by $a$ on global sections is surjective, because the subspaces of $U_6$ corresponding to $p_1$, $p_2$ are transverse. Since $H^1(\text{Gr}(3,U_6), N_{\text{Gr}(3,U_6)/\text{Gr}(3,V_{10})}) = 0$, the desired vanishing follows from the long exact sequence associated with this exact sequence.

**Proof of Theorem 7.20.** By Propositions 7.21 and 7.22, and by (the proof of) [KLSV] Theorem (0.1), we obtain, as in the proof of Theorem 6.14 after a suitable finite base change, a smooth family of polarized hyperkähler fourfolds with (smooth) central fiber birationally isomorphic to $S^2$ with the polarization $6L - 5\delta$. It follows from Remark 3.2 that this central fiber is isomorphic to $(\mathcal{M}_S(0,L,1), 6L - 5\delta)$. \hfill \qed

8. The divisor $\mathcal{D}_{30}$

Let $(S,L)$ be a general polarized K3 surface of genus 16. Unfortunately, little geometric information on $S$ is available and we were not able to find a trivector on some 10-dimensional vector space $V_{10}$ to relate $S^{[2]}$ to Debarre–Voisin varieties. We will however construct on $S^{[2]}$ a canonical rank 4-vector bundle with the same numerical invariants as the restriction of the tautological quotient bundle of $\text{Gr}(6,V_{10})$ to a Debarre–Voisin variety.

8.1. The rank-4 vector bundle $\mathcal{D}_4$ over $S^{[2]}$. By Mukai’s work ([Mu3]), there is a rigid rank-2 vector bundle $\mathcal{F}$ on $S$ with $c_1(\mathcal{F}) = L$ and Euler characteristic $\chi(S, \mathcal{F}) = 10$. Moreover, $\mathcal{F}$ is globally generated and the vector space $W_{10} := H^0(S, \mathcal{F})$ has dimension 10.

With the notation of Section 4.1 we let $\mathcal{T}_\mathcal{F}$ be the tautological rank-4 vector bundle on $S^{[2]}$ associated with $\mathcal{F}$. We have $c_1(\mathcal{T}_\mathcal{F}) = L - 2\delta$ and $H^0(S^{[2]}, \mathcal{T}_\mathcal{F}) = W_{10}$.

Consider now the tautological rank-6 vector bundle $\mathcal{T}_{\text{Sym}^2 \mathcal{F}}$ constructed on $S^{[2]}$ from the rank-3 vector bundle $\text{Sym}^2 \mathcal{F}$ over $S$.

**Lemma 8.1.** The natural evaluation map

\[ \text{ev}^+: \text{Sym}^2 \mathcal{T}_\mathcal{F} \to \mathcal{T}_{\text{Sym}^2 \mathcal{F}} \]

is surjective. Its kernel $\mathcal{D}_4$ is a rank-4 vector bundle over $S^{[2]}$ with $c_1(\mathcal{D}_4) = 2L - 7\delta$.\hfill \Box
Proof. Consider as in Section 4.1 the double cover \( p: \tilde{S} \times S \to S^{[2]} \) defined by the blow up \( \tilde{S} \times S \) of \( S \times S \) along its diagonal. Let \( q_1 \) be the first projection to \( S \), so that \( \mathcal{T}_S = p_*(\mathcal{I}_S^1) \). Tensorize the canonical surjection \( p^* \mathcal{T}_S \to q_1^* \mathcal{F} \) by the vector bundle \( q_1^* \mathcal{F} \) to obtain the exact sequence
\[
(p^* \mathcal{T}_S) \otimes q_1^* \mathcal{F} \to q_1^*(\mathcal{F} \otimes \mathcal{F}) \to 0.
\]
Its pushforward by the finite morphism \( p \) gives with the projection formula a surjection
\[
ev: \mathcal{T}_S \otimes \mathcal{T}_S \to \mathcal{T}_{S^2} \to \mathcal{T}_{S^2} \to.
\]
The map \( \ev^+ \) being the invariant part of \( \ev \), it is also surjective. Its kernel \( \mathcal{Q}_4 \) is therefore a vector bundle of rank 4 and we have \( c_1(\Sym^2 \mathcal{T}_S) = 5c_1(\mathcal{T}_S) = 5L - 10\delta \) and \( c_1(\Sym^2 \mathcal{T}_S) = 3L - 3\delta \), so \( c_1(\mathcal{Q}_4) = 2L - 7\delta \).

Lemma 8.2. The vector space \( H^0(S^{[2]}, \mathcal{Q}_4) \) has dimension at least 10 and is canonically isomorphic to the kernel
\[
V_{10} := \text{Ker}(\Sym^2 \mathcal{W}_{10} \to H^0(S, \Sym^2 \mathcal{F})).
\]

We expect this map to be onto, so that \( V_{10} \) would have dimension 10.

Proof. By [D2, Theorem 1] or [Kr, Theorem 6.6], the canonical maps
\[
H^0(S, \mathcal{F}) \to H^0(S^{[2]}, \mathcal{T}_S) \quad \quad H^0(S, \Sym^2 \mathcal{F}) \to H^0(S^{[2]}, \Sym^2 \mathcal{T}_S)
\]

are isomorphisms. By definition of \( \mathcal{Q}_4 \), we have an exact sequence
\[
0 \to H^0(S^{[2]}, \mathcal{Q}_4) \to H^0(S^{[2]}, \Sym^2 \mathcal{T}_S) \to H^0(S^{[2]}, \Sym^2 \mathcal{T}_S)
\]
whose middle term is, since (64) is bijective, isomorphic to \( \Sym^2 H^0(S, \mathcal{F}) = \Sym^2 \mathcal{W}_{10} \). We therefore conclude \( H^0(S^{[2]}, \mathcal{Q}_4) \to V_{10} \).

The Euler characteristic \( \chi(S, \Sym^2 \mathcal{F}) \) is 45 and we will show that \( h^0(S, \Sym^2 \mathcal{F}) = 45 \) on a specific K3 surface \( S \) of genus 16 introduced by Mukai in [Mu3, §6]. This surface has an elliptic fibration \( S \to \mathbb{P}^1 \) with general fiber \( A_1 \) and Mukai shows that \( \mathcal{F} \) fits in an exact sequence
\[
0 \to \mathcal{O}_S(A_1) \oplus \mathcal{O}_S(A_1) \to \mathcal{F} \to \mathcal{O}_Z(5z) \to 0,
\]
where \( Z \subseteq S \) is a smooth rational curve and \( z \) is the class of a point on \( Z \). As \( \mathcal{F} \) is globally generated and \( H^2(S, \mathcal{F}(A_1)) = 0 \) (obtained from (65)), we get \( H^2(S, \mathcal{F} \otimes \mathcal{F}) = 0 \) and
\[
(1^1(S, \mathcal{F}(A_1)) = 0 \Rightarrow (1^1(S, \mathcal{F} \otimes \mathcal{F}) = 0).
\]
Mukai showed that on this particular surface, one has \( 1^1(S, \mathcal{F}) = 1^2(S, \mathcal{F}) = 0 \), hence
\[
1^1(S, \mathcal{F}(A_1)) \cong 1^1(S, \mathcal{F}(A_1)) \cong 1^2(S, \mathcal{F}(A_1 - H)) \cong H^0(S, \mathcal{F}(A_1 - H))^\vee,
\]
where \( \mathcal{O}_S(H) := \bigwedge^2 \mathcal{F} = L \) is the polarization. Moreover, we have \( Z \sim H - 2A_1, A_1 \cdot H = 8 \), and \( H^2 = 30 \), and the sequence (65) gives
\[
0 \to \mathcal{O}_S(2A_1 - H) \oplus \mathcal{O}_S(2A_1 - H) \to \mathcal{F}(A_1 - H) \to \mathcal{O}_Z(-z) \to 0.
\]
This implies \( H^0(S, \mathcal{F}(A_1 - H)) = 0 \), hence \( 1^1(S, \mathcal{F} \otimes \mathcal{F}) = 0 \). Thus \( h^0(S, \Sym^2 \mathcal{F}) = 45 \) and \( \dim(V_{10}) \geq 10 \).

From the previous two lemmas, we obtain the following result, where we use, as in Remark 6.1, the package Schubert2 of Macaulay2 ([GS]) to compute the numerical invariants of the vector bundle \( \mathcal{Q}_4 \) on \( S^{[2]} \) (the code can be found in [X]).
Proposition 8.3. Let $(S, L)$ be a general polarized $K3$ surface of genus 16. The vector bundle $\mathcal{Q}$ induces a rational map $S^{[2]} \dasharrow \text{Gr}(6, V_{10})$ which corresponds to the polarization given in the last column of Table 1. Moreover, the vector bundle $\mathcal{Q}$ has the same Segre numbers as the rank-4 tautological quotient bundle on Debarre–Voisin varieties $K_\sigma \subseteq \text{Gr}(6, 10)$.

8.2. Geometric interpretation. Let $X$ be the image in $\mathbb{P}(W_{10}^\vee)$ of the scroll $\mathbb{P}(\mathcal{F}_x^\vee)$ by the projection from $S \times \mathbb{P}(W_{10}^\vee)$ to $\mathbb{P}(W_{10}^\vee)$.

We have $V_{10} = H^0(\mathbb{P}(W_{10}^\vee), \mathcal{I}_X(2))$, where $\mathcal{I}_X$ is the ideal sheaf of $X$ in $\mathbb{P}(W_{10}^\vee)$. We want to describe, for general points $x, y \in S$, the 6-dimensional vector space $\mathcal{I}_{6\{x,y\}}$ defined by the exact sequence

$$0 \to \mathcal{I}_{6\{x,y\}} \to V_{10} \to \mathcal{Q}_{4\{x,y\}} \to 0.$$  

Proposition 8.4. The vector space $\mathcal{I}_{6\{x,y\}}$ is the space of quadratic forms vanishing on $X$ and on the projective subspace $P_3 = \mathbb{P}(\mathcal{F}_x^\vee \oplus \mathcal{F}_y^\vee)$ of $\mathbb{P}(W_{10}^\vee)$.

Proof. The fiber over $\{x, y\}$ of the evaluation map defined in Lemma 8.1 gives an exact sequence

$$0 \to \mathcal{I}_{6\{x,y\}} \to V_{10} \to \text{Sym}^2(\mathcal{F}_x^\vee \oplus \mathcal{F}_y^\vee) \to \text{Sym}^2(\mathcal{F}_x^\vee) \oplus \text{Sym}^2(\mathcal{F}_y^\vee) \to 0,$$

hence $\mathcal{I}_{6\{x,y\}}$ consists of elements of $V_{10}$ that also vanish on $\mathbb{P}(\mathcal{F}_x^\vee \oplus \mathcal{F}_y^\vee)$. \qed

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