Programme thématique

« Points rationnels, courbes rationnelles et courbes entières sur les variétés algébriques »

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Moduli spaces for some Fano manifolds
Moduli spaces of cubic using period maps.

Smooth cubics in $\mathbb{P}^n$ have a GIT affine moduli space $\mathcal{M}_n$ (singular cubics are parameterized by a $\text{SL}(n+1)$-invariant divisor), with comp. $\mathcal{C}_n$.

These moduli spaces may be studied via the period map, (in low dimensions).

Best example is $n = 4$.

Hodge structure is of $K3$ type

$$H^4(X, \mathbb{C}) \cong H^{1,3} \oplus H^{3,1} \oplus H^{1,1}$$

This Hodge structure is determined by the lattice

$$H^{1,3}_{\text{period domain}}$$

signature $(20, 2)$

$$\Omega = \{ [\omega] \in \mathbb{P}^{20} \mid \omega \cdot \omega = 0, \quad \omega \cdot \omega > 0 \}$$

bounded symmetric domain of type $IV$ disc $(20)$

period map

$$\mathcal{C}_4 \rightarrow \Omega = \mathcal{T} \setminus \mathcal{Q}$$

$\mathcal{Q}$ subgroup of finite index of $\mathcal{O}(\Lambda)$

A lattice $H^4(X, \mathbb{Z}) \cong 2^{20} \oplus 2 \oplus E_8 \oplus A_2 (2, 1, 2)$

- $\mathcal{T}$ injective (easy)
- $\mathcal{C}$ injective (Voisin, hard; original proof had a gap filled using subsequent work of Laza)
- compactify $\mathcal{C}$ to study its image (Laza, 2007)
Image is complement of divisors $D_2$ and $D_6$ for special cubic fourfolds (Horrocks) of discriminant 2 and 6, i.e., $H^4(X, \mathbb{Z}) \cap H^{2,2}(X)$ contains rank 2 lattices of divisors 2 and 6.

This identifies $C_4$ with explicit complement of 2 hypersurfaces in $\mathbb{P}$ (Batyrev-Borodzik computation has obtained by adding 6 indecomposable components of dim $\leq 3$).

To extend $\rho$, one needs to blow-up $\mathbb{P}^3$ at a particular point (the "determinantal cubic" $x_0 x_1 x_2 x_3 x_4 = 0$).

Cubic semi-stable orbit corresponding to cubics with 2 div. sing locus.

Cubics of other dimensions

- $n=3$: Hodge structure has weight one $H^3(X)$
- $n=5$: dual lattice intersection Jacobi variety does give an injective map $E_3 \hookrightarrow A_5$ but image has large codimension.

Associated with $X$ a cubic fourfold with only 3 automorphisms.

\[ C_3 \hookrightarrow \mathbb{P}^4 \setminus \{x^3 = 0\} \]

of one point $C_3 = \mathbb{P}^1 \setminus \{y = 0\}$.

\[ (\text{on previous curve blown-up in } C_4) \]

\[ \text{www.alsace.cnrs.fr} \]
Add to \( n = 4 \)

\( F(X) \) is an irreducible hor. sympl. (IMS) variety
with "same period "

\[ \mathcal{M} \leftarrow \text{moduli space of these IMS} \]

\( 2 \) Verbitsky (with a degree 6 polarized form)

\( \mathcal{D}_2 \) contains \( S^{[2]} \) (form a divisor)

whose image in \( \mathcal{D}_2 \) is \( S \) of degree 2

These IMS "correspond" to "ell' cubics" - they

are not varieties of lines on a cubic surface

[ \( F(X) \) can be \( S^{[2]} \) for infinitely many

polarized \( S \); see BD or Hassett ]
$m=2$ no interesting Hodge structure

Associated

2. Quadratic line complexes

Varieties $G(2,n) \cap \mathbb{Q}$

Classically $m=4$

Here $m=5$ \( \rightarrow \) index 3

Hodge structure has level 1

intermediate Jacobian, dim 6

too large!

We use the same trick and consider

$\mathbb{Z} \rightarrow G(2,5)$ branched along $X$

(Fano of dim 6, index 4, \( p=1 \))

\( \rightarrow \) all are like that)

Hodge structure of K3 type

on $H^*(Z)$

$H^6(Z)_{\text{vir}} = H^{2,4} \oplus H^{3,3} \oplus H^{4,2}$

\( \oplus \) \( \oplus \) \( \oplus \)

(but different lattice $20 \oplus 2E_8(-1) \oplus 2A_1(-1)$

Moduli space: cofinite GIT quotient

$\mathbb{P} : \mathbb{Z} \rightarrow 0 = \Pi \backslash \mathbb{Q}$

$\oplus \oplus \oplus \oplus$

$T_p$ always surjective (\( \Rightarrow \) $p$ dominant)

We would like to use this map to describe $Z$

As for cubics, we can associate an IH$S$

(II$\theta$- Manivel). This is a bit more complicated
\[ I_G(2) \quad \text{quadrice containing } G, \text{ all of rank } 6 \]

\[ I_X(2) = I_G(2) \oplus C_0 \]
(if \( X \) smooth, we may take \( C_0 = 0 \) smooth)

\[ \text{Discriminant} = 4 |I_G(2)| + \chi \]

\[ \chi_X \in \mathbb{P}^5 \text{ is a } (\text{possibly non-integral}) \]

\[ \text{sechc c EPM sechc} \]

**BRIEF DEFINITION**

\[ A_{10} \subset \Lambda^3 V_6 \quad \text{Logarithmic} \]

\[ \gamma_A = \{ v \in P(V_6) \mid (\Lambda^2 V_6 \wedge v) \cap A \neq \emptyset \} \]

is either all or a (singular sechc)

\[ \text{Sing } (\gamma_A) = \gamma_A[2] \cup \left[ \bigcup_{[V_3]} \gamma_A \right] \]

\[ G = \{ V_3 \mid [\Lambda^3 V_3] \in A \} \]

\[ X_A \rightarrow \gamma_A \quad \text{dim}(\cdots) \geq 2 \]

\[ \subset \text{IHS for } A \text{ general} \]

**There is a GIT moduli space for EPM sechcs:**

\[ \text{EPM} = \text{LG}(\Lambda^3 V_6) \times \text{SL}(V_6) \quad \text{(O'Grady)} \]

**EPM** (com. to smooth double \( \text{EPM} \)) is the complement of 2 (ample) divisors

\[ \Sigma = \{ A \mid \gamma_A[2] \neq \emptyset \} \]

\[ \Delta = \{ A \mid \gamma_A[3] \neq \emptyset \} \]

**Theorem**

The association \( X \mapsto \gamma_X \) induces a morphism

\[ \Phi: Z \rightarrow \text{EPM} - \Sigma \quad \text{(hence possibly finite singular locus)} \]

Fibers are

\[ P(V_6^*), \gamma_A^* / \text{Aut} \]

\[ \subset \text{dual } \text{EPM} = \{ H \mid \Lambda^3 H \cap A \neq \emptyset \} \]

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On the other hand (by Verbitsky again),

$\text{EPW} \xrightarrow{\text{open embedding}} \Sigma$

In fact some remains true for

$\text{EPW} - \Sigma \xrightarrow{\alpha} \text{th} \Sigma$

$\Delta$ is sent onto divisor $D_{10}^{[2]}$

$x_\Delta \rightarrow S_{[2]}$ 

degree 10

$\text{period of } S$

\begin{align*}
\text{Corollary} & \quad \mathcal{Z} \text{ is an affine bundle in } \\
& \quad \mathbb{P}^5 - \text{EPW over the image of } P
\end{align*}

$R_{\mathbb{P}^8}$ $p_5$ I should be the period map of $\mathcal{Z}$

$\mathcal{Z}$ What is the image of $p_5$ (or $p_5 - \mathcal{Z}$)?

\begin{align*}
\text{Conf} & \quad (\text{Dubois} \text{, O'Grady}) \\
& \quad \text{The image of } P \text{ is the complement of}
\end{align*}

Known: it is contained

$\mathcal{O}_2$

$\mathcal{O}_4$

$\mathcal{O}_8$ is the image of normal $X$ or, equivalently, of $\Sigma$

$\mathcal{O}_2$ is the image of the orbit $A(U_3)$

$\mathcal{O}_4$

These EPW sections are $3x$ secant quadrics.

We will explain the corresponding $X$

(both are secant stable non stable points)

when $X_t$ approaches $X$, the period should approach period

of $S$ quantic in $\mathbb{P}^3$)
Those EPW sextics are 2x del Pezzo cubic in $\mathbb{P}(V_6)$ not stable period of $S$ K3 of degree 2 I don’t know the corresponding $X$ (should be singular along $V_3(\mathbb{P}^2) \subset \mathbb{G}(2, V_5)$

Tangent line complex

$X \subset \mathbb{G}(2, V_5)$ lines tangent to $X$
singular along $\mathbb{G}(2, V_5) = \text{lines in } q$

$X \times 3$ smooth quadric

3. Other dimensions

We consider linear sections of quadratic line complexes

$\mathbb{G}(2, V_5) \cap \mathbb{P}^8$
$\cap \mathbb{P}^7$
$\cap \mathbb{P}^6$

General K3 of degree 10

Easy to show that $X_4 \rightarrow X_5 \rightarrow \mathbb{G} \cap \mathbb{P}^8$ branched along quotient of quadratic line complexes