A NON LINEAR ESTIMATE ON THE LIFE SPAN OF SOLUTIONS OF THE THREE DIMENSIONAL NAVIER-STOKES EQUATIONS

JEAN-YVES CHEMIN AND ISABELLE GALLAGHER

Abstract. The purpose of this article is to establish bounds from below for the life span of regular solutions to the incompressible Navier-Stokes system, which involve norms not only of the initial data, but also of nonlinear functions of the initial data. We provide examples showing that those bounds are significant improvements to the one provided by the classical fixed point argument. One of the important ingredients is the use of a scale-invariant energy estimate.

1. Introdution

In this article our aim is to give bounds from below for the life span of solutions to the incompressible Navier-Stokes system in the whole space $\mathbb{R}^3$. We are not interested here in the regularity of the initial data: we focus on obtaining bounds from below for the life span associated with regular initial data. Here regular means that the initial data belongs to the intersection of all Sobolev spaces of non negative index. Thus all the solutions we consider are regular ones, as long as they exist.

Let us recall the incompressible Navier-Stokes system, together with some of its basic features. The incompressible Navier-Stokes system is the following:

$$(NS) \quad \begin{cases} \partial_t u - \Delta u + u\cdot\nabla u = -\nabla p \\ \operatorname{div} u = 0 \quad \text{and} \quad u_{|t=0} = u_0 \,, \end{cases}$$

where u is a three dimensional, time dependent vector field and p is the pressure, determined by the incompressibility condition $\operatorname{div} u = 0$:

$$-\Delta p = \operatorname{div}(u\cdot\nabla u) = \sum_{1\leq i,j\leq 3} \partial_i\partial_j(u^i u^j)\,.$$

This system has two fundamental properties related to its physical origin:

- scaling invariance
- dissipation of kinetic energy.

The scaling property is the fact that if a function u satisfies (NS) on a time interval $[0,T]$ with the initial data u_0, then the function u_λ defined by

$$u_\lambda(t,x) \stackrel{\text{def}}{=} \lambda u(\lambda^2 t, \lambda x)$$

satisfies (NS) on the time interval $[0,\lambda^{-2}T]$ with the initial data $\lambda u_0(\lambda\,\cdot)$. This property is far from being a characteristic property of the system (NS). It is indeed satisfied by all systems of the form

$$(GNS) \quad \begin{cases} \partial_t u - \Delta u + Q(u,u) = 0 \\ u_{|t=0} = u_0 \end{cases} \qquad \text{with} \quad Q^i(u,u) \stackrel{\text{def}}{=} \sum_{1\leq j,k\leq 3} A^i_{j,k}(D)(u^j u^k)$$

where the $A^i_{j,k}(D)$ are smooth homogenenous Fourier multipliers of order 1. Indeed denoting by $\mathbb{P}$ the projection onto divergence free vector fields

$$\mathbb{P} \stackrel{\text{def}}{=} \operatorname{Id} - (\partial_i\partial_j\Delta^{-1})_{ij}$$

the Navier-Stokes system takes the form

$$\left\{\begin{array}{c}\partial_t u - \Delta u + \mathbb{P}\,\mathrm{div}(u\otimes u) = 0\\ u_{|t=0} = u_0\,,\end{array}\right.$$

which is of the type (GNS). For this class of systems, the following result holds. The definition of homogeneous Sobolev spaces $\dot H^s$ is recalled in the Appendix.

Proposition 1.1. *Let u_0 be a regular three dimensional vector field. A positive time T exists such that a unique regular solution to (GNS) exists on $[0,T]$. Let $T^\star(u_0)$ be the maximal time of existence of this regular solution. Then, for any γ in the interval $]0,1/2[$, a constant c_γ exists such that*

$$T^\star(u_0) \geq c_\gamma \|u_0\|_{\dot H^{\frac12+2\gamma}}^{-\frac1\gamma}\,. \tag{1}$$

In the case when $\gamma = 1/4$ for the particular case of (NS), this type of result goes back to the seminal work of J. Leray (see [8]). Let us point out that the same type of result can be proved for the $L^{3+\frac{6\gamma}{1-2\gamma}}$ norm.

Proof. This result is obtained by a scaling argument. Let us define the following function

$$\underline{T}_{\dot H^{\frac12+2\gamma}}(r) \stackrel{\mathrm{def}}{=} \inf\bigl\{T^\star(u_0)\,,\ \|u_0\|_{\dot H^{\frac12+2\gamma}} = r\bigr\}\,.$$

We assume that at least one smooth initial data u_0 develops singularites, which means exactly that $T^\star(u_0)$ is finite. Let us mention that this lower bound is in fact a minimum (see [10]). Actually the function $\underline{T}_{\dot H^{\frac12+2\gamma}}$ may be computed using a scaling argument. Observe that

$$\|u_0\|_{\dot H^{\frac12+2\gamma}} = r \Longleftrightarrow \|r^{-\frac{1}{2\gamma}} u_0(r^{-\frac{1}{2\gamma}}\cdot)\|_{\dot H^{\frac12+2\gamma}} = 1\,.$$

As we have $T^\star(u_0) = r^{-\frac1\gamma} T^\star\bigl(r^{-\frac{1}{2\gamma}} u_0(r^{-\frac{1}{2\gamma}}\cdot)\bigr)$, we infer that $\underline{T}_{\dot H^{\frac12+2\gamma}}(r) = r^{-\frac1\gamma}\underline{T}_{\dot H^{\frac12+2\gamma}}(1)$ and thus that

$$T^\star(u_0) \geq c_\gamma \|u_0\|_{\dot H^{\frac12+2\gamma}}^{-\frac1\gamma} \quad \text{with} \quad c_\gamma \stackrel{\mathrm{def}}{=} \underline{T}_{\dot H^{\frac12+2\gamma}}(1)\,.$$

The proposition is proved. □

Now let us investigate the optimality of such a result, in particular concerning the norm appearing in the lower bound (1). Useful results and definitions concerning Besov spaces are recalled in the Appendix; the Besov norms of particular interest in this text are the $\dot B^{-1}_{\infty,2}$ norm which is given by

$$\|a\|_{\dot B^{-1}_{\infty,2}} \stackrel{\mathrm{def}}{=} \Bigl(\int_0^\infty \|e^{t\Delta}a\|^2_{L^\infty}\,dt\Bigr)^{\frac12}$$

and the Besov norms $\dot B^{-\sigma}_{\infty,\infty}$ for $\sigma > 0$ which are

$$\|a\|_{\dot B^{-\sigma}_{\infty,\infty}} \stackrel{\mathrm{def}}{=} \sup_{t>0} t^{\frac\sigma2}\|e^{t\Delta}a\|_{L^\infty}\,.$$

It has been known since [6] that a smooth initial data in $\dot H^{\frac12}$ (corresponding of course to the limit case $\gamma = 0$ in Proposition 1.1) generates a smooth solution for some time $T > 0$. Let us point out that in dimension 3, the following inequality holds

$$\|a\|_{\dot B^{-1}_{\infty,2}} \lesssim \|a\|_{\dot H^{\frac12}}\,.$$

The norms $\dot B^{-\sigma}_{\infty,\infty}$ are the smallest norms invariant by translation and having a given scaling. More precisely, we have the following result, due to Y. Meyer (see Lemma 9 in [9]).

Proposition 1.2. *Let $d \geq 1$ and let $(E, \|\cdot\|_E)$ be a normed space continuously included in $\mathcal{S}'(\mathbb{R}^d)$, the space of tempered distributions on $\mathbb{R}^d$. Assume that E is stable by translation and by dilation, and that a constant C_0 exists such that*

$$\forall (\lambda, e) \in]0, \infty[\times \mathbb{R}^d\,,\ \forall a \in E\,, \quad \|a(\lambda \cdot - e)\|_E \leq C_0 \lambda^{-\sigma} \|a\|_E\,.$$

Then a constant C_1 exists such that

$$\forall a \in E\,,\ \|a\|_{\dot{B}^{-\alpha}_{\infty,\infty}} \leq C_1 \|a\|_E\,.$$

Proof. Let us simply observe that, as E is continuously included in $\mathcal{S}'(\mathbb{R}^d)$, a constant C exists such that for all a in E,

$$\big|\langle a, e^{-|\cdot|^2}\rangle\big| \leq C\|a\|_E\,.$$

Then by invariance by translation and dilation of E, we infer immediately that

$$\|e^{t\Delta}a\|_{L^\infty} \leq C_1 t^{-\frac{\sigma}{2}} \|a\|_E$$

which proves the proposition. □

Now let us state a first improvement to Proposition 1.1 where the life span is bounded from below in terms of the $\dot{B}^{-1+2\gamma}_{\infty,\infty}$ norm of the initial data.

Theorem 1.1. *With the notations of Proposition* 1.1*, for any γ in the interval* $]0, 1/2[$*, a constant c'_γ exists such that*

$$T^\star(u_0) \geq T_{\mathrm{FP},\gamma}(u_0) \stackrel{\mathrm{def}}{=} c'_\gamma \|u_0\|^{-\frac{1}{\gamma}}_{\dot{B}^{-1+2\gamma}_{\infty,\infty}}\,. \tag{2}$$

This theorem is proved in Section 2; the proof relies on a fixed point theorem in a space included in the space of L^2 in time functions, with values in L^∞.

Let us also recall that if a scaling 0 norm of a regular initial data is small, then the solution of (NS) associated with u_0 is global. This a consequence of the Koch and Tataru theorem (see [7]) which can be translated as follows in the context of smooth solutions.

Theorem 1.2. *A constant c_0 exists such that for any regular initial data u_0 satisfying*

$$\|u_0\|_{BMO^{-1}} \stackrel{def}{=} \sup_{t>0} t^{\frac{1}{2}} \|e^{t\Delta}u_0\|_{L^\infty} + \Big(\sup_{\substack{x\in\mathbb{R}^3\\ R>0}} \frac{1}{R^3}\int_0^{R^2}\int_{B(x,R)} |e^{t\Delta}u_0(y)|^2 dy dt\Big)^{\frac{1}{2}} \leq c_0\,,$$

the associate solution of (GNS) is globally regular.

Let us remark that

$$\|u_0\|_{\dot{B}^{-1}_{\infty,\infty}} \leq \|u_0\|_{BMO^{-1}} \leq \|u_0\|_{\dot{B}^{-1}_{\infty,2}}.$$

We shall explain in Section 2 how to deduce Theorem 1.2 from the Koch and Tataru theorem [7].

The previous results are valid for the whole class of systems (GNS). Now let us present the second main feature of the incompressible Navier-Stokes system, which is not shared by all systems under the form (GNS) as it relies on a special structure of the nonlinear term (which must be skew-symmetric in L^2): the dissipation estimate for the kinetic energy. For regular solutions of (NS) there holds

$$\frac{1}{2}\frac{d}{dt}\|u(t)\|^2_{L^2} + \|\nabla u(t)\|^2_{L^2} = 0$$

which gives by integration in time

$$\forall t \geq 0\,, \quad \mathcal{E}\big(u(t)\big) \stackrel{\mathrm{def}}{=} \frac{1}{2}\|u(t)\|^2_{L^2} + \int_0^t \|\nabla u(t')\|_{L^2} dt' = \frac{1}{2}\|u_0\|^2_{L^2}\,. \tag{3}$$

T. Tao pointed out in his paper [11] that the energy estimate is not enough to prevent possible singularities from appearing. Our purpose here is to investigate if this energy estimate can improve the lower bound (2) of the life span for regular initial data. We recall indeed that for smooth initial data, all Leray solutions — meaning solutions in the sense of distributions satisfying the energy inequality

$$\mathcal{E}\big(u(t)\big) \leq \frac{1}{2}\|u_0\|_{L^2}^2 \tag{4}$$

coincide with the smooth solution as long as the latter exists.

What we shall use here is a rescaled version of the energy dissipation inequality in the spirit of [5], on the fluctuation $w \stackrel{\text{def}}{=} u - u_{\rm L}$ with $u_{\rm L}(t) \stackrel{\text{def}}{=} e^{t\Delta}u_0$.

Proposition 1.3. *Let u be a regular solution of (NS) associated with some initial data u_0. Then the fluctuation w satisfies, for any positive t*

$$\mathcal{E}\Big(\frac{w(t)}{t^{\frac14}}\Big)+\int_0^t \frac{\|w(t')\|_{L^2}^2}{t'^{\frac32}}\,dt' \lesssim Q_{\rm L}^0 \exp\|u_0\|^2_{\dot B^{-1}_{\infty,2}} \quad \textit{with} \quad Q_{\rm L}^0 \stackrel{\text{def}}{=} \int_0^\infty t^{\frac12}\|\,\mathbb{P}(u_{\rm L}\cdot\nabla u_{\rm L})(t)\|_{L^2}^2\,dt\,.$$

Our main result is then the following.

Theorem 1.3. *There is a constant $C>0$ such that the following holds. For any regular initial data of (NS),*

$$T^*(u_0) > T_{\rm L}(u_0) \stackrel{\text{def}}{=} C\big(Q_{\rm L}^0\big)^{-2}\Big(\|\partial_3 u_0\|^2_{\dot B^{-\frac32}_{\infty,\infty}} Q_{\rm L}^0 + \sqrt{Q_{\rm L}^0 Q_{\rm L}^1}\Big)^{-2} \exp\big(-4\|u_0\|^2_{\dot B^{-1}_{\infty,2}}\big)\,, \tag{5}$$

with

$$Q_{\rm L}^1 \stackrel{\text{def}}{=} \int_0^\infty t^{\frac32}\big\|\partial_3^2\big(\mathbb{P}(u_{\rm L}\cdot\nabla u_{\rm L})\big)(t)\big\|_{L^2}^2\,dt\,.$$

The main two features of this result are that

- the statement involves non linear quantities associated with the initial data, namely norms of $\mathbb{P}(u_{\rm L}\cdot\nabla u_{\rm L})$;
- one particular (arbitrary) direction plays a specific role.

This theorem is proved in Section 4.

The following statement shows that the lower bound on $T^*(u_0)$ given in Theorem 1.3 is, for some classes of initial data, a significant improvement.

Theorem 1.4. *Let (γ,η) be in $]0,1/2[\times]0,1[$. There is a constant C and a family $(u_{0,\varepsilon})_{\varepsilon\in]0,1[}$ of regular initial data such that with the notation of Theorems 1.1 and 1.3,*

$$T_{\rm FP}(u_{0,\varepsilon}) = C\varepsilon^2|\log\varepsilon|^{-\frac{1}{\gamma}} \quad \textit{and} \quad T_{\rm L}(u_{0,\varepsilon}) \geq C\varepsilon^{-2+\eta}\,.$$

This theorem is proved in Section 5. The family $(u_{0,\varepsilon})_{\varepsilon\in]0,1[}$ is closely related to the family used in [3] to exhibit families of initial data which do not obey the hypothesis of the Koch and Tataru theorem and which nevertheless generate global smooth solutions. However it it too large to satisfy the assumptions of Theorem 2 in [3] so it is not known if the associate solution is global.

In the following we shall denote by C a constant which may change from line to line, and we shall sometimes write $A \lesssim B$ for $A \leq CB$.

2. Proof of Theorem 1.1

Let u_0 be a smooth vector field and let us solve (GNS) by means of a fixed point method. We define the bilinear operator B by

$$\partial_t B(u,v) - \Delta B(u,v) = -\frac{1}{2}\big(Q(u,v)+Q(v,u)\big)\,, \quad \text{and } B(u,v)|_{t=0}=0\,. \tag{6}$$

One can decompose the solution u to (GNS) into

$$u = u_{\rm L} + B(u,u)\,.$$

Resorting to the Littlewood-Paley decomposition defined in the Appendix, let us define for any real number γ and any time $T>0$, the quantity

$$\|f\|_{E^\gamma_T} \stackrel{\rm def}{=} \sup_{j\in\mathbb{Z}} 2^{-j(1-2\gamma)}\big(\|\Delta_j f\|_{L^\infty([0,T]\times\mathbb{R}^3)} + 2^{2j}\|\Delta_j f\|_{L^1([0,T];L^\infty(\mathbb{R}^3))}\big)\,.$$

Using Lemma 2.1 of [2] it is easy to see that

$$\|u_{\rm L}\|_{E^\gamma_\infty} \lesssim \|u_0\|_{\dot B^{-1+2\gamma}_{\infty,\infty}}\,,$$

so Theorem 1.1 will follow from the fact that B maps $E^\gamma_T\times E^\gamma_T$ into E^γ_T with the following estimate:

$$\|B(u,v)\|_{E^\gamma_T} \le C_\gamma T^\gamma \|u\|_{E^\gamma_T}\|v\|_{E^\gamma_T}\,. \tag{7}$$

So let us prove (7). Using again Lemma 2.1 of [2] along with the fact that the $A^i_{k,\ell}(D)$ are smooth homogeneous Fourier multipliers of order 1, we have

$$\|\Delta_j B(u,v)(t)\|_{L^\infty} \lesssim \int_0^t e^{-c2^{2j}(t-t')}2^j\Big\|\Delta_j\big(u(t')\otimes v(t')+v(t')\otimes u(t')\big)\Big\|_{L^\infty}dt'\,.$$

We then decompose (component-wise) the product $u\otimes v$ following Bony's paraproduct algorithm: for all functions a and b the support of the Fourier transform of $S_{j'+1}a\Delta_{j'}b$ and $S_{j'}b\Delta_{j'}a$ is included in a ball $2^{j'}B$ where B is a fixed ball of $\mathbb{R}^3$, so one can write for some fixed constant $c>0$

$$ab = \sum_{2^{j'}\ge c2^j}\big(S_{j'+1}a\Delta_{j'}b + \Delta_{j'}aS_{j'}b\big)$$

so thanks to Young's inequality in time one can write

$$\begin{aligned}
&2^{-j(1-2\gamma)}\big(\|\Delta_j B(u,v)\|_{L^\infty([0,T]\times\mathbb{R}^3)} + 2^{2j}\|\Delta_j B(u,v)\|_{L^1([0,T];L^\infty(\mathbb{R}^3))}\big)\\
&\qquad\qquad\lesssim \mathcal{B}^1_j(u,v)+\mathcal{B}^2_j(u,v) \quad \text{with}\\
&\mathcal{B}^1_j(u,v) \stackrel{\rm def}{=} 2^{2j\gamma}\sum_{2^{j'}\ge\max\{c2^j,T^{-\frac12}\}}\|S_{j'+1}u\|_{L^\infty([0,T]\times\mathbb{R}^3)}\|\Delta_{j'}v\|_{L^1([0,T];L^\infty(\mathbb{R}^3))}\\
&\qquad\qquad + 2^{2j\gamma}\sum_{c2^j\le 2^{j'}<T^{-\frac12}}\|S_{j'+1}u\|_{L^\infty([0,T]\times\mathbb{R}^3)}\|\Delta_{j'}v\|_{L^1([0,T];L^\infty(\mathbb{R}^3))} \quad \text{and}\\
&\mathcal{B}^2_j(u,v) \stackrel{\rm def}{=} 2^{2j\gamma}\sum_{2^{j'}\ge\max\{c2^j,T^{-\frac12}\}}\|S_{j'}v\|_{L^\infty([0,T]\times\mathbb{R}^3)}\|\Delta_{j'}u\|_{L^1([0,T];L^\infty(\mathbb{R}^3))}\\
&\qquad\qquad + 2^{2j\gamma}\sum_{c2^j\le 2^{j'}<T^{-\frac12}}\|S_{j'}v\|_{L^\infty([0,T]\times\mathbb{R}^3)}\|\Delta_{j'}u\|_{L^1([0,T];L^\infty(\mathbb{R}^3))}\,.
\end{aligned} \tag{8}$$

In each of the sums over $c2^j\le 2^{j'}<T^{-\frac12}$ we write

$$\|f\|_{L^1([0,T];L^\infty(\mathbb{R}^3))} \le T\|f\|_{L^\infty([0,T]\times\mathbb{R}^3)}$$

and we can estimate the two terms $\mathcal{B}^1_j(u,v)$ and $\mathcal{B}^2_j(u,v)$ in the same way: for $\ell\in\{1,2\}$ there holds indeed

$$
\begin{aligned}
\mathcal{B}^\ell_j(u,v) &\leq \|u\|_{E^\gamma_T}\|v\|_{E^\gamma_T}\Big(2^{2j\gamma}\sum_{2^{j'}\geq \max\{c2^j,T^{-\frac12}\}}2^{-4j'\gamma}+T2^{2j(1-\gamma)}\sum_{c\leq 2^{j'-j}<(2^{2j}T)^{-\frac12}}2^{2(j'-j)(1-2\gamma)}\Big)\\
&\leq \|u\|_{E^\gamma_T}\|v\|_{E^\gamma_T}\Big(T^\gamma+T2^{2j(1-\gamma)}\sum_{c\leq 2^{j'-j}<(2^{2j}T)^{-\frac12}}2^{2(j'-j)(1-2\gamma)}\Big)\,.
\end{aligned}
$$

Once noticed that

$$
T2^{2j(1-\gamma)}\sum_{c\leq 2^{j'-j}<(2^{2j}T)^{-\frac12}}2^{2(j'-j)(1-2\gamma)}\leq \mathbf{1}_{\{2^{2j}T\leq C\}}(T2^{2j})^\gamma 2^{-2j\gamma}\lesssim T^\gamma
$$

the estimate (7) is proved and Theorem 1.1 follows. □

3. Proof of Theorem 1.2

As the solutions given by the Fujita-Kato theorem [6] and the Koch-Tataru theorem [7] are unique in their own class, they are unique in the intersection and thus coincide as long as the Fujita-Kato solution exists. Thus Theorem 1.2 is a question of propagation of regularity, which is provided by the following lemma (which proves the theorem).

Lemma 3.1. *A constant c_0 exists which satisfies the following. Let u be a regular solution of (GNS) on $[0,T[$ associated with a regular initial data u_0 such that*

$$
\|u\|_{\mathrm{K}}\stackrel{\mathrm{def}}{=}\sup_{t\in[0,T[}t^{\frac12}\|u(t)\|_{L^\infty}\leq c_0\,.
$$

Then $T^\star(u_0)>T$.

Proof. The proof is based on a paralinearization argument (see [2]). Observe that for any T less than $T^\star(u_0)$, u is a solution on $[0,T[$ of the *linear* equation

$$
(PGNS)\quad\begin{cases}\partial_t v-\Delta v+\mathcal{Q}(u,v)=0\\ v_{|t=0}=u_0\end{cases}\quad\text{with}
$$

$$
\mathcal{Q}(u,v)\stackrel{\mathrm{def}}{=}\sum_{j\in\mathbb{Z}}Q(S_{j+1}u,\Delta_j v)+\sum_{j\in\mathbb{Z}}Q(\Delta_j v,S_j u)\,.
$$

In the same spirit as (6), let us define $PB(u,v)$ by

$$
\partial_t PB(u,v)-\Delta PB(u,v)=-\mathcal{Q}(u,v)\quad\text{and}\quad PB(u,v)|_{t=0}=0\,. \tag{9}
$$

A solution of $(PGNS)$ is a solution of

$$
v=u_{\mathrm{L}}+PB(u,v)\,.
$$

Let us introduce the space F_T of continuous functions with values in $\dot H^{\frac12}$, which are elements of $L^4([0,T];\dot H^1)$, equipped with the norm

$$
\|v\|_{F_T}\stackrel{\mathrm{def}}{=}\Big(\sum_{j\in\mathbb{Z}}2^j\|\Delta_j v\|^2_{L^\infty([0,T[;L^2)}\Big)^{\frac12}+\|v\|_{L^4([0,T[;\dot H^1)}\,.
$$

Notice that the first part of the norm was introduced in [4] and is a larger norm than the supremum in time of the $\dot H^{\frac12}$ norm. Moreover there holds

$$
\|u_{\mathrm{L}}\|_{F_T}\lesssim\|u_0\|_{\dot H^{\frac12}}\,.
$$

Let us admit for a while the following inequality:

$$
\|PB(u,v)\|_{F_T}\lesssim\|u\|_{\mathrm{K}}\|v\|_{F_T}\,. \tag{10}
$$

Then it is obvious that if $\|u\|_{\rm K}$ is small enough for some time $[0,T[$, the linear equation $(PGNS)$ has a unique solution in F_T (in the distribution sense) which satisfies in particular, if c_0 is small enough,

$$\|v\|_{F_T} \leq C\|u_0\|_{\dot H^{\frac12}} + \frac12 \|v\|_{F_T}\,.$$

As u is a regular solution of $(PGNS)$, it therefore satisfies

$$\forall t<T\,,\ \|u\|_{L^4([0,t];\dot H^1)} \leq 2C\|u_0\|_{\dot H^{\frac12}}$$

which implies that $T^\star(u_0)>T$, so the lemma is proved provided we prove Inequality (10).

Let us observe that for any j in $\mathbb{Z}$,

$$\partial_t\Delta_j PB(u,v) - \Delta\Delta_j PB(u,v) = -\Delta_j\mathcal{Q}(u,v)\,. \tag{11}$$

By definition of $\mathcal{Q}$, we have

$$\big\|\Delta_j\mathcal{Q}(u,v)(t)\big\|_{L^2} \leq \sum_{j'\in\mathbb{Z}}\sum_{1\leq i,k,\ell\leq 3}\big(\big\|\Delta_j A^i_{k,\ell}(D)\big(S_{j'+1}u\Delta_{j'}v\big)\big\|_{L^2} + \big\|\Delta_j A^i_{k,\ell}(D)\big(\Delta_{j'}vS_{j'}u\big)\big\|_{L^2}\big)\,.$$

As $A^i_{k,\ell}(D)$ are smooth homogeneous Fourier multipliers of order 1, we infer that for some fixed nonnegative integer N_0

$$\begin{aligned}\big\|\Delta_j\mathcal{Q}(u,v)(t)\big\|_{L^2} &\lesssim 2^j\sum_{j'\geq j-N_0}\big(\big\|S_{j'+1}u(t)\Delta_{j'}v(t)\big\|_{L^2} + \big\|\Delta_{j'}v(t)S_{j'}u(t)\big\|_{L^2}\big)\\ &\lesssim 2^j\sum_{j'\geq j-N_0}\big(\|S_{j'+1}u(t)\|_{L^\infty}\|\Delta_{j'}v(t)\|_{L^2} + \big\|\Delta_{j'}v(t)\|_{L^2}\|S_{j'}u(t)\big\|_{L^\infty}\big)\\ &\lesssim 2^j\|u(t)\|_{L^\infty}\sum_{j'\geq j-N_0}\|\Delta_{j'}v(t)\|_{L^2}\,.\end{aligned}$$

Using Relation (11) and the definition of the norm on F_T, we infer that

$$\begin{aligned}\|\Delta_j PB(u,v)(t)\|_{L^2} &\leq \int_0^t e^{-c2^{2j}(t-t')}\big\|\Delta_j\mathcal{Q}(u,v)(t')\|_{L^2}dt'\\ &\lesssim 2^j\int_0^t e^{-c2^{2j}(t-t')}\|u(t')\|_{L^\infty}\sum_{j'\geq j-N_0}\|\Delta_{j'}v(t')\|_{L^2}dt'\\ &\lesssim 2^j\|u\|_{\rm K}\|v\|_{F_T}\sum_{j'\geq j-N_0}c_{j'}2^{-\frac{j'}{2}}\int_0^t e^{-c2^{2j}(t-t')}\frac{1}{\sqrt{t'}}dt'\,,\end{aligned}$$

where $(c_j)_{j\in\mathbb{Z}}$ denotes a generic element of the sphere of $\ell^2(\mathbb{Z})$. Thus we have, for all t less than T,

$$2^{\frac j2}\|\Delta_j PB(u,v)(t)\|_{L^2} \lesssim \|u\|_{\rm K}\|v\|_{F_T}\sum_{j'\geq j-N_0}c_{j'}2^{-\frac{j'-j}{2}}\int_0^t 2^j e^{-c2^{2j}(t-t')}\frac{1}{\sqrt{t'}}dt'\,.$$

Thanks to Young's inequality, we have $\displaystyle\sum_{j'\geq j-N_0}c_{j'}2^{-\frac{j'-j}{2}}\lesssim c_j$ and we deduce that

$$2^{\frac j2}\|\Delta_j PB(u,v)(t)\|_{L^2} \lesssim c_j\|u\|_{\rm K}\|v\|_{F_T}\int_0^t 2^j e^{-c2^{2j}(t-t')}\frac{1}{\sqrt{t'}}dt'. \tag{12}$$

As we have

$$\int_0^t 2^j e^{-c2^{2j}(t-t')}\frac{1}{\sqrt{t'}}dt' \lesssim \int_0^t\frac{1}{\sqrt{t-t'}}\frac{1}{\sqrt{t'}}dt'\,,$$

we infer finally that

$$\sum_{j\in\mathbb{Z}} 2^j \|\Delta_j PB(u,v)\|^2_{L^\infty([0,T];L^2)} \lesssim \|u\|^2_{\mathrm{K}}\|v\|^2_{F_T}\,. \tag{13}$$

Moreover returning to Inequality (12), we have

$$2^j\|\Delta_j PB(u,v)\|_{L^4([0,T];L^2)} \lesssim c_j \|u\|_{\mathrm{K}}\|v\|_{F_T}\Big\|\int_0^t 2^{\frac{3j}{2}} e^{-c2^{2j}(t-t')}\frac{1}{\sqrt{t'}}dt'\Big\|_{L^4(\mathbb{R}+)}\,.$$

The Hardy-Littlewood-Sobolev inequality implies that

$$\Big\|\int_0^t 2^{\frac{3j}{2}} e^{-c2^{2j}(t-t')}\frac{1}{\sqrt{t'}}dt'\Big\|_{L^4(\mathbb{R}+)} \lesssim 1\,.$$

Since thanks to the Minkowski inequality there holds

$$\|PB(u,v)\|^2_{L^4([0,T];\dot H^1)} \leq \sum_{j\in\mathbb{Z}} 2^{2j}\|\Delta_j PB(u,v)\|^2_{L^4([0,T];L^2)}\,,$$

together with Inequality (13) this concludes the proof of Inequality (10) and thus the proof of Lemma 3.1. □

4. Proof of Theorem 1.3

The plan of the proof of Theorem 1.3 is the following: as previously we look for the solution of (NS) under the form

$$u = u_{\mathrm{L}} + w$$

where we recall that $u_{\mathrm{L}}(t) = e^{t\Delta}u_0$. Moreover we recall that the solution u satisfies the energy inequality (4). By construction, the fluctuation w satisfies

$$\text{(NSF)}\qquad \partial_t w - \Delta w + (u_{\mathrm{L}}+w)\cdot\nabla w + w\cdot\nabla u_{\mathrm{L}} = -u_{\mathrm{L}}\cdot\nabla u_{\mathrm{L}} - \nabla p\,,\quad \operatorname{div} w = 0\,.$$

Let us prove that the life span of w satisfies the lower bound (5). The first step of the proof consists in proving Proposition 1.3, stated in the introduction. This is achieved in Section 4.1. The next step is the proof of a similar energy estimate on $\partial_3 w$ — note that contrary to the scaled energy estimate of Proposition 1.3, the next result is useful in general only locally in time. It is proved in Section 4.2.

Proposition 4.1. *With the notation of Proposition* 1.3 *and Theorem* 1.3, *the fluctuation w satisfies the following estimate:*

$$\mathcal{E}\big(\partial_3 w\big)(t) \lesssim \Big(Q^0_{\mathrm{L}}\big(t^{\frac12}\sup_{t'\in(0,t)}\|\partial_3 w(t)\|^4_{L^2} + \|\partial_3 u_0\|^2_{\dot B^{-\frac32}_{\infty,\infty}}\big) + \sqrt{Q^0_{\mathrm{L}}Q^1_{\mathrm{L}}}\Big)\exp\big(2\|u_0\|^2_{\dot B^{-1}_{\infty,2}}\big)\,.$$

Combining both propositions, one can conclude the proof of Theorem 1.3. This is performed in Section 4.3.

4.1. The rescaled energy estimate on the fluctuation: proof of Proposition 1.3.

An L^2 energy estimate on (NSF) gives

$$\frac12\frac{d}{dt}\|w(t)\|^2_{L^2} + \|\nabla w(t)\|^2_{L^2} = -\sum_{1\leq j,k\leq 3}\int_{\mathbb{R}^3} w^j\partial_j u^k_{\mathrm{L}} w^k(t,x)dx - \big(\mathbb{P}(u_{\mathrm{L}}\cdot\nabla u_{\mathrm{L}})\big|w\big)(t)\,.$$

¿From this, after an integration by parts and using the fact that the divergence of w is zero, we infer that

$$\frac12\frac{d}{dt}\Big(\frac{\|w(t)\|^2_{L^2}}{t^{\frac12}}\Big) + \frac{\|w(t)\|^2_{L^2}}{2t^{\frac32}} + \frac{\|\nabla w(t)\|^2_{L^2}}{t^{\frac12}} \leq \frac{\|w(t)\|_{L^2}\|u_{\mathrm{L}}(t)\|_{L^\infty}\|\nabla w(t)\|_{L^2}}{t^{\frac12}} + \frac{\|\,\mathbb{P}(u_{\mathrm{L}}\cdot\nabla u_{\mathrm{L}})(t)\|_{L^2}\|w(t)\|_{L^2}}{t^{\frac12}}\,.$$

Let us observe that

$$\frac{\|\mathbb{P}(u_{\rm L}\cdot\nabla u_{\rm L})(t)\|_{L^2}\|w(t)\|_{L^2}}{t^{\frac12}} = t^{\frac14}\|\mathbb{P}(u_{\rm L}\cdot\nabla u_{\rm L})(t)\|_{L^2}\frac{\|w(t)\|_{L^2}}{t^{\frac34}}\,\cdot$$

Using a convexity inequality, we infer that

$$\frac{d}{dt}\Big(\frac{\|w(t)\|_{L^2}^2}{t^{\frac12}}\Big)+\frac{\|w(t)\|_{L^2}^2}{2t^{\frac32}}+\frac{\|\nabla w(t)\|_{L^2}^2}{t^{\frac12}}\leq\frac{\|w(t)\|_{L^2}^2\|u_{\rm L}(t)\|_{L^\infty}^2}{t^{\frac12}}+t^{\frac12}\|u_{\rm L}(t)\cdot\nabla u_{\rm L}(t)\|_{L^2}^2\,.$$

Thus we deduce that

$$\begin{aligned}\frac{d}{dt}\Big(\frac{\|w(t)\|_{L^2}^2}{t^{\frac12}}\exp\Big(-\int_0^t\|u_{\rm L}(t')\|_{L^\infty}^2dt'\Big)\Big)+\exp\Big(-\int_0^t\|u_{\rm L}(t')\|_{L^\infty}^2dt'\Big)\Big(\frac{\|w(t)\|_{L^2}^2}{2t^{\frac32}}+\frac{\|\nabla w(t)\|_{L^2}^2}{t^{\frac12}}\Big)\\ \leq\exp\Big(-\int_0^t\|u_{\rm L}(t')\|_{L^\infty}^2dt'\Big)t^{\frac12}\|\mathbb{P}(u_{\rm L}\cdot\nabla u_{\rm L})(t)\|_{L^2}^2\,,\end{aligned}$$

from which we infer by the definition of the $\dot B^{-1}_{\infty,2}$ norm and of $Q^0_{\rm L}$ that

$$\forall t\geq 0\,,\quad \frac{\|w(t)\|_{L^2}^2}{t^{\frac12}}+\int_0^t\Big(\frac{\|w(t')\|_{L^2}^2}{2t'^{\frac32}}+\frac{\|\nabla w(t)\|_{L^2}^2}{t'^{\frac12}}\Big)dt'\leq Q^0_{\rm L}\,\exp\|u_0\|^2_{\dot B^{-1}_{\infty,2}}\,\cdot \tag{14}$$

Proposition 1.3 follows. □

4.2. **Proof of Proposition 4.1.** Now let us investigate the evolution of $\partial_3 w$ in L^2. Applying the partial differentiation ∂_3 to (NSF), we get

$$\begin{aligned}&\partial_t\partial_3 w-\Delta\partial_3 w+(u_{\rm L}+w)\cdot\nabla\partial_3 w+\partial_3 w\cdot\nabla u_{\rm L}\\ &\qquad=-\partial_3 u_{\rm L}\cdot\nabla w-\partial_3 w\cdot\nabla w-w\cdot\nabla\partial_3 u_{\rm L}-\partial_3(u_{\rm L}\cdot\nabla u_{\rm L})-\nabla\partial_3 p\,.\end{aligned} \tag{15}$$

The difficult terms to estimate are those which do not contain explicitly $\partial_3 w$. So let us define

$$\begin{aligned}(a)&\stackrel{\rm def}{=}-\big(\partial_3 u_{\rm L}\cdot\nabla w\big|\partial_3 w\big)_{L^2}\,,\\ (b)&\stackrel{\rm def}{=}-\big(w\cdot\nabla\partial_3 u_{\rm L}\big|\partial_3 w\big)_{L^2}\quad\text{and}\\ (c)&\stackrel{\rm def}{=}-\big(\partial_3(u_{\rm L}\cdot\nabla u_{\rm L})\big|\partial_3 w\big)_{L^2}\,.\end{aligned}$$

The third term is the easiest. By integration by parts and using the Cauchy-Schwarz inequality along with (14) we have

$$\begin{aligned}\Big|\int_0^\infty(c)(t)dt\Big|&=\Big|\int_0^\infty\int_{\mathbb{R}^3}\partial_3^2\big(\mathbb{P}(u_{\rm L}\cdot\nabla u_{\rm L})(t,x)\big)\cdot w(t,x)dxdt\Big|\\ &\leq\Big(\int_0^\infty t^{\frac32}\big\|\partial_3^2\,\mathbb{P}(u_{\rm L}\cdot\nabla u_{\rm L})(t)\big\|_{L^2}^2dt\Big)^{\frac12}\Big(\int_0^\infty\frac{\|w(t)\|_{L^2}^2}{t^{\frac32}}dt\Big)^{\frac12}\\ &\leq\sqrt{Q^0_{\rm L}Q^1_{\rm L}}\,\exp\Big(\frac12\|u_0\|^2_{\dot B^{-1}_{\infty,2}}\Big)\,.\end{aligned}$$

Now let us estimate the contribution of (a) and (b). By integration by parts, we get, thanks to the divergence free condition on $u_{\rm L}$,

$$(a)=\big(\partial_3 u_{\rm L}\otimes w\big|\nabla\partial_3 w\big)_{L^2}\quad\text{and}\quad(b)=\big(w\otimes\partial_3 u_{\rm L}\big|\nabla\partial_3 w\big)_{L^2}\,.$$

The two terms can be estimated exactly in the same way since they are both of the form

$$\int_{\mathbb{R}^3}w(t,x)\partial_3 u_{\rm L}(t,x)\nabla\partial_3 w(t,x)dx\,.$$

We have

$$\begin{aligned}\Big|\int_{\mathbb{R}^3} w(t,x)\partial_3 u_{\rm L}(t,x)\nabla\partial_3 w(t,x)dx\Big| &\leq \|w(t)\|_{L^2}\|\partial_3 u_{\rm L}(t)\|_{L^\infty}\|\nabla\partial_3 w\|_{L^2}\\ &\leq \frac{1}{100}\|\nabla\partial_3 w\|_{L^2}^2+100\|w(t)\|_{L^2}^2\|\partial_3 u_{\rm L}(t)\|_{L^\infty}^2\,.\end{aligned}$$

The first term will be absorbed by the Laplacian. The second term can be understood as a source term. By time integration, we get indeed

$$\begin{aligned}\int_0^T \|w(t)\|_{L^2}^2\|\partial_3 u_{\rm L}(t)\|_{L^\infty}^2 dt &\leq \int_0^T \frac{\|w(t)\|_{L^2}^2}{t^{\frac32}}\big(t^{\frac34}\|\partial_3 u_{\rm L}(t)\|_{L^\infty}\big)^2 dt\\ &\leq \|\partial_3 u_0\|^2_{\dot B^{-\frac32}_{\infty,\infty}}\int_0^\infty \frac{\|w(t)\|_{L^2}^2}{t^{\frac32}}dt\,,\end{aligned}$$

so it follows, thanks to Proposition 1.3, that

$$\begin{aligned}\int_0^T\int_{\mathbb{R}^3} w(t,x)\partial_3 u_{\rm L}(t,x)\nabla\partial_3 w(t,x)dxdt \leq \frac{1}{100}\int_0^T \|\nabla\partial_3 w(t)\|_{L^2}^2\,dt\\ + C\|\partial_3 u_0\|^2_{\dot B^{-\frac32}_{\infty,\infty}}\,Q^0_{\rm L}\,\exp\|u_0\|^2_{\dot B^{-1}_{\infty,2}}\,.\end{aligned}$$

The contribution of the quadratic term in (15) is estimated as follows: writing, for any function a,

$$\|a\|_{L^p_{\rm h}L^q_{\rm v}}\stackrel{\rm def}{=}\Big(\int\|a(x_1,x_2,\cdot)\|^p_{L^q(\mathbb{R})}\,dx_1dx_2\Big)^{\frac1p}\,,$$

we have by Hölder's inequality

$$\begin{aligned}\Big|\int_{\mathbb{R}^3}\partial_3 w(t,x)\cdot\nabla w(t,x)\partial_3 w(t,x)dx\Big| &\leq \|\partial_3 w(t)\|^2_{L^2_{\rm v}L^4_{\rm h}}\|\nabla w\|_{L^\infty_{\rm v}L^2_{\rm h}}\\ &\leq \|\partial_3 w(t)\|_{L^2}\|\nabla_{\rm h}\partial_3 w(t)\|_{L^2}\|\nabla w(t)\|^{\frac12}_{L^2}\|\nabla\partial_3 w(t)\|^{\frac12}_{L^2}\,,\end{aligned}$$

where we have used the inequalities

$$\|a\|_{L^\infty_{\rm v}L^2_{\rm h}}\lesssim\|\partial_3 a\|^{\frac12}_{L^2}\|a\|^{\frac12}_{L^2}\quad\text{and}\quad\|a\|_{L^2_{\rm v}L^4_{\rm h}}\lesssim\|a\|^{\frac12}_{L^2}\|\nabla_{\rm h}a\|^{\frac12}_{L^2}\tag{16}$$

with $\nabla_{\rm h}\stackrel{\rm def}{=}(\partial_1,\partial_2)$. The first inequality comes from

$$\begin{aligned}\|a(\cdot,x_3)\|^2_{L^2_{\rm h}} &= \frac12\int_{-\infty}^{x_3}\big(\partial_3 a(\cdot,z)\big|a(\cdot,z)\big)_{L^2_{\rm h}}dz\\ &\leq \frac12\int_{\mathbb{R}}\|\partial_3 a(\cdot,z)\|_{L^2_{\rm h}}\|a(\cdot,z)\|_{L^2_{\rm h}}dz\\ &\leq \|\partial_3 a\|_{L^2}\|a\|_{L^2}\end{aligned}$$

while the second simply comes from the embedding $\dot H^{\frac12}_{\rm h}\subset L^4_{\rm h}$ and an interpolation. By Young's inequality it follows that

$$\begin{aligned}\Big|\int_{\mathbb{R}^3}\partial_3 w(t,x)\cdot\nabla w(t,x)\partial_3 w(t,x)dx\Big| &\leq \frac{1}{100}\|\nabla\partial_3 w\|^2_{L^2}+C\|\nabla w(t)\|^2_{L^2}\|\partial_3 w(t)\|^4_{L^2}\\ &\leq \frac{1}{100}\|\nabla\partial_3 w(t)\|^2_{L^2}\\ &\quad+\Big(\sup_{t'\in[0,t]}\|\partial_3 w(t')\|^4_{L^2}\Big)t^{\frac12}\frac{\|\nabla w(t)\|^2_{L^2}}{t^{\frac12}}\,,\end{aligned}$$

from which we infer by Proposition 1.3 that

$$\Big|\int_{\mathbb{R}^3}\partial_3 w(t,x)\cdot\nabla w(t,x)\partial_3 w(t,x)dx\Big|\leq\frac{1}{100}\|\nabla\partial_3 w(t)\|_{L^2}^2 +\Big(\sup_{t'\in[0,t]}\|\partial_3 w(t')\|_{L^2}^4\Big)t^{\frac12}Q^0_{\rm L}\,\exp\|u_0\|^2_{\dot B^{-1}_{\infty,2}}\,.$$

Finally there holds after an integration by parts

$$\begin{aligned}\int_{\mathbb{R}^3}\partial_3 w(t,x)\cdot\nabla u_{\rm L}(t,x)\partial_3 w(t,x)dx &\leq \|\partial_3 w(t)\|_{L^2}\|u_{\rm L}(t)\|_{L^\infty}\|\nabla\partial_3 w(t)\|_{L^2}\\ &\leq \frac{1}{100}\|\nabla\partial_3 w(t)\|_{L^2}^2+C\|\partial_3 w(t)\|_{L^2}^2\|u_{\rm L}(t)\|_{L^\infty}^2\,,\end{aligned}$$

so plugging all these estimates together we infer thanks to Gronwall's inequality that

$$\begin{aligned}&\sup_{t\in[0,T]}\|\partial_3 w(t)\|_{L^2}^2+\int_0^T\|\nabla\partial_3 w(t)\|_{L^2}^2\,dt\\ &\qquad\lesssim\Big(T^{\frac12}Q^0_{\rm L}\sup_{t'\in[0,t]}\|\partial_3 w(t')\|_{L^2}^4+\|\partial_3 u_0\|^2_{\dot B^{-\frac32}_{\infty,\infty}}Q^0_{\rm L}+\sqrt{Q^0_{\rm L}Q^1_{\rm L}}\Big)\exp\Big(2\|u_0\|^2_{\dot B^{-1}_{\infty,2}}\Big)\,.\end{aligned}$$

Proposition 4.1 is proved. □

4.3. **End of the proof of Theorem 1.3.**

4.3.1. *Control of the fluctuation.* To make notation lighter let us set

$$M_{\rm L}\stackrel{\rm def}{=}\Big(\|\partial_3 u_0\|^2_{\dot B^{-\frac32}_{\infty,\infty}}Q^0_{\rm L}+\sqrt{Q^0_{\rm L}Q^1_{\rm L}}\Big)\exp\big(2\|u_0\|^2_{\dot B^{-1}_{\infty,2}}\big)\,.$$

Proposition 4.1 provides the existence of a constant K such that the following a priori estimate holds

$$\begin{aligned}&\sup_{t\in[0,T]}\|\partial_3 w(t)\|_{L^2}^2+\int_0^T\|\nabla\partial_3 w(t)\|_{L^2}^2\,dt\\ &\qquad\leq KT^{\frac12}Q^0_{\rm L}\sup_{t\in[0,T]}\|\partial_3 w(t)\|_{L^2}^4\exp\big(2\|u_0\|^2_{\dot B^{-1}_{\infty,2}}\big)+KM_{\rm L}\,.\end{aligned}$$

Let T^* be the maximal time of existence of u, hence of w, and recalling that $w(t=0)=0$, set T_1 to be the maximal time T for which

$$\sup_{t\in[0,T]}\|\partial_3 w(t)\|_{L^2}^2\leq 2KM_{\rm L}\,.$$

Then on $[0,T_1]$ there holds

$$\begin{aligned}\sup_{t\in[0,T]}\|\partial_3 w(t)\|_{L^2}^2+\int_0^T\|\nabla\partial_3 w(t)\|_{L^2}^2\,dt &\leq 4K^3T_1^{\frac12}Q^0_{\rm L}M_{\rm L}^2+KM_{\rm L}\\ &\leq KM_L\big(1+4K^2T_1^{\frac12}Q^0_{\rm L}M_{\rm L}\big)\,.\end{aligned}$$

This implies that

$$T_1\geq T_*\quad\text{with}\quad T_*\stackrel{\rm def}{=}\Big(\frac{1}{8K^2Q^0_{\rm L}M_{\rm L}}\Big)^2\,,$$

and on $[0,T_*]$ there holds

$$\sup_{t\in[0,T]}\|\partial_3 w(t)\|_{L^2}^2+\int_0^T\|\nabla\partial_3 w(t)\|_{L^2}^2\,dt\leq\frac32 KM_{\rm L}\,. \tag{17}$$

4.3.2. *End of the proof of the theorem.* Under the assumptions of Theorem 1.3 we know that there exists a unique solution u to (NS) on some time interval $[0,T^*)$, which satisfies the energy estimate. Let us prove that this time interval contains $[0,T_*]$. Since the initial data u_0 belongs to L^2, we may assume that u is a global Leray solution, meaning that

$$\forall t\geq 0\,,\quad \mathcal{E}\big(u(t)\big)\leq \frac{1}{2}\|u_0\|_{L^2}^2\,. \tag{18}$$

Moreover one clearly has

$$\sup_{t\geq 0}\|\partial_3 u_{\rm L}(t)\|_{L^2}^2+\int_0^\infty \|\nabla\partial_3 u_{\rm L}(t)\|_{L^2}^2\,dt\leq \|\partial_3 u_0\|_{L^2}^2$$

so together with (17) this implies that on $[0,T_*]$,

$$\sup_{t\in[0,T]}\|\partial_3 w(t)\|_{L^2}^2+\int_0^T \|\nabla\partial_3 u(t)\|_{L^2}^2 dt\lesssim \|\partial_3 u_0\|_{L^2}^2+M_{\rm L}\,. \tag{19}$$

Let us prove that these estimates provide a control on u in $\dot H^1$ on $[0,T_*]$. After differentiation of (NS) with respect to the horizontal variables and an energy estimate, we get for any ℓ in $\{1,2\}$ and after an integration by parts

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\partial_\ell u(t)\|_{L^2}^2+\|\nabla\partial_\ell u(t)\|_{L^2}^2 &= -\int_{\mathbb{R}^3}\partial_\ell(u\cdot\nabla u)\cdot\partial_\ell u\,(t,x)\,dx\\ &\leq \|u\|_{L^\infty_{\rm v}L^4_{\rm h}}\|\nabla u(t)\|_{L^2_{\rm v}L^4_{\rm h}}\|\partial_\ell^2 u(t)\|_{L^2}\,.\end{aligned}$$

Similarly to (16) we have

$$\begin{aligned}\|u\|_{L^\infty_{\rm v}L^4_{\rm h}}^2 &\lesssim \|u\|^2_{L^\infty_{\rm v}\dot H^{\frac12}_{\rm h}}\\ &\lesssim \int_{-\infty}^{x_3}\big(\partial_3 u(\cdot,z)\big|u(\cdot,z)\big)_{\dot H^{\frac12}_{\rm h}}dz\\ &\lesssim \|\partial_3 u\|_{L^2}\|\nabla_{\rm h}u\|_{L^2}\end{aligned}$$

so using (16) we infer that

$$\begin{aligned}\Big|\int_{\mathbb{R}^3}\partial_\ell(u\cdot\nabla u)\cdot\partial_\ell u\,(t,x)\,dx\Big| &\leq C\|\partial_3 u(t)\|_{L^2}^{\frac12}\|\nabla_{\rm h}u(t)\|_{L^2}^{\frac12}\|\nabla u(t)\|_{L^2}^{\frac12}\|\nabla\nabla_{\rm h}u(t)\|_{L^2}^{\frac12}\|\partial_\ell^2 u(t)\|_{L^2}\\ &\leq \frac{1}{100}\|\nabla\nabla_{\rm h}u(t)\|_{L^2}^2+C\|\partial_3 u\|_{L^2}^2\|\nabla_{\rm h}u\|_{L^2}^2\|\nabla u(t)\|_{L^2}^2\,.\end{aligned}$$

We obtain

$$\frac{d}{dt}\|\nabla_{\rm h}u(t)\|_{L^2}^2+\|\nabla\nabla_{\rm h}u(t)\|_{L^2}^2\lesssim \|\partial_3 u\|_{L^2}^2\|\nabla_{\rm h}u\|_{L^2}^2\|\nabla u(t)\|_{L^2}^2\,,$$

and Gronwall's inequality implies that

$$\|\nabla_{\rm h}u(t)\|_{L^2}^2+\int_0^t\|\nabla\nabla_{\rm h}u(t')\|_{L^2}^2dt'\leq \|\nabla_{\rm h}u_0\|_{L^2}^2\exp\Big(\int_0^t\|\partial_3 u(t')\|_{L^2}^2\|\nabla u(t')\|_{L^2}^2dt'\Big)\,.$$

The fact that we control $\|\nabla u\|_{L^2_t(L^2_x)}$ and $\|\partial_3 u\|_{L^\infty_t(L^2_x)}$ thanks to (18) and (19) implies that on $[0,T_*]$ there holds

$$\sup_{t\in[0,T]}\|\nabla u(t)\|_{L^2}^2+\int_0^T\|\nabla^2 u(t)\|_{L^2}^2\,dt\leq \|\nabla u_0\|_{L^2}^2\exp\Big(\|u_0\|_{L^2}(KM_{\rm L})^{\frac12}\Big)\,.$$

This means that there is a unique, smooth solution at least on $[0,T_*]$, and Theorem 1.3 is proved. □

5. Comparison of both life spans: proof of Theorem 1.4

Let us introduce the notation

$$f_\varepsilon(x_1,x_2,x_3) \stackrel{\text{def}}{=} \cos\Big(\frac{x_1}{\varepsilon}\Big) f\Big(x_1, \frac{x_2}{\varepsilon^\alpha}, x_3\Big)\,,$$

where ε is a given number, assumed to be small, and α is a fixed parameter in the open interval $]0,1[$. We assume the initial data is given by the following expression

$$u_{0,\varepsilon}(x) = \frac{A_\varepsilon}{\varepsilon}\big(0, \varepsilon^\alpha(-\partial_3\phi)_\varepsilon, (\partial_2\phi)_\varepsilon\big) \tag{20}$$

where ϕ is a smooth compactly supported function and the parameter $A_\varepsilon \gg 1$ will be tuned later.

Let us recall that Lemma 3.1 of [3] claims in particular that

$$\forall \sigma>0\,,\ \|f_\varepsilon\|_{\dot B^{-\sigma}_{p,1}} \le C_\sigma \varepsilon^{\sigma+\frac{\alpha}{p}} \quad\text{and}\quad \|f_\varepsilon\|_{\dot B^{-\sigma}_{\infty,\infty}} \ge c_\sigma\varepsilon^\sigma. \tag{21}$$

This implies that

$$\|u_{0,\varepsilon}\|_{\dot B^{-1+2\gamma}_{\infty,\infty}} \lesssim A_\varepsilon \varepsilon^{-2\gamma}\,,\ \|u_{0,\varepsilon}\|_{\dot B^{-1}_{\infty,\infty}} \sim \|u_{0,\varepsilon}\|_{\dot B^{-1}_{\infty,2}} \sim A_\varepsilon \quad\text{and}\quad \|\partial_3 u_{0,\varepsilon}\|_{\dot B^{-\frac{3}{2}}_{\infty,\infty}} \lesssim A_\varepsilon \varepsilon^{\frac12}\,. \tag{22}$$

With the notation of Theorem 1.1 there holds therefore.

$$T_{\rm FP}(u_{0,\varepsilon}) \ge C\varepsilon^2 A_\varepsilon^{-\frac{1}{\gamma}}\,.$$

Let us now compute $T_{\rm L}(u_{0,\varepsilon})$. Recalling that $u_{\rm L}(t) = e^{t\Delta}u_{0,\varepsilon}$, we can write

$$\begin{aligned} u_L^1\partial_1 u_L^1 + u_L^2\partial_2 u_L^1 &= \Big(\frac{A_\varepsilon}{\varepsilon}\Big)^2 e^{t\Delta} f_\varepsilon e^{t\Delta} g_\varepsilon \quad\text{and}\\ u_L^1\partial_1 u_L^2 + u_L^2\partial_2 u_L^2 &= \Big(\frac{A_\varepsilon}{\varepsilon}\Big)^2 e^{t\Delta} \widetilde f_\varepsilon e^{t\Delta} \widetilde g_\varepsilon. \end{aligned}$$

where f, g, $\widetilde f$, $\widetilde g$ are smooth compactly supported functions. Now let us estimate

$$\int_0^\infty t^{\frac12}\big\|e^{t\Delta} f_\varepsilon\, e^{t\Delta} g_\varepsilon\big\|_{L^2}^2 dt\,.$$

for f and g given smooth compactly supported functions. We write

$$\begin{aligned} \int_0^\infty t^{\frac12}\big\|e^{t\Delta} f_\varepsilon\, e^{t\Delta} g_\varepsilon\big\|_{L^2}^2 dt &= \int_0^\infty t^{\frac32}\big\|e^{t\Delta} f_\varepsilon\, e^{t\Delta} g_\varepsilon\big\|_{L^2}^2 \frac{dt}{t}\\ &\le \int_0^\infty \big(t^{\frac38}\|e^{t\Delta} f_\varepsilon\|_{L^4}\big)^2 \big(t^{\frac38}\|e^{t\Delta} g_\varepsilon\|\big)_{L^4}^2 \frac{dt}{t} \end{aligned}$$

thanks to the Hölder inequality. The Cauchy-Schwarz inequality and the definition of Besov norms imply that

$$\begin{aligned} \int_0^\infty t^{\frac12}\big\|e^{t\Delta} f_\varepsilon e^{t\Delta} g_\varepsilon\big\|_{L^2}^2 dt &\le \Big(\int_0^\infty \big(t^{\frac38}\|e^{t\Delta} f_\varepsilon\|_{L^4}\big)^4 \frac{dt}{t}\Big)^{\frac12} \Big(\int_0^\infty \big(t^{\frac38}\|e^{t\Delta} g_\varepsilon\|_{L^4}\big)^4 \frac{dt}{t}\Big)^{\frac12}\\ &\le \|f_\varepsilon\|^2_{\dot B^{-\frac34}_{4,4}} \|g_\varepsilon\|^2_{\dot B^{-\frac34}_{4,4}}\,. \end{aligned}$$

It is easy to check that

$$\|f_\varepsilon\|_{\dot B^{-\frac34}_{4,4}} \lesssim \varepsilon^{\frac{3+\alpha}{4}}\,,$$

so it follows (since $\mathbb{P}$ is a homogeneous Fourier multiplier of order 0) that

$$Q^0_{\rm L} \lesssim A_\varepsilon^4 \varepsilon^{\alpha-1}\,. \tag{23}$$

For the initial data (20), differentations with respect to the vertical variable ∂_3 have no real influence on the term $u_{\rm L}(t)\cdot\nabla u_{\rm L}(t)$. Indeed, we have

$$\partial_3^2\big(u_{\rm L}(t)\cdot\nabla u_{\rm L}(t)\big)=\partial_3^2 u_{\rm L}(t)\cdot\nabla u_{\rm L}(t)+2\partial_3 u_{\rm L}(t)\cdot\partial_3\nabla u_{\rm L}(t)+u_{\rm L}(t)\cdot\partial_3^2\nabla u_{\rm L}(t)$$

and it is then obvious that $\partial_3^2\big(u_{\rm L}(t)\cdot\nabla u_{\rm L}(t)\big)$ is a sum of term of the type

$$\Big(\frac{A_\varepsilon}{\varepsilon}\Big)^2 e^{t\Delta}f_\varepsilon\, e^{t\Delta}g_\varepsilon\,.$$

Then following the lines used to estimate the term Q^0_L, we write

$$\begin{aligned}\int_0^\infty t^{\frac32}\big\|e^{t\Delta}f_\varepsilon e^{t\Delta}g_\varepsilon\big\|^2_{L^2}dt &\leq \Big(\int_0^\infty \big(t^{\frac58}\|e^{t\Delta}f_\varepsilon\|_{L^4}\big)^4\frac{dt}{t}\Big)^{\frac12}\Big(\int_0^\infty \big(t^{\frac58}\|e^{t\Delta}g_\varepsilon\|_{L^4}\big)^4\frac{dt}{t}\Big)^{\frac12}\\ &\leq \|f_\varepsilon\|^2_{\dot B^{-\frac54}_{4,4}}\|g_\varepsilon\|^2_{\dot B^{-\frac54}_{4,4}}\,.\end{aligned}$$

It is easy to check that

$$\|f_\varepsilon\|_{\dot B^{-\frac54}_{4,4}}\lesssim \varepsilon^{\frac{5+\alpha}{4}}\,,$$

so it follows that

$$Q^1_{\rm L}\lesssim A^4_\varepsilon\varepsilon^{\alpha+1}\,.$$

Together with (22) and (23), we infer that

$$\begin{aligned}Q^0_{\rm L}\Big(\|\partial_3 u_0\|^2_{\dot B^{-\frac32}_{\infty,\infty}}Q^0_{\rm L}+\sqrt{Q^0_{\rm L}Q^1_{\rm L}}\Big)\exp\big(4\|u_0\|^2_{\dot B^{-1}_{\infty,2}}\big) &\lesssim A^4_\varepsilon\varepsilon^{\alpha-1}\big(A^6_\varepsilon\varepsilon^\alpha+A^4_\varepsilon\varepsilon^\alpha\big)\exp(C_0A^2_\varepsilon)\\ &\lesssim A^{10}_\varepsilon\varepsilon^{2\alpha-1}\exp(C_0A^2_\varepsilon)\end{aligned}$$

because A_ε is larger than 1. Let us choose some κ in $]0,\eta[$ and then

$$A_\varepsilon\stackrel{\rm def}{=}\Big(\frac{C_0}{-\kappa\log\varepsilon}\Big)^{\frac12}.$$

Then with the notation of Theorem 1.3 we have

$$T_{\rm L}=CA_\varepsilon^{-20}\varepsilon^{2(1-2\alpha+\kappa)}\,.$$

Let us choose κ' in $]\kappa,\eta[$. By definition of A_ε we get that

$$T_L\geq C\varepsilon^{2(1-2\alpha+\kappa')}$$

Choosing $\alpha=1-\dfrac{\eta-\kappa'}{4}$ concludes the proof of Theorem 1.4. □

Appendix A. A Littlewood-Paley toolbox

Let us recall some well-known results on Littlewood-Paley theory (see for instance [1] for more details).

Definition A.1. *Let $\phi\in\mathcal S(\mathbb R^3)$ be such that $\widehat\phi(\xi)=1$ for $|\xi|\leq1$ and $\widehat\phi(\xi)=0$ for $|\xi|>2$. We define, for $j\in\mathbb Z$, the function $\phi_j(x)\stackrel{\rm def}{=}2^{3j}\phi(2^jx)$, and the Littlewood–Paley operators*

$$S_j\stackrel{\rm def}{=}\phi_j*\cdot\quad\text{and}\quad\Delta_j\stackrel{\rm def}{=}S_{j+1}-S_j\,.$$

Homogeneous Sobolev spaces are defined by the norm

$$\|a\|_{\dot H^s}\stackrel{\rm def}{=}\Big(\sum_{j\in\mathbb Z}2^{2js}\|\Delta_ja\|^2_{L^2}\Big)^{\frac12}\,.$$

This norm is equivalent to

$$\|a\|_{\dot H^s}\sim\Big(\int_{\mathbb R^3}|\xi|^{2s}|\mathcal Fa(\xi)|^2\,d\xi\Big)^{\frac12}\,,$$

where $\mathcal{F}$ is the Fourier transform. Finally let us recall the definition of Besov norms of negative index.

Definition A.2. *Let σ be a positive real number and (p,q) in $[1,\infty]^2$. Let us define the homogeneous Besov norm $\|\cdot\|_{\dot B^{-\sigma}_{p,q}}$ by*

$$\|a\|_{\dot B^{-\sigma}_{p,q}} = \big\|t^{\frac{\sigma}{2}}\|e^{t\Delta}a\|_{L^p}\big\|_{L^q\left(\mathbb{R}^+;\frac{dt}{t}\right)}\,.$$

Let us mention that thanks to the properties of the heat flow, for $p_1 \leq p_2$ and $q_1 \leq q_2$, we have the following inequality, valid for any regular function a

$$\|a\|_{\dot B^{-\sigma-3\left(\frac{1}{p_1}-\frac{1}{p_2}\right)}_{p_2,q}} \lesssim \|a\|_{\dot B^{-\sigma}_{p_1,q}} \quad \text{and} \quad \|a\|_{\dot B^{-\sigma}_{p,q_2}} \lesssim \|a\|_{\dot B^{-\sigma}_{p,q_1}}\,.$$

An equivalent definition using the Littlewood-Paley decomposition is

$$\|a\|_{\dot B^{-\sigma}_{p,q}} \sim \Big(\sum_{j\in\mathbb{Z}} 2^{-j\sigma q}\|\Delta_j a\|^q_{L^p}\Big)^{\frac{1}{q}}\,.$$

References

[1] H. Bahouri, J.-Y. Chemin and R. Danchin, *Fourier Analysis and Nonlinear Partial Differential Equations*, Grundlehren der mathematischen Wissenschaften, **343**, Springer-Verlag Berlin Heidelberg, 2011.

[2] J.-Y. Chemin, Théorèmes d'unicité pour le système de Navier-Stokes tridimensionnel. *Journal d'Analyse Mathématique*, **77**, 1999, pages 27–50.

[3] J.-Y. Chemin and I. Gallagher, Wellposedness and stability results for the Navier-Stokes equations in $\mathbb{R}^3$, *Annales de l'Institut Henri Poincaré, Analyse Non Linéaire*, **26**, 2009, pages 599–624.

[4] J.-Y. Chemin and N. Lerner, Flot de champs de vecteurs non-lipschitziens et équations de Navier-Stokes, *Journal of Differential Equations*, **121**, 1995, pages 314–328.

[5] J.-Y. Chemin and F. Planchon, Self-improving bounds for the Navier-Stokes equations, *Bulletin de la Société Mathématique de France*, **140**, 2012, pages 583–597.

[6] H. Fujita and T. Kato: On the Navier–Stokes initial value problem I, *Archive for Rational Mechanics and Analysis*, **16** (1964), 269–315.

[7] H. Koch and D. Tataru: Well-posedness for the Navier-Stokes equations, *Advances in Mathematics*, **157**, 2001, pages 22–35.

[8] J. Leray: Essai sur le mouvement d'un liquide visqueux emplissant l'espace, *Acta Mathematica*, **63**, 1933, pages 193–248.

[9] Y. Meyer, *Wavelets, paraproducts and Navier-Stokes equations*, Current developments in mathematics, International Press, Boston, MA, 1997.

[10] E. Poulon: About the possibility of minimal blow up for Navier-Stokes solutions with data in $H^s(\mathbb{R}^3)$, arXiv:1505.06197

[11] T. Tao: Finite time blowup for an averaged three-dimensional Navier-Stokes equation, *Journal of the American Mathematical Society*, **29**, 2016, pages 601–674.

(J.-Y. Chemin) Laboratoire Jacques Louis Lions - UMR 7598, Université Pierre et Marie Curie, Boîte courrier 187, 4 place Jussieu, 75252 Paris Cedex 05, France

E-mail address: chemin@ann.jussieu.fr

(I. Gallagher) DMA, École normale suprieure, CNRS, PSL Research University, 75005 Paris, and UFR de mathématiques, Université Paris-Diderot, Sorbonne Paris-Cité, 75013 Paris, France.

E-mail address: gallagher@math.ens.fr