Corrigendum to “FANO VARIETIES”

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As Benjamin Nill pointed out, there is a slight imprecision in the notation and the proof of Theorem 8 of [D]. One has to treat the case $a = 0$ separately. The conclusion of the Theorem holds as stated. Here is a corrected proof.\footnote{Note that Cinzia Casagrande proved later in [C] a strong version of the Batyrev conjecture: the Picard number of a normal projective $\mathbb{Q}$-factorial Gorenstein toric Fano variety $X$ of dimension $n$ is at most $2n$, with equality if and only if $n$ is even and $X$ is isomorphic to the $n/2$-th self-product of the blow-up of $\mathbb{P}^2$ at three noncollinear points.}

**Theorem 1** The Picard number of a smooth toric Fano variety of dimension $n$ is at most

$$2 + 2\sqrt{(n^2 - 1)(2n - 1)}$$

**Proof.** Let $u$ be a vertex of $P$. The half line $\mathbb{R} - u$ meets the boundary of $P$ at a point that belongs to some face of dimension $k \leq n - 1$ (choose $k$ minimal) hence is in the relative interior of the convex hull of some vertices $u_1, \ldots, u_d$, with $d \leq k + 1$ by Helly’s theorem. One can therefore write

$$0 = \lambda u + \lambda_1 u_1 + \cdots + \lambda_d u_d$$

with $\lambda_1 \geq \cdots \geq \lambda_d > 0$ and $\lambda > 0$. Let $a \in \{0, \ldots, d\}$, let $V_a(Q)$ be the set of vertices of $Q$ that are on the facets $F_u, F_{u_1}, \ldots, F_{u_a}$ or on a facet adjacent to them, and let $v$ be a vertex of $Q$ in the complement $V_a(Q)^c$ of $V_a(Q)$ in the set $V(Q)$ of all vertices of $Q$.

Note first that for each $i \in \{2, \ldots, d\}$, the intersection $F_{u_1} \cap F_{u_i}$ is a face of $Q$ of dimension at least $n - k - 1$, hence contains at least $n - k$ vertices. Hence for each $b > 0$,

$$F_{u_1} \cup \cdots \cup F_{u_b} \text{ contains at most } n + (b - 1)k \text{ vertices.}$$
Now by Remark 5(2), we have
\[ \langle u, v \rangle \geq 1 \quad \langle u_i, v \rangle \geq 1 \]
for \(1 \leq i \leq a\), hence
\[ -\lambda_{a+1} \langle u_{a+1}, v \rangle - \cdots - \lambda_d \langle u_d, v \rangle = \lambda \langle u, v \rangle + \sum_{i=1}^{a} \lambda_i \langle u, v_i \rangle \]
\[ \geq \lambda + \sum_{i=1}^{a} \lambda_i > \sum_{i=1}^{a} \lambda_i \]
This implies that the integer \( \langle u_i, v \rangle \) must be equal to \(-1\) for at least \(a+1\) indices \(i\) in \(\{a+1, \ldots, d\}\), i.e., that \(v\) must be on at least \(a+1\) faces among \(F_{u_{a+1}}, \ldots, F_{u_d}\).

For \(a = 0\), we obtain that \(\mathcal{V}_0(Q)^c\) is contained in \(F_{u_1} \cup \cdots \cup F_{u_d}\), hence
\[ \text{Card}(\mathcal{V}_0(Q)^c) \leq n + (d - 1)k \]
by (1). For \(a > 0\), consider the set
\[ I_a = \{(v, i) \in \mathcal{V}_a(Q)^c \times \{a+1, \ldots, d\} \mid v \in F_{u_i}\} \]
The fiber of \(i\) for the second projection \(I_a \to \{a+1, \ldots, d\}\) consists of vertices that are on \(F_{u_i}\) but that are not on the intersection \(\bigcap_{j=1}^{d} F_{u_j}\), which is an \((n - k - 1)\)-dimensional face of \(Q\), hence has \(n - k\) vertices. We obtain
\[ \text{Card}(I_a) \leq k(d - a) \]
Since we proved that each fiber of the first projection has at least \(a + 1\) elements, we get
\[ \text{Card}(\mathcal{V}_a(Q)^c) \leq \frac{\text{Card}(I_a)}{a + 1} \leq \frac{k(d - a)}{a + 1} \]
Using (1) again, we obtain
\[
\text{Card}(\mathcal{V}_a(Q)) \leq n + (a - 1)k \quad \text{vertices on } F_u
\]
\[+ n + (a - 1)k \quad \text{vertices on } F_{u_1} \cup \cdots \cup F_{u_a}\]
\[+ (a + 1)n \quad \text{vertices adjacent to } F_u, F_{u_1}, \ldots, F_{u_a}\]
and
\[
\text{Card}(\mathcal{V}_0(Q)) \leq n \text{ vertices on } F_u \\
+ n \text{ vertices adjacent to } F_u
\]

All in all, we obtain
\[
\text{Card}(\mathcal{V}(Q)) \leq 2n + (a - 1)k + (a + 1)n + \frac{k(d - a)}{a + 1}
\]
\[
= 3n - 2k + a(k + n) + \frac{k(d + 1)}{a + 1}
\]
for all \( a \in \{0, \ldots, d\} \). Taking
\[
a = \left\lceil \sqrt{\frac{k(d + 1)}{k + n}} \right\rceil
\]
we get
\[
\text{Card}(\mathcal{V}(Q)) \leq 3n - 2k + 2\sqrt{k(d + 1)(k + n)}
\]
Therefore,
\[
\text{Card}(\mathcal{V}(Q)) \leq \max_{1 \leq d \leq k + 1 \leq n} \left( 3n - 2k + 2\sqrt{k(d + 1)(k + n)} \right)
\]
\[
= n + 2 + 2\sqrt{(n^2 - 1)(2n - 1)}
\]

\[\square\]

References
