Problem 1

Recall that a topological space $X$ is irreducible if it is non-empty and is not the union of two strict closed subsets. In other words, if $X_1$ and $X_2$ are closed subsets of $X$ and $X = X_1 \cup X_2$, then $X = X_1$ or $X = X_2$.

a) Let $X$ be a topological space and let $V \subset X$ be a subset (endowed with the induced topology). Prove that $V$ is irreducible if and only if its closure $\overline{V}$ is irreducible.

b) Let $X$ and $Y$ be topological spaces and let $u : X \to Y$ be a continuous map. If $X$ is irreducible, prove that $u(X)$ is irreducible.

Problem 2

Let $k$ be an infinite (not necessarily algebraically closed) field. Let $C \subset k^2$ be the vanishing set $V(X^2 - Y^3)$.

a) Prove that the ideal of $C$ is the ideal in $k[X, Y]$ generated by $X^2 - Y^3$ and that $C$ is irreducible (Hint: use the “parametrization” $k \to C$ given by $t \mapsto (t^3, t^2)$ and express $A(C) = k[X, Y]/I(C)$ as a subring of $k[T]$).

b) Prove that $C$ is not isomorphic to $k$ (Hint: prove that $A(C)$ is not a principal ideal domain).

c) How do these these results generalize to the vanishing set $V(X^r - Y^s)$, where $r$ and $s$ are relatively prime positive integers?

Problem 3

Let $k$ be an infinite (not necessarily algebraically closed) field, let $u : \mathbb{P}^1_k \to \mathbb{P}^3_k$ be the regular map defined by $u(s, t) = (s^3, s^2t, st^2, t^3)$, and set $C := u(\mathbb{P}^1_k)$.

a) Prove that no 4 distinct points of $C$ are contained in a hyperplane in $\mathbb{P}^3_k$.

b) Prove that any quadric in $\mathbb{P}^3_k$ (i.e., any subset of $\mathbb{P}^3_k$ defined by a non-zero homogeneous polynomial of degree 2) that contains 7 distinct points of $C$ contains $C$.

c) Prove that $C$ is the vanishing set in $\mathbb{P}^3_k$ of the (homogeneous) ideal $I$ in $k[T_0, T_1, T_2, T_3]$ generated by the homogeneous polynomials $T_0T_2 - T_1T_3$, $T_1T_2 - T_0T_3$, which can be neatly expressed as the $2 \times 2$-minors of the matrix

$$
\begin{pmatrix}
T_0 & T_1 & T_2 \\
T_1 & T_2 & T_3
\end{pmatrix}.
$$

d) Prove that the ideal of $C$ is $I$ (Hint: prove that any polynomial $P \in k[T_0, T_1, T_2, T_3]$ is congruent modulo $I$ to a polynomial of the type $A(T_0, T_1, T_3) + T_2B(T_3)$ and that if $P$ vanishes on $C$, one has $B = 0$; then, use a similar method to show that $A$ is divisible by $T_1^3 - T_0^2T_3$).

e) (Extra credit) How do these results generalize to the regular map $u : \mathbb{P}^1_k \to \mathbb{P}^n_k (n \geq 3)$ defined by $u(s, t) = (s^n, s^{n-1}t, \ldots, st^{n-1}, t^n)$?