UNIVERSAL DYNAMICS FOR THE DEFOCUSING LOGARITHMIC SCHRÖDINGER EQUATION

RÉMI CARLES AND ISABELLE GALLAGHER

Abstract. We consider the nonlinear Schrödinger equation with a logarithmic nonlinearity, whose sign is such that no non-trivial stationary solution exists. Explicit computations show that in the case of Gaussian initial data, the presence of the nonlinearity affects the large time behavior of the solution: the dispersion is faster than usual by a logarithmic factor in time and the positive Sobolev norms of the solution grow logarithmically in time. Moreover after rescaling in space by the dispersion rate, the modulus of the solution converges to a universal Gaussian profile (whose variance is independent of the initial variance). In the case of general initial data, we show that these properties remain, in a weaker sense. One of the key steps of the proof consists in using the Madelung transform to reduce the equation to a variant of the isothermal compressible Euler equation, whose large time behavior turns out to be governed by a parabolic equation involving a Fokker–Planck operator.

1. Introduction

1.1. **Setting.** We are interested in the following equation

$$i\partial_t u + \frac{1}{2}\Delta u = \lambda \ln\left(|u|^2\right) u\,, \quad u_{|t=0} = u_0\,, \tag{1.1}$$

with $x \in \mathbb{R}^d$, $d \geqslant 1$, $\lambda \in \mathbb{R}\setminus\{0\}$. It was introduced as a model of nonlinear wave mechanics and in nonlinear optics ([6], see also [7, 33, 34, 35, 40]). The mathematical study of this equation goes back to [14, 12] (see also [13]). The sign $\lambda < 0$ seems to be the more interesting from a physical point of view, and this case has been studied formally and rigorously (see [16, 34] for instance). On the other hand, the case $\lambda > 0$ seems to have been little studied mathematically, except as far as the Cauchy problem is concerned (see [14, 28]). In this article, we address the large time properties of the solution in the case $\lambda > 0$, revealing several new features in the context of Schrödinger equations, and more generally Hamiltonian dispersive equations.

We recall that the mass, angular momentum and energy are (formally) conserved, in the sense that defining

$$M(u(t)) := \|u(t)\|^2_{L^2(\mathbb{R}^d)}\,,$$

$$J(u(t)) := \operatorname{Im}\int_{\mathbb{R}^d} \bar{u}(t,x)\nabla u(t,x)dx\,,$$

$$E(u(t)) := \frac{1}{2}\|\nabla u(t)\|^2_{L^2(\mathbb{R}^d)} + \lambda\int_{\mathbb{R}^d} |u(t,x)|^2 \ln|u(t,x)|^2 dx\,,$$

RC is supported by the French ANR projects SchEq (ANR-12-JS01-0005-01), BECASIM (ANR-12-MONU-0007-04) and BoND (ANR-13-BS01-0009-01).

then formally,

$$\frac{d}{dt}M(u(t)) = \frac{d}{dt}J(u(t)) = \frac{d}{dt}E(u(t)) = 0\,.$$

The last identity reveals the Hamiltonian structure of (1.1).

Remark 1.1 (Effect of scaling factors). Unlike what happens in the case of an homogeneous nonlinearity (classically of the form $|u|^p u$), replacing u with κu ($\kappa > 0$) in (1.1) has only little effect, since we have

$$i\partial_t(\kappa u) + \frac{1}{2}\Delta(\kappa u) = \lambda \ln\left(|\kappa u|^2\right)\kappa u - 2\lambda(\ln\kappa)\kappa u\,.$$

The scaling factor thus corresponds to a purely time-dependent gauge transform:

$$\kappa u(t,x)e^{2it\lambda\ln\kappa}$$

solves (1.1) (with initial datum κu_0). In particular, the L^2-norm of the initial datum does not influence the dynamics of the solution.

Note that whichever the sign of λ, the energy E has no definite sign. The distinction between focusing or defocusing nonlinearity is thus a priori ambiguous. We shall see however that in the case $\lambda < 0$, no solution is dispersive, while for $\lambda > 0$, solutions have a dispersive behavior (with a non-standard rate of dispersion). This is why we choose to call *defocusing* the case $\lambda > 0$.

1.2. **The focusing case.** In [14] (see also [13]), the Cauchy problem is studied in the case $\lambda < 0$. Define

$$W := \left\{u \in H^1(\mathbb{R}^d)\,,\, x \mapsto |u(x)|^2 \ln|u(x)|^2 \in L^1(\mathbb{R}^d)\right\}.$$

Proposition 1.2 (Théorème 2.1 from [14], see also Theorem 9.3.4 from [13])**.** *Let the initial data u_0 belong to W. In the case when $\lambda < 0$, there exists a unique, global solution $u \in C(\mathbb{R}; W)$ to* (1.1)*. In particular, for all $t \in \mathbb{R}$, $|u(t,\cdot)|^2 \ln|u(t,\cdot)|^2$ belongs to $L^1(\mathbb{R}^d)$, and the mass $M(u)$ and the energy $E(u)$ are independent of time.*

In the case when $\lambda < 0$, it can be proved that there is no dispersion for large times. Indeed the following result holds.

Lemma 1.3 (Lemma 3.3 from [12])**.** *Let $\lambda < 0$ and $k < \infty$ such that*

$$L_k := \left\{u \in W,\ \|u\|_{L^2(\mathbb{R}^d)} = 1,\ E(u) \leqslant k\right\} \neq \emptyset\,.$$

Then

$$\inf_{\substack{u\in L_k\\ 1\leqslant p\leqslant\infty}} \|u\|_{L^p(\mathbb{R})} > 0\,.$$

This lemma, along with the conservation of the energy for (1.1), indicates that in the case $\lambda < 0$, the solution to (1.1) is not dispersive: typically, its L^∞ norm is bounded from below. Actually in the case of Gaussian initial data, some solutions are even known to be periodic in time, as proved in [16] (and already noticed in [6]).

Theorem 1.4 ([16])**.** *In the case $\lambda < 0$, the* Gausson $\exp(2i\omega t + \omega + d/2 + \lambda|x|^2)$ *is a solution to* (1.1) *for any period $\omega \in \mathbb{R}$.*

We emphasize that several results address the existence of stationary solutions to (1.1) in the case $\lambda < 0$, and the orbital stability of the Gausson; see e.g. [6, 12, 16, 4].

1.3. **Main results.** Throughout the rest of this paper, we assume $\lambda > 0$.

1.3.1. *The Cauchy problem.* Define, for $0 < \alpha \leqslant 2$, the weighted L^2 space

$$V_\alpha := \Big\{ u \in L^2(\mathbb{R}^d) \, , \, x \mapsto \langle x \rangle^{\alpha/2} u(x) \in L^2(\mathbb{R}^d) \Big\} \, ,$$

where $\langle x \rangle := \sqrt{1 + |x|^2}$, with norm

$$\|u\|_{V_\alpha} := \| \langle x \rangle^{\alpha/2} u(x) \|_{L^2(\mathbb{R}^d)} \, .$$

Note that for any $\alpha > 0$, $V_\alpha \cap H^1 \subset W$. The Cauchy problem for (1.1) is investigated in [28], where in three space dimensions, the existence of a unique solution in $L^\infty(\mathbb{R}; H^1(\mathbb{R}^3)) \cap C(\mathbb{R}; L^2(\mathbb{R}^3))$ is proved as soon as the initial data belongs to $V_1 \cap H^1(\mathbb{R}^3)$. Actually it is possible to improve slightly that result into the following theorem.

Theorem 1.5. *Let the initial data u_0 belong to $V_\alpha \cap H^1(\mathbb{R}^d)$ with $0 < \alpha \leqslant 2$. In the case when $\lambda > 0$, there exists a unique, global solution $u \in L^\infty_{\rm loc}(\mathbb{R}; V_\alpha \cap H^1)$ to (1.1). Moreover the mass $M(u)$, the angular momentum $J(u)$, and the energy $E(u)$ are independent of time. If in addition $u_0 \in H^2(\mathbb{R}^d)$, then $u \in L^\infty_{\rm loc}(\mathbb{R}; H^2)$.*

The main focus of this paper concerns large time asymptotics of the solution. The situation is very different from the $\lambda < 0$ case described above (see Theorem 1.4). Indeed we can prove that (some) solutions tend to zero in L^∞ for large time, while the H^1 norm is always unbounded.

1.3.2. *Large time behavior: the Gaussian case.* As noticed already in [6], an important feature of (1.1) is that the evolution of initial Gaussian data remains Gaussian. Since (1.1) is invariant by translation in space, we may consider centered Gaussian initial data. The following result is a crucial guide for the general case. We define from now on the function

$$\ell(t) := \frac{\ln \ln t}{\ln t} \cdot \tag{1.2}$$

Theorem 1.6. *Let $\lambda > 0$, and consider the initial data*

$$u_0(x) = b_0 \exp\Big(-\frac{1}{2} \sum_{j=1}^d a_{0j} x_j^2 \Big) \tag{1.3}$$

with $b_0, a_{0j} \in \mathbb{C}$, $\alpha_{0j} = \operatorname{Re} a_{0j} > 0$. Then the solution u to (1.1) is given by

$$u(t,x) = b_0 \prod_{j=1}^d \frac{1}{\sqrt{r_j(t)}} \exp\Big(i\phi_j(t) - \alpha_{0j} \frac{x_j^2}{2 r_j^2(t)} + i \frac{\dot r_j(t)}{r_j(t)} \frac{x_j^2}{2} \Big)$$

for some real-valued functions ϕ_j, r_j depending on time only, such that, as $t \to \infty$,

$$r_j(t) = 2t\sqrt{\lambda \alpha_{0j} \ln t} \Big(1 + \mathcal{O}\big(\ell(t)\big) \Big), \quad \dot r_j(t) = 2\sqrt{\lambda \alpha_{0j} \ln t} \Big(1 + \mathcal{O}\big(\ell(t)\big) \Big). \tag{1.4}$$

In particular, as $t \to \infty$,

$$\|u(t)\|_{L^\infty(\mathbb{R}^d)} \sim \frac{1}{\big(t\sqrt{\ln t} \big)^{d/2}} \frac{\|u_0\|_{L^2}}{\big(2\lambda\sqrt{2\pi} \big)^{d/2}} \, .$$

On the other hand u belongs to $L^\infty_{\rm loc}(\mathbb{R}; H^1(\mathbb{R}^d))$ and as $t \to \infty$

$$\|\nabla u(t)\|^2_{L^2(\mathbb{R}^d)} \underset{t\to\infty}{\sim} 2\lambda d \|u_0\|^2_{L^2(\mathbb{R}^d)} \ln t.$$

Remark 1.7 (Numerical simulations). The asymptotic expansions for r_j and its derivative show that the convergences mentioned above are fairly slow. Therefore, performing reliable large time numerical simulations on (1.1) is a challenging issue.

At least three aspects of this result differ from the more standard Schrödinger equations, as discussed in more detail below:

- The dispersion is of order $(t\sqrt{\ln t})^{-d/2}$, as opposed to $t^{-d/2}$ in the case of the free Schrödinger equation $i\partial_t u + \frac{1}{2}\Delta u = 0$, or of defocusing nonlinear Schrödinger equations with sufficiently short range nonlinearity. The nonlinearity therefore has an effect on the dispersion rate.
- Even though the solution is dispersive, its H^1-norm is unbounded.
- Up to a rescaling, the modulus of u converges for large time to a *universal* Gaussian profile,
$$(2t\sqrt{\lambda \ln t})^{d/2} \left| u\left(t, x \times 2t\sqrt{\lambda \ln t}\right)\right| \underset{t\to\infty}{\longrightarrow} \frac{\|u_0\|_{L^2}}{\pi^{d/4}} e^{-|x|^2/2},$$
that is, regardless of the value of the variance of the Gaussian initial datum (a more precise statement is given in Corollary 1.12 below).

Indeed, in the linear case

$$i\partial_t u_{\rm free} + \frac{1}{2}\Delta u_{\rm free} = 0\,, \quad u_{{\rm free}|t=0} = u_0\,, \tag{1.5}$$

the integral representation

$$u_{\rm free}(t,x) = \frac{1}{(2i\pi t)^{d/2}} \int_{\mathbb{R}^d} e^{i\frac{|x-y|^2}{2t}} u_0(y)dy$$

readily yields the well-known dispersive estimate

$$\|u_{\rm free}(t)\|_{L^\infty(\mathbb{R}^d)} \lesssim \frac{1}{|t|^{d/2}} \|u_0\|_{L^1(\mathbb{R}^d)}\,.$$

Moreover writing

$$e^{i\frac{t}{2}\Delta} = M_t D_t \mathcal{F} M_t\,,$$

where M_t stands for the multiplication by the function $e^{i\frac{|x|^2}{2t}}$, and

$$\left(D_t f\right)(x) := \frac{1}{(it)^{d/2}} f\left(\frac{x}{t}\right), \quad \mathcal{F}f(\xi) := \hat{f}(\xi) := \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} f(x) e^{-ix\cdot\xi} dx\,,$$

we have $e^{i\frac{t}{2}\Delta} \underset{t\to\infty}{\sim} M_t D_t \mathcal{F}$, that is

$$\|u_{\rm free}(t) - A(t)u_0\|_{L^2(\mathbb{R}^d)} \underset{t\to\pm\infty}{\longrightarrow} 0, \quad A(t)u_0(x) := \frac{1}{(it)^{d/2}} \hat{u}_0\left(\frac{x}{t}\right) e^{i\frac{|x|^2}{2t}}\,, \tag{1.6}$$

a formula which has proven very useful in the nonlinear (long range) scattering theory (see e.g. [24, 31]).

In the case of the defocusing nonlinear Schrödinger equation with power-like nonlinearity,

$$i\partial_t u + \frac{1}{2}\Delta u = |u|^{2\sigma} u\,, \quad u_{|t=0} = u_0\,, \tag{1.7}$$

if σ is sufficiently large (say $\sigma > 2/d$ if $u_0 \in H^1(\mathbb{R}^d)$, even though this bound can be lowered if in addition $\hat{u}_0 \in H^1(\mathbb{R}^d)$), then there exists $u_+ \in H^1(\mathbb{R}^d)$ such that in $L^2(\mathbb{R}^d)$,

$$u(t,x) \underset{t\to\infty}{\sim} e^{i\frac{t}{2}\Delta}u_+(x) \underset{t\to\infty}{\sim} \frac{1}{(it)^{d/2}}\hat{u}_+\left(\frac{x}{t}\right)e^{i\frac{|x|^2}{2t}},$$

where the last relation stems from (1.6). Therefore, Theorem 1.6 shows that unlike in the free case (1.5) or in the above nonlinear case (1.7), the dispersion is modified (it is even enhanced the larger the λ), and the asymptotic profile $\hat{u}_+$ (with $u_+ = u_0$ in the free case), which depends on the initial profile, is replaced by a universal one (up to a normalizing factor),

$$\frac{\|u_0\|_{L^2}}{\pi^{d/4}}e^{-|x|^2/2}.$$

Finally, we note that the H^1-norm of u is unbounded for large time in the case of Theorem 1.6, a point which is due to the fact that the dispersive rate of u is larger that the oscillatory rate, by a logarithmic factor, since

$$\frac{\dot{r}_j(t)}{r_j(t)} \underset{t\to\infty}{\sim} \frac{1}{t}.$$

1.3.3. *Long time behavior: the general case.* We show that the three features described above remain in a fairly general framework, up to weakening some aspects of the statement. Before stating the general result, let us introduce the universal dispersion rate τ through the following lemma.

Lemma 1.8 (Universal dispersion)**.** *Consider the ordinary differential equation*

$$\ddot{\tau} = \frac{2\lambda}{\tau}, \quad \tau(0) = 1, \quad \dot{\tau}(0) = 0. \tag{1.8}$$

It has a unique solution $\tau \in C^2(0,\infty)$, and it satisfies, as $t \to \infty$,

$$\tau(t) = 2t\sqrt{\lambda \ln t}\Big(1 + \mathcal{O}\big(\ell(t)\big)\Big), \quad \dot{\tau}(t) = 2\sqrt{\lambda \ln t}\Big(1 + \mathcal{O}\big(\ell(t)\big)\Big).$$

Let us now turn to the case of general initial data. Denote by

$$\gamma(x) := e^{-|x|^2/2}$$

the Gaussian with variance one discussed above. In view of Remark 1.1, we may suppose $\|u_0\|_{L^2(\mathbb{R}^d)} = \|\gamma\|_{L^2(\mathbb{R}^d)}$, an assumption that we make in the next statement in order to lighten the notations.

Theorem 1.9. *Let $u_0 \in V_2 \cap H^1 = H^1 \cap \mathcal{F}(H^1)$, with $\|u_0\|_{L^2(\mathbb{R}^d)} = \|\gamma\|_{L^2(\mathbb{R}^d)}$, and rescale the solution provided by Theorem 1.5 to $v = v(t,y)$ by setting*

$$u(t,x) = \frac{1}{\tau(t)^{d/2}}v\left(t, \frac{x}{\tau(t)}\right)\exp\left(i\frac{\dot{\tau}(t)}{\tau(t)}\frac{|x|^2}{2}\right). \tag{1.9}$$

There exists C such that for all $t \geqslant 0$,

$$\int_{\mathbb{R}^d}\left(1 + |y|^2 + \big|\ln|v(t,y)|^2\big|\right)|v(t,y)|^2dy + \frac{1}{\tau(t)^2}\|\nabla_y v(t)\|^2_{L^2(\mathbb{R}^d)} \leqslant C. \tag{1.10}$$

We have moreover

$$\int_{\mathbb{R}^d}\begin{pmatrix}1\\ y\\ |y|^2\end{pmatrix}|v(t,y)|^2dy \underset{t\to\infty}{\longrightarrow} \int_{\mathbb{R}^d}\begin{pmatrix}1\\ y\\ |y|^2\end{pmatrix}\gamma^2(y)dy. \tag{1.11}$$

As a consequence,

$$\|\nabla u(t)\|^2_{L^2(\mathbb{R}^d)} \underset{t\to\infty}{\sim} 2\lambda d\|u_0\|^2_{L^2(\mathbb{R}^d)} \ln t\,.$$

Finally,

$$|v(t,\cdot)|^2 \underset{t\to\infty}{\rightharpoonup} \gamma^2 \quad \text{weakly in } L^1(\mathbb{R}^d)\,. \tag{1.12}$$

Remark 1.10. If the initial data is not normalized in $L^2(\mathbb{R}^d)$ then the result (1.12) becomes

$$|v(t,\cdot)|^2 \underset{t\to\infty}{\rightharpoonup} \frac{\|u_0\|^2_{L^2}}{\pi^{d/2}}\gamma^2 \quad \text{weakly in } L^1(\mathbb{R}^d)\,.$$

To the best of our knowledge, this is the first time that a universal profile is observed for the large time behavior of solutions to a dispersive, Hamiltonian equation. This profile is reached in a weak sense only in Theorem 1.9, as far as the convergence is concerned, but also because the modulus of the solution only is captured. This indicates that a lot of information remains encoded in the oscillations of the solution. See Section 1.4 for more on the large time asymptotics.

Remark 1.11. As a straightforward consequence, we infer the slightly weaker property that $|v(t,\cdot)|^2$ converges to γ^2 in Wasserstein distance:

$$W_2\left(\frac{|v(t,\cdot)|^2}{\|u_0\|^2_{L^2}},\frac{\gamma^2}{\pi^{d/2}}\right)\underset{t\to\infty}{\longrightarrow} 0,$$

where we recall that the Wasserstein distance is defined, for ν_1 and ν_2 probability measures, by

$$W_p(\nu_1,\nu_2) = \inf\left\{\left(\int_{\mathbb{R}^d\times\mathbb{R}^d}|x-y|^p d\mu(x,y)\right)^{1/p};\quad (\pi_j)_\sharp\mu=\nu_j\right\},$$

where μ varies among all probability measures on $\mathbb{R}^d\times\mathbb{R}^d$, and $\pi_j:\mathbb{R}^d\times\mathbb{R}^d\to\mathbb{R}^d$ denotes the canonical projection onto the j-th factor (see e.g. [45]).

In the Gaussian case, the Csiszár-Kullback inequality enables us to obtain the strong convergence of $|v|^2$ to γ^2 in L^1. This is made precise in the next statement.

Corollary 1.12 (Strong convergence in the Gaussian case)**.** *Suppose that the initial data u_0 is a Gaussian as in* (1.3)*, with $\|u_0\|_{L^2(\mathbb{R}^d)} = \|\gamma\|_{L^2(\mathbb{R}^d)}$. Then, with v given by* (1.9)*, the relative entropy of $|v|^2$ goes to zero for large time:*

$$\int_{\mathbb{R}^d}|v(t,y)|^2\ln\left|\frac{v(t,y)}{\gamma(y)}\right|^2 dy \underset{t\to\infty}{\longrightarrow} 0\,,$$

and the convergence of $|v|^2$ to γ^2 is strong in L^1:

$$\big\||v(t,\cdot)|^2-\gamma^2\big\|_{L^1(\mathbb{R}^d)}\underset{t\to\infty}{\longrightarrow} 0\,.$$

Also, a straightforward consequence of Theorem 1.9 is the unboundedness of all Sobolev norms above L^2:

Corollary 1.13. *Let $u_0\in V_2\cap H^1 = H^1\cap\mathcal{F}(H^1)$, and $0<s<1$. The solution to* (1.1) *satisfies, as $t\to\infty$,*

$$\|u(t)\|_{\dot H^s(\mathbb{R}^d)} \gtrsim (\ln t)^{s/2}\,,$$

where $\dot H^s(\mathbb{R}^d)$ denotes the standard homogeneous Sobolev space.

1.4. **Comments.** As already pointed out, the nonlinearity is responsible for the new dispersive rate, as it introduces a logarithmic factor. In particular, no scattering result relating the dynamics of (1.1) to the free dynamics $e^{i\frac{t}{2}\Delta}$ must be expected. This situation can be compared with the more familiar one with low power nonlinearity, where a long range scattering theory is (sometimes) available. If $\sigma \leqslant 1/d$ in (1.7), then u cannot be be compared with a free evolution for large time, in the sense that if for some $u_+ \in L^2(\mathbb{R}^d)$,

$$\|u(t) - e^{i\frac{t}{2}\Delta}u_+\|_{L^2(\mathbb{R}^d)} \underset{t\to\infty}{\longrightarrow} 0\,,$$

then $u = u_+ = 0$ ([5]). In the case $\sigma = 1/d$, $d = 1,2,3$, a nonlinear phase modification of $e^{i\frac{t}{2}\Delta}$ must be incorporated in order to describe the asymptotic behavior of u ([43, 31]). The same is true when (1.7) is replaced with the Hartree equation [25, 41, 42]. In all these cases, as well as for some quadratic nonlinearities in dimension 3 ([30]), the dispersive rate of the solution remains the same as in the free case, of order $t^{-d/2}$. Note however that a similar logarithmic perturbation of the dispersive rate was observed in [32], for the equation

$$i\partial_t u + \frac{1}{2}\partial_x^2 u = i\lambda u^3 + |u|^2 u\,, \quad x \in \mathbb{R}\,, \tag{1.13}$$

with $\lambda \in \mathbb{R}$, $0 < |\lambda| < \sqrt{3}$. More precisely, the authors construct small solutions satisfying the bounds

$$\frac{1}{\sqrt{t}(\ln t)^{1/4}} \lesssim \sup_{|x|\leqslant\sqrt{t}} |u(t,x)| \lesssim \frac{1}{\sqrt{t}(\ln t)^{1/4}}\,, \quad \text{as } t\to\infty\,.$$

An important difference with (1.1) though is that the L^2-norm of the solution of (1.13) is not preserved by the flow, and that (1.13) has no Hamiltonian structure.

On the other hand, in view of the large time behavior of $|u|^2$, it would seem sensible to compare u with a solution to

$$i\partial_t u_{\rm lin} + \frac{1}{2}\Delta u_{\rm lin} = -\lambda\frac{|x|^2}{\tau(t)^2}u_{\rm lin}\,, \tag{1.14}$$

with nontrivial effects altering the phase. The fundamental solution associated with this equation is given by a generalized Mehler formula, or even by a generalized lens transform since the potential is isotropic, relating the solution to (1.14) with the solution of the free equation (1.5); see [9]. Resuming the computations from [9, Section 4], we see that the solution to (1.14) is given in terms of the free solution to (1.5) through the formula

$$u_{\rm lin}(t,x) = \frac{1}{\tau(t)^{d/2}}u_{\rm free}\left(\int_0^t \frac{ds}{\tau(s)^2}, \frac{x}{\tau(t)}\right)\exp\left(i\frac{\dot\tau(t)}{\tau(t)}\frac{|x|^2}{2}\right),$$

a formula which is comparable with the one relating u and v in Theorem 1.9, up to the change of time variable. With the same change of unknown function as in Theorem 1.9, we have

$$v_{\rm lin}(t,y) = u_{\rm free}\left(\int_0^t \frac{ds}{\tau(s)^2}, y\right).$$

In view of Lemma 1.8, strongly in $L^1(\mathbb{R}^d)$,

$$|v_{\rm lin}(t,y)|^2 \underset{t\to\infty}{\longrightarrow} |u_{\rm free}\,(T,y)|^2\,, \quad T := \int_0^\infty \frac{ds}{\tau(s)^2} < \infty.$$

Recall that (1.14) is expected to approximate the solution to (1.1) for large time, up to unknown oscillatory factors: therefore, one would have to impose at least $|u_{\rm free}(T,y)| = \gamma(y)$ (up to the normalization in L^2). We do not investigate this issue in this paper.

Note also that the non-standard dispersion for (1.1) suggests that global in time Strichartz estimates should be available for u, possibly with a slight improvement. However, our approach, based rather on energy estimates, does not provide such information.

Another non-standard dispersion rate was observed in [8] in the case of

$$i\partial_t u + \frac{1}{2}\Delta u = -\frac{|x|^2}{2}u + |u|^{2\sigma}u\,.$$

There, the solution disperses exponentially fast in time, but this is a *linear effect*: the solution to

$$i\partial_t u + \frac{1}{2}\Delta u = -\frac{|x|^2}{2}u$$

disperses exponentially in time, and the consequence is that any nonlinearity of the form $|u|^{2\sigma}u$, $\sigma > 0$, becomes negligible for large time. Similarly, the H^1-norm of the solution then grows exponentially in time, but then again, this is true in the linear case. On the other hand, the unboundedness of the H^1-norm of the solution to (1.1) is due to nonlinear effects only. Note that the reduction (1.14) draws a parallel between (1.1) and the above linear model: the effect of the time decaying factor $1/\tau^2$ is to modify the dispersion, but in a moderate way compared to the case without τ.

A natural question, which we do not address here, is to ask whether the large time behavior of a power-like perturbation of (1.1),

$$i\partial_t u + \frac{1}{2}\Delta u = \lambda \ln\left(|u|^2\right)u + |u|^{2\sigma}u\,, \tag{1.15}$$

can be described in the large time limit. Since the logarithmic nonlinearity has a strong effect on the dynamics, the question would be to compare the solution to (1.15) with a solution to (1.1) as $t \to \infty$ (and not with a solution to the free Schrödinger equation). As already suggested above, at least on a formal level, this might be compared with the equation

$$i\partial_t u + \frac{1}{2}\Delta u = -\lambda\frac{|x|^2}{\tau(t)^2}u + \frac{1}{\tau(t)^{d\sigma}}e^{-\sigma|x|^2/\tau(t)^2}u\,.$$

Finally let us remark that the convergence to a universal profile is reminiscent of what happens for the linear heat equation on $\mathbb{R}^d$, or, in a nonlinear setting, of the works [20, 21, 22] on the Navier-Stokes equations: there it is proved that up to a rescaling which corresponds to the natural scaling of the equations, the vorticity converges strongly to a Gaussian which is known as the Oseen vortex. The main argument, as in the present case, is the reduction to a Fokker-Planck equation.

1.5. **Outline of the paper.** In Section 2, we give the proof of Theorem 1.5. Section 3 contains the explicit computations in the Gaussian case, leading to Theorem 1.6. The main step consists in a reduction to ordinary differential equations, and the computations concerning the main ODEs are gathered in this section, including the proof of Lemma 1.8. The first part of Theorem 1.9, that is everything except (1.12), is proved in Section 4 and relies on energy estimates with appropriate weights in space. The weak limit (1.12) is proved in Section 5. The main idea consists in using a Madelung transform, which leads to the study of a hyperbolic system which is a variant of the isothermal, compressible Euler equation. A rescaling in the time variable reduces the study to a non autonomous perturbation of the Fokker-Planck equation, and the a priori estimates obtained in Section 4 imply that a weak limit of the solution satisfies the Fokker-Planck equation for large times. Since it is known that the large time behavior of the solution to the Fokker-Planck equation is the centered Gaussian, a tightness argument on the rescaled solution concludes the proof. The proofs of the corollaries are given in the final Section 6.

Acknowledgements. The authors wish to thank Kleber Carrapatoso, Laurent Desvillettes, Erwan Faou, Matthieu Hillairet and Cédric Villani for enlightening discussions.

2. Cauchy problem: proof of Theorem 1.5

In this section we sketch the proof of the existence of a unique weak solution, which follows very standard ideas.

2.1. Uniqueness.

The uniqueness of the solution in L^2 follows easily from the following lemma.

Lemma 2.1 (Lemma 9.3.5 from [13]). *We have*

$$\left|\operatorname{Im}\left(\left(z_2\ln|z_2|^2 - z_1\ln|z_1|^2\right)(\bar z_2-\bar z_1)\right)\right| \leqslant 4|z_2-z_1|^2, \quad \forall z_1,z_2\in\mathbb{C}.$$

Consider indeed u_1 and u_2 two solutions of (1.1) in $L^\infty(\mathbb{R};L^2(\mathbb{R}^d))$. Then the function $u := u_1 - u_2$ satisfies

$$i\partial_t u + \frac{1}{2}\Delta u = \lambda\big(\ln\left(|u_1|^2\right)u_1 - \ln\left(|u_2|^2\right)u_2\big)$$

and an energy estimate gives directly

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|u(t)\|^2_{L^2(\mathbb{R}^d)} &= \lambda\operatorname{Im}\int_{\mathbb{R}^d}\big(\ln\left(|u_1|^2\right)u_1 - \ln\left(|u_2|^2\right)u_2\big)(\bar u_1-\bar u_2)(t)\,dx\\ &\leqslant 4\lambda\|u(t)\|^2_{L^2(\mathbb{R}^d)}\end{aligned}$$

thanks to Lemma 2.1. Uniqueness (and in fact stability in L^2) follows directly, by integration in time.

2.2. Existence.

To prove the existence of a weak solution we proceed by approximating the equation as follows: consider for all $\varepsilon\in(0,1)$ the equation

$$i\partial_t u_\varepsilon + \frac{1}{2}\Delta u_\varepsilon = \lambda\ln\left(\varepsilon+|u_\varepsilon|^2\right)u_\varepsilon, \quad u_{\varepsilon|t=0}=u_0. \tag{2.1}$$

Equation (2.1) is easily solved in $C(\mathbb{R};L^2(\mathbb{R}^d))$ since it is subcritical in L^2 (see [13]). It remains therefore to prove uniform bounds for $u_\varepsilon(t)$ in $V_\alpha\cap H^1(\mathbb{R}^d)$, which will provide compactness in space for the sequence u_ε. Since time compactness (in $H^{-2}(\mathbb{R}^d)$) is a direct consequence of the equation, the Ascoli theorem will then give the result. Actually once a bound in $L^\infty_{\rm loc}(\mathbb{R};H^1(\mathbb{R}^d))$ is derived, then the $L^\infty_{\rm loc}(\mathbb{R};V_\alpha)$ bound can be obtained directly thanks to the following computation: define

$$I_{\varepsilon,\alpha}(t) := \int_{\mathbb{R}^d}\langle x\rangle^\alpha|u_\varepsilon|^2(t,x)\,dx.$$

Then multiplying the equation by $\langle x\rangle^\alpha \overline{u}_\varepsilon$ and integrating in space provides

$$\begin{aligned}\frac{d}{dt}I_{\varepsilon,\alpha}(t) &= \alpha \operatorname{Im}\int \frac{x\cdot\nabla u_\varepsilon}{\langle x\rangle^{2-\alpha}}\,\overline{u}_\varepsilon(t)\,dx\\ &\leqslant \alpha \|\,\langle x\rangle^{\alpha-1}\,u_\varepsilon(t)\|_{L^2(\mathbb{R}^d)}\|\nabla u_\varepsilon(t)\|_{L^2(\mathbb{R}^d)}\\ &\leqslant \alpha \|\,\langle x\rangle^{\alpha/2}\,u_\varepsilon(t)\|_{L^2(\mathbb{R}^d)}\|\nabla u_\varepsilon(t)\|_{L^2(\mathbb{R}^d)}\,.\end{aligned}$$

where the last estimate stems form the property $\alpha\leqslant 2$. Therefore,

$$\|u_\varepsilon(t)\|^2_{V_\alpha}\leqslant \|u_0\|^2_{V_\alpha}+\alpha\int_0^t \|u_\varepsilon(t')\|_{V_\alpha}\|\nabla u_\varepsilon(t')\|_{L^2(\mathbb{R}^d)}\,dt'\,.$$

So it remains to compute the $H^1(\mathbb{R}^d)$ norm of $u_\varepsilon(t)$. This is quite easy since the problem becomes linear in ∇u_ε. Indeed for any $1\leqslant j\leqslant d$ one has

$$i\partial_t\partial_j u_\varepsilon+\frac12\Delta\partial_j u_\varepsilon=\lambda\ln\left(\varepsilon+|u_\varepsilon|^2\right)\partial_j u_\varepsilon+2\lambda\frac{1}{\varepsilon+|u_\varepsilon|^2}\operatorname{Re}(\bar u_\varepsilon\partial_j u_\varepsilon)u_\varepsilon \tag{2.2}$$

which is again subcritical in L^2 since $\left|\frac{1}{\varepsilon+|u_\varepsilon|^2}2\operatorname{Re}(\bar u_\varepsilon\partial_j u_\varepsilon)u_\varepsilon\right|\leqslant 2|\partial_j u_\varepsilon|$. We therefore conclude that u_ε belongs to $L^\infty_{\rm loc}(\mathbb{R};H^1(\mathbb{R}^d))$. The conservation of mass, angular momentum, and energy is established in the same way as in [14] (see also [13]). The first part of Theorem 1.5 follows.

2.3. **Higher regularity.** As in [14], the idea is to consider time derivatives. This fairly general idea in the context of nonlinear Schrödinger equations (see [13]) is all the more precious in the present framework that the logarithmic nonlinearity is very little regular. In particular, we emphasize that if $u_0\in H^k(\mathbb{R}^d)$, $k\geqslant 3$, we cannot guarantee in general that this higher regularity is propagated.

To complete the proof of Theorem 1.5, assume that $u_0\in V_\alpha\cap H^2$, for some $\alpha>0$. We already know that a unique, global, weak solution $u\in L^\infty_{\rm loc}(\mathbb{R};V_\alpha\cap H^1)$ is obtained by the procedure described in the previous subsection, that is, as the limit of u_ε solution to (2.1). The idea is that for all $T>0$, there exists $C=C(T)$ independent of $\varepsilon\in(0,1)$ such that

$$\sup_{-T\leqslant t\leqslant T}\|\partial_t u_\varepsilon(t)\|_{L^2(\mathbb{R}^d)}\leqslant C\,.$$

Indeed, we know directly from (2.1) that

$$\partial_t u_{\varepsilon|t=0}=\frac{i}{2}\Delta u_0-i\lambda\ln\left(\varepsilon+|u_0|^2\right)u_0\in L^2(\mathbb{R}^d)\,,$$

uniformly in ε, in view of the pointwise estimate

$$\left|\ln\left(\varepsilon+|u_0|^2\right)u_0\right|\leqslant C\left(|u_0|^{1+\eta}+|u_0|^{1-\eta}\right)\,,$$

where $\eta>0$ can be chosen arbitrarily small, and C is independent of $\varepsilon\in(0,1)$. Then we can replace the spatial derivative ∂_j in (2.2) with the time derivative ∂_t, and infer that $\partial_t u_\varepsilon\in L^\infty_{\rm loc}(\mathbb{R};L^2(\mathbb{R}^d))$, uniformly in ε: by passing to the limit (up to a subsequence), $\partial_t u\in L^\infty_{\rm loc}(\mathbb{R};L^2(\mathbb{R}^d))$. Using the equation (1.1), we conclude that $\Delta u\in L^\infty_{\rm loc}(\mathbb{R};L^2(\mathbb{R}^d))$. This concludes the proof of Theorem 1.5. □

3. Propagation of Gaussian data: proof of Theorem 1.6

In this section we prove Theorem 1.6, by reducing the study to that of a system of ordinary differential equations.

3.1. **From** (1.1) **to ordinary differential equations.**

3.1.1. *The Gaussian structure.* As noticed in [6], the flow of (1.1) preserves any initial Gaussian structure. We consider the data given by (1.3), and we seek the solution u to (1.1) under the form

$$u(t,x) = b(t)\exp\Big(-\frac{1}{2}\sum_{j=1}^d a_j(t)x_j^2\Big)\,, \tag{3.1}$$

with $\operatorname{Re} a_j(t)>0$. With u of this form, (1.1) becomes equivalent to

$$i\partial_t u + \frac{1}{2}\Delta u = \lambda\Big(\ln|b(t)|^2 - \sum_{j=1}^d \operatorname{Re} a_j(t)x_j^2\Big)u\,,\quad u_{|t=0}=u_0\,.$$

This is a linear Schrödinger equation with a time-dependent harmonic potential, and an initial Gaussian. It is well-known in the context of the propagation of coherent states (see [29, 15]) that the evolution of a Gaussian wave packet under a time-dependent harmonic oscillator is a Gaussian wave packet. Therefore, it is consistent to look for a solution to (1.1) of this form. Notice in particular that

$$\|u(t)\|_{L^p(\mathbb{R}^d)} = (2\pi)^{d/(2p)}\frac{|b(t)|}{\Big(\prod_{j=1}^d \operatorname{Re} a_j(t)\Big)^{1/(2p)}}\,,\quad 1\leqslant p\leqslant\infty\,, \tag{3.2}$$

and

$$\|\nabla u(t)\|^2_{L^2(\mathbb{R}^d)} = \frac{1}{2}\pi^{d/2}\frac{|b(t)|^2}{\Big(\prod_{j=1}^d \operatorname{Re} a_j(t)\Big)^{1/2}}\sum_{j=1}^d\frac{|a_j(t)|^2}{(\operatorname{Re} a_j(t))}\,. \tag{3.3}$$

To prove Theorem 1.6 we therefore need to find the asymptotic behavior in time of $b(t)$ and $a_j(t)$.

3.1.2. *The ODEs.* Plugging (3.1) into (1.1), we obtain

$$i\dot b - i\sum_{j=1}^d \dot a_j\frac{x_j^2}{2}b - \sum_{j=1}^d\frac{a_jb}{2} + \sum_{j=1}^d a_j^2\frac{x_j^2}{2}b = \lambda\Big(\ln\left(|b|^2\right) - \sum_{j=1}^d\left(\operatorname{Re} a_j\right)x_j^2\Big)b\,.$$

Equating the constant in x and the factors of x_j^2, we get

$$i\dot a_j - a_j^2 = 2\lambda\operatorname{Re} a_j\,,\quad a_{j|t=0}=a_{0j}\,, \tag{3.4}$$

$$i\dot b - \sum_{j=1}^d\frac{a_jb}{2} = \lambda b\ln\left(|b|^2\right)\,,\quad b_{|t=0}=b_0\,. \tag{3.5}$$

We can express the solution to (3.5) directly as a function of the a_j's: indeed

$$b(t) = b_0\exp\Big(-i\lambda t\ln\left(|b_0|^2\right) - \frac{i}{2}\sum_{j=1}^d A_j(t) - i\lambda\sum_{j=1}^d\operatorname{Im}\int_0^t A_j(s)sds\Big)\,,$$

where we have set

$$A_j(t) := \int_0^t a_j(s)ds\,.$$

We also infer from (3.4) that $y := \operatorname{Re} a_j$ solves $\dot y = 2y \operatorname{Im} a_j$, hence

$$\operatorname{Re} a_j(t) = \operatorname{Re} a_{0j} \exp\left(2\int_0^t \operatorname{Im} a_j(s)ds\right).$$

Since the equations (3.4) are decoupled as j varies, we simply consider from now on

$$i\dot a - a^2 = 2\lambda \operatorname{Re} a\,,\quad a_{|t=0} = a_0 = \alpha_0 + i\beta_0\,, \tag{3.6}$$

which amounts to assuming $d = 1$ in (1.1). Note that β_0 is actually zero in our context but it is not more difficult to deal with that more general case. Following [37], we seek a of the form

$$a = -i\frac{\dot\omega}{\omega}\cdot$$

Then (3.6) becomes

$$\ddot\omega = 2\lambda\omega \operatorname{Im}\frac{\dot\omega}{\omega}\cdot$$

Introducing the polar decomposition $\omega = re^{i\theta}$, we get

$$\begin{cases} \ddot r - (\dot\theta)^2 r = 2\lambda r\dot\theta \\ \ddot\theta r + 2\dot\theta\dot r = 0\,. \end{cases}$$

Notice that

$$\dot\theta_{|t=0} = \alpha_0\,,\quad \left(\frac{\dot r}{r}\right)_{|t=0} = -\beta_0\,.$$

We therefore have a degree of freedom to set $r(0)$, and we decide $r(0) = 1$ so

$$\dot\theta(0) = \operatorname{Re} a_0 = \alpha_0\,,\quad \dot r(0) = -\operatorname{Im} a_0 = -\beta_0\,.$$

The second equation yields

$$\frac{d}{dt}\left(r^2\dot\theta\right) = r\left(2\dot r\dot\theta + r\ddot\theta\right) = 0\,,$$

so $r^2\dot\theta$ is constant and we can express the problem in terms of r only: we write

$$a(t) = \frac{\alpha_0}{r(t)^2} - i\frac{\dot r(t)}{r(t)}\,, \tag{3.7}$$

with

$$\ddot r = \frac{\alpha_0^2}{r^3} + 2\lambda\frac{\alpha_0}{r}\,,\quad r(0) = 1\,,\ \dot r(0) = -\beta_0\,. \tag{3.8}$$

Multiplying by $\dot r$ and integrating, we infer

$$(\dot r)^2 = \beta_0^2 + \alpha_0^2\left(1 - \frac{1}{r^2}\right) + 4\lambda\alpha_0 \ln r\,. \tag{3.9}$$

Back to the solution u, in the case when $d = 1$ then writing in view of (3.2) and (3.7)

$$\|u(t)\|_{L^\infty(\mathbb{R}^d)} = |b(t)| = |b_0| \exp\left(\frac{1}{2}\int_0^t \operatorname{Im} a(s)ds\right) = \frac{|b_0|}{\sqrt{r(t)}}$$

we find that the study of $r(t)$ is enough to find the dispersion rate of $u(t)$. Once the rate in one space dimension is known, the result in d space dimensions follows directly.

Moreover recalling (3.3), we have

$$\begin{aligned}\|\nabla u(t)\|^2_{L^2(\mathbb{R}^d)} &= \frac{1}{2}\pi^{d/2}\frac{|b(t)|^2}{\left(\prod_{j=1}^d \operatorname{Re} a_j(t)\right)^{1/2}}\sum_{j=1}^d \frac{|a_j(t)|^2}{(\operatorname{Re} a_j(t))}\\ &= \frac{\pi^{d/2}|b_0|^2}{2\left(\prod_{j=1}^d r_j(t)\right)\left(\prod_{j=1}^d \operatorname{Re} a_j(t)\right)^{1/2}}\sum_{j=1}^d \frac{|a_j(t)|^2}{(\operatorname{Re} a_j(t))}\\ &= \frac{\pi^{d/2}|b_0|^2}{2\sqrt{\prod_{j=1}^d \alpha_{0j}}}\sum_{j=1}^d \left((\dot r_j)^2+\frac{\alpha_0^2}{r_j^2}\right)\frac{1}{\alpha_{0j}}\\ &= c+2\lambda\frac{\pi^{d/2}|b_0|^2}{2\sqrt{\prod_{j=1}^d \alpha_{0j}}}\sum_{j=1}^d \ln r_j(t)\,.\end{aligned}$$

As soon as $r_j(t)\to\infty$ when $|t|\to\infty$, the H^1 norm therefore becomes unbounded. This is proved to be the case below (with an explicit rate): actually it can be seen from the rate provided in Lemma 3.8 below that the energy remains bounded because the unbounded contributions of both parts of the energy cancel exactly.

3.2. **Study of $r(t)$.** The aim of this paragraph is to prove the following result. Recall notation (1.2).

Lemma 3.1. *Let r solve* (3.8). *Then as $t\to\infty$, there holds*

$$r(t)=2t\sqrt{\lambda\alpha_0\ln t}\left(1+\mathcal{O}\big(\ell(t)\big)\right).$$

The proof of the lemma is achieved in three steps: first we prove, in Paragraph 3.2.1, that $r(t)\to\infty$ as $t\to\infty$. In view of that result it is natural to approximate the solution to (3.8) by

$$\ddot r_{\text{eff}}=2\lambda\frac{\alpha_0}{r_{\text{eff}}}\,,\quad r_{\text{eff}}(T)=r(T)\,,\ \dot r_{\text{eff}}(T)=\dot r(T)\,, \tag{3.10}$$

for $T\gg 1$. This is proved in Paragraph 3.2.2, along with a first estimate on the large time behavior of r_{eff}. The conclusion of the proof is achieved in Paragraph 3.2.3, by proving Lemma 1.8.

3.2.1. *First step: $r(t)\to\infty$.* We readily see from (3.9) that r is bounded from below:

$$\exists\delta>0\,,\quad r(t)\geqslant\delta\,,\quad \forall t\in\mathbb{R}\,.$$

Indeed, if it were not so, there would exist a sequence t_n such that $r(t_n)\to 0$: for n large, the right hand side of (3.9) then becomes negative, hence a contradiction.

Now let us prove that $r(t)\to+\infty$ as $t\to+\infty$. Assume first that $\dot r(0)>0$. Then (3.8) yields $\ddot r\geqslant 0$, hence $\dot r(t)\geqslant\dot r(0)$ for all $t\geqslant 0$, and

$$r(t)\geqslant \dot r(0)t+1\underset{t\to+\infty}{\longrightarrow}+\infty\,. \tag{3.11}$$

On the other hand, for $\dot r(0)\leqslant 0$, assume that r is bounded, $r(t)\leqslant M$. Then (3.8) yields

$$\ddot r(t)\geqslant\frac{\alpha_0^2}{M^3}+2\lambda\frac{\alpha_0}{M}\,,$$

hence a contradiction for t large enough. We infer that for T sufficiently large, there holds $r(T) \geqslant 1$ and $\dot{r}(T) > 0$. The first case then implies $r(t) \to +\infty$.

Note that we have proved in particular that

$$\exists T \geqslant 1\,, \quad \dot{r}(T) > 0 \quad \text{and} \quad \forall t \geqslant T\,, \quad r(t) \geqslant \dot{r}(T)(t-T)+1\,. \tag{3.12}$$

3.2.2. *Second step: $r(t) \sim r_{\text{eff}}(t)$ with a rough bound.* Let us prove the following result.

Lemma 3.2. *There is T large enough so that defining r_{eff} the solution of* (3.10) *then as $t \to \infty$, there holds*

$$|r_{\text{eff}}(t)| = 2t\sqrt{\lambda\alpha_0 \ln t} + \epsilon(t\sqrt{\ln t})\,, \quad \textit{and} \quad |r(t) - r_{\text{eff}}(t)| \leqslant C(T)t\,, \quad \forall t \geqslant T\,,$$

where $\epsilon(t)/t$ goes to zero as t goes to infinity.

Proof. Let us start by studying r_{eff}. Multiplying (3.10) by $\dot{r}_{\text{eff}}$ and integrating, we get

$$\begin{aligned}(\dot{r}_{\text{eff}}(t))^2 &= (\dot{r}(T))^2 + 4\lambda\alpha_0 \ln r_{\text{eff}}(t) - 4\lambda\alpha_0 \ln r(T)\\ &= 4\lambda\alpha_0 \ln r_{\text{eff}}(t) + \beta_0^2 + \alpha_0^2\left(1 - \frac{1}{r(T)^2}\right)\,,\end{aligned}$$

where we have used (3.9) at time $t = T$. Denote by

$$C_0 := \beta_0^2 + \alpha_0^2\left(1 - \frac{1}{r(T)^2}\right) \approx \beta_0^2 + \alpha_0^2 = |a_0|^2\,,$$

since $T \gg 1$. By similar arguments as in the proof of (3.12) in Paragraph 3.2.1, we have $\dot{r}_{\text{eff}}(t) > 0$ for all $t \geqslant T$, and

$$r_{\text{eff}}(t) \geqslant \dot{r}(T)(t-T) + 1$$

hence

$$\dot{r}_{\text{eff}}(t) = \sqrt{4\lambda\alpha_0 \ln r_{\text{eff}}(t) + C_0}\,.$$

Separating the variables,

$$\frac{dr_{\text{eff}}}{\sqrt{4\lambda\alpha_0 \ln r_{\text{eff}} + C_0}} = dt\,,$$

so we naturally consider the anti-derivative

$$I := \int \frac{dr}{\sqrt{4\lambda\alpha_0 \ln r + C_0}}\,.$$

The change of variable

$$y := \sqrt{4\lambda\alpha_0 \ln r + C_0}$$

yields

$$I = \frac{1}{2\lambda\alpha_0}\int e^{(y^2 - C_0)/(4\lambda\alpha_0)}dy\,.$$

Since for x large (Dawson function, see e.g. [1]),

$$\int e^{x^2}dx \sim \frac{1}{2x}e^{x^2}\,,$$

we infer

$$I \sim \frac{r}{\sqrt{4\lambda\alpha_0 \ln r + C_0}}\,.$$

In particular,

$$\frac{r_{\rm eff}(t)}{\sqrt{4\lambda\alpha_0 \ln r_{\rm eff}(t) + C_0}} \underset{t\to+\infty}{\sim} t\,,$$

hence

$$\frac{r_{\rm eff}(t)}{\sqrt{\ln r_{\rm eff}(t)}} \underset{t\to+\infty}{\sim} 2t\sqrt{\lambda\alpha_0}\,.$$

This relation is inverted through, for some $\kappa\in\mathbb{R}$,

$$r_{\rm eff}(t) \sim t\,(\ln t)^{\kappa}\, 2\sqrt{\lambda\alpha_0}\,,$$

and we find that necessarily, $\kappa = 1/2$: we conclude that

$$r_{\rm eff}(t) \underset{t\to+\infty}{\sim} 2t\sqrt{\lambda\alpha_0 \ln t}\,.$$

Now let us prove that r can be well approximated by $r_{\rm eff}$. We define $h := r - r_{\rm eff}$ and we want to prove that if T is chosen large enough, then $h(t) \lesssim t$ when $t\to\infty$. We have

$$\begin{aligned}\dot h(t) &= \sqrt{4\lambda\alpha_0 \ln r(t) + \beta_0^2 + \alpha_0^2\Big(1-\frac{1}{r(t)^2}\Big)}\\ &\quad - \sqrt{4\lambda\alpha_0 \ln r_{\rm eff}(t) + \beta_0^2 + \alpha_0^2\Big(1-\frac{1}{r(T)^2}\Big)}\\ &\leqslant \sqrt{4\lambda\alpha_0\Big|\ln\frac{r(t)}{r_{\rm eff}(t)}\Big| + \alpha_0^2\Big(\frac{1}{r(T)^2}-\frac{1}{r(t)^2}\Big)}\,.\end{aligned}$$

Given $\varepsilon\in(0,1/2)$, let $T\geqslant 1$ be large enough so that for all $t\geqslant T$

$$r_{\rm eff}(t) \geqslant t\sqrt{\lambda\alpha_0 \ln t} \tag{3.13}$$

and

$$\alpha_0^2\Big(\frac{1}{r(T)^2}-\frac{1}{r(t)^2}\Big) \leqslant \varepsilon^2\,. \tag{3.14}$$

We shall also need that

$$\left(2\frac{(\lambda\alpha_0)^{\frac14}}{\sqrt{\ln T}} + \varepsilon\right) \leqslant \frac12\,. \tag{3.15}$$

Then noticing that

$$\begin{aligned}\Big|\ln\frac{r(t)}{r_{\rm eff}(t)}\Big| &= \Big|\ln\Big(1+\frac{h(t)}{r_{\rm eff}(t)}\Big)\Big|\\ &\leqslant \frac{|h(t)|}{r_{\rm eff}(t)}\\ &\leqslant \frac{|h(t)|}{t\sqrt{\lambda\alpha_0\ln t}} \leqslant \frac{|h(t)|}{t\sqrt{\lambda\alpha_0 \ln T}}\end{aligned}$$

as soon as $t\geqslant T$ thanks to (3.13), we infer that

$$\forall t\geqslant T\,,\quad \dot h(t) \leqslant \varepsilon + 2\sqrt{\lambda\alpha_0}\left(\frac{|h(t)|}{t\sqrt{\lambda\alpha_0\ln T}}\right)^{\frac12}\,,\quad \text{with}\quad h(T)=0\,.$$

Our goal is to prove that the function $t \mapsto h(t)/t$ is bounded for large t, so let $T^* > T$ be the maximal time such that

$$\forall t \in [T, T^*)\,, \quad |h(t)| \leqslant t\,.$$

Then for $t \in [T, T^*)$,

$$\dot h(t) \leqslant \varepsilon + 2(\lambda\alpha_0)^{\frac14} \frac{1}{\sqrt{\ln T}}$$

so thanks to (3.15)

$$h(t) \leqslant \Big(\varepsilon + 2(\lambda\alpha_0)^{\frac14} \frac{1}{\sqrt{\ln T}}\Big)(t-T) \leqslant \frac{t}{2}\,,$$

which contradicts the maximality of T^*. The result follows, and Lemma 3.2 is proved. □

3.2.3. *Third step: $r(t) \sim r_{\text{eff}}(t)$ with improved bound.* Let us end the proof of Lemma 3.1. By (3.9) and as in the previous paragraph, we have for T sufficiently large so that $\dot r(t) \geqslant \dot r(T) > 0$ for $t \geqslant T$:

$$\dot r = \sqrt{C_0 + \alpha_0^2\left(\frac{1}{r(T)^2} - \frac{1}{r^2}\right) + 4\lambda\alpha_0 \ln r}\,,$$

with the same constant C_0 as above: recall that

$$\dot r_{\text{eff}} = \sqrt{C_0 + 4\lambda\alpha_0 \ln r_{\text{eff}}}\,.$$

To lighten notation let us recall that $h := r - r_{\text{eff}}$ and let us define

$$R_{\text{eff}} := C_0 + 4\lambda\alpha_0 \ln r_{\text{eff}}\,.$$

Then using a Taylor expansion for $\dot r$, we have:

$$\begin{aligned}\dot r &= \sqrt{R_{\text{eff}} + \alpha_0^2\big(\frac{1}{r(T)^2} - \frac{1}{r^2}\big) + 4\lambda\alpha_0 \ln\big(1 + \frac{h}{r_{\text{eff}}}\big)} \\ &= \sqrt{R_{\text{eff}}}\sqrt{1 + \frac{1}{R_{\text{eff}}}\alpha_0^2\big(\frac{1}{r(T)^2} - \frac{1}{r^2}\big) + 4\frac{\lambda\alpha_0}{R_{\text{eff}}} \ln\big(1 + \frac{h}{r_{\text{eff}}}\big)}\,.\end{aligned}$$

On the one hand we know that $R_{\text{eff}} \to \infty$ and by Lemma 3.2 we have $h \lesssim t$ and $r_{\text{eff}} \underset{t\to\infty}{\sim} t\sqrt{\ln t}$ so we infer that

$$\dot r \underset{t\to\infty}{\sim} \sqrt{R_{\text{eff}}}\Big(1 + \frac{1}{2R_{\text{eff}}}\Big(\alpha_0^2\big(\frac{1}{r(T)^2} - \frac{1}{r^2}\big) + 4\lambda\alpha_0 \ln\big(1 + \frac{h}{r_{\text{eff}}}\big)\Big)\Big)\,.$$

As a consequence

$$\dot r - \dot r_{\text{eff}} \underset{t\to\infty}{\sim} \frac{1}{2\sqrt{R_{\text{eff}}}}\Big(\alpha_0^2\big(\frac{1}{r(T)^2} - \frac{1}{r^2}\big) + 4\lambda\alpha_0 \ln\big(1 + \frac{h}{r_{\text{eff}}}\big)\Big)$$

and since $h/r_{\text{eff}} = \mathcal{O}(1/\sqrt{\ln t})$ we infer that

$$\dot r - \dot r_{\text{eff}} \underset{t\to\infty}{\sim} \frac{C(T)}{\sqrt{\lambda \ln t}}\,.$$

By integration, and comparison of diverging integrals, we find

$$h(t) \underset{t\to\infty}{\sim} C_1 \frac{t}{\sqrt{\ln t}}\,,$$

hence

$$r(t) = 2t\sqrt{\lambda\alpha_0 \ln t}\left(1 + \mathcal{O}\big(\ell(t)\big)\right),$$

as soon as we know that this holds for r_{eff}. Lemma 3.1 is therefore proved, up to the study of the universal dispersion τ.

3.3. **Study of the universal dispersion $\tau(t)$: proof of Lemma 1.8.** It remains to prove Lemma 1.8. By scaling, we may assume $\lambda = 1$, to lighten the notations. Introduce the approximate solution

$$\tau_{\text{eff}}(t) := 2t\sqrt{\ln t}.$$

We have clearly

$$\sqrt{\ln t} = \sqrt{\ln \tau_{\text{eff}}}\left(1 + \mathcal{O}\left(\frac{\ln\ln t}{\ln \tau_{\text{eff}}}\right)\right).$$

In view of a comparison with (1.8), which reads

$$\dot{\tau} = 2\sqrt{\ln t}\,,$$

write

$$\dot{\tau}_{\text{eff}} = 2\sqrt{\ln t} + \frac{1}{\sqrt{\ln t}} = 2\sqrt{\ln \tau_{\text{eff}}}\left(1 + \mathcal{O}\left(\frac{\ln\ln t}{\ln \tau_{\text{eff}}}\right)\right) = 2\sqrt{\ln \tau_{\text{eff}}} + \mathcal{O}\left(\frac{\ln\ln t}{\sqrt{\ln t}}\right).$$

Thus,

$$\begin{aligned}
\dot{\tau} - \dot{\tau}_{\text{eff}} &= 2\left(\sqrt{\ln \tau} - \sqrt{\ln \tau_{\text{eff}}}\right) + \mathcal{O}\left(\frac{\ln\ln t}{\sqrt{\ln t}}\right)\\
&= 2\sqrt{\ln \tau_{\text{eff}} + \ln\frac{\tau}{\tau_{\text{eff}}}} - 2\sqrt{\ln \tau_{\text{eff}}} + \mathcal{O}\left(\frac{\ln\ln t}{\sqrt{\ln t}}\right).
\end{aligned}$$

Since we already know from Lemma 3.2 that $\tau/\tau_{\text{eff}} \to 1$, we obtain

$$\dot{\tau} - \dot{\tau}_{\text{eff}} = \mathcal{O}\left(\frac{\ln\ln t}{\sqrt{\ln t}}\right), \quad \text{and} \quad \tau - \tau_{\text{eff}} = \mathcal{O}\left(t\frac{\ln\ln t}{\sqrt{\ln t}}\right),$$

by integration. This proves Lemma 1.8. □

Back to the previous section, we simply note that

$$\dot{r}_{\text{eff}} - \sqrt{\alpha_0}\dot{\tau} = \sqrt{C_0 + 4\lambda\alpha_0 \ln r_{\text{eff}}} - \sqrt{4\lambda\alpha_0 \ln \tau}\,,$$

with $C_0 \neq 0$ in general, so the same computation as above yields

$$\dot{r}_{\text{eff}} - \sqrt{\alpha_0}\dot{\tau} = \mathcal{O}\left(\frac{1}{\sqrt{\ln r_{\text{eff}}}}\right) = \mathcal{O}\left(\frac{1}{\sqrt{\ln t}}\right),$$

hence

$$r_{\text{eff}} - \sqrt{\alpha_0}\tau = \mathcal{O}\left(\frac{t}{\sqrt{\ln t}}\right),$$

by integration. This completes the proof of Lemma 3.1. □

4. General case: preparation for the proof of Theorem 1.9

In this section we prove (1.10) and (1.11) of Theorem 1.9.

4.1. **First a priori estimates.** Recall that by definition, v is related to u through the relation

$$u(t,x) = \frac{1}{\tau(t)^{d/2}} v\left(t, \frac{x}{\tau(t)}\right) \exp\left(i\frac{\dot\tau(t)}{\tau(t)}\frac{|x|^2}{2}\right), \tag{4.1}$$

where τ is the solution to

$$\ddot\tau = \frac{2\lambda}{\tau}, \quad \tau(0)=1, \quad \dot\tau(0)=0.$$

Then v solves

$$i\partial_t v + \frac{1}{2\tau(t)^2}\Delta_y v = \lambda v \ln\left|\frac{v}{\gamma}\right|^2 - \lambda v \ln\tau, \quad v_{|t=0} = u_0,$$

where we recall that $\gamma(y) = e^{-|y|^2/2}$. Using a gauge transform (by replacing v with $ve^{i\theta(t)}$ for $\dot\theta = \lambda\ln\tau$), we may assume that the last term is absent, and we focus our attention on

$$i\partial_t v + \frac{1}{2\tau(t)^2}\Delta_y v = \lambda v \ln\left|\frac{v}{\gamma}\right|^2, \quad v_{|t=0} = u_0. \tag{4.2}$$

Because we now have a non-autonomous equation, the Hamiltonian structure of (1.1) is lost. We compute

$$E(t) := \operatorname{Im}\int_{\mathbb{R}^d} \bar v(t,y)\partial_t v(t,y)dy = E_{\rm kin}(t) + \lambda E_{\rm ent}(t),$$

where

$$E_{\rm kin}(t) := \frac{1}{2\tau(t)^2}\|\nabla_y v(t)\|_{L^2}^2$$

is the kinetic energy and

$$E_{\rm ent}(t) := \int_{\mathbb{R}^d} |v(t,y)|^2 \ln\left|\frac{v(t,y)}{\gamma(y)}\right|^2 dy$$

can be considered as a relative entropy. The transform (4.1) is unitary on $L^2(\mathbb{R}^d)$ so the conservation of mass for u trivially corresponds to the conservation of mass for v :

$$\|v(t)\|_{L^2} = \|u_0\|_{L^2} = \|\gamma\|_{L^2}, \tag{4.3}$$

and we will show, as stated in (1.11), that

$$\int |y|^2|v(t,y)|^2dy \underset{t\to\infty}{\longrightarrow} \int |y|^2\gamma(y)^2dy.$$

Thanks to (4.3), the Csiszár-Kullback inequality yields

$$E_{\rm ent}(t) \gtrsim \left\||v(t)|^2 - \gamma^2\right\|_{L^1(\mathbb{R}^d)},$$

hence in particular $E_{\rm ent} \geqslant 0$, which is another way of justifying the term “defocusing” for the case $\lambda > 0$. We easily compute

$$\dot E = -2\frac{\dot\tau}{\tau}E_{\rm kin}. \tag{4.4}$$

Ideally, we would like to prove directly $E(t) \underset{t\to\infty}{\longrightarrow} 0$. The property $E(t)\to 0$ can be understood as follows:

- $E_{\rm kin}\to 0$ means that v oscillates in space more slowly than τ, hence that the main spatial oscillations of u have been taken into account in (4.1) (as a matter of fact, the boundedness of $E_{\rm kin}$ suffices to reach this conclusion).
- $E_{\rm ent}\to 0$ implies $|v(t)|^2\to\gamma^2$ strongly in $L^1(\mathbb{R}^d)$.

It turns out than in the case of Gaussian initial data, we can infer from Section 3 that indeed $E(t)\to 0$, each term going to zero logarithmically in time (see Section 6 for the case of $E_{\rm ent}$). In the general case, we cannot reach this conclusion. Note however that if we had $E_{\rm kin}\gtrsim 1$, then integrating (4.4) we would get $E(t)\to-\infty$ as $t\to\infty$, hence a contradiction. Therefore,

$$\exists t_k\to\infty\,,\quad E_{\rm kin}(t_k)\to 0\,.$$

Remark 4.1 (Probabilistic approach). In view of a comparison with standard questions in probability related to logarithmic Sobolev inequalities (see e.g. [3]), let us rewrite (4.2) in terms of $g:=v/\gamma$, and equip $\mathbb{R}^d$ with the Gaussian measure $\gamma^2(y)dy$. Up to an irrelevant factor $\pi^{d/2}$, this is a probability measure, and (4.2) becomes

$$i\partial_t g=\frac{1}{2\tau(t)^2}\mathcal{L}g+\lambda g\ln|g|^2\,,\quad \mathcal{L}:=-\Delta_y+2y\cdot\nabla_y+d-|y|^2\,.$$

As a matter of fact, if instead of considering (1.8), one considers the adimensionalized version of (3.8),

$$\ddot{\tilde\tau}=\frac{1}{\tilde\tau}+\frac{2\lambda}{\tilde\tau}\,,\quad \tilde\tau(0)=1\,,\ \dot{\tilde\tau}(0)=0\,,$$

then $\mathcal{L}$ is replaced by $\tilde{\mathcal{L}}:=-\Delta_y+2y\cdot\nabla_y+d$, a Fokker-Planck operator which plays a central role to prove the last point of Theorem 1.9. The goal is then to understand the large time behavior of $|g|^2$ (which is expected to converge to 1) in $L^1(d\gamma^2)$. The Csiszár-Kullback inequality now reads

$$\big\||g(t)|^2-1\big\|_{L^1(d\gamma^2)}\leqslant 2\int_{\mathbb{R}^d}|g(t,y)|^2\ln|g(t,y)|^2\gamma(y)^2dy\,.$$

The logarithmic Sobolev inequality yields

$$\int_{\mathbb{R}^d}|g(t,y)|^2\ln|g(t,y)|^2\gamma(y)^2dy\lesssim\int_{\mathbb{R}^d}|\nabla_y|g(t,y)||^2\,\gamma(y)^2dy\,,$$

but then again, the last term seems delicate to control.

We now prove the first part of Theorem 1.9, that is, (1.10) which is recast and complemented in the next lemma.

Lemma 4.2. *Under the assumptions of Theorem* 1.9*, there holds*

$$\sup_{t\geqslant 0}\left(\int_{\mathbb{R}^d}\left(1+|y|^2+\left|\ln|v(t,y)|^2\right|\right)|v(t,y)|^2dy+\frac{1}{\tau(t)^2}\|\nabla_y v(t)\|^2_{L^2(\mathbb{R}^d)}\right)<\infty$$

and

$$\int_0^\infty\frac{\dot\tau(t')}{\tau^3(t')}\|\nabla_y v(t')\|^2_{L^2(\mathbb{R}^d)}dt'<\infty. \tag{4.5}$$

Proof. Write

$$E_{\rm ent}=\int_{\mathbb{R}^d}|v|^2\ln|v|^2+\int_{\mathbb{R}^d}|y|^2|v|^2\,,$$

and

$$\int_{\mathbb{R}^d} |v|^2 \ln |v|^2 = \int_{|v|>1} |v|^2 \ln |v|^2 + \int_{|v|<1} |v|^2 \ln |v|^2.$$

We have

$$E_+ := E_{\rm kin} + \lambda \int_{|v|>1} |v|^2 \ln |v|^2 + \lambda \int_{\mathbb{R}^d} |y|^2 |v|^2 \leqslant E(0) + \lambda \int_{|v|<1} |v|^2 \ln \frac{1}{|v|^2} \cdot$$

The last term is controlled by

$$\int_{|v|<1} |v|^2 \ln \frac{1}{|v|^2} \lesssim \int_{\mathbb{R}^d} |v|^{2-\varepsilon} ,$$

for all $\varepsilon > 0$. We conclude thanks to the estimate

$$\int_{\mathbb{R}^d} |v|^{2-\varepsilon} \lesssim \|v\|_{L^2}^{2-(1+d/2)\varepsilon} \|yv\|_{L^2}^{d\varepsilon/2} ,$$

for $\varepsilon > 0$ sufficiently small ($0 < \varepsilon < \frac{4}{d+2}$), which can be readily proved by an interpolation method (cutting the integral into $|y| < R$ and $|y| > R$, using Hölder inequality and optimizing over R; see e.g. [11]). This implies

$$E_+ \lesssim 1 + E_+^{d\varepsilon/4} ,$$

and thus $E_+ \in L^\infty(\mathbb{R})$.

Finally, (4.5) follows from (4.4), since $E(t) \geqslant 0$ for all $t \geqslant 0$. □

4.2. **Convergence of some quadratic quantities.** Let us prove (1.11), as stated in the next lemma.

Lemma 4.3. *Under the assumptions of Theorem* 1.9*, there holds*

$$\int_{\mathbb{R}^d} \begin{pmatrix} 1 \\ y \\ |y|^2 \end{pmatrix} |v(t,y)|^2 dy \underset{t\to\infty}{\longrightarrow} \int_{\mathbb{R}^d} \begin{pmatrix} 1 \\ y \\ |y|^2 \end{pmatrix} \gamma^2(y) dy \, .$$

Proof. Introduce

$$I_1(t) := \operatorname{Im} \int_{\mathbb{R}^d} \overline{v}(t,y) \nabla_y v(t,y) dy \, , \quad I_2(t) := \int_{\mathbb{R}^d} y |v(t,y)|^2 dy \, .$$

We compute:

$$\dot{I}_1 = -2\lambda I_2 \, , \qquad \dot{I}_2 = \frac{1}{\tau^2(t)} I_1 \, . \tag{4.6}$$

Set $\tilde{I}_2 := \tau I_2$: we have $\ddot{\tilde{I}}_2 = 0$, hence (unless the data are well prepared in the sense that $I_1(0) = 0$)

$$I_2(t) = \frac{1}{\tau(t)} \left(\dot{\tilde{I}}_2(0) t + \tilde{I}_2(0) \right) = \frac{1}{\tau(t)} \left(-I_1(0) t + I_2(0) \right) \underset{t\to\infty}{\sim} \frac{c}{\sqrt{\ln t}} \, ,$$

and

$$I_1(t) \underset{t\to\infty}{\sim} \tilde{c} \frac{t}{\sqrt{\ln t}} \cdot$$

In particular,

$$\int_{\mathbb{R}^d} y |v(t,y)|^2 dy \underset{t\to\infty}{\longrightarrow} 0 = \int_{\mathbb{R}^d} y \gamma(y)^2 dy \, .$$

In order to obtain estimates for higher order quadratic observables, we perform systematic estimates based on multiplier methods. We multiply the equation by $(y, i\nabla_y)^\alpha \bar{v}$ with $|\alpha| = 2$, then we integrate in space, and take the imaginary part of the outcome. Denote

$$H_1(t) := \int |y|^2 |v(t,y)|^2 dy\,,$$

$$J_2(t) := \operatorname{Im} \int v(t,y) y \cdot \nabla_y \bar{v}(t,y) dy\,,$$

$$J_3(t) := \int |\nabla_y v(t,y)|^2 dy\,.$$

We shall see that the definition of H_1 has to be altered, hence the discrepancy in the notations.

• Multiplier $|y|^2\bar{v}$. We write

$$\frac{1}{2}\frac{d}{dt}\int |y|^2|v|^2 + \frac{1}{2\tau^2}\operatorname{Im}\int |y|^2 \bar{v}\Delta_y v = 0\,,$$

hence, after integrating by parts,

$$\dot{H}_1 = -\frac{2}{\tau^2} J_2\,.$$

• Multiplier $iy \cdot \nabla_y \bar{v}$. We readily have

$$-\operatorname{Im}\int y\cdot\nabla_y\bar{v}\partial_t v + \frac{1}{2\tau^2}\operatorname{Re}\int y\cdot\nabla_y\bar{v}\Delta_y v = \lambda \operatorname{Re}\int v y\cdot\nabla_y \bar{v}\ln\left|\frac{v}{\gamma}\right|^2\,.$$

We notice

$$\frac{d}{dt}\operatorname{Im}\int vy\cdot\nabla_y\bar{v} = 2\operatorname{Im}\int \partial_t v\, y\cdot\nabla_y\bar{v} - d\operatorname{Im}\int v\partial_t\bar{v}\,,$$

$$-\operatorname{Im}\int v\partial_t\bar{v} = -\frac{1}{2\tau^2}J_3 - \lambda E_{\rm ent}\,,$$

$$\nabla_y \ln\left|\frac{v}{\gamma}\right|^2 = \frac{\nabla_y v}{v} + \frac{\nabla_y\bar{v}}{\bar{v}} + 2y\,,$$

and get

$$\dot{J}_2 = -\frac{1}{\tau^2}J_3 - d\lambda\|v\|_{L^2}^2 + 2\lambda H_1\,.$$

• Multiplier $\Delta_y\bar{v}$. We have

$$\operatorname{Re}\int \Delta_y\bar{v}\partial_t v = \lambda\operatorname{Im}\int v\Delta_y\bar{v}\ln\left|\frac{v}{\gamma}\right|^2\,.$$

After integrating by parts, we infer

$$\dot{J}_3 = 4\lambda J_2 + 2\lambda\operatorname{Im}\int\frac{\bar{v}}{v}\left(\nabla_y v\right)^2\,.$$

Unfortunately, the system of equations that we get is not closed, because of the last term in the above relation. As we are not able to obtain fine estimates for this term (the estimate for $\|\nabla_y v(t)\|_{L^2}^2$ is too rough), we follow another strategy.

We expect to have

$$H_1(t) \underset{t\to\infty}{\longrightarrow} \int_{\mathbb{R}^d} |y|^2 e^{-|y|^2} dy = \frac{d}{2}\|\gamma\|_{L^2}^2 = \frac{d}{2}\|v\|_{L^2}^2\,.$$

Therefore, the right unknown is not H_1, but

$$H_1(t) - \int |y|^2\gamma^2 = H_1(t) - \frac{d}{2}\int \gamma^2 = H_1(t) - \frac{d}{2}\|v\|_{L^2}^2 .$$

So we finally denote

$$J_1(t) := \int_{\mathbb{R}^d} |y|^2 |v(t,y)|^2 dy - \frac{d}{2}\|v\|_{L^2}^2 .$$

To summarize, we have computed:

$$\dot{J}_1 = -\frac{2}{\tau^2} J_2 , \tag{4.7}$$

$$\dot{J}_2 = -\frac{1}{\tau^2} J_3 + 2\lambda J_1 , \tag{4.8}$$

$$\dot{J}_3 = 4\lambda J_2 + 2\lambda \operatorname{Im}\int \frac{\bar{v}}{v}\left(\nabla_y v\right)^2 . \tag{4.9}$$

We already know from Lemma 4.2 that

$$J_1 = \mathcal{O}(1) , \quad J_3 = \mathcal{O}(\tau^2) = \mathcal{O}(t^2 \ln t) ,$$

hence, by integrating (4.8), we find

$$J_2 = \mathcal{O}(t) .$$

In order to exploit these informations, we go back to the conserved quantities for u and translate them into estimates on v.

- Mass: $\displaystyle\frac{d}{dt}\|u(t)\|_{L^2}^2 = 0$;
- Angular momentum: $\displaystyle\frac{d}{dt}\operatorname{Im}\int \bar{u}\nabla u = 0$;
- Energy:

$$\frac{d}{dt}\left(\frac{1}{2}\|\nabla u(t)\|_{L^2}^2 + \lambda \int_{\mathbb{R}^d} |u(t,x)|^2 \ln |u(t,x)|^2 dx\right) = 0 .$$

We recall the mass conservation for v stated in (4.3). The conservation of angular momentum for u yields, in terms of v:

$$\frac{d}{dt}\left(\dot{\tau} I_2 + \frac{1}{\tau} I_1\right) = 0 .$$

This is indeed a consequence of (4.6). When using the energy however, we get some interesting new piece of information. Substituting (4.1) into the conservation of the energy of u, we get

$$\frac{d}{dt}\left(\frac{1}{2\tau^2}J_3 + \frac{(\dot{\tau})^2}{2}\int |y|^2|v|^2 - \frac{\dot{\tau}}{\tau}J_2 + \lambda\int |v|^2\ln|v|^2 - \lambda d\ln\tau\int |v|^2\right) = 0 .$$

Recall that we know that $J_2 = \mathcal{O}(t)$. Therefore, in the above expression, all the terms are bounded functions of time, but two:

$$\frac{(\dot{\tau})^2}{2}\int |y|^2|v|^2 \quad \text{and} \quad -\lambda d \ln\tau \int |v|^2 .$$

We infer

$$\frac{(\dot{\tau})^2}{2}\int |y|^2|v|^2 - \lambda d\ln\tau\int |v|^2 = \mathcal{O}(1) .$$

Integrating (1.8), we find

$$\frac{(\dot\tau)^2}{2} = 2\lambda \ln \tau\,,$$

hence

$$\int |y|^2|v|^2 - \frac{d}{2}\|v\|_{L^2}^2 = \mathcal{O}\left(\frac{1}{\ln \tau}\right).$$

Since $\tau(t) \underset{t\to\infty}{\sim} 2t\sqrt{\lambda \ln t}$ thanks to Lemma 1.8, we obtain

$$J_1 = \mathcal{O}\left(\frac{1}{\ln t}\right) \quad \text{as } t\to\infty\,.$$

The lemma is proved. □

At this stage, we therefore have proved Theorem 1.9, up to the final point regarding the asymptotic profile for $|v|^2$.

Remark 4.4. The pseudo-conformal conservation law (see [13]) writes, in the case of (1.1):

$$\frac{d}{dt}\left(\frac{1}{2}\|(x+it\nabla)u\|_{L^2}^2 + \lambda t^2\int |u|^2\ln|u|^2\right) = -\lambda dt\int |u|^2 + 2\lambda t\int |u|^2\ln|u|^2\,.$$

It is equivalent to the virial evolution:

$$\begin{aligned}
&\frac{d}{dt}\int |x|^2|u|^2 = 2\,\mathrm{Im}\int \bar u x\cdot\nabla u,\\
&\frac{d}{dt}\,\mathrm{Im}\int \bar u x\cdot\nabla u = \int|\nabla u|^2 + \lambda d\int |u|^2\,.
\end{aligned}$$

With these two equations, we recover (4.7) and (4.8), but without bringing any new information. Similarly one can write Morawetz estimates, that bring new information which we have not been able to exploit: following [26, 44] we write an estimate with the usual weight $|x|$. After some computations we come up with

$$\int_0^t \|u(t')\|_{L^4}^4 dt' + \int_0^t \left\|\nabla|u|^2(t')\right\|_{L^2}^2 dt' \lesssim \sqrt{\ln t}\,,$$

which if compared with the Gaussian case is to be expected. However translating that estimate for v and considering an extra weight of the form $(\dot\tau)^{-1-\alpha}$ gives

$$\int_0^\infty \frac{1}{t(\ln t)^{1+\alpha}}\|v(t)\|_{L^4}^4 dt + \int_0^\infty \frac{1}{t^3(\ln t)^{2+\alpha}}\left\|\nabla|v|^2(t)\right\|_{L^2}^2 dt < \infty\,,\quad \forall\alpha>0,$$

which was not known before.

5. End of the proof of Theorem 1.9

In this section we conclude the proof of Theorem 1.9 by obtaining the weak convergence to a universal profile as stated in (1.12).

5.1. **Hydrodynamical approach.** We recall that the Madelung transform is a classical tool (see e.g. [39, 36, 23], or the survey [10]) to relate the (nonlinear) Schrödinger equation to fluid dynamics equations, via the change of unknown

$$v(t,y) = a(t,y)e^{i\phi(t,y)} \quad a,\phi \in \mathbb{R}\,. \tag{5.1}$$

Formally one obtains in our case the system of equations

$$\begin{cases} \partial_t \phi + \dfrac{1}{2\tau^2}|\nabla_y \phi|^2 + \lambda \ln \left|\dfrac{a}{\gamma}\right|^2 = \dfrac{1}{2\tau^2}\dfrac{\Delta_y a}{a} \\ \partial_t a + \dfrac{1}{\tau^2}\nabla_y \phi \cdot \nabla_y a + \dfrac{1}{2\tau^2} a \Delta_y \phi = 0\,, \end{cases}$$

which is easily related to the compressible Euler equations by using the change of unknown

$$\rho(t,y) := a^2 \quad \Lambda := a\nabla\phi\,, \quad J := a\Lambda\,. \tag{5.2}$$

Note that in the explicit case of Gaussian initial data studied in Section 3, ρ is bounded in Sobolev spaces uniformly in time, whereas Λ and J are unbounded as $t \to \infty$. In terms of these hydrodynamical variables, the above system becomes

$$\begin{cases} \partial_t \rho + \dfrac{1}{\tau^2}\nabla \cdot J = 0 \\ \partial_t J + \dfrac{1}{\tau^2}\nabla \cdot (\Lambda \otimes \Lambda) + \lambda \nabla \rho + 2\lambda y \rho = \dfrac{1}{4\tau^2}\Delta \nabla \rho - \dfrac{1}{\tau^2}\nabla \cdot (\nabla\sqrt{\rho} \otimes \nabla \sqrt{\rho}) \\ \partial_j J^k - \partial_k J^j = 2\Lambda^k \partial_j \sqrt{\rho} - 2\Lambda^j \partial_k \sqrt{\rho}\,, \quad j,k \in \{1,\dots,d\}\,. \end{cases} \tag{5.3}$$

Note that in the case where the initial data for (5.3) are well prepared, in the sense that they stem from the polar decomposition of an initial wave function as in (5.1)–(5.2), then the approach presented in [10, Section 5] can readily be adapted to show that (5.3) holds true in the distributional sense. We shall however retain simply one property related to this system: as soon as we have a solution v to (4.2), it can be decomposed as in (5.1)–(5.2) so as to produce a solution to (5.3). The most delicate issue to prove this is to give a suitable meaning to the phase ϕ when v vanishes; we refer to [10, Section 5] for details.

We shall prove that

$$\rho(t) \underset{t\to\infty}{\rightharpoonup} \gamma^2 \quad \text{weakly in } L^1(\mathbb{R}^d)\,.$$

This will stem from the fact that the weak limit of ρ evolves according to a Fokker–Planck operator. We note that a formal link between the hydrodynamical formulation of (1.1) and the Fokker–Planck equation can be found in [38, 27].

5.2. **Heuristics.** Let us explain the heuristics of the proof, which will be made rigorous in the next section. Formally only retaining the higher order terms (in terms of growth in time) in (5.3) we are led to studying the following simple model

$$\begin{cases} \partial_t \rho + \dfrac{1}{\tau^2}\nabla \cdot J = 0 \\ \partial_t J + \lambda \nabla \rho + 2\lambda y \rho = 0\,. \end{cases} \tag{5.4}$$

Note that in the explicit case of the evolution of a Gaussian (recall the computations of Section 3), we can check that in the above simplification, we have indeed eliminated negligible terms. By elimination of J, (5.4) implies that

$$\partial_t\left(\tau^2 \partial_t \rho\right) = \lambda \nabla \cdot (\nabla + 2y)\,\rho = \lambda L \rho\,,$$

where

$$L := \Delta_y + \nabla_y \cdot (2y\,\cdot)$$

is a Fokker–Planck operator. On the other hand,

$$\partial_t\left(\tau^2\partial_t\rho\right) = \tau^2\partial_t^2\rho + 2\dot\tau\tau\partial_t\rho\,,$$

so since $\tau^2 \ll (\dot\tau\tau)^2$, it is natural to change scales in time and define s such that

$$\frac{\dot\tau\tau}{\lambda}\partial_t = \partial_s\,,$$

or in other words define the following change of variables:

$$s = \int \frac{1}{\lambda\dot\tau\tau} = \int \frac{\ddot\tau}{2\dot\tau} = \frac{1}{2}\ln\dot\tau(t)\,. \tag{5.5}$$

Notice that

$$s \sim \frac{1}{2}\ln\ln t\,, \quad t\to\infty\,. \tag{5.6}$$

Then again discarding formally lower order terms we find

$$\partial_s\rho = L\rho\,,$$

for which it is well-known (see for instance [21]) that in large times the solution converges strongly to an element of the kernel of L, hence a Gaussian. Notice that the convergence is exponentially fast in s variables, so returning to t variables produces a logarithmic decay due to (5.6): we recover the logarithmic convergence rate observed in the Gaussian case (Section 3).

The difficulty to make this argument rigorous is the justification that the lower order terms may indeed be discarded, since we have very little control on higher norms on v to guarantee compactness in space of the solution: we have more precisely a sharp control of the momenta of v, but rather poor estimates in H^1. More precisely, we do expect v to oscillate rapidly in time (in view of the Gaussian case), but $\sqrt{\rho}$ should be bounded in H^1, a property that does not seem easy to prove (because of the prefactor $1/\tau^2$ in the equation). This is the main obstacle to proving strong convergence to a Gaussian in the general case, and explains why in the end we only obtain a week convergence result in L^1. This is made precise in the next section.

5.3. **End of the proof.** Let us follow the steps of the previous paragraph, this time neglecting no term. First, we consider a variant of the hydrodynamical formulation of (4.2), by recalling that the two nonlinear terms in (5.3) correspond exactly to $\tau^{-2}\nabla\cdot|\nabla v|^2$, after the polar decomposition of v. Therefore, we simply use the fact that if $v=\sqrt{\rho}e^{i\phi}$, then we have

$$\begin{cases} \partial_t\rho + \dfrac{1}{\tau^2}\nabla\cdot J = 0 \\ \partial_t J + \lambda\nabla\rho + 2\lambda y\rho = \dfrac{1}{4\tau^2}\Delta\nabla\rho - \dfrac{1}{\tau^2}\nabla\cdot|\nabla v|^2\,. \end{cases} \tag{5.7}$$

By elimination of J,

$$\partial_t\left(\tau^2\partial_t\rho\right) = -\partial_t\nabla\cdot J = \lambda L\rho - \frac{1}{4\tau^2}\Delta^2\rho - \frac{1}{\tau^2}\Delta|\nabla v|^2\,,$$

with again $L := \Delta + \nabla \cdot (2y\, \cdot)$. With the change of variable (5.5) we introduce the notation $\tilde\rho(s(t),y) := \rho(t,y)$, and we find for $\tilde\rho$ the following equation:

$$\partial_s\tilde\rho - \frac{2\lambda}{(\dot\tau)^2}\partial_s\tilde\rho + \frac{\lambda}{(\dot\tau)^2}\partial_s^2\tilde\rho = L\tilde\rho - \frac{1}{4\lambda\tau^2}\Delta^2\tilde\rho - \frac{1}{\lambda\tau^2}\Delta|\nabla\tilde v|^2\,, \tag{5.8}$$

where one should keep in mind that the functions τ and $\dot\tau$ also have undergone the change of time variable. In terms of s, Lemma 1.8 yields

$$\dot\tau(s) \underset{s\to\infty}{\sim} 2\sqrt{\lambda}e^{2s}\,,\quad \tau(s) \underset{s\to\infty}{\sim} 2\sqrt{\lambda}e^{2s+e^{4s}}\,.$$

To make the discussion at the end of the previous subsection more precise, we comment on the various terms in (5.8):

- The term $\frac{2\lambda}{(\dot\tau)^2}\partial_s\tilde\rho$ is essentially harmless in the large time limit, for it could be handled by a slight modification of the time variable, for instance.
- The term $\frac{\lambda}{(\dot\tau)^2}\partial_s^2\tilde\rho$ is expected to be negligible in the large time limit. However, it makes (5.8) second order in time: one would like to take advantage of the smoothing properties of e^{sL}, by using Duhamel's formula typically, but this approach is delicate in this context. Note that it has been established before that in similar situations, the parabolic behavior gives the leading order large time dynamics, even if the coefficient of $\partial_s^2\tilde\rho$ is not asymptotically vanishing ([19]): the proof of this fact relies on energy estimates whose analogue in the case of (5.8) we could not establish.
- By using the Fourier transform in space, it is easy to compute the fundamental solution of
$$\partial_s\tilde\rho = L\tilde\rho - \frac{1}{4\lambda\tau^2}\Delta^2\tilde\rho\,,$$
in the same way as [20] (without the last term).
- A possible idea to prove that the solution to (5.8) converges (strongly) to the Gaussian γ^2 as $s\to\infty$ would be to use the spectral decomposition of L, as given for instance in [20]. The main issue is then that we can control the last term in (5.8) in L^1-based spaces, as opposed to L^2 where the spectral decomposition is available. Note that this term is the only one that prevents (5.8) from being a linear, homogeneous, equation.

In terms of s, the time integrability property of $E_{\rm kin}$ provided in (4.5) becomes

$$\int_0^\infty \left(\frac{\dot\tau(s)}{\tau(s)}\right)^2 \|\nabla\tilde v(s)\|^2_{L^2}ds < \infty\,. \tag{5.9}$$

On the other hand, Lemma 4.2 yields

$$\sup_{s\geqslant 0}\int_{\mathbb{R}^d}\tilde\rho(s,y)\left(1+y^2+|\ln\tilde\rho(s,y)|\right)dy < \infty\,. \tag{5.10}$$

Mimicking the general approach of e.g. [17, 18], for $s\in[-1,2]$ and $s_n\to\infty$, set

$$\tilde\rho_n(s,y) := \tilde\rho(s+s_n,y)\,.$$

From (5.10) along with the de la Vallée-Poussin and Dunford–Pettis Theorems, we get up to extracting a subsequence

$$\tilde\rho_n \rightharpoonup \tilde\rho_\infty \quad \text{in } L^p_s(-1,2;L^1_y)\,,$$

for all $p\in[1,\infty)$. Up to another subsequence,

$$\tilde\rho_n(0) \rightharpoonup \tilde\rho_{0,\infty} \quad \text{in } L^1_y\,.$$

In view of (5.8):

$$\begin{cases} \partial_s\tilde\rho_\infty = L\tilde\rho_\infty \text{ in } \mathcal{S}'\left((-1,2)\times\mathbb{R}^d\right), \\ \tilde\rho_{\infty|s=0} = \tilde\rho_{0,\infty}\in L^1 . \end{cases} \tag{5.11}$$

We now go back to (5.7) and show that $\tilde\rho_\infty$ is independent of s. In the s variable, we have

$$\begin{cases} \partial_s\tilde\rho + \dfrac{\dot\tau}{\lambda\tau}\nabla\cdot\tilde J = 0 \\ \partial_s\tilde J + \tau\dot\tau\left(\nabla+2y\right)\tilde\rho - \dfrac{\dot\tau}{4\lambda\tau}\nabla\Delta\tilde\rho = -\dfrac{\dot\tau}{\lambda\tau}\nabla|\nabla\tilde v|^2 . \end{cases} \tag{5.12}$$

Since $J = \operatorname{Im}\bar v\nabla_y v$, (5.9) implies

$$\frac{\dot\tau}{\tau}\tilde J \in L^2_s L^1_y .$$

With $\tilde J_n(s) := \tilde J(s+s_n)$, we have

$$\tilde J_n \underset{n\to\infty}{\longrightarrow} 0 \quad \text{in } L^2(-1,2;W^{-1,1}),$$

hence

$$\partial_s\tilde\rho_\infty = 0 . \tag{5.13}$$

Putting (5.11) and (5.13) together, we have

$$L\tilde\rho_{\infty|s=1} = 0 ,$$

and since $\tilde\rho_{\infty|s=1}$ is a smooth function, we infer $\tilde\rho_\infty = \alpha\gamma^2$, $0\leqslant\alpha\leqslant 1$.

Using (5.10) again, we see that the family $(\tilde\rho(1+s_n,\cdot))_n$ is tight, and so $\alpha=1$. The limit being unique, no extraction of a subsequence is needed, and we conclude

$$\tilde\rho(s) \underset{s\to\infty}{\rightharpoonup} \gamma^2 \quad \text{weakly in } L^1(\mathbb{R}^d) .$$

Theorem 1.9 is proved. □

6. Proof of the corollaries

6.1. **Proof of Corollary 1.12.** In view of the tensorization in Theorem 1.6, we prove Corollary 1.12 in the case $d=1$ to lighten the notations, and we assume

$$u_0(x) = b_0\exp\left(-a_0(x-x_0)^2/2\right),$$

with $b_0, a_0\in\mathbb{C}$, $\operatorname{Re}a_0 = \alpha_0 > 0$. We start with an initial center x_0 to show that in terms of v, the center is eventually zero (like in [21]). Recall that we have

$$u(t,x) = b_0\frac{1}{\sqrt{r(t)}}e^{i\phi(t)}\exp\left(-\alpha_0\frac{(x-x_0)^2}{2r^2(t)} + i\frac{\dot r(t)}{r(t)}\frac{(x-x_0)^2}{2}\right),$$

with r solution to (3.8), $r(0)=1$, $\dot r(0) = -\operatorname{Im}a_0$. We thus have

$$\begin{aligned} v(t,y) = b_0\sqrt{\frac{\tau(t)}{r(t)}}e^{i\phi(t)}\exp\left(-\alpha_0\frac{\tau^2}{r^2}\frac{y^2}{2} + \alpha_0\frac{\tau}{r^2}yx_0 - \alpha_0\frac{x_0^2}{2r^2}\right) \\ \times\exp\left(i\left(\frac{\dot r}{r}-\frac{\dot\tau}{\tau}\right)\tau^2\frac{y^2}{2} - i\frac{\dot r}{r}\tau y x_0 + i\frac{\dot r}{r}\frac{x_0^2}{2}\right). \end{aligned}$$

In particular,

$$|v(t,y)|^2 = |b_0|^2 \frac{\tau(t)}{r(t)} \exp\left(-\alpha_0 \frac{\tau^2}{r^2} y^2 + 2\alpha_0 \frac{\tau}{r^2} y x_0 - \alpha_0 \frac{x_0^2}{r^2}\right).$$

On the other hand,

$$\|u_0\|_{L^2} = |b_0| \left(\frac{\pi}{\alpha_0}\right)^{1/4} = \pi^{1/4},$$

where the last equality corresponds to our assumption motivated by Remark 1.1. Therefore, the relative entropy is

$$\begin{aligned} E_{\rm ent}(t) &= \int_{\mathbb{R}} |v(t,y)|^2 \ln\left(\frac{|v(t,y)|^2}{\gamma^2(y)}\right) dy \\ &= \ln\left(\sqrt{\alpha_0}\frac{\tau(t)}{r(t)}\right) \|u_0\|_{L^2}^2 - \left(\alpha_0 \frac{\tau(t)^2}{r(t)^2} - 1\right) \int_{\mathbb{R}} y^2 |v(t,y)|^2 dy \\ &\quad + 2\alpha_0 x_0 \frac{\tau(t)}{r^2(t)} \int_{\mathbb{R}} y|v(t,y)|^2 dy - \alpha_0 \frac{x_0^2}{r^2(t)} \|u_0\|_{L^2}^2 \underset{t\to\infty}{\longrightarrow} 0, \end{aligned}$$

where we have used the properties of the solutions to (3.8) and (1.8), established in Section 3. The end of the corollary simply stems from the standard Csiszár-Kullback inequality

$$\big\||v(t,\cdot)|^2 - \gamma^2\big\|_{L^1} \lesssim E_{\rm ent}.$$

6.2. **Proof of Corollary 1.13.** Fix $0 < s < 1$. In view of Remark 1.1, we may assume that $\|u_0\|_{L^2} = \|\gamma\|_{L^2}$ and use the conclusion of Theorem 1.9, since we are not tracking the multiplicative constants. The convergence in the Wasserstein distance W_2 (Remark 1.11) implies (see e.g. [45, Theorem 7.12])

$$\int |y|^{2s} |v(t,y)|^2 dy \underset{t\to\infty}{\longrightarrow} \int |y|^{2s} \gamma^2(y) dy. \tag{6.1}$$

The idea is then to apply a fractional derivative to (1.9), that is

$$u(t,x) = \frac{1}{\tau(t)^{d/2}} v\left(t, \frac{x}{\tau(t)}\right) \exp\left(i \frac{\dot\tau(t)}{\tau(t)} \frac{|x|^2}{2}\right).$$

In order to shortcut this step, we recall a lemma employed in a somehow similar situation, even though in the context of semi-classical limit. We therefore simplify the initial statement and leave out the dependence on the semi-classical parameter:

Lemma 6.1 (Lemma 5.1 from [2])**.** *There exists C such that if $u \in H^1(\mathbb{R}^d)$ and w is such that $\nabla w \in L^\infty(\mathbb{R}^d)$,*

$$\||w|^s u\|_{L^2} \leqslant \|u\|_{\dot H^s} + \|(\nabla - iw)u\|_{L^2}^s \|u\|_{L^2}^{1-s} + C\left(1 + \|\nabla w\|_{L^\infty}\right) \|u\|_{L^2}.$$

In [2], w corresponds to the gradient of rapid oscillations carried by an exponential, so we naturally introduce

$$w(t,x) = \frac{\dot\tau(t)}{\tau(t)} x.$$

In the present framework, Lemma 6.1 yields:

$$(\dot\tau)^s \||y|^s v(t)\|_{L^2} \leqslant \|u(t)\|_{\dot H^s} + \left\|\frac{1}{\tau}\nabla v(t)\right\|_{L^2}^s \|u_0\|_{L^2}^{1-s} + C\left(1 + \frac{\dot\tau}{\tau}\right) \|u_0\|_{L^2}.$$

The result follows readily: the behavior of the left hand side is given by Lemma 1.8 and (6.1), and all the terms of the right hand side are bounded, but the first one.

References

[1] M. Abramowitz and I. A. Stegun, *Handbook of mathematical functions with formulas, graphs, and mathematical tables*, vol. 55 of National Bureau of Standards Applied Mathematics Series, For sale by the Superintendent of Documents, U.S. Government Printing Office, Washington, D.C., 1964.

[2] T. Alazard and R. Carles, *Loss of regularity for super-critical nonlinear Schrödinger equations*, Math. Ann., 343 (2009), pp. 397–420.

[3] C. Ané, S. Blachère, D. Chafaï, P. Fougères, I. Gentil, F. Malrieu, C. Roberto, and G. Scheffer, *Sur les inégalités de Sobolev logarithmiques*, vol. 10 of Panoramas et Synthèses [Panoramas and Syntheses], Société Mathématique de France, Paris, 2000. With a preface by Dominique Bakry and Michel Ledoux.

[4] A. H. Ardila, *Logarithmic Schrödinger equation: on the orbital stability of the Gausson solution*. preprint, archived at `http://arxiv.org/abs/1607.01479`, 2016.

[5] J. E. Barab, *Nonexistence of asymptotically free solutions for nonlinear Schrödinger equation*, J. Math. Phys., 25 (1984), pp. 3270–3273.

[6] I. Białynicki-Birula and J. Mycielski, *Nonlinear wave mechanics*, Ann. Physics, 100 (1976), pp. 62–93.

[7] H. Buljan, A. Šiber, M. Soljačic, T. Schwartz, M. Segev, and D. N. Christodoulides, *Incoherent white light solitons in logarithmically saturable non-instantaneous nonlinear media*, Phys. Rev. E, 68 (2003), p. 036697.

[8] R. Carles, *Nonlinear Schrödinger equations with repulsive harmonic potential and applications*, SIAM J. Math. Anal., 35 (2003), pp. 823–843.

[9] ———, *Nonlinear Schrödinger equation with time dependent potential*, Commun. Math. Sci., 9 (2011), pp. 937–964.

[10] R. Carles, R. Danchin, and J.-C. Saut, *Madelung, Gross-Pitaevskii and Korteweg*, Nonlinearity, 25 (2012), pp. 2843–2873.

[11] R. Carles and L. Miller, *Semiclassical nonlinear Schrödinger equations with potential and focusing initial data*, Osaka J. Math., 41 (2004), pp. 693–725.

[12] T. Cazenave, *Stable solutions of the logarithmic Schrödinger equation*, Nonlinear Anal., 7 (1983), pp. 1127–1140.

[13] ———, *Semilinear Schrödinger equations*, vol. 10 of Courant Lecture Notes in Mathematics, New York University Courant Institute of Mathematical Sciences, New York, 2003.

[14] T. Cazenave and A. Haraux, *Équations d'évolution avec non linéarité logarithmique*, Ann. Fac. Sci. Toulouse Math. (5), 2 (1980), pp. 21–51.

[15] M. Combescure and D. Robert, *Coherent states and applications in mathematical physics*, Theoretical and Mathematical Physics, Springer, Dordrecht, 2012.

[16] P. d'Avenia, E. Montefusco, and M. Squassina, *On the logarithmic Schrödinger equation*, Commun. Contemp. Math., 16 (2014), pp. 1350032, 15.

[17] L. Desvillettes, *Convergence to equilibrium in large time for Boltzmann and B.G.K. equations*, Arch. Rational Mech. Anal., 110 (1990), pp. 73–91.

[18] E. Feireisl and H. Petzeltová, *Large-time behaviour of solutions to the Navier-Stokes equations of compressible flow*, Arch. Ration. Mech. Anal., 150 (1999), pp. 77–96.

[19] T. Gallay and G. Raugel, *Scaling variables and asymptotic expansions in damped wave equations*, J. Differential Equations, 150 (1998), pp. 42–97.

[20] T. Gallay and C. E. Wayne, *Invariant manifolds and the long-time asymptotics of the Navier-Stokes and vorticity equations on* $\mathbf{R}^2$, Arch. Ration. Mech. Anal., 163 (2002), pp. 209–258.

[21] ———, *Global stability of vortex solutions of the two-dimensional Navier-Stokes equation*, Comm. Math. Phys., 255 (2005), pp. 97–129.

[22] ———, *Long-time asymptotics of the Navier-Stokes equation in* $\mathbb{R}^2$ *and* $\mathbb{R}^3$ *[Plenary lecture presented at the 76th Annual GAMM Conference, Luxembourg, 29 March–1 April 2005]*, ZAMM Z. Angew. Math. Mech., 86 (2006), pp. 256–267.

[23] I. Gasser and P. A. Markowich, *Quantum hydrodynamics, Wigner transforms and the classical limit*, Asymptot. Anal., 14 (1997), pp. 97–116.

[24] J. Ginibre, *An introduction to nonlinear Schrödinger equations*, in Nonlinear waves (Sapporo, 1995), R. Agemi, Y. Giga, and T. Ozawa, eds., GAKUTO International Series, Math. Sciences and Appl., Gakkōtosho, Tokyo, 1997, pp. 85–133.

[25] J. Ginibre and G. Velo, *Long range scattering and modified wave operators for some Hartree type equations. III. Gevrey spaces and low dimensions*, J. Differential Equations, 175 (2001), pp. 415–501.

[26] ———, *Quadratic Morawetz inequalities and asymptotic completeness in the energy space for nonlinear Schrödinger and Hartree equations*, Quart. Appl. Math., 68 (2010), pp. 113–134.

[27] P. Guerrero, J. L. López, and J. Montejo-Gámez, *A wavefunction description of quantum Fokker-Planck dissipation: derivation and numerical approximation of transient dynamics*, J. Phys. A, 47 (2014), pp. 035303, 21.

[28] P. Guerrero, J. L. López, and J. Nieto, *Global H^1 solvability of the 3D logarithmic Schrödinger equation*, Nonlinear Anal. Real World Appl., 11 (2010), pp. 79–87.

[29] G. A. Hagedorn, *Semiclassical quantum mechanics. I. The $\hbar \to 0$ limit for coherent states*, Comm. Math. Phys., 71 (1980), pp. 77–93.

[30] N. Hayashi, T. Mizumachi, and P. I. Naumkin, *Time decay of small solutions to quadratic nonlinear Schrödinger equations in 3D*, Differential Integral Equations, 16 (2003), pp. 159–179.

[31] N. Hayashi and P. Naumkin, *Domain and range of the modified wave operator for Schrödinger equations with a critical nonlinearity*, Comm. Math. Phys., 267 (2006), pp. 477–492.

[32] ———, *Logarithmic time decay for the cubic nonlinear Schrödinger equations*, Int. Math. Res. Not. IMRN, (2015), pp. 5604–5643.

[33] E. F. Hefter, *Application of the nolinear Schrödinger equation with a logarithmic inhomogeneous term to nuclear physics*, Phys. Rev. A, 32 (1985), p. 1201.

[34] E. S. Hernandez and B. Remaud, *General properties of Gausson-conserving descriptions of quantal damped motion*, Physica A, 105 (1980), p. 130.

[35] W. Krolikowski, D. Edmundson, and O. Bang, *Unified model for partially coherent solitons in logaritmically nonlinear media*, Phys. Rev. E, 61 (2000), p. 3122.

[36] L. Landau and E. Lifschitz, *Physique théorique ("Landau-Lifchitz"). Tome III: Mécanique quantique. Théorie non relativiste*, Éditions Mir, Moscow, 1967. Deuxième édition, Traduit du russe par Édouard Gloukhian.

[37] T. Li and D. Wang, *Blowup phenomena of solutions to the Euler equations for compressible fluid flow*, J. Differential Equations, 221 (2006), pp. 91–101.

[38] J. L. López and J. Montejo-Gámez, *A hydrodynamic approach to multidimensional dissipation-based Schrödinger models from quantum Fokker-Planck dynamics*, Phys. D, 238 (2009), pp. 622–644.

[39] E. Madelung, *Quanten theorie in Hydrodynamischer Form*, Zeit. F. Physik, 40 (1927), p. 322.

[40] S. D. Martino, M. Falanga, C. Godano, and G. Lauro, *Logarithmic Schrödinger-like equation as a model for magma transport*, Europhys. Lett., 63 (2003), p. 472.

[41] K. Nakanishi, *Modified wave operators for the Hartree equation with data, image and convergence in the same space*, Commun. Pure Appl. Anal., 1 (2002), pp. 237–252.

[42] ———, *Modified wave operators for the Hartree equation with data, image and convergence in the same space. II*, Ann. Henri Poincaré, 3 (2002), pp. 503–535.

[43] T. Ozawa, *Long range scattering for nonlinear Schrödinger equations in one space dimension*, Comm. Math. Phys., 139 (1991), pp. 479–493.

[44] F. Planchon and L. Vega, *Bilinear virial identities and applications*, Ann. Sci. Éc. Norm. Supér. (4), 42 (2009), pp. 261–290.

[45] C. Villani, *Topics in optimal transportation*, vol. 58 of Graduate Studies in Mathematics, American Mathematical Society, Providence, RI, 2003.

CNRS, Institut Montpelliérain Alexander Grothendieck, Univ. Montpellier, CC51, Place E. Bataillon, 34095 Montpellier, France

E-mail address: Remi.Carles@math.cnrs.fr

Institut de Mathématiques de Jussieu, Université Paris-Diderot, UFR de Mathématiques, Bâtiment Sophie Germain, 75205 Paris Cedex 13, France

E-mail address: gallagher@math.univ-paris-diderot.fr